

Revealed preference and indifferent selection*

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Abstract

It is shown that preferences can be constructed from observed choice behavior in a way that is robust to *indifferent selection* (i.e., the agent is indifferent between two alternatives but, nevertheless, is only observed selecting one of them). More precisely, a suggestion by [Savage \(1954\)](#) to reveal indifferent selection by considering small monetary perturbations of alternatives is formalized and generalized to a purely topological framework: preferences over an arbitrary topological space can be uniquely derived from observed behavior under the assumptions that they are *continuous* and *nonsatiated* and that a strictly preferred alternative is always chosen, and indifferent selection is then characterized by discontinuity in choice behavior. Two particular cases are then analyzed: *monotonic* preferences over a partially ordered set, and preferences representable by a continuous *pseudo-utility* function.

Keywords. Revealed preference, indifference, choice behavior, continuity, nonsatiation, monotonicity, pseudo-utility.

JEL Classification. D11.

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Introduction

How does an agent choose between two *indifferent* alternatives a and a' ? On the one hand, indifference means that she views this choice as (*ex ante*) immaterial for her well-being, so she might as well select a , or a' , or randomize between them, or delegate her choice. On the other hand, for an outside observer who only has data about the agent's choice behavior, say the agent chooses a over a' , it is of importance to know whether the agent actually strictly prefers a to a' or she is indifferent between a and a' . In the latter case, we say that the agent makes an *indifferent selection*.

To illustrate the importance of indifferent selection, consider a social planner who has to select between two social alternatives a and a' . The alternatives involve two agents and the social planner has data about their respective choice behavior, from which he seeks to infer their respective preferences. The first agent strictly prefers a to a' and, accordingly, chooses a over a' ; the second agent is indifferent between a and a' but nevertheless selects a' over a . Then a Pareto-dominates a' , but if the social planner assumes that his observations about the agent's behavior fully reflect their preferences, he will think that the second agent strictly prefers a' to a and, hence, that a and a' are Pareto-noncomparable. Thus, neglecting indifferent selection can block Pareto-improvements (e.g., if g is the *status quo*).

How, then, can one disentangle between strict preference and indifference based on behavioral data? The usual revealed preference approach merely rules out indifferent selection by assuming that if the agent is indifferent between a and a' , then she will be observed randomizing between a and a' . However, this assumption is, however, hard to justify: why could not an indifferent agent decide to select a , say, instead of resorting to a randomization device, or to randomize subjectively rather than observably? In his pioneering work on decision making under uncertainty, [Savage \(1954, p17\)](#) noted the problem of indifferent selection and informally suggested a more satisfactory solution: he argued that indifference could be revealed by considering small monetary perturbations of alternatives. Namely, if the agent is indifferent between a and a' , then adjoining any (however small) monetary bonus to a should make it chosen over a' and, similarly, adjoining any bonus to a' should make it chosen over a . On the other hand, if she strictly prefers a to a' , then adjoining a small enough bonus to a' should not make it chosen over a .

This paper formalizes [Savage \(1954\)](#)'s argument in a general topological framework. More precisely, it is shown that if preferences over an arbitrary topological space are assumed to be *continuous* and *nonsatiated*, and that a strictly preferred alternative is always chosen, then preferences can be uniquely derived from observed behavior, and indifferent selection is characterized by discontinuity in behavior. To make the model fully behavioral, necessary and sufficient conditions on behavior are then provided under which there indeed exists a preference relation satisfying these assumptions.

Two applications of this general model are provided. First, if the set of alternatives is naturally endowed with a partial order representing some notion of *objective betterness* ([Cubitt and Sugden, 2001](#)), e.g., more money is better, then the nonsatiation condition is naturally strengthened to *monotonicity* with respect to this partial order. In this particular setup, [Savage \(1954\)](#)'s informal argument is explicitly recovered. Second (and going back to the general topological setup), in the case where the derived preferences can be represented by a continuous utility function, this function turns out to be a *pseudo-utility* representation ([Moulin, 1988](#); [Subiza and Peris, 1998](#)) of observed behavior. Thus, choice behavior can yield a continuous pseudo-utility function even though it might not be continuous nor even representable by any utility function in the usual sense. This gives rise to generalizations of classical representation results for continuous preferences over a topological space ([Eilenberg, 1941](#); [Debreu, 1954](#); [Rader, 1963](#)).

The paper is organized as follows. [Section 1](#) introduces the setup. [Section 2](#) presents an example illustrating all subsequent results. [Section 3](#) contains the general results for preferences over an arbitrary topological space. [Section 4](#) analyzes the particular case of monotonic preferences over a partially ordered set. [Section 5](#) analyzes the particular case of preferences representable by a pseudo-utility function.

1 Setup

Let \mathbb{R} and \mathbb{Q} denote the set of real and rational numbers, respectively. Consider an agent facing an arbitrary set \mathcal{A} of choice *alternatives*. The agent's *preferences* over \mathcal{A} are modeled by means of a binary relation \succsim on \mathcal{A} (i.e., $\succsim \subseteq \mathcal{A} \times \mathcal{A}$), with $a \succsim a'$ indicating that she weakly prefers a to a' . As usual, we say that \succsim is:

- *reflexive* if, for all $a \in \mathcal{A}$, $a \succcurlyeq a$,
- *complete* if, for all $a, a' \in \mathcal{A}$, $[a \succcurlyeq a' \text{ or } a' \succcurlyeq a]$,
- *antisymmetric* if, for all $a, a' \in \mathcal{A}$, $[a \succcurlyeq a' \text{ and } a' \succcurlyeq a] \Rightarrow a = a'$,
- *transitive* if, for all $a, a', a'' \in \mathcal{A}$, $[a \succcurlyeq a' \text{ and } a' \succcurlyeq a''] \Rightarrow a \succcurlyeq a''$.

From \succcurlyeq are derived the binary relations \succ and \sim on \mathcal{A} defined by, for all $a, a' \in \mathcal{A}$,

$$\begin{aligned} a \succ a' &\Leftrightarrow [a \succcurlyeq a' \text{ and not } a' \succcurlyeq a] && \text{(strict preference),} \\ a \sim a' &\Leftrightarrow [a \succcurlyeq a' \text{ and } a' \succcurlyeq a] && \text{(indifference).} \end{aligned}$$

We distinguish between preferences modeling the agent's choice behavior and preferences modeling her tastes. Formally, the agent is endowed with two distinct preference relations on \mathcal{A} :

- a *behavioral* preference relation, denoted by \succcurlyeq_B , with $a \succcurlyeq_B a'$ indicating that she would select a if she had to choose between a and a' ,
- a *cognitive* preference relation, denoted by \succcurlyeq_C , with $a \succcurlyeq_C a'$ indicating that she likes a at least as much as a' .

We assume that the agent's choice behavior does not contradict her tastes in the sense of choosing a strictly less liked alternative:

Definition. A binary relation \succcurlyeq_B on a set \mathcal{A} is **compatible** with a binary relation \succcurlyeq_C on \mathcal{A} if, for all $a, a' \in \mathcal{A}$, $a \succ_C a' \Rightarrow a \succ_B a'$.

Although situations in which $a \succ_C a'$ and $a' \succ_B a$ can be conceived (e.g., addiction), compatibility of behavior with tastes seems natural in most economic settings. Note that no constraint is imposed on choice behavior in situations of cognitive indifference, which allows for *indifferent selection*, i.e., $a \sim_C a'$ and $[a \succ_B a' \text{ or } a' \succ_B a]$. This is unlike many economic models that implicitly assume $\succcurlyeq_B = \succcurlyeq_C$ (Mandler, 2005) and, hence, $\sim_B = \sim_C$, thereby ruling out indifferent selection (e.g., when determining equilibrium and analyzing welfare by means of a single preference relation per agent). Whereas the conceptual distinction between these two preference concepts is well-understood in the literature (Mas-Colell, Whinston, and Green, 1995, p5), behavioral preferences falling in the revealed preference tradition (Samuelson, 1938) and cognitive preferences in the ordinalist tradition (Pareto, 1906), their formal distinction is a key feature of the present analysis.

Note that assuming $\succsim_B = \succsim_C$ first requires that one allows for behavioral indifference between distinct alternatives (otherwise cognitive indifference is ruled out). Thus, the revealed preference literature often finds it convenient to interpret behavioral preference as indicating that an alternative is “choosable” rather than actually chosen, and to operationalize this interpretation by enabling the agent to resort to some observable randomization (or delegation) device rather than selecting one single alternative by herself. However, even if randomization is allowed and identified with behavioral indifference, it is hard to justify why the agent could not decide to select a single alternative by herself rather than randomize when she is cognitively indifferent (in fact, cognitive indifference precisely means that she views such a selection as harmless), or to randomize subjectively rather than observably. The starting point of the present analysis is the formal separation between the two preference concepts and the degree of freedom between them that is allowed by the compatibility assumption. Although it becomes possible, in this framework, to rule out $a \sim_B a'$ *per se*, in line with a fully behavioral interpretation of preference, this turns out to be technically unnecessary and, consequently, \succsim_B is not assumed to be antisymmetric.

Both \succsim_B and \succsim_C are assumed to be complete. Completeness of behavioral preferences means that choice between any two alternatives in \mathcal{A} is conceivable and is generally considered an innocuous assumption. Completeness of cognitive preferences, although also a standard assumption, is often judged restrictive: it means that the agent is able to come up with a judgment about which of any two alternatives in \mathcal{A} she likes better. The role of this assumption here is to single out indifferent selection as the only possible source of discrepancy between behavioral and cognitive preferences:

Lemma 1. Let \succsim_B and \succsim_C be two complete binary relations on a set \mathcal{A} such that \succsim_B is compatible with \succsim_C . Then $\succsim_C = \succsim_B \cup \sim_C$.

Proof. By compatibility, one has $\succ_C \subseteq \succ_B$ and, hence, $\succsim_C = \succ_C \cup \sim_C \subseteq \succ_B \cup \sim_C$. Conversely, completeness of \succsim_C and compatibility together imply $\succsim_B \subseteq \succsim_C$ and, hence, $\succ_B \cup \sim_C \subseteq \succ_C \cup \sim_C = \succsim_C$. \square

As appears from [Lemma 1](#), under our completeness and compatibility assumptions, cognitive and behavioral preferences only differ for alternatives a and a' such

that $a \sim_C a'$ and $[a \succ_B a' \text{ or } a' \succ_B a]$ (for a similar analysis allowing for incomplete preferences, see Danan, 2003). Finally, it should be noted that neither cognitive nor behavioral preferences are assumed to be transitive.

The present analysis assumes that \succ_B is observable but not \succ_C , and addresses the question whether \succ_C can be recovered from \succ_B . This is in line with a revealed preference approach: the agent's tastes, which are unobservable mental states, are revealed by her observable choice behavior (if choice behavior is modeled, more generally, by a choice function, then the existence of a complete behavioral preference relation is characterized by Sen (1971)'s Axioms α and γ). In the present setup, the problem boils down to that of behaviorally identifying indifferent selection.

2 An example

As an illustration, suppose alternatives are commodity bundles made of two goods, and the agent is observed choosing lexicographically by, first, maximizing the total quantity of goods and, second, maximizing the quantity of good 1. Formally, $\mathcal{A} = \mathbb{R}^2$ and, for all $(x_1, x_2), (x'_1, x'_2) \in \mathbb{R}^2$,

$$(x_1, x_2) \succ_B (x'_1, x'_2) \Leftrightarrow [x_1 + x_2 > x'_1 + x'_2 \text{ or } [x_1 + x_2 = x'_1 + x'_2 \text{ and } x_1 \geq x'_1]].$$

Note that \succ_B is antisymmetric and, therefore, can receive a fully behavioral interpretation. What can we say about the agent's cognitive preferences? First, it is possible that her tastes fully coincide with her observed behavior and that she never makes an indifferent selection, i.e., $\succ_C = \succ_B$. But it might also be the case that her tastes do not select between any alternatives and that her behavioral preferences fully result from indifferent selection, i.e., $\succ_C = \mathcal{A} \times \mathcal{A}$. Between these two extremes, it is conceivable, e.g., that only the total quantity of goods matters according to her tastes whereas maximizing the quantity of good 1 is only used to achieve indifferent selection, i.e., for all $(x_1, x_2), (x'_1, x'_2) \in \mathbb{R}^2$,

$$(x_1, x_2) \succ_C (x'_1, x'_2) \Leftrightarrow x_1 + x_2 \geq x'_1 + x'_2.$$

Thus, additional assumptions must be imposed in order to pin down the agent's cognitive preferences.

Now, note that, among the three possible cognitive preference relations defined above, the first is discontinuous and the second is locally satiated (i.e., it has a local maximum). As it turns out, the latter is the unique continuous and locally nonsatiated cognitive preference relation with which \succcurlyeq_B is compatible. Furthermore, the difference between \succcurlyeq_B (which is locally nonsatiated and discontinuous) and this third cognitive preference relation resides in the continuity of the latter: indifferent selection is characterized by discontinuity of the behavioral preference relation. As shown in [Section 3](#), this argument can be generalized to preferences over an arbitrary topological space \mathcal{A} .

In this example, nonsatiation of \succcurlyeq_C (the third one) takes a particular form: not only is any bundle strictly dispreferred to some other bundle in each of its neighborhoods, but there are some specific directions along which these other bundles are found. Namely, $(x'_1, x'_2) \succ_C (x_1, x_2)$ whenever $(x'_1, x'_2) > (x_1, x_2)$, i.e., \succcurlyeq_C is monotonic with respect to the vector order \geq on \mathbb{R}^2 . If we assume that one of the goods is money, then we explicitly recover [Savage \(1954\)](#)'s informal argument that indifferent selection can be revealed by adjoining small monetary bonuses to alternatives: it is characterized by a strict behavioral preference that is reversed by adjoining an arbitrarily small bonus by the unchosen alternative. [Section 4](#) generalizes this argument to preferences over any partially ordered set \mathcal{A} satisfying some unboundedness and denseness properties.

Finally, note that \succcurlyeq_C can be represented by the continuous utility function $u : \mathcal{A} \rightarrow \mathbb{R}$ defined by, for all $(x_1, x_2) \in \mathbb{R}^2$, $u(x_1, x_2) = x_1 + x_2$, and that u is unique up to a strictly increasing and continuous transformation. This function, then, also represent \succcurlyeq_B , but in a weaker sense (namely, $(x_1, x_2) \succ_B (x'_1, x'_2)$ whenever $u(x_1, x_2) > u(x'_1, x'_2)$, but information on indifferent selection is lost). Note that this continuous utility function can be derived even though \succcurlyeq_B is neither continuous nor even representable by any utility function in the usual sense. [Section 5](#) proceeds in this fashion to generalize classical representation results for continuous preferences over a topological space \mathcal{A} satisfying some countability and connectedness properties.

3 Main results

Now turning to the general case, we assume \mathcal{A} is a topological space (Munkres, 2000) and denote by $\text{cl}(\cdot)$ and $\text{int}(\cdot)$ the closure and interior operators on \mathcal{A} , respectively. Given an alternative $a \in \mathcal{A}$ and a binary relation B on \mathcal{A} ($B = \succcurlyeq, \succ, \sim$), we define the following subsets of \mathcal{A} :

$$\begin{aligned} A(B, a) &= \{a' \in \mathcal{A} : a' B a\}, \\ A(a, B) &= \{a' \in \mathcal{A} : a B a'\}. \end{aligned}$$

We say that a binary relation \succcurlyeq on \mathcal{A} is:

- *upper semi-continuous* if, for all $a \in \mathcal{A}$, $A(\succcurlyeq, a)$ is closed,
- *weakly upper nonsatiated* if, for all $a \in \mathcal{A}$, $a \in \text{cl}(A(\succ, a))$,
- *strongly upper nonsatiated* if, for all $a, a' \in \mathcal{A}$, $a \succcurlyeq a' \Rightarrow a \in \text{cl}(A(\succ, a'))$.

Upper semi-continuity is defined as usual (if \succcurlyeq is complete, then it is equivalent to require that $A(a, \succ)$ be open). Weak upper nonsatiation also corresponds to the standard notion of local nonsatiation of preferences: no alternative in \mathcal{A} is a local maximum for \succcurlyeq . If \succcurlyeq is transitive, then this local property has the following global implication: if $a \succcurlyeq a'$, then any neighborhood of a contains an alternative a'' such that $a'' \succ a$ and, hence, $a'' \succ a'$, so $a \in \text{cl}(A(\succ, a'))$. This latter property is independently stated as strong upper nonsatiation because transitivity is not assumed here (it indeed implies weak upper nonsatiation provided that \succcurlyeq is reflexive). Note that if \succcurlyeq is strongly upper nonsatiated, then it has “thin indifference curves” in the sense that, for all $a \in \mathcal{A}$, $A(\sim, a)$ is nowhere dense.

Upper semi-continuity and strong upper nonsatiation turn out to be sufficient for pinning down the agent’s cognitive preference relation:

Theorem 1. Let \succcurlyeq_B and \succcurlyeq_C be two complete binary relations on a topological space \mathcal{A} such that \succcurlyeq_B is compatible with \succcurlyeq_C . If \succcurlyeq_C is upper semi-continuous and strongly upper nonsatiated, then, for all $a, a' \in \mathcal{A}$,

$$a \succcurlyeq_C a' \Leftrightarrow a \in \text{cl}(A(\succcurlyeq_B, a')). \quad (1)$$

Proof. Let $a, a' \in \mathcal{A}$. If $a \succcurlyeq_C a'$, then $a \in \text{cl}(A(\succcurlyeq_C, a'))$ because \succcurlyeq_C is strongly upper nonsatiated and, hence, $a \in \text{cl}(A(\succcurlyeq_B, a'))$ because \succcurlyeq_B is compatible with

\succsim_C , so $a \in \text{cl}(A(\succsim_B, a'))$. Conversely, if $a \in \text{cl}(A(\succsim_B, a'))$, then $a \in \text{cl}(A(\succsim_C, a'))$ because \succsim_C is complete and \succsim_B is compatible with \succsim_C and, hence, $a \in A(\succsim_C, a')$ because \succsim_C is upper semi-continuous, i.e., $a \succsim_C a'$. \square

To understand how indifferent selection is behaviorally identified, consider two alternatives a and a' such that $a \succ_B a'$ and say that (a, a') is an *upper semi-discontinuity point* for \succsim_B if $a' \in \text{cl}(A(\succsim_B, a))$. It then follows from Equation 1 that the agent makes an indifferent selection between a and a' (i.e., $a \sim_C a'$) if and only if (a, a') is an upper semi-discontinuity point for \succsim_B . More precisely, if (a, a') is an upper semi-discontinuity point for \succsim_B , then one must have $a \sim_C a'$ by upper semi-continuity of \succsim_C (this rules out, e.g., $\succsim_C = \succsim_B$ in Section 2's example). Conversely, if (a, a') is not an upper semi-discontinuity point for \succsim_B , then one cannot have $a \sim_C a'$ by strong upper nonsatiation of \succsim_C (this rules out, e.g., $\succsim_C = \mathcal{A} \times \mathcal{A}$ in Section 2's example).

Now, in order to obtain a fully behavioral result, we need to characterize those behavioral preference relations \succsim_B for which the cognitive preference relation \succsim_C defined by Equation 1 is indeed upper semi-continuous and strongly upper nonsatiated. Because \succsim_C only differs from \succsim_B by the fact that it has more indifferences (see Lemma 1), satiation of \succsim_B would imply satiation of \succsim_C , so it is necessary that \succsim_B be strongly upper nonsatiated. On the other hand, upper semi-continuity of \succsim_B is not necessary, as shown by Section 2's example. However, strong upper nonsatiation of \succsim_B alone is not sufficient. For example, let $\mathcal{A} = \mathbb{R}$, define the function $f : \mathbb{R} \rightarrow \mathbb{R}$ by, for all $x \in \mathbb{R}$, $f(x) = x$ if $x \in \mathbb{Q}$ and $f(x) = -x$ otherwise, and assume the agent's behavioral preference relation \succsim_B is given by, for all $x, x' \in \mathbb{R}$, $x \succsim_B x' \Leftrightarrow f(x) \geq f(x')$. Then \succsim_B is strongly upper nonsatiated but Equation 1 implies $x \sim_C 0$ for all $x \in \mathbb{R}$, so \succsim_C is not strongly upper nonsatiated. The problem is that \succsim_B is too discontinuous. More precisely, $(x, 0)$ is an upper semidiscontinuity point for all $x \in \mathbb{Q}$ and \mathbb{Q} is dense in \mathbb{R} . On the contrary, for any alternative $a' \in \mathcal{A}$, we need (a, a') to be an upper semi-discontinuity point for only a nowhere dense set of alternatives a for $A(\sim_C, a')$ to be nowhere dense.

To capture this latter property, say that a binary relation \succ on a topological space \mathcal{A} is *upper archimedean* if, for all $a, a' \in \mathcal{A}$ such that $a \succ a'$ and for all neighborhood V of a , there exist $\hat{a} \in V$ and a neighborhood V' of a' such that, for all $\hat{a}' \in V'$, $\hat{a} \succ \hat{a}'$. Intuitively, small variations in alternatives have a small effect on

preferences: if $a \succ a'$ and one moves from a to some close (and strictly preferred, by nonsatiation) \hat{a} , then $\hat{a} \succ \hat{a}'$ for all \hat{a}' sufficiently close to a' . It follows that (\hat{a}, a') cannot be an upper semi-discontinuity point for any \hat{a} sufficiently close to a , ruling out the preceding example. Any upper semi-continuous binary relation is upper archimedean, but the converse does not hold, as shown by [Section 2](#)'s example. We can now state:

Theorem 2. Let \succcurlyeq_B be a complete binary relation on a topological space \mathcal{A} . Then \succcurlyeq_B is upper archimedean and strongly upper nonsatiated if and only if there exists a complete, upper semi-continuous and strongly upper nonsatiated binary relation \succcurlyeq_C on \mathcal{A} such that \succcurlyeq_B is compatible with \succcurlyeq_C . Moreover, \succcurlyeq_C is unique.

Proof. Uniqueness follows from [Theorem 1](#). Assume that \succcurlyeq_B is upper archimedean and strongly upper nonsatiated. The binary relation \succcurlyeq_C on \mathcal{A} defined by [Equation 1](#) is complete by completeness of \succcurlyeq_B and upper semi-continuous by definition. Moreover, for all $a, a' \in \mathcal{A}$, $a \succcurlyeq_B a' \Rightarrow a \succcurlyeq_C a'$ and, hence, \succcurlyeq_B is compatible with \succcurlyeq_C . Now, let $a, a' \in \mathcal{A}$ such that $a \succcurlyeq_C a'$, and let V be a neighborhood of a . Then there exists $\bar{a} \in V$ such that $\bar{a} \succcurlyeq_B a'$. Hence, by strong upper nonsatiation of \succcurlyeq_B , there exists $\tilde{a} \in V$ such that $\tilde{a} \succ_B a'$. Hence, by upper archimedeanity of \succcurlyeq_B , there exist $\hat{a} \in V$ and a neighborhood V' of a' such that $\hat{a} \succ_B \hat{a}'$ for all $\hat{a}' \in V'$. It follows that $\hat{a} \succ_C a'$ by [Equation 1](#), so \succcurlyeq_C is strongly upper nonsatiated.

Conversely, assume that there exists a complete, upper semi-continuous and strongly upper nonsatiated binary relation \succcurlyeq_C on \mathcal{A} such that \succcurlyeq_B is compatible with \succcurlyeq_C . Let $a, a' \in \mathcal{A}$ such that $a \succcurlyeq_B a'$. Then $a \succcurlyeq_C a'$ by completeness of \succcurlyeq_C and compatibility and, hence, $a \in \text{cl}(A(\succcurlyeq_C, a'))$ by strong upper nonsatiation of \succcurlyeq_C . Hence $a \in \text{cl}(A(\succcurlyeq_B, a'))$ by compatibility, so \succcurlyeq_B is strongly upper nonsatiated. Now, let $a, a' \in \mathcal{A}$ such that $a \succ_B a'$, and let V be a neighborhood of a . By the preceding argument, there exists $\hat{a} \in V$ such that $\hat{a} \succ_C a'$. Hence, by completeness of \succcurlyeq_B , there exists a neighborhood V' of a' such that $\hat{a} \succ_B \hat{a}'$ for all $\hat{a}' \in V'$, so \succcurlyeq_B is upper archimedean. \square

Finally, let us note that [Theorem 1](#) and [Theorem 2](#) also hold if upper semi-continuity, strong upper nonsatiation, and upper archimedeanity are replaced by their lower analogs. More precisely, say that a binary relation \succcurlyeq on a topological space \mathcal{A} is:

- *lower semi-continuous* if, for all $a \in \mathcal{A}$, $A(a, \succsim)$ is closed,
- *weakly lower nonsatiated* if, for all $a \in \mathcal{A}$, $a \in \text{cl}(A(a, \succ))$,
- *strongly lower nonsatiated* if, for all $a, a' \in \mathcal{A}$, $a \succ a' \Rightarrow a' \in \text{cl}(A(a, \succ))$,
- *lower archimedean* if, for all $a, a' \in \mathcal{A}$ such $a \succ a'$ and for all neighborhood V' of a' , there exist $\hat{a}' \in V'$ and a neighborhood V of a such that, for all $\hat{a} \in V$, $\hat{a} \succ \hat{a}'$.

Equation 1 then becomes:

$$a \succ_C a' \Leftrightarrow a' \in \text{cl}(A(a, \succ_B)).$$

Moreover, the upper and lower versions of the results can straightforwardly be combined: say that a binary relation \succsim on a topological space \mathcal{A} is *continuous* if it is both upper semi-continuous and lower semi-continuous, *weakly* (resp., *strongly*) *nonsatiated* if it is both weakly (resp., strongly) upper nonsatiated and weakly (resp., strongly) lower nonsatiated, and *archimedean* if it is both upper archimedean and lower archimedean.

4 Monotonic preferences

In his pioneering work on decision making under uncertainty, Savage (1954) mentioned the problem of indifferent selection, and informally suggested to solve it by adjoining small monetary bonuses to alternatives. As he argued, cognitive weak preference for a over a' could then be behaviorally identified with the observation that any (however small) bonus adjoined to a makes it chosen over a' . This argument can be formally recovered as a special case of Section 3's analysis. More precisely, the essential point about money here is that any strictly positive monetary bonus can be considered as an objective improvement, i.e., it can be assumed that the agents always prefers more money.

In order to capture this notion of objective betterness (Cubitt and Sugden, 2001), we assume that \mathcal{A} is a partially ordered set (poset), and denote by \geq^* its partial order relation (i.e., \geq^* is a reflexive, antisymmetric, and transitive binary relation on \mathcal{A}). If $a \geq^* a'$, we say that a *dominates* a' , i.e., a is objectively strictly better than a' . We also assume that the poset \mathcal{A} is *unbounded* (i.e., for all $a \in \mathcal{A}$, there exist $a', a'' \in \mathcal{A}$ such that $a' \geq^* a \geq^* a''$) and *dense* (i.e., for all $a, a' \in \mathcal{A}$ such that $a \geq^* a'$, there exists $a'' \in \mathcal{A}$ such that $a \geq^* a'' \geq^* a'$).

The monetary incentives setup corresponds to the following special case: assume $\mathcal{A} = \mathcal{B} \times \mathbb{R}$, where \mathcal{B} is an arbitrary set of basic alternatives (on which one wants to elicit the agent's preferences), and \mathbb{R} stands for the set of monetary payoffs. Thus the alternative $(b, \varepsilon) \in \mathcal{A}$ corresponds to the basic alternative b with an adjoined monetary bonus ε (note that \mathbb{R} can be reduced to any open interval). Define the partial order \geq^* on \mathcal{A} by, for all $(b, \varepsilon), (b', \varepsilon') \in \mathcal{B} \times \mathbb{R}$,

$$(b, \varepsilon) \geq^* (b', \varepsilon') \Leftrightarrow [b = b' \text{ and } \varepsilon \geq \varepsilon'].$$

Then \geq^* is unbounded (meaning that a positive or negative monetary bonus can be adjoined to any alternative in \mathcal{A}) and dense (reflecting perfect divisibility of money). Note that, more generally, \mathbb{R} could be replaced by any set naturally endowed with an unbounded and dense order (i.e., complete partial order), e.g., representing quality of some good (higher quality is better), or time (earlier is better). Besides these special cases in which incentives correspond to a separate dimension of alternatives, the general setup applies if \mathcal{A} is, e.g., a commodity space (\geq^* being the natural vector order on \mathbb{R}^n) or a space of lotteries or acts whose outcomes are commodity bundles (\geq^* being the first-order stochastic dominance relation).

Given a binary relation \succsim on a poset \mathcal{A} , define the binary relation \succsim^* on \mathcal{A} by, for all $a, a' \in \mathcal{A}$,

$$a \succsim^* a' \Leftrightarrow A(>^*, a) \subseteq A(\succsim, a'),$$

i.e., $a \succsim^* a'$ if and only if any alternative dominating a is strictly preferred to a' , and say that \succsim is:

- *weakly upper monotonic* if, for all $a \in \mathcal{A}$, $a \succsim^* a$,
- *strongly upper monotonic* if, for all $a, a' \in \mathcal{A}$, $a \succ a' \Rightarrow a \succsim^* a'$.

As for nonsatiation, weak upper monotonicity is the usual requirement that dominating alternatives are strictly preferred, and strong upper monotonicity (which implies the weak version provided that \succsim is reflexive) follows from it if \succsim is transitive but must be independently stated otherwise. It is intuitively straightforward that monotonicity implies nonsatiation: by unboundedness of \mathcal{A} , any alternative $a \in \mathcal{A}$ is dominated by some a' and by denseness of \mathcal{A} this a' can be taken arbitrarily close to a , so no a cannot be a local maximum if \succsim is monotonic. To establish

this statement (and others) formally, we endow \mathcal{A} with the partial order topology (i.e., the topology generated by the subbasis $\bigcup_{a \in \mathcal{A}} \{A(>^*, a), A(a, >^*)\}$, which is well-defined by unboundedness of \mathcal{A}). We then obtain:

Lemma 2. Let \succsim be a complete and strongly upper monotonic binary relation on an unbounded and dense poset \mathcal{A} . Then:

- a. \succsim is strongly upper nonsatiated,
- b. \succsim is upper semi-continuous if and only if, for all $a, a' \in \mathcal{A}$, $a \succsim^* a' \Rightarrow a \succsim a'$,
- c. \succsim is upper archimedean if and only if, for all $a, a', \hat{a} \in \mathcal{A}$ such that $[\hat{a} >^* a$ and $a \succ a']$, there exists $\hat{a}' \in \mathcal{A}$ such that $[\hat{a} \succ \hat{a}'$ and $\hat{a}' >^* a']$.

Proof. *a.* Let $a, a' \in \mathcal{A}$ such that $a \succsim a'$ and let V be a neighborhood of a . By unboundedness of \mathcal{A} , there exists $\tilde{a} \in \mathcal{A}$ dominating a such that V contains all $\hat{a} \in \mathcal{A}$ such that $[\tilde{a} >^* \hat{a}$ and $\hat{a} >^* a]$. By denseness of \mathcal{A} , there exists such a \hat{a} . By strong upper monotonicity of \succsim , $\hat{a} \succ a'$ and, hence, $a \in \text{cl}(A(\succ, a'))$.

b. Assume that \succsim is upper semi-continuous, let $a, a' \in \mathcal{A}$ such that $a \succsim^* a'$, and let V be a neighborhood of a . By the preceding argument, there exists $\hat{a} \in V$ such that $\hat{a} >^* a$ and, hence, $\hat{a} \succ a'$, so $a \succsim a'$. Conversely, assume that, for all $a, a' \in \mathcal{A}$, $a \succsim^* a' \Rightarrow a \succsim a'$, and let $a, a' \in \mathcal{A}$ such that $a \in \text{cl}(A(\succsim, a'))$. For all $\hat{a} \in \mathcal{A}$ such that $\hat{a} >^* a$, $A(\hat{a}, >^*)$ is a neighborhood of a and, hence, there exists $\tilde{a} \in A(\hat{a}, >^*)$ such that $\tilde{a} \succsim a'$. By strong upper monotonicity of \succsim , this implies $\hat{a} \succ a'$. Hence $a \succsim^* a'$ and, hence $a \succsim a'$.

c. Assume that \succsim is upper archimedean, and let $a, a', \hat{a} \in \mathcal{A}$ such that $[\hat{a} >^* a$ and $a \succ a']$. Then $A(\hat{a}, >^*)$ is a neighborhood of a and, hence, there exist $\tilde{a} \in A(\hat{a}, >^*)$ and a neighborhood V' of a' such that $\tilde{a} \succ \tilde{a}'$ for all $\tilde{a}' \in V'$. Hence $\hat{a} \succ \tilde{a}'$ for all $\tilde{a}' \in V'$ by strong upper monotonicity of \succsim and, in particular, $\hat{a} \succ \hat{a}'$ for some $\hat{a}' \in V'$ such that $\hat{a}' >^* a'$. Conversely, assume that, for all $a, a', \hat{a} \in \mathcal{A}$ such that $[\hat{a} >^* a$ and $a \succ a']$, there exists $\hat{a}' \in \mathcal{A}$ such that $[\hat{a} \succ \hat{a}'$ and $\hat{a}' >^* a']$. Let $a, a' \in \mathcal{A}$ such that $a \succ a'$ and let V be a neighborhood of a . Then there exists $\hat{a} \in V$ such that $\hat{a} >^* a$ and, hence, there exists $\hat{a}' \in \mathcal{A}$ such that $[\hat{a} \succ \hat{a}'$ and $\hat{a}' >^* a']$. It follows that $A(\hat{a}', >^*)$ is a neighborhood of a' . Now, suppose there exists $\tilde{a}' \in A(\hat{a}', >^*)$ such that not $\hat{a} \succ \tilde{a}'$. Then $\tilde{a}' \succsim \hat{a}$ by completeness of \succsim and, hence, $\hat{a}' \succ \hat{a}$ by strong upper monotonicity of \succsim , a contradiction. \square

Note that the weak version of part **a** also holds: if \succsim is weakly upper monotonic, then it is weakly upper nonsatiated. Parts **b** and **c** give restatements of the preceding

section's topological properties given the order structure assumed in the present section. In particular, archimedeaness now more transparently reflects the property that small changes in alternatives have a small effect on preference: if $a \succ a'$ and some (however small) bonus is adjoined to a (yielding the alternative \hat{a}), then it must be possible to adjoin some (small enough) bonus to a' (yielding the alternative \hat{a}') so as to preserve the relation $\hat{a} \succ \hat{a}'$. We obtain analogs of [Theorem 1](#) and [Theorem 2](#) in this setup:

Theorem 3. Let \succcurlyeq_B and \succcurlyeq_C be two complete binary relations on an unbounded and dense poset \mathcal{A} such that \succcurlyeq_B is compatible with \succcurlyeq_C . If \succcurlyeq_C is upper semi-continuous and strongly upper monotonic, then $\succcurlyeq_C = \succcurlyeq_B^*$.

Proof. Let $a, a' \in \mathcal{A}$. If $a \succcurlyeq_C a'$, then $a \succcurlyeq_C^* a'$ by strong upper monotonicity of \succcurlyeq_C and, hence, $a \succcurlyeq_B^* a'$ by compatibility. Conversely, if $a \succcurlyeq_B^* a'$, then $\hat{a} \succcurlyeq_C a'$ for all $\hat{a} \in \mathcal{A}$ such that $\hat{a} >^* a$ by completeness of \succcurlyeq_C and compatibility. Now, for all $\hat{a} \in \mathcal{A}$ such that $\hat{a} >^* a$, there exists $\tilde{a} \in \mathcal{A}$ such that $\hat{a} >^* \tilde{a} >^* a$ by denseness of \mathcal{A} and, hence, $\hat{a} \succcurlyeq_C a'$ by strong upper monotonicity of \succcurlyeq_C . Hence $a \succcurlyeq_C^* a'$ and, hence, $a \succcurlyeq_C a'$ by upper semi-continuity of \succcurlyeq_C . \square

Theorem 4. Let \succcurlyeq_B be a complete binary relation on an unbounded and dense poset \mathcal{A} . Then \succcurlyeq_B is upper archimedean and strongly upper monotonic if and only if there exists a complete, upper semi-continuous and strongly upper monotonic binary relation \succcurlyeq_C on \mathcal{A} such that \succcurlyeq_B is compatible with \succcurlyeq_C . Moreover, \succcurlyeq_C is unique.

Proof. Assume that \succcurlyeq_B is upper archimedean and strongly upper monotonic. By [Theorem 2](#), it is sufficient to show that the binary relation \succcurlyeq_C on \mathcal{A} defined by [Equation 1](#) is strongly upper monotonic. Let $a, a', \hat{a} \in \mathcal{A}$ such that $a \succcurlyeq_C a'$ and $\hat{a} >^* a$. Then $A(\hat{a}, >^*)$ is a neighborhood of a and, hence, there exists $\bar{a} \in A(\hat{a}, >^*)$ such that $\bar{a} \succcurlyeq_B a'$. By denseness of \mathcal{A} , there exists $\tilde{a} \in A(\hat{a}, >^*)$ such that $\tilde{a} >^* \bar{a}$ and, hence $\tilde{a} \succcurlyeq_B a'$ by strong upper monotonicity of \succcurlyeq_B . Hence, by upper archimedeaness of \succcurlyeq_B , there exist $\check{a} \in A(\hat{a}, >^*)$ and a neighborhood V' of a' such that $\check{a} \succcurlyeq_B \hat{a}'$ for all $\hat{a}' \in V'$. By strong upper monotonicity of \succcurlyeq_B , it follows that $\hat{a} \succcurlyeq_B \hat{a}'$ for all $\hat{a}' \in V'$. Hence not $a' \succcurlyeq_C a$, i.e., $a \succcurlyeq_C a'$ by completeness of \succcurlyeq_C .

Conversely, assume that there exists a complete, upper semi-continuous and strongly upper monotonic binary relation \succcurlyeq_C on \mathcal{A} such that \succcurlyeq_B is compatible with

\succ_C . By [Theorem 2](#), it is sufficient to show that \succ_B is strongly upper monotonic. Let $a, a' \in \mathcal{A}$ such that $a \succ_B a'$. Then $a \succ_C a'$ by completeness of \succ_C and compatibility and, hence $a \succ_C^* a'$ by strong upper monotonicity of \succ_C . Hence $a \succ_B^* a'$ by compatibility. \square

As in the general case, lower versions of weak and strong monotonicity can be defined and the corresponding results follow. Moreover, the upper and lower results can be combined, but note that strong upper monotonicity and strong lower monotonicity are equivalent for a complete binary relation (and similarly for weak monotonicity).

5 Pseudo-utility

It is well-known ([Cantor, 1895](#)) that a complete binary relation \succ on a set \mathcal{A} is representable by a *utility* function (i.e., there exists a function $u : \mathcal{A} \rightarrow \mathbb{R}$ such that, for all $a, a' \in \mathcal{A}$, $u(a) \geq u(a') \Leftrightarrow a \succ a'$) if and only if it is transitive and *separable* (i.e., there exists a countable subset A of \mathcal{A} such that, for all $a, a' \in \mathcal{A}$ such that $a \succ a'$, there exists $\hat{a} \in A$ such that $a \succ \hat{a} \succ a'$). Now, in the present setup, starting from a complete binary relation \succ_B on a topological space \mathcal{A} , we derive a complete binary relation \succ_C and, if \succ_C is transitive and separable, it can be represented by a utility function u , even though \succ_B might be neither transitive nor separable. Furthermore, because \succ_C is continuous, we know that u can be taken continuous provided that \mathcal{A} satisfies certain topological properties ([Eilenberg, 1941](#); [Debreu, 1954](#); [Rader, 1963](#)), even though \succ_B might be discontinuous. Thus, the present analysis can yield generalizations of classical results on representation of presences by continuous utility functions.

Because \succ_B is compatible with \succ_C but might not be equal to \succ_C , a utility function u representing \succ_C does not in general represent \succ_B in the classical sense, but in the following, weaker sense ([Moulin, 1988](#); [Subiza and Peris, 1998](#)):

Definition. A function $u : \mathcal{A} \rightarrow \mathbb{R}$ is a **pseudo-utility** representation of a binary relation \succ_B on a set \mathcal{A} if, for all $a, a' \in \mathcal{A}$, $u(a) > u(a') \Rightarrow a \succ_B a'$.

A pseudo-utility representation u is not a full utility representation because it loses information on the preference relation between alternatives $a, a' \in \mathcal{A}$ such

that $u(a) = u(a')$ (i.e., in the present framework, on indifferent selection). For this reason, a constant u is a pseudo-utility representation of any binary relation and, therefore, additional restrictions must be imposed for the representation to be of interest. Subiza and Peris (1998) assume that \mathcal{A} is a topological space (endowed with some finite measure) and require u to be continuous and *nontrivial* (i.e., if $[a \succ_B \hat{a}$ and $\hat{a} \succ_B a']$ for all \hat{a} in some open subset of \mathcal{A} , then $u(a) > u(a')$). In the present setup, by nonsatiation of \succ_C , a stronger requirement than nontriviality can be imposed: say that u is *locally unbounded above* if, for all open subset V of \mathcal{A} , $\arg \max_{a \in V} u(a) = \emptyset$.

Lemma 3. Let \succ_B be a binary relation on a topological space \mathcal{A} and u be a pseudo-utility representation of \succ_B . If u is locally unbounded above, then it is nontrivial.

Proof. Assume that u is locally unbounded above, and let $a, a' \in \mathcal{A}$ such that $[a \succ_B \hat{a}$ and $\hat{a} \succ_B a']$ for all \hat{a} in some open subset V of \mathcal{A} . Then $u(a) \geq u(\hat{a})$ and $u(\hat{a}) \geq u(a')$ because u is a pseudo-utility representation of \succ_B . Moreover, by local unboundedness of u , it is not possible that $u(a) = u(\hat{a})$ for all $\hat{a} \in V$. Hence there must exist $\hat{a} \in V$ such that $u(a) > u(\hat{a})$ and, hence, $u(a) > u(a')$, so $a \succ_B a'$. \square

In order to establish pseudo-utility representation results, we need to characterize those behavioral preference relations \succ_B on \mathcal{A} for which the cognitive preference relation \succ_C on \mathcal{A} defined by Equation 1 is transitive. Say that a binary relation \succ on a topological space \mathcal{A} is *upper closure-transitive* if, for all $a, a', a'' \in \mathcal{A}$, $[a \in \text{cl}(A(\succ, a'))$ and $a' \in \text{cl}(A(\succ, a''))] \Rightarrow a \in \text{cl}(A(\succ, a''))$. In the present setup, upper closure-transitivity is weaker than transitivity, but still yields equivalence between weak and strong nonsatiation:

Lemma 4. Let \succ be a complete, upper archimedean, and weakly upper nonsatiated binary relation on a topological space \mathcal{A} . Then:

- a. if \succ is transitive, then it is upper closure-transitive,
- b. if \succ is upper closure-transitive, then it is strongly upper nonsatiated.

Proof. a. Assume that \succ is transitive, let $a, a', a'' \in \mathcal{A}$ such that $[a \in \text{cl}(A(\succ, a'))$ and $a' \in \text{cl}(A(\succ, a''))]$, and let V be a neighborhood of a . Then there exists $\hat{a} \in V$ such that $\hat{a} \succ a'$. By weak upper nonsatiation of \succ , there exists $\tilde{a} \in V$ such that $\tilde{a} \succ \hat{a}$ and, hence, $\tilde{a} \succ a'$ by transitivity of \succ . Hence, by upper archimedeaness of

\succsim , there exist $\bar{a} \in V$ and a neighborhood V' of a' such that $\bar{a} \succ \hat{a}'$ for all $\hat{a}' \in V'$. Moreover, there exists $\tilde{a}' \in V'$ such that $\tilde{a}' \succsim a''$ and, hence, $\bar{a} \succ \hat{a}''$ by transitivity of \succsim , so $a \in \text{cl}(A(\succsim, a''))$.

b. Assume that \succsim is upper closure-transitive, let $a, a' \in \mathcal{A}$ such that $a \succsim a'$, and let V be a neighborhood of a . Then, by weak upper nonsatiation of \succsim , there exists $\hat{a} \in V$ such that $\hat{a} \succ a$. Hence, by upper archimedeaness of \succsim , there exist $\tilde{a} \in V$ and a neighborhood \bar{V} of a such that $\tilde{a} \succ \bar{a}$ for all $\bar{a} \in \bar{V}$. Now, suppose there does not exist $\check{a} \in V$ such that $\check{a} \succ a'$. Then $a' \succsim \check{a}$ for all $\check{a} \in V$ by completeness of \succsim and, hence, $a \in \text{cl}(A(\succsim, \check{a}))$ for all $\check{a} \in V$ by upper closure-transitivity of \succsim , a contradiction. Hence there exists $\check{a} \in V$ such that $\check{a} \succ a'$, so $a \in \text{cl}(A(\succ, a'))$. \square

Note that the converse of part **a** does not hold: upper closure-transitivity is strictly weaker than transitivity. For example, consider the commodity space $\mathcal{A} = \mathbb{R}^2$ of [Section 2](#) and the behavioral preference relation \succsim_B on \mathcal{A} defined therein, and amend \succsim_B by assuming $(2, 0) \succ_B (1, 1) \succ_B (0, 2) \succ_B (2, 0)$. Then \succsim_B is not transitive, but it is upper closure-transitive (upper closure-transitivity of \succsim_B is equivalent to transitivity of the third cognitive relation \succsim_C defined in the example, independently of the latter amendment of \succsim_B).

We obtain a generalization of [Rader \(1963\)](#)'s representation result on a second-countable topological space:

Theorem 5. Let \succsim_B be a complete binary relation on a second-countable topological space \mathcal{A} . Then \succsim_B is upper archimedean, weakly upper nonsatiated and upper closure-transitive if and only if there exists an upper semi-continuous and locally unbounded above pseudo-utility representation u of \succsim_B . Moreover, u is unique up to a strictly increasing transformation $f : u(\mathcal{A}) \rightarrow \mathbb{R}$ such that $f \circ u$ is upper semi-continuous.

Proof. Assume that \succsim_B is upper archimedean, weakly upper nonsatiated and upper closure-transitive. Then, by [Theorem 2](#) and [Lemma 4](#), there exists a complete, upper semi-continuous and weakly upper nonsatiated binary relation \succsim_C on \mathcal{A} such that \succsim_B is compatible with \succsim_C . Moreover, \succsim_C is transitive by [Theorem 1](#) and [Lemma 4](#). Because \mathcal{A} is second-countable, it follows from [Rader \(1963\)](#)'s result that there exists an upper semi-continuous utility representation u of \succsim_C . Moreover, u is locally unbounded above by weak upper nonsatiation of \succsim_C . Finally, u is a pseudo-utility representation of \succsim_B because \succsim_B is compatible with \succsim_C .

Conversely, assume that there exists an upper semi-continuous and locally unbounded above pseudo-utility representation u of \succsim_B . Define the binary relation \succsim_C on \mathcal{A} by, for all $a, a' \in \mathcal{A}$, $a \succsim_C a' \Leftrightarrow u(a) \geq u(a')$. Then u is a utility representation of \succsim_C and, hence, \succsim_C is complete, transitive, and upper semi-continuous. Moreover, \succsim_C is weakly upper nonsatiated because u is locally unbounded above, and \succsim_B is compatible with \succsim_C because u is a pseudo-utility representation of \succsim_B . Hence \succsim_B is upper archimedean and weakly upper nonsatiated by [Theorem 2](#) and [Lemma 4](#). Moreover, \succsim_B is upper closure-transitive by [Theorem 1](#) and [Lemma 4](#).

For the uniqueness part, let u be an upper semi-continuous and locally unbounded above pseudo-utility representation of \succsim_B and let $f : u(\mathcal{A}) \rightarrow \mathbb{R}$. Clearly, if f is strictly increasing then $f \circ u$ is also a locally unbounded above pseudo-utility representation of \succsim_B . Conversely, if $f \circ u$ is a locally unbounded above pseudo-utility representation of \succsim_B then, for all $a, a' \in \mathcal{A}$, $u(a) > u(a') \Rightarrow a \succ a' \Rightarrow f \circ u(a) \geq f \circ u(a')$. Moreover, suppose there exist $a, a' \in \mathcal{A}$ such that $[u(a) > u(a') \text{ and } f \circ u(a) = f \circ u(a')]$. Then $V = \{\hat{a} \in \mathcal{A} : u(a) > u(\hat{a})\}$ is an open subset of \mathcal{A} by upper semi-continuity of u and $a' \in \arg \max_{\hat{a} \in V} f \circ u(\hat{a})$, a contradiction because $f \circ u$ is locally unbounded above. Hence f is strictly increasing. \square

In order to get a lower version of this result, say that a binary relation \succsim on a topological space \mathcal{A} is *lower closure-transitive* if, for all $a, a', a'' \in \mathcal{A}$, $[a' \in \text{cl}(A(a, \succsim)) \text{ and } a'' \in \text{cl}(A(a', \succsim))] \Rightarrow a'' \in \text{cl}(A(a, \succsim))$, and say that a function $u : \mathcal{A} \rightarrow \mathbb{R}$ is *locally unbounded below* if, for all open subset V of \mathcal{A} , $\arg \min_{a \in V} u(a) = \emptyset$. The analog of [Theorem 5](#) then holds. Moreover, the two results can be combined, yielding a generalization of [Debreu \(1954\)](#)'s representation result. Say that \succsim is *closure-transitive* if it is both upper closure transitive and lower closure-transitive, and that u is *locally unbounded* if it is locally unbounded above and locally unbounded below.

Theorem 6. Let \succsim_B be a complete binary relation on a second-countable topological space \mathcal{A} . Then \succsim_B is archimedean, weakly nonsatiated and closure-transitive if and only if there exists a continuous and locally unbounded pseudo-utility representation u of \succsim_B . Moreover, u is unique up to a strictly increasing transformation $f : u(\mathcal{A}) \rightarrow \mathbb{R}$ such that $f \circ u$ is continuous.

We conclude with a generalization of [Eilenberg \(1941\)](#)'s representation result on a first-countable and connected topological space:

Theorem 7. Let \succsim_B be a complete binary relation on a first-countable and connected topological space \mathcal{A} . Then \succsim_B is archimedean, weakly nonsatiated and closure-transitive if and only if there exists a continuous and locally unbounded pseudo-utility representation u of \succsim_B . Moreover, u is unique up to a strictly increasing and continuous transformation.

The proofs of [Theorem 6](#) and [Theorem 7](#) are similar to that of [Theorem 5](#) and, therefore, omitted. Note that in [Theorem 7](#), connectedness of \mathcal{A} yields a stronger uniqueness result ([Wakker, 1991](#)).

Conclusion

This paper shows that preferences can be derived from choice behavior in a way that is robust to indifferent selection. More precisely, a suggestion by [Savage \(1954\)](#) to reveal indifferent selection by considering small monetary perturbations of alternatives is formalized in a general topological setup, and is found to essentially rely on an assumption of continuity of preferences.

Although [Savage \(1954\)](#)'s argument is well known, it is seldom used to elicit indifference in practice. Rather, the experimental literature generally resorts to such devices as randomization or delegation. A possible defense of this standard practice is that the monetary perturbation method, although more satisfactory in theory, is impossible to implement because there is no such thing as an infinitely small monetary bonus in practice. Nevertheless, using some small bonus as an approximation could still represent an improvement over the usual elicitation methods. For example, if an experimental study provides evidence of some strict preference pattern violating a standard axiom, then such evidence could be strengthened by checking for robustness of the pattern to the adjunction of this small bonus.

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