TESTING $I(1)$ AGAINST $I(d)$ ALTERNATIVES IN THE PRESENCE OF DETERMINISTIC COMPONENTS

BY JUAN J. DOLADO[a], JESUS GONZALO[a], AND LAURA MAYORAL[b] *

[a]Dept. of Economics, Universidad Carlos III de Madrid.

[b]Dept. of Economics, Universitat Pompeu Fabra.

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Abstract

This paper discusses the role of deterministic components in the DGP and in the auxiliary regression model which underlies the implementation of the Fractional Dickey-Fuller (FDF) test for $I(1)$ against $I(d)$ processes with $d \in [0, 1)$. This is an important test in many economic applications because $I(d)$ processess with $d < 1$ are mean-reverting although, when $0.5 \leq d < 1$, like $I(1)$ processes, they are nonstationary. We show how simple is the implementation of the FDF in these situations, and argue that it has better properties than LM tests. A simple testing strategy entailing only asymptotically normally-distributed tests is also proposed. Finally, an empirical application is provided where the FDF test allowing for deterministic components is used to test for long-memory in the per capita GDP of several OECD countries, an issue that has important consequences to discriminate between growth theories, and on which there is some controversy.

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1. INTRODUCTION

The goal of this paper is twofold. First, we extend an existing statistical procedure for detecting a unit root against mean-reverting fractional alternatives in time series free of deterministic components to the case where they may exhibit a trending behavior or have a non-zero mean. Second, we compare the behavior of this test to that of other tests available in the literature. In particular, we focus on the Fractional Dickey-Fuller (FDF, henceforth) test proposed by Dolado, Gonzalo and Mayoral (2002, DGM hereafter) who have generalized the traditional DF test of $I(1)$ against $I(0)$ processes without deterministic components to the broader framework of testing $I(1)$ against $I(d)$ with $d \in [0, 0.5) \cup (0.5, 1)$.[1] Relying upon the DF approach, the underlying idea is to test for the statistical significance of the coefficient ϕ in the potentially unbalanced regression $\Delta y_t = \phi\ \Delta^d y_{t-1} + \varepsilon_t$, where ε_t is an $i.i.d.$ disturbance, L is the lag operator and $\Delta = (1 - L)$. The regressor $\Delta^d y_{t-1}$ is constructed by applying the truncated binomial expansion of the filter $(1 - L)^d$ to y_{t-1}, so that $\Delta^d y_t = \sum_{i=0}^{t-1} \pi_i(d)\ y_{t-i}$ where $\pi_i(d)$ is the i-th coefficient in that expansion.

The FDF test is based upon the t-ratio of $\hat{\phi}_{ols}$, $t_\phi(d)$, so that non-rejection of H_0: $\phi = 0$ against H_1: $\phi < 0$, implies that the process is $I(1)$, namely, $\Delta y_t = \varepsilon_t$. Conversely, rejection of the null implies that the process is $I(d)$, $0 \leq d < 1$, namely, $\Delta^d y_t = C(L)\varepsilon_t$, where the lag polynomial $C(L)$ has all its roots outside the unit circle. The distribution of $t_\phi(d)$ depends on whether d is assumed (arbitrarily) pre-fixed (if a simple alternative is considered) or estimated (when considering a composite alternative), and the distance $1 - d$. When d is pre-fixed as in the standard DF case (where $d = 0$), the asymptotic distribution of the $t_\phi(d)$ is a $N(0, 1)$ variate when $0.5 < d < 1$, whilst it is nonstandard, i.e., a functional of Fractional Brownian motion (fBM), when $0 \leq d < 0.5$.[2] In particular, for $d = 0$, $t_\phi(d)$ follows the well-known DF distribution, otherwise the critical values become less negative than the standard DF case as $d \uparrow 0.5$. By contrast, whenever d is pre-estimated using any (trimmed) $T^{1/2}$-consistent

[1] Although the case where $d = 0.5$ was treated in DGM, it constitutes a discontinuity point in the analysis of fractionally integrated processes, splitting the class of $I(d)$ processes into stationary (for $d < 0.5$) and nonstationarity (for $d \geq 0.5$). Moreover the behaviour of $\{y_t\}$ differs between $d = 0.5$ and $d > 0.5$;cf. Liu (1998). For this reason, as is often the case in most of the literature, we ignore this possibility. To simplify the notation in the sequel, however, we will refer to the permissable range of d under the alternative as $0 \leq d < 1$.

[2] The intuition for these results is that whenever the values of d under the null and the alternative hypothesis are close (i.e., when d belongs to the nonstationary range or when d is estimated using a trimmed $T^{1/2}$-consistent estimator) asymptotic normality follows under the null hypothesis, whereas when they distant (i.e., when d belongs to the stationary range) the limiting distributions are nonstandard.

estimator[3], $\widehat{d}$, of $d \in [0,1)$, the asymptotic distribution of $t_\phi\ (\widehat{d})$ becomes pivotal and is always $N(0,1)$ for any value of d within the pre-specified range.

The advantages of this test, in parallel with the DF approach, rely on its simplicity and good performance in finite samples, both in terms of size and power. Specifically, when compared to other well-known tests for long memory, like the Lagrange Multiplier (LM) test developed by Robinson (1994) in the frequency domain and its time domain version by Tanaka (1999), the FDF test presents the advantage of not requiring the correct specification of a parametric model. For this reason, although the FDF test is not the asymptotically uniformly most powerful invariant (UMPI) test (see Tanaka, 1999) under a sequence of local alternatives approaching the null at the $T^{-1/2}$ rate in a parametric model with gaussian errors, it fares very well in terms of power relative to both parametric and semiparametric tests in the frequency and time domains, and even better than the UIMP test when errors are non-gaussian, as discussed at length in DGM.[4]

Following the development of unit root tests in the past, where the canonical zero-mean AR(1) model was subsequently augmented with deterministic components (including drifts, and linear, nonlinear and broken trends), our goal in this paper is to investigate how the limiting distribution of the FDF test changes when some deterministic components are considered in the DGP and in the maintained hypothesis. In particular, we will restrict our analysis in this paper to the role of a *drift* and/or a *linear trend* since many (macro) economic time series exhibit this type of trending behavior in their levels. However, we will briefly discuss how to extend the testing procedure to more general cases.

In the $I(1)$ vs. $I(0)$ framework, a constant and a linear time trend are typically included in the auxiliary regression model in such cases so that, if a unit root exists, the constant term

[3] A trimming such as the one proposed in DGM (2002, formula (33)) may be necessary in small samples to avoid estimates of d above 1. Also note that Lobato and Velasco (2003) have addressed the issue of optimality of the FDF test where the DGP is a pure $I(d)$, $0 \leq d < 1$ process with no deterministic components and found that $T^{1/2}$-consistency in the estimation of d can be relaxed to $T^{1/4}\log(T)$-consistency. Since this condition holds for many semiparametric estimators with an appropriate choice of the bandwidth parameter (see Velasco, 1999) the range of estimators that can be used to implement the FDF test is much larger. However, investigating how this generalization extends to the presence of deterministic components exceeds the scope of this paper. Thus, in the sequel we will restrict our results to $T^{1/2}$- consistency although we conjecture that, under weaker conditions, their results may still hold.

[4] As shown in DGM (2002), the proposed test has also better power properties than those based on a direct estimation of d in semiparametric or parametric models since the former often yield large confidence intervals whilst the precision of the latter hinges on the correct specification of the model.

becomes a trend under the null hypothesis. As DF (1981) showed, including the linear time trend in the maintained model allows one to achieve an invariant test to the presence of a drift in the true data generation process. When dealing with $I(d)$ processes, the standard approach in the literature to account for deterministic components (henceforth denoted by $\mu(t)$) is to consider the additive model $y_t = \mu(t) + I(d)$, so that $E[\Delta^d(y_t - \mu(t))] = 0$ (see Robinson, 1994 and Tanaka, 1999). In this setup, our *first contribution* in this paper is to derive the corresponding (numerically) invariant FDF test of the null $d = 1$ against the alternative $0 \leq d < 1$ when $\mu(t) = \alpha + \beta t$. As will be shown below, invariance of the FDF test to the values of α and β is achieved by including the nonlinear trend $\Delta^d \mu(t)$ in the maintained hypothesis where such a variable is constructed in the same way as the regressor $\Delta^d y_{t-1}$. As when $\mu(t) = 0$, pre-fixed values of d imply that the asymptotic distributions of the invariant FDF test differ according to whether $0 \leq d < 0.5$ or $0.5 < d < 1$ whereas they are always $N(0,1)$ when d is estimated using a (trimmed) $T^{1/2}$-consistent estimator. As a by-product of this analysis, using similar arguments to those in DGM, our *second contribution* is to provide new theoretical results and Monte-Carlo evidence showing that the power of the FDF test in finite samples compares very well with the power of the LM test except when both d is extremely close to unity.

Lastly, we wish to stress that, despite focusing on the case where the error term in the DGP is *i.i.d*, the asymptotic results obtained here remain valid when the disturbance is allowed to be autocorrelated, as it happens in the (augmented) DF case (ADF henceforth). In this respect, DGM (Theorems 6 and 7) have proved that, in order to remove the correlation, it is sufficient to augment the set of regressors in the auxiliary regression described above with k lags of the dependent variable such that $k \uparrow \infty$ as $T \uparrow \infty$, and $k^3/T \uparrow 0$, as in Said and Dickey (1984). As discussed below, this procedure turns out to be much simpler than accounting for serial correlation in the LM test. Moreover, as in the zero-mean case, we will show that the FDF test is more powerful in most cases, without being subject to large size distortions. An empirical application dealing with testing the possibility that long GNP *per capita* series for several OECD countries may follow mean-reverting $I(d)$ processes serves to illustrate our proposed methodology.

The rest of the paper is structured as follows. Sections 2 analyzes the derivation of invariant FDF tests when the null hypothesis is a random walk with or without drift. Section 3 focuses on its comparison with the LM tests discussed above. Section 4 discusses an empirical application of the previous tests. Finally, Section 5 draws some concluding remarks.

Proofs of theorems and lemmae are collected in Appendix 1 while sets of non-standard critical values for the FDF test with pre-fixed $d \in [0, 0.5)$ appear in Appendix 2.

In the sequel, the definition of a $I(d)$ process that we will adopt is that of an (asymptotically) stationary process when $d < 0.5$, and of a non-stationary (truncated) process when $d > 0.5$. Those definitions are similar to those used in, e.g., Robinson (1994) and Tanaka (1999) and are summarized in Appendix A of DGM . Moreover, the following conventional notation is adopted throughout the paper: $\Gamma(.)$ denotes the gamma function, $\{\pi_i(d)\}$ represents the sequence of coefficients associated to the expansion of Δ^d in powers of L and are defined as

$$\pi_i(d) = \frac{\Gamma(i-d)}{\Gamma(-d)\Gamma(i+1)}.$$

The indicator function is denoted by $1_{(.)}$ and I_n is the identity matrix of order n; $W_d(.)$ and $B(.)$ represent standard Type II-fBM corresponding to the limit distributions of the standardized partial sums of asymptotically stationary (truncated) $I(d)$ processes as defined in Marinucci and Robinson (1999) and standard BM, respectively. Finally, $\stackrel{w}{\rightarrow}$ and $\stackrel{p}{\rightarrow}$ denote weak convergence and convergence in probability, respectively.

2. DEFINITION OF THE INVARIANT FDF TEST

2.1 The i.i.d. case

Employing the methodology in DGM we assume, like in Robinson (1994), that the process y_t is generated as the sum of a deterministic component, $\mu(t)$, and an $I(d)$ component, u_t, so that

$$y_t = \mu(t) + u_t, \tag{1}$$

where

$$u_t = \frac{\varepsilon_t 1_{(t>0)}}{\Delta - \phi \Delta^d L}. \tag{2}$$

For simplicity, ε_t is assumed to be an *i.i.d.* error term.[5] Our interest is in $H_0 : \phi = 0$ (y_t is $I(1)$) vs. $H_1 : \phi < 0$ (y_t is $I(d)$). The null and alternative hypotheses can be rewritten as

[5] This assumption will be later relaxed in subsection 2.2.

$$H_0 : \Delta[y_t - \mu(t)] = \varepsilon_t, \tag{3}$$

against

$$H_1 : \left(\Delta - \phi\Delta^d L\right)[y_t - \mu(t)] = \varepsilon_t, \tag{4}$$

In DGM it is shown that $\left(\Delta - \phi\Delta^d L\right) = \Pi(L)\Delta^d$, where $\Pi(L) = (\Delta^{1-d} - \phi L)$ has all its roots outside the unit circle if $-2\phi^{1-d} < 0$, and verifies $\Pi(0) = 1$ and $\Pi(1) = -\phi$. Thus, under H_0, y_t is I(1) whereas, under H_1, denoting $C(L) = \Pi(L)^{-1}$, with $C(0) = 1$ and $C(1) = -1/\phi$, y_t becomes $I(d)$ and follows the process $\Delta^d[y_t - \mu(t)] = C(L)\varepsilon_t$, where $C(L)$ has its roots outside the unit circle. As is standard in the context of unit root tests, the asymptotic distributions of the proposed statistics will depend on the nature of the deterministic components included in $\mu(t)$. In the sequel, we will restrict our attention to the most popular cases treated in the literature, namely, when $\mu(t)$ is a linear time trend, $\mu(t) = \alpha + \beta t$ or, alternatively, is just a constant term, $\mu(t) = \alpha$. Hence, in the more general case, equation (1) becomes $(y_t - \alpha - \beta t) = u_t$. Premultiplying this expression by the polynomial $(\Delta - \phi\Delta^d L)$ we get the following auxiliary regression model (denoted hereafter as RM) as the maintained hypothesis

$$RM\ 1\text{: } \Delta y_t = \beta - \phi\alpha\tau_{t-1}(d) - \phi\beta\tau_{t-1}(d-1) + \phi\Delta^d y_{t-1} + \varepsilon_t, \tag{5}$$

with

$$\tau_{t-1}(\varrho) = \sum_{i=0}^{t-1} \pi_i(\varrho),$$

where the coefficients $\pi_i(\varrho)$ belong to the binomial expansion of $(1-L)^\varrho$ in powers of L. Note that $\Delta^d t = \Delta^d \Delta^{-1} 1_{\{t>0\}}$ so that, in line with the notation used above, such a trend is labelled as $\tau_{t-1}(d-1)$ in the sequel. Both nonlinear time trends capture the trending behavior of the series under the alternative. Notice that the DF case when $d = 0$ is embedded in this setup since $\tau_{t-1}(0) = 1$ and $\tau_{t-1}(-1) = t - 1$, giving rise to a constant and a linear time trend in the maintained hypothesis. As for the intermediate cases, Figure 1 plots a range of the time trends $\tau_{t-1}(d-1)$ generated with different values of $d \in [0, 1)$. As d becomes larger the trend becomes more concave and its slope becomes flatter.

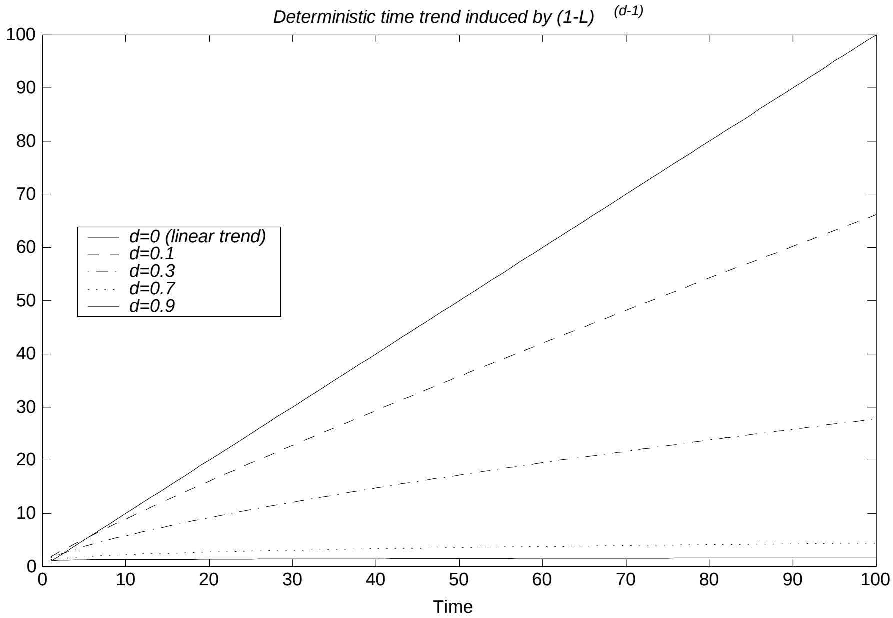

Figure 1

The test of $H_0 : I(1)$ relies upon $\phi = 0$ in model (5). Thus, when H_0 is true, the process becomes

$$DGP\ 1 : \Delta y_t = \beta + \varepsilon_t,\ t \geq 1, \tag{6}$$

whereas, under H_1, it is an $I(d)$ process with a linear time trend like in (4).

If the presence of the linear trend in the level of the series is discarded from the outset (as e.g. when modelling interest or exchange rates) then $\alpha \neq 0$ and $\beta = 0$ in (1), giving rise to

$$DGP\ 2 : \Delta y_t = \varepsilon_t,\ t \geq 1, \tag{7}$$

so that the corresponding auxiliary regression model regression becomes

$$RM\ 2\ : \ \Delta y_t = -\phi\alpha\tau_{t-1}(d) + \phi\Delta^d y_{t-1} + \varepsilon_t. \tag{8}$$

As in the traditional DF framework, it can be shown that the t-ratio on the OLS estimator of ϕ in either (5) or (8), denoted as $t_{\hat{\phi}^{\tau}_{ols}}$ and $t_{\hat{\phi}^{\mu}_{ols}}$, respectively, is numerically invariant to the (unknown) values of α and β.

In the following theorem, the asymptotic properties of the test under the null hypothesis are presented.

Theorem 1 *Under the null hypothesis that* y_t *is generated by* DGP 1 (DGP 2)*, the OLS estimator of* ϕ *in* RM 1, $\hat{\phi}_{ols}^{\tau}$, *(of* ϕ *in* RM 2, $\hat{\phi}_{ols}^{\mu}$, *when* $\beta = 0$*) is a consistent estimator of* $\phi = 0$ *and converges to its true value* ($\phi = 0$) *at a rate* T^{1-d} *when* $0 \leq d < 0.5$*, and at the standard rate* $T^{1/2}$ *when* $0.5 < d < 1$*. The asymptotic distributions of the associated* $t-$*ratios,* $t_{\hat{\phi}_{ols}^{\tau}}$ *and* $t_{\hat{\phi}_{ols}^{\mu}}$ *are given by*

$$t_{\hat{\phi}_{ols}^{i}} \xrightarrow{w} \Lambda_i(d) \qquad \textit{if } \; 0 < d < 0.5, \; \textit{for } i = \{\mu, \tau\},$$

and,

$$t_{\hat{\phi}_{ols}^{i}} \xrightarrow{w} N(0,1) \qquad \textit{if } 0.5 < d < 1, \; \textit{for } i = \{\mu, \tau\},$$

where $\Lambda_i(d), i = \{\mu, \tau\}$ *are functionals of fBM (see Appendix 1) that depend on* d *but not on the other parameters of the model.*

Finally, if d is estimated, instead of assuming an (arbitrary) pre-fixed value under the alternative, then *RM 1* would be as follows

$$\Delta y_t = \alpha_1 + \alpha_2 \tau_{t-1}(\widehat{d_T}) + \alpha_3 \tau_{t-1}(\widehat{d_T} - 1) + \phi \Delta^{\widehat{d}_T} y_{t-1} + \varepsilon_t, \tag{9}$$

where $\widehat{d_T}$ is a (trimmed) $T^{1/2}-$consistent estimate of d.[6] Likewise, if no trend is allowed under H_0, then the model becomes

$$\Delta y_t = \alpha_1 \tau_{t-1}(\widehat{d_T}) + \phi \Delta^{\widehat{d}_T} y_{t-1} + \varepsilon_t. \tag{10}$$

As discussed in DGM (2002), among the different estimation procedures available in the time domain which yield $T^{1/2}$-consistent estimates of d in the permissible range, the ML estimators derived by Beran (1995) and Tanaka (1999) or the Minimum Distance estimators derived by Galbraith and Zinde-Walsh (1997) and Mayoral (2004) can be used. Then, the following result holds.

Theorem 2 *Let* $\widehat{d_T}$ *be a (trimmed)* $T^{1/2}$*-consistent estimator of* d*,* $0 \leq d < 1$*, such that* $T^{1/2}(\widehat{d_T} - d) \xrightarrow{w} \xi$*, where* ξ *is a non-degenerate random variable. Then, under the null hypothesis*

[6]Effectively, if $\widetilde{d_T}$ is a $\mathrm{T}^{1/2}$-consistent estimator of d, $\widehat{d_T} = \widetilde{d_T}$, if $\widetilde{d_T} < 1 - c$, and $\widehat{d_T} = 1 - c$, if $\widetilde{d_T} \geq 1$, where $c > 0$ is a (fixed) value in the neighborhood of zero that ensures that $\widehat{d_T}$ is strictly smaller than unity.

that y_t *is generated by* DGP 1 (DGP 2)*, the asymptotic distribution of the* $t - ratios$ *on the OLS coefficient associated to* ϕ *in* (9) *and in* (10), $t_{\hat{\phi}^{\tau}_{ols}}(\widehat{d})$ *and* $t_{\hat{\phi}^{\mu}_{ols}}(\widehat{d})$*, are given by*

$$t_{\hat{\phi}^{i}_{ols}}(\widehat{d}) \xrightarrow{w} N(0,1), \; i = \{\mu, \tau\}.$$

To check how the previous asymptotic results perform in finite samples, Tables A2a-A2b in Appendix 2 report the empirical critical values of $t_{\hat{\phi}^{\tau}_{ols}}$ and $t_{\hat{\phi}^{\mu}_{ols}}$ in *RM 1* and *RM 2* for different significance levels and different values of d. The results are based on a Monte-Carlo study with a number of replications $N = 10,000$ of *DGP 2* (since the test is invariant to the value of α and β) where $\sigma_{\varepsilon} = 1$ and $T = 100, 400, 1000$. As the theory predicts, the critical values for $d \in [0, 0.5)$ are clearly different from those corresponding to a one-sided test using a standardized $N(0,1)$ distribution ($-1.28, -1.64$ and -2.33, respectively, for the three significance levels reported below). By contrast, when $d \in (0.5, 1)$, the critical values resemble much more those of a $N(0,1)$ distribution. This is the case for values of $d > 0.6$ and samples sizes $T \geq 100$, although for $T = 100$ the test is slightly under-sized. Nonetheless, in the case where d is estimated using Mayoral's (2004) Minimum Distance (MD) estimator, which satisfies the requirements above, the empirical sizes at the 5% nominal level are 5.18 %, 5.12 %and 4.98 % for $T = 100, 400$ and 1000, respectively. In this case, moreover, the power of the FDF test in model RM1 when $d = 0.9$ happens to be fairly satisfactory (26.7%, 65.4% and 94.3% for T= 100, 400, and 1000, respectively). The rejection frequencies when using RM2 turn out to be very similar and are not reported. Thus, on the basis of these results, we recommend estimating d.

Finally, it is convenient to finish this section with a brief discussion about the implications of running the FDF test including the same deterministic components $\mu(t)$ as in the DGP, instead of the non-linear trends $\tau_{t-1}(d)$ and $\tau_{t-1}(d-1)$ defined above.[7] To illustrate the effects of such a way of proceeding, let us assume that, under the null, the DGP is a random walk with a drift, i.e., $\Delta y_t = \alpha + \varepsilon_t$ and that, under the alternative the regression model becomes[8]:

$$\Delta y_t = \alpha + \phi \Delta^d y_{t-1} + e_t, \tag{11}$$

[7]Note that, statistically speaking, the specification of the maintained hypothesis including the same deterministic terms as the DGP is unsound since the unconditional expectation of y_t differs under the null and the alternative.

[8]The following reasoning equally holds when both the DGP and the auxiliary regression model contain the linear time trend $\mu(t) = \alpha + \beta t$.

where $\Delta^d y_{t-1} = \Delta^{d-1}\alpha + \Delta^{d-1}\varepsilon_{t-1} = \alpha\tau_{t-1}(d-1) + \Delta^{d-1}\varepsilon_{t-1}$. As explained above, $\tau_{t-1}(d-1) = t-1$ when $d = 0$ and equals 1 when d =1. In the intermediate cases, when $d \in (0,1)$, it is easy to prove that $\tau_{t-1}(d-1)$ is of order $O(T^{1-d})$ since, by Stirling's approximation, we get that $\pi_i\,(d-1) = \Gamma(i+1-d)/[\Gamma(1-d)\Gamma(i+1)] \sim i^{-d}/\ \Gamma(1-d)$. Hence, the sum from 1 to T of those terms will yield the previous order of magnitude.[9] This implies that the variance of the deterministic and stochastic components of $\Delta^d y_{t-1}$ are $O(\alpha T^{3-2d})$ for the former and $O_p(\sigma^2 T^{2(1-d)})$, when $d \in (0,\,0.5)$, and $O_p(\sigma^2 T)$, when $d \in (0.5,\,1)$, for the latter. Hence, the dominating component in (11) is the nonlinear trend induced by $\Delta^d y_{t-1}$ implying that the t-ratio $t_{\hat{\phi}_{ols}}(d)$ will always be asymptotically distributed as $N(0,\,1)$, even when $d \in (0,\,0.5)$. This result mimics the one derived by West (1988) in the $I(1)$ vs. $I(0)$ framework. Moreover, as Hylleberg and Mizon (1989) noticed in that case, the finite sample behavior of the test will depend on the relative size of α and σ^2. When α is sufficiently large relative to σ^2 the asymptotic $N(0,1)$ approximation will hold in finite sample whereas in the opposite case, its behavior will be dominated by $\Delta^{d-1}\varepsilon_{t-1}$, so that, as shown in Theorem 1 above, the finite sample distribution will be close to the non-standard distribution $\Lambda_\mu(d)$ when $d \in (0,\,0.5)$. However, the main drawback of implementing the FDF test in (11) is that if the series has a linear trend ($\alpha \neq 0$) then the power of the test will be very low for values of d sufficiently smaller than unity. Indeed, for $d = 0$, the power is null.

2.2 Stationary case: The invariant AFDF test

Next, we generalize the DGP considered in (1) by assuming that u_t follows an stationary linear *AR(p)* process, namely, $A(L)u_t = \varepsilon_t$ where $A(L) = 1 - a_1 L - ...a_p L^p$ with $A(z) \neq 0$ for $|z| \leq 1$.[10] Following the same procedure as before, the auxiliary model for the process *without* deterministic components ($\mu(t) = 0$) becomes

$$A(L)\Delta y_t = \phi A(1)\Delta^d y_{t-1} + \phi \widetilde{A}(L)\Delta^{d+1} y_{t-1} + \varepsilon_t, \qquad (12)$$

where $A(L) = A(1) + \widetilde{A}(L)\Delta$ with $\widetilde{A}(L)$ having its roots outside the unit circle.

Rewriting (12) as $\Delta y_t = \phi A(1)\Delta^d y_{t-1} + [1 - A(L) + \phi\widetilde{A}(L)\Delta^d L]\Delta y_t + \varepsilon_t$, yields

[9] Note that $d = 1$ implies $O(1)$ whereas $d = 0$ implies $O(T)$, in accord with the previous discussion of the two extreme cases.

[10] The assumption of a finite lag *AR(p)* model is made for illustrative purposes since the results based on DGM (Theorem 7), to be discussed below, are valid for any stationary ARMA process.

$$\Delta y_t = \phi A(1)\Delta^d y_{t-1} + \Psi(L)\Delta y_t + \varepsilon_t,$$

such that $\Psi(L) = [1 - A(L) + \phi \widetilde{A}(L)\Delta^d L]$, with $\Psi(0) = 0$ (since $A(0) = 1$) and $\Psi(1) = 1 - A(1) < \infty$, implies that $\Psi(L) = B(L)L$. Thus

$$\Delta y_t = \phi A(1)\Delta^d y_{t-1} + B(L)\Delta y_{t-1} + \varepsilon_t, \tag{13}$$

where, making use of the arguments in Theorem 7 of DGM, it can be proved that the infinite lag polynomial $B(L)$ can be approximated by an AR (k) polynomial, $B_k(L)$, such that $k^3/T \uparrow 0$ when $k \uparrow \infty$ and $T \uparrow \infty$ (see Said and Dickey, 1984). For example, in the $AR(1)$ case, namely, when $A(L) = 1 - aL$, then $A(1) = 1 - a$ and $B(L) = 1 + (1-L)^d$. Therefore, in the case where the disturbance is serially correlated, the FDF test of $H_0 : d = 1$ vs. $H_0 : 0 \leq d < 1$, known as the Augmented FDF (AFDF) test, is based on the t-ratio on the coefficient of $\Delta^d y_{t-1}$ in the auxiliary model

$$\Delta y_t = \phi A(1)\Delta^d y_{t-1} + B_k(L)\Delta y_{t-1} + \varepsilon_t. \tag{14}$$

Note that if $A(1) \simeq 0$, i.e., the AR polynomial has a root close to unity, then a test on $\phi = 0$ will have little power when $\phi A(1) \simeq 0$ even if $\phi \neq 0$, as it happens in the standard AFDF test. Finally, along the lines of the derivation of (9) and (10), if we now consider the case where $\mu(t) = \alpha + \beta t$, the AFDF test will be implemented in the following auxiliary model

$$\Delta y_t = \alpha_1 + \alpha_2 \tau_{t-1}(d) + \alpha_3 \tau_{t-1}(d-1) + \phi A(1)\Delta^d y_{t-1} + B_k(L)\Delta y_{t-1} + \varepsilon_t, \tag{15}$$

where $\alpha_3 = 0$ if $\mu(t) = \alpha$.

3. FDF VS. LM TESTS

As discussed in the Introduction, the closest competitor to the FDF test is Tanaka's (1999) LM test in the time domain. This test, denoted as τ_T, considers the null hypothesis of $d = d_0$ against the alternative of $d = d_0 + \theta$ where $\theta \neq 0$ for the DGP $\Delta^{d+\theta}[y_t - \mu(t)] = \varepsilon_t$. Thus, in line with the hypotheses considered in this paper, we will focus on the particular case where $d_0 = 1$ and $0 < \theta \leq 1$. Assuming that $\varepsilon_t \sim N(0, \sigma^2)$, the log-likelihood function can be written as

$$L\,(\theta,\sigma) = -\frac{T}{2}\ln(\sigma^2) - \frac{1}{2\sigma^2}\sum_{t=1}^{T}[(1-L)^{1+\theta}y_t]^2. \tag{16}$$

Then, taking the derivative of the log-likelihood function w.r.t. θ, evaluated at $\theta = 0$, and making use of the result $\sum_{j=1}^{\infty} j^{-2} = \pi^2/6$, yields the following score-LM test

$$\tau_T = \sqrt{\frac{6}{\pi^2}}\,T^{1/2}\sum_{j=1}^{T-1} j^{-1}\widehat{\rho}_j \overset{w}{\to} N\,(0,1)\,, \tag{17}$$

where $\widehat{\rho}_j = \sum_{t=j+1}^{T} \Delta\widetilde{y}_t\Delta\widetilde{y}_{t-j} \,/ \sum_{t=1}^{T}(\Delta\widetilde{y}_t)^2$, and $\Delta\widetilde{y}_t$ are the OLS residuals from regressing Δy_t on $\Delta\mu(t)$. Therefore, if just a constant term is considered, then $\Delta\widetilde{y}_t = \Delta y_t$; likewise, with a linear trend, $\Delta\widetilde{y}_t = \Delta y_t - \overline{\Delta y}$ where $\overline{\Delta y}$ denotes the sample mean of Δy_t.

As Breitung and Hassler (2002) have shown, an alternative simpler way to compute the score test is as the t-ratio (t_γ) of $\widehat{\gamma}_{ols}$ in the regression

$$\Delta\widetilde{y}_t = \gamma x^*_{t-1} * e_t, \tag{18}$$

where $x^*_{t-1} = \sum_{j=1}^{t-1} j^{-1}\Delta\widetilde{y}_{t-j}$. Intuitively, since $t_\gamma = \sum(\Delta\widetilde{y}_t x^*_{t-1})/\widehat{\sigma}_e(\sum(x^*_{t-1})^2)^{1/2}$ and, under $H_0 : \theta{=}0$, $\widehat{\sigma}_e$ tends to σ and plim $T^{-1}\sum(x^*_{t-1})^2 = \pi^2/6$, then t_γ has the same limiting distribution as τ_T.

An advantage of the τ_T test if that, by working under the null, the regressor $\Delta^d y_{t-1}$ in the FDF test does not need to be constructed, albeit one needs to construct x^*_{t-1} . Furthermore, Tanaka (1999) has proved that, under a sequence of local alternatives of the type $\theta = -T^{1/2}\delta$ with $\delta > 0$, τ_T (or t_γ) is the UMPI test. In such a case its limiting distribution becomes $N(-\delta\pi^2/6,\, 1)$ whereas DGM (Th. 3) obtain that the corresponding distribution of the FDF test is $N(-\delta,\, 1)$. Since $\pi^2/6 > 1$, the non-centrality parameter of the LM test is larger and hence it is more powerful. To the best of our knowledge, however, the case of fixed alternatives has not been studied in the literature and therefore, in the sequel, we deal with this case.

Suppose that the alternative holds, namely, the DGP is now $\Delta^d y_t = \varepsilon_t$ with $d \in (0,1)$.[11] Then, $\Delta^d y_t = \Delta^{-\theta}\varepsilon_t$ where $\theta = d - 1 < 0$. Then the following result holds.

[11] The case where $d = 0$ is excluded since $\Gamma(-1) \uparrow \infty$. In other words, the standard formulae (see Baillie, 1996) for the autocorrelations of a pure I(θ) process is only valid when $\theta > -1$, i.e., $d > 0$.

Theorem 3 *: If* $\Delta y_t = \Delta^{-\theta}\varepsilon_t$ *where* $d \in (0,1)$ *and hence* $\theta = d-1 < 0$*, then*

$$T^{-1/2} t_{\hat{\phi}_{ols}}(d) \xrightarrow{p} \frac{(d-1)\Gamma(2-d)}{[\Gamma(3-2d)-(d-1)^2\Gamma^2(2-d)]^{1/2}} = c_{FDF}(d),$$

$$T^{-1/2}\tau_T \xrightarrow{p} \sqrt{\frac{6}{\pi^2}}\frac{\Gamma(2-d)}{\Gamma(d-1)}\sum_{j=1}^{\infty} j^{-2(2-d)} = c_{LM}(d),$$

where $\sum_{j=1}^{T-1} j^{-2(2-d)}$ *corresponds to Riemann's zeta function which is summable since* $2(2-d) > 2$*, and* $c_{FDF}\ (d)$ $(c_{LM}(d))$ *denotes the non-centrality parameter under the fixed alternative* $\theta \neq 0$ *of the FDF (LM) test.*

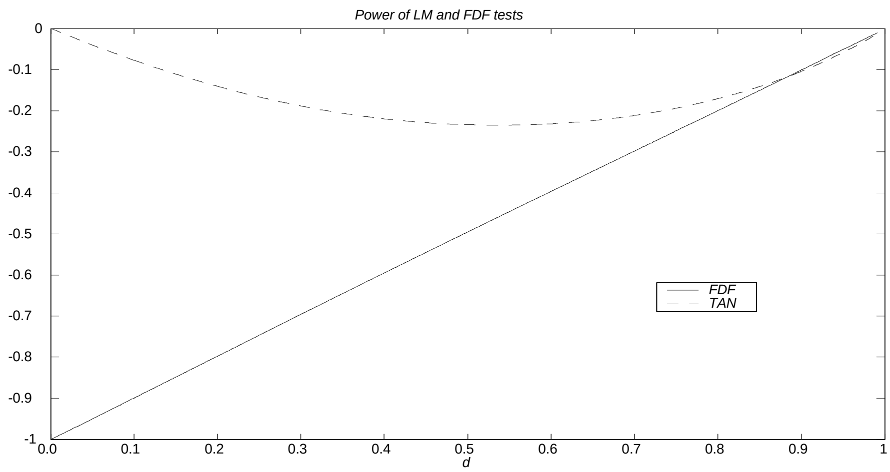

Figure 2

Figure 2 displays the two above-mentioned non-centrality parameters for $d \in (0,1)$.[12] The most striking finding is that, whereas the depicted values of $c_{FDF}(d)$ are monotonically decreasing in d, those of $c_{LM}(d)$ are non-monotonic, and that $c_{FDF} > c_{LM}$ for almost all values of d except for those very close to 1 and even in this case, the distance in favor of c_{LM} is very small. The intuition for the non-monotonicity of $c_{LM}\ (d)$ is the presence of $\Gamma(d-1)$ in the denominator of c_{LM} in Theorem 3. As $d \uparrow 0$, $\Gamma(d-1)$ gets larger in absolute value and therefore c_{LM} becomes closer to zero, a feature which does not affect c_{FDF}. Moreover, the result

[12]Notice that Theorem 3 excludes the point $d = 0$. For $d = 0$, it is easy to show that $C_{FDF} = -1$. As for C_{LM}, notice that $\Delta y_t = \Delta\varepsilon_t$ and therefore the only non-zero correlation is $\rho_1 = -0.5$. Thus $c_{LM} = -0.5\sqrt{6/\pi^2} \simeq -0.13$.

in Theorem 3 is an asymptotic one and, as will be shown below, for realistic sample sizes, the rejection rates of the FDF test under the alternative are also larger than those of the LM test, except in cases where d is very close to unity and the error term is normally distributed. Thus, for fixed alternatives, the FDF is bound to be more powerful than the LM test.

In the case where $A(L)\,u_t = \varepsilon_t$, Tanaka (1999) has proved that $\tau_T = \widehat{\omega}^{-1}T^{1/2}\Sigma j^{-1}\widehat{\rho}_j \overset{w}{\to} N(0,1)$, where $\omega^2 = (\pi^2/6) - (\kappa_1...\kappa_p)'\Phi^{-1}(\kappa_1\,...\kappa_p)$ with $\kappa_i = \Sigma\,_{j=1}^{\infty} j^{-1}c_{j-i}$ and c_j are the coefficients of L^j in the expansion of $1/A(L)$. If $A(L) = 1 - aL$, $\omega^2 = (\pi^2/6) - (a^{-2} - 1)(\ln(1 - a))^2$ whereas for more general $AR\,(p)$ processes, the computation gets very involved. Breitung and Hassler (2002), however, argue that computation of the LM test in the t_γ helps to account for serial correlation. Using the approach by Agiakglou and Newbold (1994), they advocate implementing regression (18), augmented with p lags of Δy_t, but this time replacing Δy_t and $x^*_{t-1}$with the residuals obtained from the estimation of an $AR(p)$ process for Δy_t.

Monte-Carlo evidence in favor of the FDF test was provided by DGM in the case where there are not deterministic components (see Tables I and II in DGM). In what follows we provide some additional simulations when $\mu(t) = \alpha + \beta t$. Table 1 presents the rejection frequencies for local alternatives at the 5% level of the FDF and LM tests in its two alternative versions τ_T and t_γ. The DGP, $\Delta^d y_t = \varepsilon_t$, is simulated 10,000 times, with $d = 1 - \delta/T^{1/2}$ for $\delta = \{0.5, 1.0, 1.5$ and $2.0\}$, $T = \{25, 50, 100, 400\}$, $\sigma = 1$ and the considered auxiliary model is *RM1.* Since in the case where d is estimated there are very small size distortions, we have used Mayoral's (2004) approach to estimate d and 5% c.v. at the lower tail of a $N(0,1)$ to construct the critical region. Bold figures signify better performance of either test. As can be observed, the most relevant finding is that, except for very large sample sizes, the FDF has larger power than the LM tests, in accord with the result derived in Theorem 3 above. Moreover, there does not seem to be a power loss in finite samples when deterministic components are included relative to the case where they are not (see DGM, 2002, Table 5). Table 2, in turn, reports the power when the DGP is $\Delta^d y_t = \varepsilon_t/(1 - 0.7L)$ for several values of d. In this case, the FDF test clearly outperforms the LM tests. Lastly, we briefly report some results on the consequences of having departures from gaussianity in the distribution of ε_t in the above-mentioned DGP. For example, when the errors follow a zero-mean standarized $\chi^2(1)$ distribution and they are *i.i.d*, the power of the FDF test run with RM1, for $d = 0.8, 0.9$ and $T = 100$, is 57.9 % and 27.5 % whereas the corresponding rejection frequencies of Tanaka's τ_T are 52.8 % and 17.2 %, respectively. Thus, the FDF test seems to fare better than the LM test in the presence on non-gaussian errors.

TABLE 1

Power of FDF and LM tests, 5% Level (RM1), Local Alternatives

DGP: $\Delta^d y_t = \varepsilon_t$, $d = 1 - \delta/T^{1/2}$												
	LM: t_γ				*LM*: τ_T				*FDF*			
$\delta \| T$	25	50	100	400	25	50	100	400	25	50	100	400
0.5	13.1	14.9	14.5	11.0	2.3	8.5	10.4	14.4	**16.8**	**16.9**	**15.3**	**14.9**
1	25.7	27.0	27.1	28.1	5.9	13.9	23.2	**30.6**	**25.8**	**27.3**	**29.6**	28.8
1.5	31.0	36.8	44.2	50.1	9.4	28.4	42.8	**54.1**	**32.9**	**37.1**	**45.6**	46.1
2	46.2	56.9	**65.0**	70.2	18.3	45.3	63.4	**73.0**	**47.2**	**58.1**	58.2	64.3

TABLE 2

Power AFDF and LM Tests, 5% Level (RM1)

DGP: $\Delta^d y_t = a\Delta^d y_{t-1} + \varepsilon_t$; $a = 0.7$						
	LM (τ_T)		*LM* (t_γ)		*AFDF*	
d	$T = 100$	$T = 400$	$T = 100$	$T = 400$	$T = 100$	$T = 400$
0.9	5.7%	12.0%	12.5%	16.1%	**19.3%**	**26.7%**
0.7	7.1%	25.5%	16.4%	36.6%	**28.3%**	**44.3%**
0.6	26.4%	71.7%	17.2%	86.9%	**39.0%**	**99.4%**
0.3	**53.2%**	96.85%	47.0%	98.2%	51.2%	**100.0%**
0.1	68.9%	**100%**	**73.7%**	**100%**	70.3%	**100.0%**

A simple strategy to test for the value of d in the presence of deterministic components

In view of the above results, a natural approach arises to test the null of $I(1)$ vs. $I(d)$ in the presence of deterministic components when d is estimated. Before commenting on this testing strategy, however, it is important to stress that an interesting consequence from our analysis is that, in contrast to the use of the standard DF test for $H_0 : d = 1$ when deterministic components are present, there is no need to use new critical values relative to the case where no deterministic components are considered. This is so since all the critical values come from the $N(0,1)$ distribution. These two features transform the problem of determining the right deterministic components into the standard issue of variable selection.

Our proposed testing strategy for $H_0 : d = 1$ vs. $H_1 : 0 \leq d < 1$ will take as starting point *RM 1* in (5). First, if the null is rejected, then the process is not $I(1)$ and the testing strategy stops. If the null is not rejected, then we can test whether the coefficient of the nonlinear trend $\tau_{t-1}(d-1)$ is significant. If it is significant, we stop. Otherwise, we estimate *RM 2* including only $\tau_{t-1}(d)$ and follow again the same strategy. In sum, our proposed strategy is easy to apply and turns out to be much simpler than those often used in applied work, as the next section illustrates.

4. EMPIRICAL ILLUSTRATION

An interesting application of the theoretical results applied above is to examine whether the time-series of GDP per capita of several OECD countries behave as $I(d)$ processes with $0.5 < d < 1$. These are series which are clearly trending upwards and therefore provide nice examples of the role of deterministic terms in the use of the FDF test. As pointed out in an interesting paper by Michelacci and Zaffaroni (2000), such a long- memory behavior could well explain the seemingly contradictory results obtained in the literature on growth and convergence that a unit root cannot be rejected in (the log of) those series and yet a 2% rate convergence rate to a steady-state level (approximated by a linear trend) is typically found in most empirical exercises of this kind (see Barro and Sala i Martín, 1995 and Jones, 1995). The explanation offered by these authors to this puzzle relies upon two well-known results in the literature on long-memory processes, namely that standard unit root tests have low power against values of d in the nonstationary range $(0.5 < d < 1)$, and that for all values of $d \in [0,\ 1)$ there is "mean reversion", in the sense that shocks do not have permanent effects. Using Maddison's (1995) data set of annual GDP per capita series for 16 OECD countries during the period 1870 - 1994 (125 observations) and a log-periodogram estimator of d due to Robinson (1995), they find that in most countries the order of fractional integration is in the interval (0.5, 1), compatible with the 2% rate of convergence found in the literature of beta-convergence and, therefore, validating in this way their explanation of the puzzle. Since that estimation procedure is restricted to the range of $I(d)$ processes with finite variance, namely, $|d| < 1/2$, the authors proceed by first detrending the data and then applying the truncated filter $(1-L)^{1/2}$ to the residuals, discarding the first 10 observations to initialize the series.

The previous results have been recently criticized by Silverberg and Verspagen (2001) on the grounds of both the use of the $(1-L)^{1/2}$ filter and Robinson's semi-parametric estimation procedure, which suffers from serious small-sample bias. Instead, they propose the use of the first-difference filter, $(1-L)$, to remove the trend and of Sowell's (1992) parametric ML estimator of ARFIMA models to tackle short-memory contamination in the estimation of d. Using those alternative procedure they find, in stark contrast to Michelacci and Zaffaroni's results, that d tends to be either not significantly different from unity or significantly above unity for most countries in an extended sample of 25 countries.

To shed light on this controversy, we apply the invariant FDF test developed in Section 2 to the logged GDP p.c. of a subset of ten of the main OECD countries which are listed in Table 3, where the estimated intercept and its standard deviation in the regression $\Delta y_t = \beta + u_t$ is reported. As can be inspected, the mean (average GDP p.c. growth rate) is always highly significant making it convenient to use *RM 1* as the maintained hypothesis. Indeed, when the ADF and the Phillips-Perron (P-P) unit root tests (not reported) were computed using a constant and a time trend in the regression model, the $I(1)$ null hypothesis could not be rejected. The KPPS test, which takes $I(0)$ as the null, yielded overall rejection confirming the high persistence of the series. Thus there are clear signs that the first-difference series have a drift and that it is likely that they are nonstationary.

TABLE 3

ESTIMATES OF $\widehat{\beta}$ AND SD($\widehat{\beta}$)

Country	Mean	St. D.
Australia	0.012	0.004
Canada	0.0195	0.005
Denmark	0.018	0.008
France	0.018	0.006
Germany	0.018	0.007
Italy	0.019	0.006
Netherlands	0.015	0.006
UK	0.013	0.003
USA	0.017	0.005
Spain	0.019	0.005

Since there were clear signs of autocorrelation in u_t, an AFDF test with intercept and linear trend according to *RM1* was applied to the series. The number of lags of the dependent variable was chosen according to the AIC criterion with a maximum lag of length $k = 4$, since $T = 125$ (95 for Spain) and $T^{1/3} = 5$. Pre-estimation of d using Sowell´s (1992) ML parametric approach for various $ARFIMA$ (p, d, q) specifications of the first-differenced data, with p and q up to four lags, allows one to select a value of d for each country on the basis of the AIC criterion. The reported estimates of d in the preferred models, $\widehat{d}_{ML}$, presented in the second and fourth columns of Table 4, add unity to the obtained estimates. Estimates were also obtained using Mayoral ´s (2004) MD estimation approach, with the series in levels, yielding the pre-estimates of d, $\widehat{d}_{MD}$, in the preferred models presented in the third and fifth columns of Table 4. Both sets of estimates tend to provide similar results. In general, the estimated values of d belong to (0.5, 1). Using the AFDF test with pre-fixed values of d, the first four columns of Table 5 show strong rejections of H_0: $d = 1$ in most cases. Likewise, for robustness, the last column reports the results of the FDF test in *RM 1* with estimated d, using the $\widehat{d}_{MD}$ estimates in Table 4 and a trimming value of $c = 0.05$ for Australia whose estimated d exceeds unity. Again, with the exception of Spain, we find strong rejections of the null. Thus, our results seem to favor nonstationary, albeit mean-reverting, values of d, in agreement with Michelacci and Zaffaroni (2000) and therefore consistent with an exogenous growth assumption.[13]

TABLE 4

ESTIMATES OF d (ML AND MD)

Country	$\widehat{d}_{ML}$	model	$\widehat{d}_{MD}$	model
Australia	0.69	$(1, d, 0)$	0.71	$(0, d, 0)$
Canada	0.50	$(1, d, 0)$	0.44	$(1, d, 0)$
Denmark	0.71	$(1, d, 0)$	0.72	$(1, d, 0)$
France	0.77	$(0, d, 1)$	0.82	$(0, d, 1)$
Germany	0.81	$(0, d, 1)$	0.80	$(0, d, 1)$
Italy	0.82	$(0, d, 1)$	0.81	$(0, d, 1)$
Netherlands	0.77	$(0, d, 1)$	0.77	$(0, d, 1)$
UK	0.60	$(1, d, 0)$	0.71	$(1, d, 0)$
USA	0.78	$(0, d, 0)$	0.73	$(1, d, 0)$
Spain	0.83	$(1, d, 0)$	0.92	$(0, d, 0)$

[13] Use of the testing strategy described in Section 3 yields similar results.

TABLE 5

AFDF TEST AGAINST FI(d)

Country | d	0.9	0.8	0.7	0.6	$\widehat{d}_{MD}$
Australia	-2.27*	-2.41*	-2.55	-2.67*	-2.54*
Canada	-2.78*	-2.87*	-2.95*	-3.05*	-4.21*
Denmark	-2.84*	-2.99*	-3.09*	-5.83*	-3.16*
France	-2.26*	-2.32*	-2.38*	-2.47*	-2.42*
Germany	-2.63*	-2.73*	-2.81*	-3.87*	-2.77*
Italy	-2.04*	-2.06*	-2.03	-2.05	-2.11*
Netherlands	-2.41*	-2.52*	-2.56*	-2.62*	-2.54*
UK	-2.31*	-2.34*	-2.36*	-2.36*	-2.41*
USA	-3.12*	-3.29*	-3.39*	-3.53*	-3.42*
Spain	-0.24	-0.39	-0.66	-0.79	-0.34

Note: (*) denotes 5%- rejection of the null hypothesis of a unit root versus a fractional one.

5. CONCLUSIONS

This paper has developed statistics for detecting the presence of a unit root in time-series data against the alternative of mean-reverting fractional processes allowing for deterministic terms, $\mu(t)$, (a constant or a constant and a time trend) in the DGP and in the auxiliary regression used to implement the FDF test. Two main findings have been obtained. *First*, if the DGP is $y_t = \mu(t) + I(d)$, with $d \in [0, 1)$, so that $E(\Delta^d y_t) = \Delta^d \mu(t)$ then inclusion of nonlinear trends of the form $\Delta^d \mu(t)$ in the regression model yields invariant tests to the parameters defining $\mu(t)$. Alternatively, if the error term in the DGP is serially correlated, the set of regressors involving $\Delta^d \mu(t)$ and $\Delta^d y_{t-1}$ should be augmented with a suitable number of lags of the dependent variable, Δy_t. This test has a non standard asymptotic distribution when d is (arbitrarily) pre-fixed in the range (0, 0.5). However, asymptotic normality holds either when $d \in$ (0.5, 1) or when d is estimated using a (trimmed) $T^{1/2}-$consistent estimator. *Second*, we provide new theoretical results regarding the gains in power under fixed alternatives of applying the FDF test instead of conventional LM tests.

Notice that the proposed approach not only is very simple but it could be easily extended to account for other different deterministic components to the linear time trend considered here.

For example, under nonlinear trends (quadratic, cubic, etc.) or structural breaks (in the mean or the slope of the linear time trend), all what is needed is to construct the corresponding $\Delta^d \mu(t)$ terms by means of the truncated binomial expansion of $(1-L)^d$ in terms of L. As in the case considered in this paper, implementation of the FDF test in those circumstances can be easily done with any standard econometric packages.

Useful extensions of the present paper's setup that are under current investigation by the authors include testing fractional integration versus $I(0)$ allowing for structural breaks (see Dolado, Gonzalo and Mayoral, 2005), testing for cointegration between two $I(d)$ series which have a non-zero drift and where a constant term or a linear trend is included in the regression model and finally, an extension of this framework to panel data.

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APPENDIX 1

In order to prove Theorem 1, the following lema would be needed.

Lemma 1 *Let y_t be a random walk process defined as in* (7)*. Under the assumptions of Section 2, the following convergences follow:*

If $0 < d < 1$, *then*

1. $T^{-(1-d)} \sum_{i=2}^{T} \tau_{t-1}(d) \to \frac{1}{(d-1)\Gamma(-d)}$.

2. $T^{-(2-d)} \sum_{i=2}^{T} \tau_{t-1}(d-1) \to \frac{1}{\Gamma(3-d)}$.

3. $T^{-(1-2d)} \sum_{i=2}^{T} \tau_{t-1}^2(d) \to C_1(d) < \infty$, for $d \in (0,\ 0.5)$, and
   $\sum_{i=2}^{T} \tau_{t-1}^2(d) \to C_2(d) < \infty$, for $d \in (0.5,\ 1)$.

4. $T^{-(3-2d)} \sum_{i=2}^{T} \tau_{t-1}^2(d-1) \to \frac{1}{(3-2d)\Gamma^2(2-d)}$.

5. $T^{-(2-2d)} \sum_{i=2}^{T} \tau_{t-1}(d) \tau_{t-1}(d-1) \to C_3(d)$.

6. $\sum_{i=2}^{T} \tau_{t-1}(d) \varepsilon_t \xrightarrow{d} N\left(0,\ \sigma^2 C_1\right)$ for $d \in (0,\ 0.5)$.
   $T^{-(1/2-d)} \sum_{i=2}^{T} \tau_{t-1}(d) \varepsilon_t \xrightarrow{w} N\left(0,\ \sigma^2 C_2\right)$ for $d \in (0.5,\ 1)$.

7. $T^{-(3/2-d)} \sum_{i=2}^{T} \tau_{t-1}(d-1) \varepsilon_t \xrightarrow{w} N\left(0,\ \sigma^2 C_4\right)$, $C_4 = \frac{1}{(3-2d)\Gamma^2(2-d)}$.

8. $T^{-(1-d)} \sum_{i=2}^{T} \Delta^d y_{t-1} \varepsilon_t \xrightarrow{w} \sigma^2 \int_0^1 W_{-d}(r)\, dB(r)$ for $d \in (0,\ 0.5)$, and
   $T^{-0.5} \sum_{i=2}^{T} \Delta^d y_{t-1} \varepsilon_t \xrightarrow{w} \sigma^2 N\left(0,\ \frac{\Gamma(2d-1)}{\Gamma(d)}\right)$ for $d \in (0.5, 1)$.

9. $T^{-(3/2-2d)} \sum_{i=2}^{T} \tau_{t-1}(d) \Delta^d y_{t-1} \xrightarrow{w} \frac{1}{\Gamma(-d)(d-1)} \int_0^1 r^{-d} W_{-d}(d)\, dr$ for $d \in (0, 0.5)$, and
   $T^{-(1-d)} \sum_{i=2}^{T} \tau_{t-1}(d) \Delta^d y_{t-1} \xrightarrow{w} 0$ for $d \in (0.5,\ 1)$.

10. $T^{-(2-d)} \sum_{i=2}^{T} \tau_{t-1}(d-1) \Delta^d y_{t-1} \xrightarrow{p} 0$ for $d \in (0.5,\ 1)$, and
    $T^{-(5/2-2d)} \sum_{i=2}^{T} \tau_{t-1}(d-1) \Delta^d y_{t-1} \xrightarrow{w} \sigma^2 \int_0^1 r^{1-d} W_{-d}(r)\, dr$ for $d \in (0,\ 0.5)$.

11. $T^{-1} \sum (\Delta^d y_{t-1})^2 \xrightarrow{p} Var(y)$ if $d \in (0.5,\ 1)$, and
    $T^{-2(1-d)} \sum (\Delta^d y_{t-1})^2 \xrightarrow{w} \sigma^2 \int_0^1 W_{-d}^2(r)\, dr$ if $d \in [0, 0.5)$,

12. $T^{-1} \sum_{i=2}^{T} \Delta^d y_{t-1} \xrightarrow{p} 0$ if $d \in (0.5,\ 1)$, and
    $T^{-(3/2-d)} \sum_{i=2}^{T} \Delta^d y_{t-1} \xrightarrow{w} \int_0^1 W_{-d}(r)\, dr$ if $d \in (0,\ 0.5)$.

Proof of Lemma 1

1. Notice that $\sum_{i=2}^{T} \tau_{t-1}(d)$ can alternatively be written as

$$\lim_{T\to\infty} \sum_{i=2}^{T} \tau_{t-1}(d) = \lim_{T\to\infty} T\pi_0(d) + (T-1)\pi_1(d) + ... \tag{19}$$

and also note that $\sum_{i=0}^{\infty} \pi_i(d) = 0$, then,

$$\lim_{T\to\infty} T^{-(1-d)} \sum_{i=2}^{T} \tau_{t-1}(d) = \lim_{T\to\infty} T^{-(1-d)} \sum_{i=0}^{T} (T-i))\pi_i$$

$$= T^d \lim_{T\to\infty} \sum_{i=0}^{T} \pi_i(d) - T^{-(1-d)} \lim_{T\to\infty} \sum_{i=1}^{T} i\pi_i(d) \tag{20}$$

$$= \frac{1}{\Gamma(-d)} \lim_{T\to\infty} T^{-(1-d)} \sum_{i=1}^{T} i^{-d} = \frac{-1}{\Gamma(-d)(1-d)}, \tag{21}$$

where the last equality follows from applying L'Hôpital's rule to the first term of (20) and noticing that it tends to zero.

2. In this case,

$$\lim_{T\to\infty} T^{-(2-d)} \sum_{i=2}^{T} \tau_{t-1}(d-1) = \frac{1}{\Gamma(1-d)} \lim_{T\to\infty} \sum_{t=1}^{T} \sum_{i=1}^{t} i^{-d} = \frac{1}{\Gamma(3-d)}.$$

3. Since $\sum_{i=0}^{\infty} \pi_i(d) = 0$, it is possible to write $\pi_0 = -\sum_{i=1}^{\infty} \pi_i(d), \pi_0 + \pi_1 = -\sum_{i=2}^{\infty} \pi_i(d)$, etc. Then,

$$\sum_{i=2}^{T} \tau_{t-1}^2(d) = -\sum_{j=1}^{T} \left(\sum_{i=j}^{\infty} \pi_i(d) \right)^2. \tag{22}$$

Since the coefficients $\{\pi_i(d)\}_{i=0}^{\infty}$ are such that $\pi_i \sim i^{-1-d}$, then $\left(\sum_{i=j}^{\infty} \pi_i(d)\right)^2 = O\left(j^{-2d}\right)$ (see, Davidson (1994, $p.$ 32)). This implies that if $d \in (0.5,\ 1)$, the quantity in (22) is summable and if $d \in (0,\ 0.5)$ it is $O\left(T^{1-2d}\right)$.

4. The proof of this result is similar to the previous ones and therefore is omitted.

5. Idem.

6. These limits are a direct application of Corollary 5.25 p.130 in White (1984).

7. Idem.

8. See DGM (2002) for the proofs of these results.

9. The first result follows from point 1 in this lemma and the results in Dolado and Marmol (2004). The second follows from noting that $T^{-(1-d)}E\left(\sum_{i=2}^{T}\tau_{t-1}(d)\,\Delta^{d}y_{t-1}\right)=0$ and that $T^{-2(1-d)}E\left(\left(\sum_{i=2}^{T}\tau_{t-1}(d)\,\Delta^{d}y_{t-1}\right)^{2}\right)\rightarrow 0.$

10. Idem.

11. See DGM for the proof of this result.

12. Idem.■

Proof of Theorem 1

For simplicity, let us consider first *RM 2* defined in equation (8). Since the nature of the asymptotic distribution depends upon the value of d used to run the regression, two cases ought to be distinguished.

I. First case: $0 \leq d < 0.5$. Define the scaling matrix

$$\Upsilon_T = \begin{pmatrix} T^{1/2-d} & 0 \\ 0 & T^{1-d} \end{pmatrix}, \tag{23}$$

and taking into account the results in Lemma 1 we easily get

$$\begin{pmatrix} T^{1/2} & 0 \\ 0 & T^{1-d} \end{pmatrix}\begin{pmatrix} \hat{\alpha} \\ \hat{\phi} \end{pmatrix} = \begin{pmatrix} C_1 & \frac{1}{\Gamma(-d)(1-d)}\sigma\int_0^1 r^{-d}W_{-d}(d)\,dr \\ \frac{1}{\Gamma(-d)(1-d)}\sigma\int_0^1 r^{-d}W_{-d}(d)\,dr & \sigma^2\int_0^1 W_{-d}^2(r)\,dr \end{pmatrix}^{-1} \begin{pmatrix} \sigma N(0,\ C_1) \\ \sigma^2\int_0^1 W_{-d}(r)\,dB(r) \end{pmatrix} + o_p(1),$$

implying that

$$t^{\mu}_{\hat{\phi}} \xrightarrow{d} \frac{C_1^{1/2}\left(\Gamma(-d)(1-d)\right)^{-1}\left[\int_0^1 W_{-d}(r)\,dB(r) - B(1)\int_0^1 r^{-d}W_{-d}(r)\,dr\right]}{\left[C_1\int_0^1 W_{-d}^2(r)\,dr - \left(\frac{1}{\Gamma(-d)(1-d)}\int_0^1 r^{-d}W_{-d}(d)\,dr\right)^2\right]^{1/2}},$$

which is a functional of fractional brownian motions and other terms just depending on d.

II. Second case: $0.5 < d < 1$. Defining the scaling matrix

$$\Upsilon_T = \begin{pmatrix} 1 & 0 \\ 0 & T^{1-d} \end{pmatrix}$$

and taking into account the results of Lemma 1 is straightforward to check that $t^{\mu}_{\hat{\phi}} \sim N(0,1)$.

Consider now RM 1 as defined in (5). To see that the parameter ϕ is numerically invariant to any linear transformation in y_t, note that the regression (5) can be equivalently written as

$$\begin{aligned} \Delta y_t &= \alpha_0 + (\alpha_1 + \phi y_0)\,\tau_{t-1}(d) + (\alpha_2 + \phi\beta)\,\tau_{t-1}(d-1) + \phi\Delta^d (y_{t-1} - y_0 + \beta(t-1)) + \varepsilon_t \\ &= \alpha_0^* + \alpha_1^* \tau_{t-1}(d) + \alpha_2^* \tau_{t-1}(d-1) + \phi^* \Delta^d \xi_{t-1}, \end{aligned} \tag{24}$$

where $\phi^* = \phi$, $\alpha_0^* = \alpha_0$, $\alpha_1^* = (\alpha_1 + \phi y_0)$, $\alpha_2^* = (\alpha_2 + \phi\beta)$ and under the null hypothesis, ξ_t is a random walk without drift and with initial condition equal to zero. Following Hamilton (1994, *p.*498), it is straightforward to see that the *OLS* estimator of ϕ and its associated t-statistic are numerically identical to the one that would be obtained if the original process was ξ_t instead of y_t. Taking into account this invariance property, it is possible to consider without loss of generality that $y_0 = \beta = 0$. Then, the rest of the proof is similar to the previous one and, therefore, is omitted.$\blacksquare$

Proof of Theorem 2

When $\widehat{d}$ is chosen such that $\widehat{d} = \widehat{d}_T$ if $\widehat{d}_T < 1-c$ and $\widehat{d} = 1-c$ if $\widehat{d}_T \geq 1-c$, where c is a (fixed) value in the neighborhood of zero, it is clear that $\hat{d} \xrightarrow{p} 1-c$, since $\hat{d}_T$ is a consistent estimator of $d\ (=1)$. Applying the mean value theorem (MVT) on $t^{\mu}_{\phi_{ols}}$ around the point $(1-c)$ yields

$$t^{\mu}_{\phi_{ols}}\left(\hat{d}\right) = t^{\mu}_{\phi_{ols}}(1-c) + \frac{\partial t^{\mu}_{\phi_{ols}}\left(\breve{d}\right)}{\partial d}\left(\hat{d} - (1-c)\right), \tag{25}$$

where $\breve{d}$ is an intermediate point between $\hat{d}$ and $(1-c)$. This implies that in order to prove that $\left(t^{\mu}_{\phi_{ols}}\left(\hat{d}\right) - t^{\mu}_{\phi_{ols}}(1-c)\right) = o_p(1)$ it has to be shown that $\frac{\partial t^{\mu}_{\phi_{ols}}(\breve{d})}{\partial d}\left(\hat{d} - (1-c)\right) = o_p(1)$. Notice that $\breve{d} \in \left(\hat{d},\ 1-c\right)$ and therefore, $\breve{d} \xrightarrow{p} (1-c)$. In order to replace $\breve{d}$ in (25) by its probability limit, $1-c$, it is needed to show that $\partial t^{\mu}_{\phi_{ols}}(d)/\partial d$ converges uniformly to a non-stochastic function in an open neighborhood of $(1-c)$ (see Amemiya, 1985). Using the same strategy as in DGM (2002), it can be shown that $T^{-1/2}\partial t^{\mu}_{\phi_{ols}}(d)/\partial d$ converges pointwise to zero. The uniform convergence follows from the pointwise convergence and an equicontinuity argument implied by the differenciability of $\partial t^{\mu}_{\phi_{ols}}(d)/\partial d$ with respect to d (cf. Davidson 1994, p. 340, and Velasco and Robinson, 2000). The result follows just by noticing that $T^{1/2}\left(\hat{d}-(1-c)\right)$ is $O_p(1)$ and therefore $\partial t^{\mu}_{\phi_{ols}}(d)/\partial d\left(\hat{d}-(1-c)\right)=o_p(1)$.

The proof for the case where a deterministic trend is included can be constructed along the same lines.■

Proof of Theorem 3

Under the alternative hypothesis, the $t_{\hat{\phi}_{ols}}$ statistic can be written as,

$$t_{\hat{\phi}_{ols}}=\frac{\sum\Delta y_t\varepsilon_{t-1}/T}{T^{-1/2}\left(\left(\sum\left(\Delta y_t-\hat{\phi}\varepsilon_{t-1}\right)^2/T\right)\sum\varepsilon^2_{t-1}/T\right)^{1/2}}$$

and therefore is easy to check that,

$$T^{-1/2}t_{\hat{\phi}_{ols}}\xrightarrow{p}\frac{(d-1)}{\left(\Gamma(3-2d)/\Gamma^2(2-d)-(1-d)^2\right)}$$

Similarly, by the LLN, the LM statistic, standarized by $T^{1/2}$, converges to

$$T^{-1/2}\tau_T\xrightarrow{p}\sum_{k=1}^{T-1}\frac{1}{k}\rho_k$$

where ρ_k is the correlation function of a pure $FI(d-1)$. It follows that (see Baillie, 1996),

$$T^{-1/2}\tau_T\xrightarrow{p}\sqrt{\frac{6}{\pi^2}\frac{\Gamma(2-d)}{\Gamma(d-1)}\sum_{j=1}^{\infty}j^{-2(2-d)}}.\blacksquare$$

APPENDIX 2

TABLE A2a

Critical Values

DGP: $\Delta y_t = \varepsilon_t$; RM 1: $\Delta y_t = \alpha_1 + \alpha_2 \tau_{t-1}(d) + \alpha_3 \tau_{t-1}(d-1) + \phi \Delta^{d_1} y_{t-1} + e_t$									
T	$T = 100$			$T = 400$			$T = 1000$		
d_1 / sig.lev.	90%	95%	99%	90%	95%	99%	90%	95%	99%
0.05	-3.277	-3.583	-4.211	-3.219	-3.524	-4.076	-3.094	-3.488	-4.006
0.10	-3.179	-3.478	-4.059	-3.116	-3.418	-4.021	-3.006	-3.360	-3.877
0.15	-3.036	-3.357	-3.985	-2.993	-3.325	-3.880	-2.931	-3.252	-3.759
0.20	-2.947	-3.157	-3.835	-2.869	-3.124	-3.769	-2.784	-3.101	-3.643
0.25	-2.792	-3.014	-3.765	-2.739	-2.993	-3.674	-2.731	-2.975	-3.548
0.30	-2.670	-2.895	-3.619	-2.597	-2.889	-3.504	-2.481	-2.882	-3.433
0.35	-2.576	-2.716	3.564	-2.468	-2.806	-3.398	-2.303	-2.781	-3.352
0.40	-2.469	-2.695	-3.432	-2.340	-2.653	-3.261	-2.214	-2.600	-3.247
0.45	-2.315	-2.586	-3.320	-2.226	-2.565	-3.229	-2.049	-2.441	-3.148
0.50	-2.202	-2.428	-3.183	-2.086	-2.402	-3.050	-1.974	-2.318	-2.978
0.55	-2.100	-2.282	-3.222	-1.847	-2.370	-3.021	-1.751	-2.279	-2.930
0.60	-2.009	-2.182	-3.001	-1.758	-2.116	-2.881	-1.621	-2.164	-2.994
0.65	-1.807	-2.102	-2.849	-1.666	-2.188	-2.811	-1.563	-1.981	-2.708
0.70	-1.753	-2.015	-2.757	-1.629	-2.056	-2.735	-1.524	-1.971	-2.673
0.75	-1.641	-1.962	-2.644	-1.568	-1.982	-2.630	-1.448	-1.969	-2.617
0.80	-1.563	-1.833	-2.564	-1.492	-1.902	-2.554	-1.376	-1.759	-2.501
0.85	-1.491	-1.750	-2.505	-1.341	-1.760	-2.495	-1.331	-1.758	-2.446
0.90	-1.441	-1.702	-2.437	-1.293	-1.750	-2.428	-1.292	-1.705	2.418
0.95	-1.381	-1.682	-2.388	-1.283	-1.710	-2.372	-1.279	-1.280	-2.331

TABLE A2b

CRITICAL VALUES

DGP: $\Delta y_t = \varepsilon_t$; RM 2: $\Delta y_t = \alpha_1 \tau_{t-1}(d) + \phi \Delta^{d_1} y_{t-1} + e_t$									
T	$T = 100$			$T = 400$			$T = 1000$		
d_1 / sig.lev.	90%	95%	99%	90%	95%	99%	90%	95%	99%
0.05	-2.508	-2.808	-3.508	-2.468	-2.751	-3.360	-2.516	-2.826	-3.383
0.10	-2.424	-2.762	-3.424	-2.406	-2.676	-3.276	-2.404	-2.703	-3.296
0.15	-2.311	-2.665	-3.311	-2.318	-2.641	-3.241	-2.334	-2.651	-3.160
0.20	-2.217	-2.542	-3.217	-2.214	-2.511	-3.111	-2.168	-2.497	-3.086
0.25	-2.099	-2.380	-3.099	-2.108	-2.419	-3.033	-2.104	-2.434	-3.055
0.30	-1.994	-2.344	-2.994	-1.951	-2.296	-2.940	-1.980	-2.296	-2.904
0.35	-1.885	-2.242	-2.885	-1.880	-2.190	-2.977	-1.816	-2.158	-2.777
0.40	-1.801	-2.1267	-2.801	-1.734	-2.070	-2.749	-1.677	-2.001	-2.625
0.45	-1.724	-2.082	-2.724	-1.640	-1.999	-2.687	-1.628	-1.974	-2.673
0.50	-1.623	-1.971	-2.643	-1.514	-1.886	-2.569	-1.537	-1.872	-2.575
0.55	-1.540	-1.913	-2.596	-1.486	-1.840	-2.541	-1.430	-1.781	-2.467
0.60	-1.456	-1.821	-2.525	-1.408	-1.743	-2.511	-1.366	-1.769	-2.423
0.65	-1.449	-1.811	-2.483	-1.370	-1.730	-2.448	-1.345	-1.751	-2.469
0.70	-1.422	-1.815	-2.439	-1.347	-1.746	-2.403	-1.314	-1.696	-2.393
0.75	-1.353	-1.793	-2.393	-1.347	-1.699	-2.386	-1.307	-1.676	-2.357
0.80	-1.341	-1.736	-2.371	-1.296	-1.681	-2.351	-1.336	-1.669	-2.342
0.85	-1.310	-1.694	-2.350	-1.290	-1.682	-2.339	-1.335	-1.673	-2.337
0.90	-1.298	-1.664	-2.347	-1.305	-1.651	-2.338	-1.324	-1.649	-2.343
0.95	-1.257	-1.654	-2.337	-1.266	-1.643	-2.406	-1.262	-1.642	-2.3339