

# Voting with Preferences over Margins of Victory\*

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## Abstract

This paper analyzes a two-alternative voting model with the distinctive feature that voters have preferences over margins of victory. We study voting contests with a finite as well as an infinite number of voters, and with and without mandatory voting. The main result of the paper is the existence and characterization of a unique equilibrium outcome in all those situations. At equilibrium, voters who prefer a larger support for one of the alternatives vote for such alternative. The model also provides a formal argument for the *conditional sincerity voting* condition in Alesina and Rosenthal (1995) and the *benefit of voting* function in Llavador (2006). Finally, we offer new insights on explaining why some citizens may vote *strategically* for an alternative different from the one declared as the most preferred.

KEYWORDS: Margin of victory, plurality, abstention, strategic voting, committee voting, elections.

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# 1 Introduction

“Yesterday, I went to visit my neighbor, who is called Samer. Samer is in his thirties, and he works in a supermarket [...] He said that he likes Fattah’s party [...] but at that time, he thought that **he wanted Fattah to win in low results. So he voted for Hamas**, who used to be the second power in the Palestinians streets.”

Quotes from Palestinians who voted for Hamas  
Behind the Wall  
[www.lifebehindthewall.org](http://www.lifebehindthewall.org)

Arguments like Samer’s opening quote are the most visible expression of a common fact: voters show interest for the support that each alternative receives in an election. In fact, an important part media attention, candidates’ post-election statements, and in general election analysis concentrate on the margin of victory rather than on the identity of the winner, which in many cases is known beforehand.

There are many explanations why voters may care about the allocation of votes, most of them boiling down to the expectation that the share of the vote received by the winner may influence policymaking: issues of legitimacy to implement unpopular policies often arise for candidates elected by narrow margins; the power of the opposition to block legislation (for example with the filibusters in the U.S. democracy) or, in general, to control government depends on its electoral support; parties’ choice of political positions are often influenced by the support they received in the last election; even the viability of that party may be affected by the relative support since, in many democracies, public funding for a party is based on the share of the vote it received. On a different context, it is common for committees to stress

the support received by a chosen alternative or elected candidate as a signal of internal agreement or disagreement with the proposed plan of action or economic strategy.

The theoretical literature on political competition has also captured this idea that the electoral support may affect policies by specifying an institutional context like a divided government (Alesina and Rosenthal, 1995), “proportional representation” (Ortuño-Ortín, 1997), or a generic policymaking function that relates policies and vote allocation (Llavador, 2006). However, those analysis either assume that citizens vote sincerely or use an *ad hoc* description of voters’ behavior. For example, Alesina and Rosenthal (1995) imposes a *conditional sincerity condition* that requires that no voter prefers “a decrease in the expected vote for the party he has voted for” (p.50). Llavador (2006) models the *benefit of voting* as the utility change implied by a marginal increase in the support for the party supported with the vote.

The current paper analyzes a voting contest where individuals have preferences over the distribution of votes. Although preferences over vote allocations could be easily derived from preferences over policies by specifying any of the institutional contexts mentioned above, we take preferences as primitives of the model and impose minimal restrictions on their functional forms to work within a more general framework.

The setting throughout the paper involves two alternatives and a set of individuals who have single-peaked preferences over vote allocations. We study first electorates with a finite number of voters (which includes committee voting) and mandatory voting. Voters must vote for one of the two alternatives and a voting equilibrium is a Nash equilibrium in which no voter

wants to change her vote. Two observations are in order. First, the restriction on the number of alternatives does not confine policies to a unidimensional policy space. Because individuals have preferences defined over vote allocations, alternatives may represent bundles of policies or positions in several issues. Thus, there is no restriction on the dimensionality of the policy space. Secondly, we do not impose continuity, concavity/convexity or symmetry conditions on preferences (see figure 1), and individuals may differ not only in their bliss point but also in the functional form of their utility.

The main result of the paper proves the existence of a unique voting equilibrium outcome. The intuition is simple. At equilibrium, those voters who want a larger support for one of the alternatives must support it with their vote, imposing in this way an upper and a lower bound on the share of votes for each alternative. We prove then that there exists a unique electoral outcome consistent with those bounds (Theorem 1). Moreover, the equilibrium electoral outcome is the unique fixed point of the *closed* survival function associated to the distribution of the electorate (see Section 3 for details and Figure 2 for illustrations of different equilibria). Our analysis also predicts that some voters who favor an electorally balanced result may cast their vote *strategically* for an alternative different from the one they would declare as their most preferred in order to compensate an excessive support for such alternative, as it is consistently reported in election surveys.<sup>1</sup>

Next we analyze electorates with a continuum of voters by studying equi-

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<sup>1</sup>For example, in the SELECTS Swiss electoral study of 2003, which is conducted after the parliamentary election, 11% of those who reported having voted give different answers to the questions “which party did you vote for?” and “which is the party you feel closest to?”. (I thank Patricia Funk for this information.)

libria as the number of citizens tend to infinity.<sup>2</sup> Section 4 shows that only one electoral outcome can be sustained as the limit of the equilibrium electoral outcomes of a sequence of finite societies whose distributions weakly converge to a continuous distribution  $F$ . Moreover, this electoral outcome is the fixed point of the survival function  $1 - F$ .

Finally, we show that we can go a step further and obtain the previous results, for finite and infinite electorates, even when we allow voters to abstain. In our setting, only those who obtain their preferred outcome may abstain: since everybody is *pivotal*, in the sense of influencing the electoral outcome, and there is no cost of voting, abstention can only arise as a strategic decision to avoid changing an electoral outcome which is optimal for the point of view of the voter.

## 2 The Model

Consider a group of  $n$  individuals who have to choose between two alternatives  $A$  and  $B$ .<sup>3</sup> Let  $S = \{A, B\}$  represent the set of strategies for each voter with representative element  $s_i \in S$ . Thus voting is mandatory in this section.<sup>4</sup> A profile of actions is a vector  $s = (s_1, \dots, s_n)$ . Each profile  $s$  determines an electoral outcome defined as the fraction of votes for each al-

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<sup>2</sup>Observe that we cannot use the same approach as before, since with a continuum of voters the action chosen by an individual does not affect the electoral outcome and hence any profile of votes is a Nash equilibrium.

<sup>3</sup>As mentioned in the introduction, each alternative may represent a bundle of policies or positions over many issues. Hence the restriction is on the number of alternatives that voters can choose from but not on the dimensionality of the policy space.

<sup>4</sup>We introduce abstention in Section 5. Although the results with and without abstention are substantially identical, we find helpful for the exposition to separate their presentation.

ternative. Given that there are only two alternatives, the electoral outcome is fully described by the fraction of votes that alternative  $A$  receives. Let  $E_n = \{0, \frac{1}{n}, \dots, \frac{n-1}{n}, 1\}$  be the **set of electoral outcomes**, where an element  $e$  of  $E_n$  represents the **fraction of the total vote** cast for alternative  $A$ , with the remaining fraction  $1 - e$  supporting  $B$ .

Define the **electoral outcome function**  $\tilde{e} : S \times \dots \times S \rightarrow E_n$  such that

$$\tilde{e}(s) = \frac{|\{s_i \in s : s_i = A\}|}{n}. \quad (1)$$

The distinguishing feature of our approach is that individuals have preferences over electoral outcomes. Let individual  $i$ 's preferences be represented by the utility function  $u_i : [0, 1] \rightarrow \mathfrak{R}$ . (Observe that voters may differ in the functional form of their utilities.) Assume that preferences are single-peaked. Denote by  $e_i = \arg \max_{e \in E_n} u_i(e)$  voter  $i$ 's preferred electoral outcome among those feasible and assume it is unique.<sup>5</sup> As figure 1 illustrates, these are very mild conditions that allow for a wide variety of preferences. In particular, we want to emphasize that we do not impose any continuity, concavity or symmetry condition on preferences.

[Figure 1 about here.]

We can identify voters by their ideal electoral outcomes. Let  $f_n : [0, 1] \rightarrow [0, 1]$  represent the probability function of voters' preferred electoral outcomes. And let  $F_n(x) = \sum_{z \leq x} f_n(z)$  be the corresponding discrete cumulative distribution function.

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<sup>5</sup>It is sufficient, although not necessary, for this condition to hold that voters have ideal policies that are feasible.

A voting equilibrium is a profile of strategies  $s^* \in S^n$  such that no citizen has incentive to change the current electoral outcome  $e^* = \tilde{e}(s^*)$  by choosing a different action from her equilibrium action  $s_i^*$ . That is, we use Nash Equilibrium in pure strategies as our concept of equilibrium.

**Definition 1** *A **voting equilibrium** is a profile of strategies  $s^* \in S^n$  such that for all  $i = 1, \dots, n$*

$$u_i(\tilde{e}(s^*)) \geq u_i(\tilde{e}(s_i, s_{-i}^*)) \quad \text{for all } s_i \in S.$$

### 3 Equilibrium results for a finite number of voters

The main result of this section shows that a unique equilibrium electoral outcome  $e^*$  exists and divides the electorate such that all voters to the left of  $e^*$  vote for  $B$  while all voters to the right vote for  $A$ . The intuition is simple. Voters with ideal outcome to the right of  $e^*$  want a larger support for alternative  $A$ , and therefore they must be voting for  $A$  at equilibrium. Similarly, those voters whose ideal policy sits to the right of  $e^*$  must be voting for  $B$ , for otherwise they could shift the electoral outcome in their favor by switching their vote.

These conditions impose upper and lower bounds on the share of votes that alternative  $A$  may receive at equilibrium. Namely, at any individually consistent outcome  $e^*$ , alternative  $A$  must receive at least  $1 - F(e^*)$  of the votes and no more than  $1 - F(e^*) + f(e^*)$ . The following lemma shows that

there exists one and only one electoral outcome satisfying these necessary conditions, becoming the only possible equilibrium outcome.

**Lemma 1** *Let  $f_n$  and  $F_n$  be the discrete PDF and CDF of a population of  $n$  individuals. Then, the correspondence  $\phi_n : [0, 1] \rightarrow [0, 1]$  with  $\phi_n(x) = [1 - F_n(x), 1 - F_n(x) + f_n(x)]$  has a unique fixed point. That is, there exists a unique  $x^* \in [0, 1]$  such that  $x^* \in [1 - F_n(x^*), 1 - F_n(x^*) + f_n(x^*)]$ .*

*Proof:*

Observe that  $\phi_n : [0, 1] \rightarrow [0, 1]$ . It follows from Kakutani's fixed point theorem that if  $\phi_n$  is closed then it has a fixed point.

Take  $x^k \rightarrow \bar{x}$ ,  $y^k \in \phi_n(x^k)$  and  $y^k \rightarrow \bar{y}$ . By construction of the correspondence  $\phi$ , we can always find a ball  $B_\epsilon(\bar{x})$  around  $\bar{x}$  such that  $\forall x \in B_\epsilon(\bar{x})$ ,  $\phi_n(x) \subseteq \phi_n(\bar{x})$ . Thus, for a sufficiently large  $N$  and for all  $k > N$ ,  $|x^k - \bar{x}| < \epsilon$  and hence  $\phi_n(\bar{x}) \supseteq \phi_n(x^k) \ni y^k$ . Since  $y^k \rightarrow \bar{y}$  and  $y^k \in \phi_n(\bar{x})$  for all sufficiently large  $k$ , then  $\bar{y} \in \phi_n(\bar{x})$  and hence  $\phi_n$  is closed. Therefore there exists  $x^*$  such that  $x^* \in \phi_n(x^*)$ .

Finally, we prove uniqueness. Observe that the survival function  $1 - F_n$  is a non-increasing function satisfying that for all  $x' < x$ ,  $1 - F_n(x) + f_n(x) \leq 1 - F_n(x')$ .

Take  $x < x^*$ . Recall that  $x^* \in \phi(x^*)$  and hence  $1 - F_n(x^*) \leq x^* \leq 1 - F_n(x^*) + f_n(x^*)$ . Then  $1 - F_n(x) \geq 1 - F_n(x^*) + f_n(x^*) \geq x^* > x$ , and  $x \notin \phi_n(x)$  for all  $x < x^*$ .

Similarly, take  $x > x^*$ . Then,  $1 - F_n(x) + f_n(x) \leq 1 - F_n(x^*) \leq x^* < x$ , and  $x \notin \phi_n(x)$  for all  $x > x^*$ .

We conclude then that  $\phi_n$  has a unique fixed point. □



The previous lemma has restricted the set of possible equilibrium outcomes to one element:  $e^*$  such that  $e^* \in \phi_n(e^*)$ . It also follows from the reasoning above that if that outcome is an equilibrium then it acts as a dividing type (not related to the *median*), such that all voters with  $e_i < e^*$  vote for  $B$ , while all voters with  $e_i > e^*$  vote for  $A$ . It still remains to show that such an equilibrium exists. But, since any vote is optimal for voters obtaining their ideal outcome ( $e_i = e^*$ ), equilibrium existence only requires to find an allocation of votes  $s^*$  satisfying the previous conditions and consistent with the electoral outcome  $e^*$ . The following theorem compiles the previous results and shows that this is the case and an equilibrium always exists. Moreover, the equilibrium is unique up to a permutation of votes among voters obtaining their preferred electoral outcome.

**Theorem 1** *Consider a  $n$ -citizen electoral game with single-peaked preferences over electoral outcomes. Then:*

1. *Let  $s^*$  be a voting equilibrium and let  $e^* = \tilde{e}(s^*)$  be its associated equilibrium electoral outcome. Then  $s_i^* = A$  for all  $i$  with  $e_i > e^*$  and  $s_i^* = B$  for all  $i$  with  $e_i < e^*$ .*
2. *There always exists a voting equilibrium  $s^* = (s_1^*, \dots, s_n^*)$ .*
3. *The electoral equilibrium outcome is unique. Namely,  $e^* = \tilde{e}(s^*)$  for all voting equilibrium  $s^*$ .*

*Proof:*

(1) Let  $s^*$  be a voting equilibrium with  $k^*$  voters voting for  $A$ . That is,  $e^* = \tilde{e}(s^*) = k^*/n$ . Suppose that  $s_i^* = A$  for some voter with  $e_i < e^*$ . If voter  $i$  decides to switch her vote to  $B$  she will change to electoral outcome

to  $\tilde{e}(B, s_{-i}^*) = (k^* - 1)/n$ . Because  $e_i$  is feasible,

$$e_i \leq \frac{k^* - 1}{n} < \frac{k^*}{n} = e^*.$$

By single-peakedness of preferences,  $u_i(\tilde{e}(B, s_{-i}^*)) > u_i(e^*)$ , contradicting the assumption that  $s^*$  is an equilibrium.

A symmetric argument explains why no voter with  $e_i > e^*$  can be voting for  $A$  at equilibrium.

(2) Let  $e^*$  be the unique fixed point of  $\phi_n$ , as defined in lemma 1. Namely,  $1 - F(e^*) \leq e^* \leq 1 - F(e^*) + f(e^*)$ . Let  $s^*$  such that  $s_i^* = A$  for all  $e_i > e^*$  and  $s_i^* = B$  for all  $e_i < e^*$ , and so satisfying the equilibrium necessary conditions presented in part (1). Let  $q_A = n(1 - F(e^*))$  and  $q_B = n(F(e^*) - f(e^*))$  represent the number of voters with  $e_i > e^*$  and  $e_i < e^*$ , respectively. Then  $q_A \leq n e^*$  and  $q_B \leq n(1 - e^*)$ .

If  $f(e^*) = 0$ , then  $q_A + q_B = n$ ,  $1 - F(e^*) = e^*$ , and  $s^*$  is an equilibrium.

If  $f(e^*) > 0$ , then there exists at least one  $e_i = e^*$  and hence  $e^*$  is feasible, that is, there exists an integer  $k$  such that  $e^* = k/n$ . Let  $k_A = n e^* - q_A$  voters with  $e_i = e^*$  vote for  $A$ , and let  $s_i^* = B$  for the remaining  $k_B = n f(e^*) - k_A$  voters with  $e_i = e^*$ . Note that any strategy that yields  $e^*$  as the electoral outcome is optimal for voters with  $e_i = e^*$ . It follows that  $n \tilde{e}(s^*) = q_A + k_A = n e^*$ , and everybody is voting optimally. Therefore,  $s^*$  is an equilibrium.

(3) Finally, uniqueness of the electoral equilibrium follows directly from part (1) and the uniqueness of the fixed point in Lemma 1.  $\square$

[Figure 2 about here.]

Observe that the equilibrium outcome is graphically very appealing (see figure 2) and easy to calculate as the unique fixed point of the correspondence  $\phi_n$  (constructed by “closing” the survival function  $1 - F_n$ ).

More importantly, Theorem 1 offers new insights on why some citizens do not vote for their preferred alternative. Election surveys consistently report a group of citizens who vote *strategically* for an alternative different from the one they declared as the most preferred.<sup>6</sup> According to the present model, some of those voters favor an electorally balanced result and decide to cast their vote to compensate an excessive support for one of the alternatives. Therefore, Theorem 1 offers two testable hypothesis: citizens voting for an alternative different from their declared as most preferred should favor (relatively) balanced results and should expect *too much* support for the alternative they prefer.

Finally, suppose that all voters prefer outcomes with their favored alternative getting all the votes, and hence, in this sense, they do not stress the margin of victory. In our settings, this implies that the electorate’s preferred outcomes concentrate on 0 and 1. It follows from the previous analysis that the electoral outcome will be  $e^* = 1 - F_n(0) = f_n(1)$ . Therefore, at equilibrium everybody votes for their declared preferred alternative. That is, *sincere voting* obtains, replicating the Nash equilibrium in weakly undominated strategies of traditional voting models.

**Corollary 1** *Let voters’ ideal electoral outcomes concentrate on 0 and 1,*

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<sup>6</sup>See footnote 1. Our own anecdotal evidence also supports the existence of such behavior. For example, during the referendum to ratify the European constitution in Spain, some Catalan nationalist parties campaigned against ratification, and it was argued that their position would have changed had not been undoubtedly clear from the very beginning that the “yes-vote” would prevail.

that is,  $f_n(0) + f_n(1) = 1$ . Then  $s_i^* = A$  for all  $i$  with  $e_i = 1$ , and  $s_i^* = B$  for all  $i$  with  $e_i = 0$ .

*Proof:*

If  $f_n(0) = 1$  or  $f_n(0) = 0$  then, since  $f_n(0) + f_n(1) = 1$ , all voters unanimously prefer that one of the alternatives receives all the votes and hence all voters vote for this alternative at equilibrium.

If  $0 < f_n(0) < 1$ , then  $F_n(e) = f_n(0)$  for all  $e < 1$ . Therefore,  $e^* = 1 - F_n(e^*) = 1 - f_n(0) \in (0, 1)$ . By Theorem 1, everybody with  $e_i > e^*$  vote for  $A$  and everybody with  $e_i < e^*$  vote for  $B$ . But, because these voters are concentrated on 1 and 0, respectively, they all vote *sincerely*.  $\square$

## 4 Continuum of voters

In this section we study equilibria in a society with a continuum of agents. The previous analysis cannot be trivially extended to this new setting since, with a continuum of agents, the action chosen by an individual does not affect the electoral outcome. Hence any profile of strategies is an equilibrium. We can analyze however the voting game as the number of agents tends to infinity.

Consider a sequence of discrete distributions  $\{F_n\}_{n=1}^{\infty}$  that weakly converges to a continuous distribution  $F$  as  $n$  goes to infinity.<sup>7</sup> We know from Theorem 1 that there exists a unique equilibrium outcome  $e_n^*$  associated to each  $F_n$ . Then we can analyze where the sequence of equilibria tends to.

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<sup>7</sup>The sequence  $\{F_n\}_{n=1}^{\infty}$  is said to *weakly converge* or *converge pointwise* to a function  $F$  on  $[0, 1]$  if the sequence  $\{F_n(x)\}_{n=1}^{\infty} \in \mathbb{R}$  converges to  $F(x)$  for each  $x \in [0, 1]$  (Sydsæter et al., 2005, p.86)

The following theorem shows that it converges to the unique fixed point  $e^*$  of the continuous survival function associated to  $F$ , namely it converges to  $e^*$  defined as  $e^* = 1 - F(e^*)$  (see Figure 3). In fact, the theorem goes further and says that no matter how we approach the continuous CDF, the sequence of electoral equilibria always converges to the same value  $e^*$ .

In other words, from the analysis of discrete distributions,  $e^*$  comes out as the natural candidate for a voting equilibrium outcome with a continuum of voters. Take any sequence of discrete distributions that weakly converges to  $F$ , then the sequence of electoral equilibria always converges to the same value  $e^*$ . Thus,  $e^*$  is the only electoral outcome that can be sustained as the limit of equilibrium electoral outcomes of a sequence of societies with a finite number of citizens whose distributions weakly converge to  $F$  as the number of citizens tend to infinity.

[Figure 3 about here.]

**Theorem 2** *Let  $F$  be a continuous distribution function. Let  $\{F_n\}_{n=1}^{\infty}$  be a sequence of discrete distribution functions that weakly converges to  $F$ , where  $F_n$  represents the CDF of a population with  $n$  agents. Letting  $e_n^*$  be the electoral equilibrium outcome associated to the distribution  $F_n$ ,  $n = 1, 2, \dots$ , the sequence  $\{e_n^*\}_{n=1}^{\infty}$  converges to  $e^*$  as  $n$  goes to  $\infty$ , where  $e^*$  is defined as the unique solution to  $e = 1 - F(e)$ .*

*Proof:*

Let  $F$  be a continuous distribution, and let  $e^* = 1 - F(e^*)$ . (Uniqueness follows from the strict monotonicity of  $F$ .) Take a sequence of discrete CDFs

$\{F_n\}_{n=1}^\infty$  that weakly converges to  $F$ , that is, taking  $e \in [0, 1]$  then

$$\forall \epsilon > 0 \quad \exists N_\epsilon \text{ such that } \forall n > N_\epsilon \quad |F_n(e) - F(e)| < \epsilon \quad (2)$$

From Theorem 1, there exists a unique voting equilibrium outcome  $e_n^*$  associated to each  $F_n$ . And we want to show that the sequence of equilibrium outcomes  $\{e_n^*\}_{n=1}^\infty$  converges to  $e^*$  as  $n$  goes to infinity. That is,

$$\forall \epsilon \quad \exists N : \forall n > N \quad |e_n^* - e^*| < \epsilon. \quad (3)$$

Take  $\epsilon > 0$  and  $n > N_{\epsilon^*}$  (see (2)), and consider the following three cases:

(i)  $F_n(e^*) = F(e^*) = 1 - e^*$ . It follows from the characterization of  $e_n^*$  that  $e_n^* = e^*$  for all  $n$ , and hence  $|e_n^* - e^*| = 0 < \epsilon$ .

(ii)  $F_n(e^*) < F(e^*) = 1 - e^*$ . Since  $F_n$  is a non-decreasing function,  $F_n(e^* + \epsilon) \geq F_n(e^*)$ . From weak convergence (2),  $F_n(e^*) > F(e^*) - \epsilon = 1 - e^* - \epsilon$ . Therefore,  $F_n(e^* + \epsilon) > F(e^*) - \epsilon = 1 - (e^* + \epsilon)$  which, jointly with  $F_n(e^*) < 1 - e^*$ , implies that  $e_n^* \in (e^*, e^* + \epsilon)$  and hence  $|e_n^* - e^*| < \epsilon$ .

(iii)  $F_n(e^*) > F(e^*) = 1 - e^*$ . Since  $F_n$  is a non-decreasing function,  $F_n(e^* - \epsilon) \leq F_n(e^*)$ . From (2),  $F_n(e^*) < F(e^*) + \epsilon = 1 - e^* + \epsilon$ . Therefore,  $F_n(e^* - \epsilon) < 1 - (e^* - \epsilon)$ . It follows that  $e_n^* \in (e^* - \epsilon, e^*)$ , and hence  $|e_n^* - e^*| < \epsilon$ .

Therefore, we have proved that (3) holds and hence that  $\{e_n^*\} \rightarrow e^*$  as  $n \rightarrow \infty$ .  $\square$

We have found that, when voting is mandatory, for any discrete or continuous distribution of the electorate, there exists a unique electoral outcome

which acts as a dividing type. This result is consistent with the intuition behind Alesina and Rosenthal’s (1995) conditional sincerity and Llavador’s (2006) benefit of voting, providing a formal analysis explaining their modeling choice of voting behavior.

In the next section we show that we can go an step further and obtain these results, with minor qualifications, even when we allow voters to abstain.

## 5 Abstention

This section relaxes the assumption of mandatory voting and let citizens choose between voting for one of the two alternatives or abstaining. Formally, we extend the set of strategies to include the possibility of abstention:  $S = \{A, B, O\}$ .

Incorporating abstention allows for many more outcomes not feasible under mandatory voting and requires adapting three concepts already introduced: the set of feasible electoral outcome, the electoral outcome function, and preferences over electoral outcomes.

If at least one individual votes, an electoral outcome is fully described by the fraction of votes that alternative  $A$  receives. Letting  $\hat{E}_k = \{0, \frac{1}{k}, \dots, \frac{k-1}{k}, 1\}$  represent possible outcomes when  $k$  agents vote, the set of electoral outcomes with abstention can be constructed as  $E_n = \bigcup_{k=1}^n \hat{E}_k$ .

Define the **electoral outcome function with abstention**  $\tilde{e} : S \times \dots \times S \rightarrow E_n$  as<sup>8</sup>

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<sup>8</sup>A precise definition should assign a value to the case when nobody vote. However, we only need one voter who prefers an outcome different from everybody abstaining to obtain a positive turnout, in which case the analysis does not depend on the value assigned to the option of zero votes. We assume in the analysis that such a person exists.

$$\tilde{e}(s) = \frac{|\{s_i \in s : s_i = A\}|}{|\{s_i \in s : s_i = A\}| + |\{s_i \in s : s_i = B\}|}. \quad (4)$$

Finally, let citizens have preferences defined over electoral outcomes and represented by the single-peaked utility function  $u_i : [0, 1] \rightarrow \mathbb{R}$ . Let a voter  $i$  have a unique preferred outcome  $e_i$  among those feasible. We want to emphasize that we have not imposed any additional restriction over preferences with respect to the analysis with mandatory voting.<sup>9</sup>

The following theorem shows that the results found under mandatory voting with finite populations are replicated when we introduce abstention. Obviously, Lemma 1 holds since it was not related in any way with the set of strategies.

**Theorem 3** *Consider a  $n$ -citizen electoral game with single-peaked preferences over electoral outcomes and abstention. Then:*

1. *Let  $s^*$  be a voting equilibrium and let  $e^* = \tilde{e}(s^*)$  be its associated equilibrium electoral outcome. Then  $s_i^* = A$  for all  $i$  with  $e_i > e^*$  and  $s_i^* = B$  for all  $i$  with  $e_i < e^*$ . Hence only voters with  $e_i = e^*$  may abstain.*
2. *There exists a voting equilibrium  $s^* = (s_1^*, \dots, s_n^*)$ .*
3. *The electoral equilibrium outcome is unique. Namely,  $e^* = \tilde{e}(s^*)$  for all voting equilibria.*

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<sup>9</sup>Observe that defining preferences over electoral outcomes implies assuming that voters have no preferences over abstention levels. Alternatively, we could have assumed that citizens hold electoral outcomes lexicographically above turnout or that everybody prefers more participation than less. Although incorporating preferences over abstention is an interesting avenue of research, we believe that our current description captures many real situations.



*Proof:*

The proof follows closely the same steps as the proof of Theorem 1. If  $e_n^*$  is a voting equilibrium outcome, then everybody with  $e_i > e_n^*$  must be voting  $s_i^* = A$  since they prefer a larger support for  $A$ . Similarly, voters with  $e_i < e_n^*$  must be voting  $s_i^* = B$ . Therefore, from Lemma 1, the unique candidate for an equilibrium outcome is  $e_n^*$ : the fixed point of  $\phi_n$ . Let  $k_A = |\{i : e_i > e_n^*\}|$  and  $k_B = |\{e_i < e_n^*\}|$ .

If there exists  $q \in \{0, 1, \dots, n\}$  such that  $e_n^* = k/n$ , the same voting profiles described in the proof of Theorem 1 are still equilibria when we allow for abstention.

Assume now that such a  $k$  does not exist. Because  $e_n^* \in \phi_n(e_n^*)$ ,  $1 - F_n(e_n^*) \leq e_n^* \leq 1 - F_n(e_n^*) + f_n(e_n^*)$ .

Suppose that  $e_n^* = 1 - F_n(e_n^*)$ , then  $e_n^* = \sum_{e > e_n^*} f_n(e) = \sum_{e > e_n^*} q(e)/n$ , where  $q(e) = n f_n(e)$  is the number of citizens with  $e_i = e$ . But this is a contradiction with the assumption that there does not exist an integer  $k$  such that  $e_n^* = k/n$ . Similarly, if  $e_n^* = 1 - F_n(e_n^*) + f_n(e_n^*)$ , then  $e_n^* = \sum_{e \geq e_n^*} f_n(e) = \sum_{e \geq e_n^*} q(e)/n$ , reaching also a contradiction.

We have proved then that  $1 - F_n(e_n^*) < e_n^* < 1 - F_n(e_n^*) + f_n(e_n^*)$ . It follows that  $f_n(e_n^*) > 0$  and so  $e_n^*$  is feasible. Then there exist integers  $q_A$  and  $q_B$  such that  $e_n^* = q_A/(q_A + q_B)$ . Define  $n_A = q_A - k_A$ ,  $n_B = q_B - k_B$ , and  $n_O = n - q_A - q_B$ . Observe that:

- i)  $n_A, n_B$ , and  $n_O$  are integers less than  $n$ .
- ii)  $n_A + n_B + n_O = n - k_A - k_B = q(e_n^*)$ . Hence letting  $n_A$  voters vote for  $A$ ,  $n_B$  vote for  $B$  and  $n_O$  abstain, we have partition the set of voters with  $e_i = e^*$ .

iii) The share of the vote for alternative  $A$  resulting from the previous allocation of votes is

$$\frac{k_A + n_A}{k_A + n_A + k_B + n_B} = \frac{q_A}{q_A + q_B} = e_n^*$$

where we have used the definition of  $n_A$  and  $n_B$ . Therefore, we have found the following equilibrium:  $s_i^* = A$  for all  $e_i > e_n^*$  and  $n_A$  voters with  $e_i = e_n^*$ ;  $s_i^* = B$  for all  $e_i < e_n^*$  and  $n_B$  voters with  $e_i = e_n^*$ ; and the remaining  $n_O$  voters with  $e_i = e_n^*$  abstain.

Finally, uniqueness of the equilibrium outcome follows from Lemma 1.  $\square$

Observe that only those who obtain their preferred outcome may abstain. In our setting, everybody is *pivotal* in the sense of influencing the electoral outcome and there is no cost of voting. Hence, abstention can only arise as a strategic decision: citizens who abstain do so to avoid changing an electoral outcome that coincides with their most preferred one.

Theorem 3 also shows that, for a finite number of agents, the equilibrium electoral outcome is unique and characterized as the fixed point of the connected survival function. Therefore, we can still apply Theorem 2 to the analysis of a society with a continuum of agents with abstention. That is,  $e^*$  (such that  $e^* = 1 - F(e^*)$ ) is the only electoral outcome that can be sustained as the limit of a sequence of equilibrium electoral outcomes from societies with a finite number of citizens who may abstain and whose distributions converge to  $F$  as the number of citizens tend to infinity.

## 6 Final remarks

This paper analyzes a two-alternative voting model with the distinctive feature that voters have preferences over margins of victory. We study voting contests with a finite as well as an infinite number of voters, and with and without mandatory voting. The main result of the paper is the existence of a unique equilibrium outcome. At equilibrium voters who prefer a larger support for one of the alternatives vote for such alternative. Uniqueness allow us to easily embed our voting model into a political competition model of divided government, proportional representation, or any other institutional framework in which the political power of the government or the opposition is affected by the electoral support it receives. In fact, our voting model provides a formal argument for the *conditional sincerity voting* condition in Alesina and Rosenthal (1995) and the *benefit of voting* function in Llavador (2006).

Future work should extend the analysis to include more than two alternatives (which requires a multi-dimensional space of electoral outcomes) and to introduce uncertainty on voter expectations. Given the clarity and generality of our current results, we are optimistic on the future understanding of these questions, which may shed much needed light on voting behavior on parliamentary elections and multiparty systems.

## References

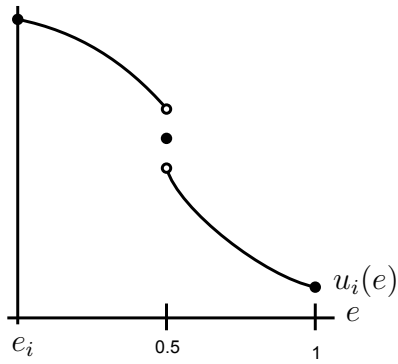
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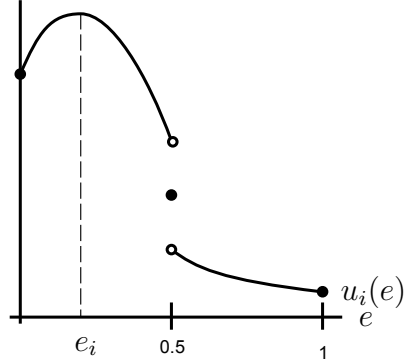
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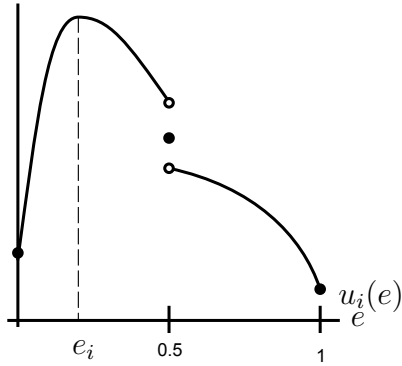
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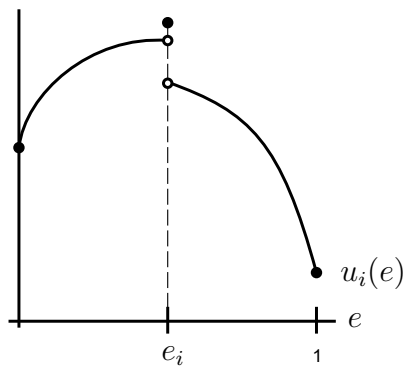
(a) Voter  $i$ 's favorite outcome is  $B$  getting 100% of the vote.



(b) Voter  $i$  prefers any outcome with  $B$  winning, but she'd rather have a little diversification of the vote.



(c) Voter  $i$  wants  $B$  to win, but she prefers  $A$  winning by a narrow margin than  $B$  getting a too large share of the vote.



(d) Voter  $i$  has a preference for close results. Anything different from a tie is "much" worse. Nevertheless, she also shows a preference for  $B$ .

Figure 1: **Examples of preferences over electoral outcomes** for an individual who shows a preference for alternative  $B$ . As the pictures illustrate, continuity, concavity, or symmetry conditions are not required for the analysis. Electoral outcomes are represented by  $e$ : the fraction of the vote cast for alternative  $A$ .

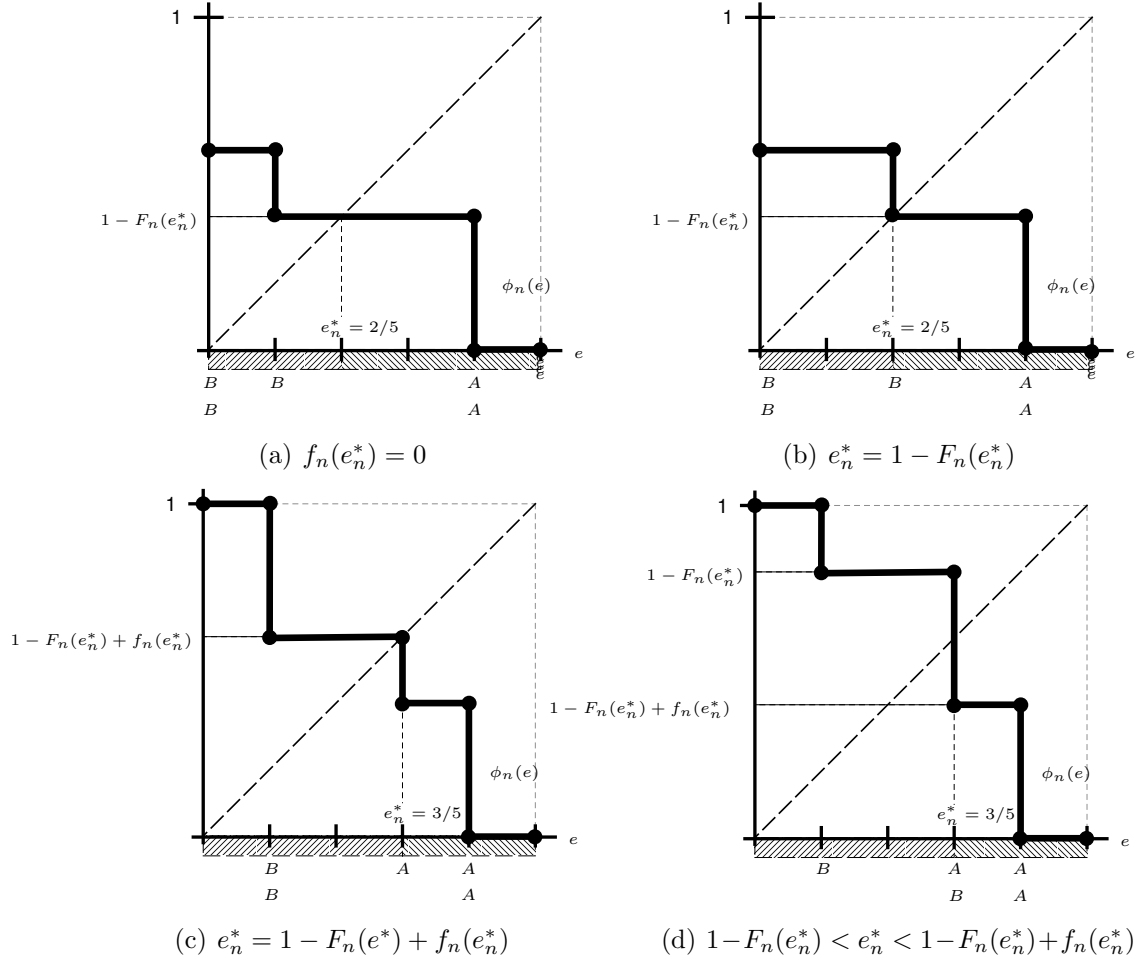


Figure 2: **Electoral equilibria for different distributions of a five-voter electorate.** Capital letters under the  $x$ -axis represent the location of voters' ideal electoral outcomes and the alternative supported with their vote in equilibrium. The equilibrium electoral outcome,  $e_n^*$  represents the fraction of the vote for alternative  $A$ . Voters with a lower (higher) ideal electoral outcome vote for  $B$  ( $A$ ). The correspondence  $\phi$  is obtained by “closing” the survival function  $1 - F_n$ :  $\phi_n(e) = [1 - F_n(e), 1 - F_n(e) + f_n(e)]$ , whose lower and upper bounds represent the fraction of people who want at least  $e\%$  of votes for  $A$  and strictly more than  $e\%$  of votes for  $A$ , respectively.

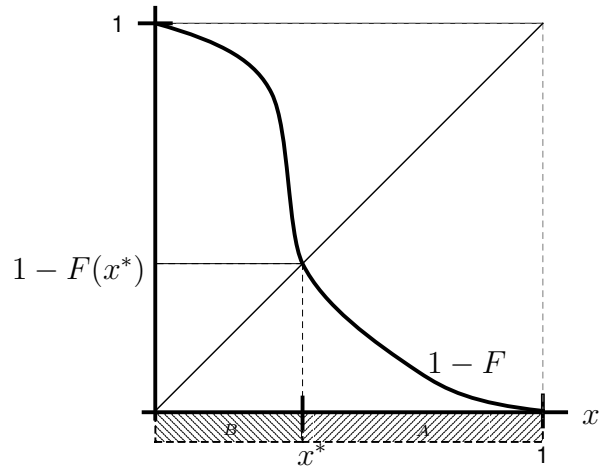


Figure 3: **Equilibrium outcome for an electorate with a continuum of voters.** The equilibrium electoral outcome  $x^*$  represents the fraction of the vote for alternative  $A$ . At equilibrium, the fraction of voters with a higher ideal electoral outcome ( $1 - F(x^*)$ ) equals the fraction of voters voting for  $A$  (i.e.  $x^*$ ).