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# The Variance Matrix of Sample Second-Order Moments in Multivariate Linear Relations *

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**Abstract**

We derive an expression for the variance matrix of the vector of (uncentered) sample second-order moments under multivariate linear relations and an independence assumption. An application of the result is presented

Key words: Moment structures, stochastic independence, non-normality, asymptotic distribution, non-central chi-square, goodness-of-fit statistic.

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# 1 Introduction

In a wide class of models for multivariate analysis, it is assumed that a vector of observable variables satisfies the following multivariate linear relation (e.g., Anderson, 1987):

(1) $$z = \sum_{i=1}^{L} B_i \delta_i,$$

where z is a p-dimensional vector of observable variables, the $\delta_i$'s are random vectors and the $B_i$'s are parameter matrices. The moment matrices $\Phi_{ii} := E\delta_i\delta_i'$, i=1,2, .., L, where "E" denotes mathematical expectation, are assumed to be finite.

An example of (1) is the following linear latent variable model:

(2a) $$\begin{cases} z = \Lambda\eta + \varepsilon \\ \eta = B\eta + \xi \end{cases},$$

where $\eta$ is an m-dimensional vector of (possibly) latent variables, $\varepsilon$ is an p-dimensional centered random vector of measurement errors, and $\xi$ is an m-dimensional random vector (usually composed of disturbance terms of structural equations and exogenous variables). Without loss of generality, we assume that (I - B) is invertible and that the last component of $\xi$, $\eta$ and z is a constant equal to 1 (with this we encompass models that restrict the means of observable variables, and not only the variances and covariances). Typically, $\Lambda$, B, $\Psi := E\,\varepsilon\varepsilon'$ and $Z := E\xi\xi'$ will be matrix-valued functions of a q-dimensional parameter vector $\theta$.

Model (2a) is, in effect, a specific case of (1), since we can write

(2b) $$z = \Lambda(I - B)^{-1}\xi + \varepsilon = \sum_{i=1}^{k} \Lambda(I - B)^{-1}J_i\xi_i + \sum_{i=1}^{k^*} T_i\varepsilon_i,$$

where the $\xi_i$'s and $\varepsilon_i$'s are k and k* subvectors of $\xi$ and $\varepsilon$, respectively, and the $J_i$'s and $T_i$'s are 0-1 matrices such that

$$\xi = \sum_{i=1}^{k} J_i\xi_i \text{ and } \varepsilon = \sum_{i=1}^{k^*} T_i\,\varepsilon_i.$$

It should be mentioned that model (2a) encompasses a variety of structural equation models like Factor Analysis and the so called LISREL models (Jöreskog and Sörbom, 1989). The gradient vector and Hessian matrix associated with model (2a), for different type of fitting functions, can be found in Neudecker and Satorra (1991).

A matrix that plays a fundamental role in assessing the asymptotic distribution of estimators and test statistics, is the following variance matrix of the vector of cross-product moments of z:

(3) $$\Gamma := \text{var}\,(\text{v}\; zz'),$$

which, throughout the paper, is assumed to be finite. Here "var" denotes variance matrix, and "v" is the operator that stacks the non-redundant elements of a symmetric matrix in a column vector. Note that v zz' = $D^+$ vec zz', where "vec" stacks the rows of a matrix as a column vector, $D^+ = (D'D)^{-1}D$ and D is the 0-1 "duplication" matrix for which vec zz' = D v zz' (for further details on the matrices D and $D^+$, see Magnus and Neudecker, 1988). Note that $\Gamma$ is a $p^* \times p^*$ matrix with $p^* := p(p+1)/2$.

In this paper we derive the expression for $\Gamma$ in terms of the matrices $B_i$'s and moment matrices of the $\delta_i$'s, under the assumption that the $\delta_i$'s of (1) are mutually independent. We do this in Section 2 of the paper. The expression to be obtained does not follow from known results on variances of quadratic forms, as (for example) the results of Browne and Neudecker (1988), Neudecker and Wansbeek (1987) and Rao and Kleffe (1988, Section 2.6), which do not consider a linear relation (1) with the $\delta_i$'s possibly non-normally distributed.

The expression obtained for $\Gamma$ is used in Section 3 to simplify the derivation of results on asymptotic robustness in moment structure analysis.

## 2 The expression for $\Gamma$ under multivariate linear relations

In relation with (1), consider

(4) $$E\, \delta_i = 0 \text{ , for } i=1,\dots, L-1,$$

(5) $$\delta_L = 1 \text{ (i.e., } \delta_L \text{ is scalar constant to 1)}$$

and the following independence assumption (IA):

(6) IA: The $\delta_i$'s are mutually independent.

Under the current setting, the following lemma applies (The proof of the lemma is sketched in the Appendix).

<u>Lemma 1</u>

When (4) to (6) hold, the variance matrix $\Gamma$ of (3) can be written as

(7) $$\Gamma = \Omega - 2\, D^+(\mu\mu' \otimes \mu\mu')\, D^{+'} + \sum_{i=1}^{L-1} [\; 2D^+ (B_i \otimes \mu)\{E\; \delta_i (v\, \delta_i\delta_i')'\}D'(B_i \otimes B_i)'\, D^{+'} +$$

$$2D^+(B_i \otimes B_i)D\, \{E\,(v\, \delta_i\delta_i')\, \delta_i'\}(B_i \otimes \mu)'\, D^{+'} +$$

$$D^+(B_i \otimes B_i)D\{(\text{var}\, v(\delta_i\delta_i') - 2D^+E(\delta_i\delta_i') \otimes E\,(\delta_i\delta_i')D^{+'}\}D'(B_i \otimes B_i)'D^{+'}\,],$$

where

(8) $$\Omega := 2\, D^+\, Ezz' \otimes Ezz'\, D^{+'}$$

and $\mu := Ez$.

Remarks

1. When z is normally distributed, (7) simplifies to what we will call the normal (N) expression of $\Gamma$ :

(9) $$\Gamma_N := 2 D^+ [ Ezz' \otimes Ezz' - \mu\mu' \otimes \mu\mu' ] D^{+'},$$

since for centered and normally distributed $\delta_i$'s it holds that (e.g., Neudecker and Wansbeek, 1987)

(10a) $$E\, \delta_i(v\, \delta_i\delta_i')' = 0$$

and

(10b) $$\text{var } v(\delta_i\delta_i') = 2D^+E(\delta_i\delta_i') \otimes E\,(\delta_i\delta_i')D^{+'}.$$

2. Given any random vector z, applying the Lemma to the following (trivial) two-terms multivariate linear relation $z = (z-\mu) + \mu$, we obtain

(11) $$\Gamma = 2\, D^+(Ezz' \otimes Ezz' - \mu\mu' \otimes \mu\mu' )D^{+'} +$$

$$2D^+ (I \otimes \mu )(E(z-\mu)(v(z-\mu)(z-\mu)')') +$$

$$2\ (E\,(v\,(z-\mu)(z-\mu)')(z-\mu)' )\,(I \otimes \mu )'\, D^{+'} +$$

$$((\text{var } v(z-\mu)(z-\mu)') - 2\, D^+E((z-\mu)(z-\mu)') \otimes E((z-\mu)(z-\mu)')D^{+'}),$$

where I is an identity matrix of appropriate dimensions. This is in accordance with an equivalent result of Rao and Kleffe (1988, Section 2.6).

Result (7) will now be used to show that, under certain conditions, $\Omega$ of (8) can substitute $\Gamma$ in the formulae for asymptotic standard errors and test statistics in moment structure analysis. Since the matrix $\Omega$ involves only the second-order moments of the data, which are easier to estimate than higher-order moments, this substitution is of high practical relevance.

## 3 Asymptotic robustness in moment structure analysis

Consider the multivariate linear relation (1) under assumptions (4) to (6), and assume additionally that the $B_i$'s are continuously differentiable functions of a t-dimensional parameter vector $\tau$ and the $\Phi_{ii}$'s are unrestricted symmetric matrices. Denote $\Sigma = Ezz'$, then we obtain a moment structure $\Sigma = \Sigma(\theta)$ where

(12) $$\theta := [\, \tau', (v\ \Phi_{11})', \ldots, (v\ \Phi_{ii})', \ldots, (v\ \Phi_{LL})']'$$

is an (unrestricted) q-dimensional parameter vector ($q \geq t$). This set up arises, for example, in (2b) when $\Lambda$ and B are matrix-valued functions of $\tau$ and the $\xi_i$'s and $\varepsilon_i$'s are mutually independent random variables with unrestricted moment matrices ( a specific model of this type is the factor analysis model).
Since (1) implies that

(13) $$E\, v(zz') = \sum_{i=1}^{L} D^{+}(B_i \otimes B_i) D\, v(\Phi_{ii}),$$

the partition (12) of $\theta$ implies that the $(p^* \times q)$ derivative matrix $\Delta := (\partial/\partial\theta')\, \sigma(\theta)$ can be written as

(14) $$\Delta = [\, \Delta_\tau, D^{+}(B_1 \otimes B_1)D, \ldots, D^{+}(B_i \otimes B_i)D, \ldots, D^{+}(B_L \otimes B_L)D],$$

where $\Delta_\tau := (\partial/\partial\tau')\sigma(\theta)$ is a $p^* \times t$ matrix and $\sigma(\theta) := v\, \Sigma(\theta)$.

Consider now a sample $z_1, z_2, \ldots, z_n$ of n independent observations of z, and let $s := v(S)$ be the reduced vector of sample moments, where

(15) $$S := \sum_{\alpha=1}^{n} (z_\alpha z_\alpha')/n$$

is the (uncentered) sample (second-order) moment matrix of z. Straightforward application of the Central Limit theorem shows that

(16) $$n^{1/2}(s - \sigma^{0}) \;{-}{-}{-}{>}_L\; N(0, \Gamma),$$

where "$--->_L$" indicates convergence in distribution, $\sigma^0$ is the asymptotic limit of s and $\Gamma$ is the $p^* \times p^*$ matrix defined in (3) above. Consider an estimate $\hat{\theta}$ of $\theta$ with the property of being $n^{1/2}$-consistent (i.e., $n^{1/2}(\hat{\theta} - \theta)$ is bounded in distribution). Typically, $\hat{\theta}$ will be the minimizer of

(17) $$F = (s - \sigma(\theta))'\, \hat{W}\, (s - \sigma(\theta)),$$

where $\hat{W}$ is a weight matrix converging in probability to a positive definite matrix, say W. It can also be the minimizer of the (pseudo) maximum likelihood function

(18) $$F_{ML} = \ln|\Sigma(\theta)| + tr\{ S\, \Sigma(\theta)^{-1}\} - \ln|S| - p.$$

Instrumental variable estimators are also $n^{1/2}$-consistent estimators of $\theta$ (e.g., Jennrich, 1987). Computer programs that produce such estimators for the class of models described in (2) are, for example, LISREL (Jöreskog and Sörbom, 1989), EQS (Bentler, 1989 ), LISCOMP (Muthén, 1987) and LINCS (Schoenberg, 1989).

Let us denote by $\hat{\tau}$ the $(t \times 1)$ subvector of $\hat{\theta}$ corresponging to $\tau$. By standard asymptotic theory, the asymptotic variance matrix of $\hat{\tau}$ is

(19) $$\text{avar}(\hat{\tau}) = n^{-1}\Theta_\tau \Gamma \Theta_\tau',$$

where $\Theta_\tau$ is the $t \times p$ leading sub-matrix of $(\Delta'W\Delta)^{-1}\Delta'W$, say $[(\Delta'W\Delta)^{-1}\Delta'W]_{t \times p^*}$ (e.g., Satorra, 1989). Note that in the case of (pseudo) maximum likelihood estimation, then $W = \partial^2 F_{ML}(\sigma,\sigma)/\partial\sigma\partial\sigma'$ and equals $\Omega$ of (8) (e.g., Neudecker and Satorra, 1991).

Once the estimate $\hat{\theta}$ is known, the vector $\hat{\sigma} := \sigma(\hat{\theta})$ of fitted moments can be computed. To test the adequacy of the model, an asymptotic chi-square goodness-of-fit test statistic can be defined as

(20) $$G = n(s - \hat{\sigma})' \hat{A} (s - \hat{\sigma}),$$

where $\hat{A}$ is a consistent estimate of $\Delta_\perp (\Delta_\perp' \Gamma \Delta_\perp)^- \Delta_\perp$ and "-" denotes a g-inverse. Under a sequence of local alternatives (Neyman, 1937; see McManus, 1991), namely

(21) $$\sigma^o := \sigma^o{}_n \quad \text{with} \quad \sqrt{n}(\sigma^o{}_n - \sigma) = \delta,$$

where $\delta$ is a finite p*-dimensional vector, standard results of Moore (1977) show that[1]

(22) $$G \dashrightarrow_L \chi^2{}_r(\lambda),$$

where $\chi^2{}_r(\lambda)$ is a non-central chi-square distribution with $r = \text{rank}(\Delta_\perp' \Gamma \Delta_\perp)$ degrees of freedom and non-centrality parameter $\lambda = \delta' \Delta_\perp' (\Delta_\perp' \Gamma \Delta_\perp)^- \Delta_\perp \delta$. Here $\Delta_\perp$ means an orthogonal complement of the derivative matrix $\Delta$. The asymptotic distribution of G will, of course, be central chi-square when the drift parameter $\delta$ of (21) equals zero (i.e., when the model is "exactly true"). In the particular case of covariance structure analysis (where z is assumed to be of zero mean), the above goodness-of-fit statistic G was introduced by Browne (1984).

Since $\Theta_t \Delta$ and $\Delta_\perp' \Delta$ equal zero, the partition (14) of $\Delta$ implies

(23) $$\Theta_t D^+(B_i \otimes B_i) D = 0, \quad i = 1,\ldots,L,$$

and

(24) $$\Delta_\perp' D^+(B_i \otimes B_i) D = 0, \quad i = 1,\ldots,L,$$

which, combined with the expression for $\Gamma$ obtained in Lemma 1 (see (7)), yield the following fundamental results:

(25) $$\Theta_\tau \Gamma \Theta_\tau' = \Theta_\tau \Omega \Theta_\tau'$$

and

(26) $$\Delta_\perp' \Gamma \Delta_\perp = \Delta_\perp' \Omega \Delta_\perp .$$

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[1]In fact, noting that

$\sqrt{n}(s - \hat{\sigma}) = \sqrt{n}(s - \sigma^o) + \sqrt{n}(\sigma^o - \sigma_o) + \sqrt{n}(\sigma_o - \hat{\sigma}) =$

$\sqrt{n}(s - \sigma^o) + \sqrt{n}(\sigma^o - \sigma_o) - \sqrt{n}\Delta(\hat{\theta} - \theta),$

under the sequence of local alternatives (17), we can write

$\sqrt{n}\, \Delta_\perp (s - \hat{\sigma}) = \sqrt{n}\, \Delta_\perp (s - \sigma^o) + \sqrt{n}\Delta_\perp(\sigma^o - \sigma_o) \dashrightarrow_L N(\Delta_\perp \delta, \Delta_\perp \Gamma \Delta_\perp').$

Result (25) and (26) have very interesting practical implications. In effect, (25) allows us to estimate the variance matrix of $\hat{\tau}$ as

$$\text{avar}(\hat{\tau}) = n^{-1}\Theta_t \hat{\Omega} \Theta_t' =$$

(27) $$= [ (\hat{\Delta}' \hat{W} \hat{\Delta})^{-1} \hat{\Delta}' \hat{W} \hat{\Omega} \hat{W} \hat{\Delta} (\hat{\Delta}' \hat{W} \hat{\Delta})^{-1} ]_{t \times t},$$

where $[\ ]_{t \times t}$ denotes the leading $t \times t$ submatrix of the matrix enclosed. Further, (26) allow us to construct an asymptotic chi-square goodness-of-fit statistic G* as

(28) $$G^* = n (s - \hat{\sigma})' \hat{\Delta}_{\perp} ( \hat{\Delta}_{\perp}' \hat{\Omega} \hat{\Delta}_{\perp})^{-} \hat{\Delta}_{\perp}' (s - \hat{\sigma}).$$

Here "$\hat{\ }$" means evaluated at $\hat{\theta}$ or simply a consistent estimate. It should be noted that a consistent estimate of $\Omega$ is easily obtained by replacing S for Ezz' in (7b).

Limited to the context of covariance structure analysis, the asymptotic chi-squaredness of G* of (28), as well as the validity of the result (27), under the current assumptions was proven in Satorra and Bentler (1990, 1991). It should be noted that when $F_{ML}$ is used, then G* of (28) and the statistic $nF_{ML}(S, \Sigma(\hat{\theta}))$ have the same asymptotic distribution and, hence, both will be asymptotically chi-square under the current assumptions. Results (25) and (26) encompass asymptotic robustness results of Amemiya and Anderson (1990), Anderson (1987), Satorra and Bentler (1990), Browne and Shapiro (1988) and Satorra (1991). The present approach for proving results of asymptotic robustness has also been exploited in Satorra (1991).

Clearly, the possibility of using a consistent estimate of $\Omega$ instead of $\Gamma$ simplifies computations considerably. In fact, when computing the variance matrix (27) of estimates of $\tau$, as well as to computing G* of (28), only the second-order moments of the data are involved. In contrast, under nonnormality of z, usual consistent estimates of $\Gamma$ involve higher-order moments of the data.

Slight modifications of the arguments above will also show the validity of (25) and (26) when some of the matrices $\Phi_{ii}$'s are restricted to be continuously differentiable functions of $\tau$, provided that condition (10) is verified for the $\delta_i$'s with restricted $\Phi_{ii}$'s (as is the case when $\delta_i$ is normally distributed).

## Appendix

This appendix sketches the proof of the Lemma

Results such as $K(A \otimes B)K = (B \otimes A)$, $K(A \otimes b) = b \otimes A$, and $\text{vec}\, b b' = b \otimes b$, where A is a matrix, b is a vector and K is the commutation matrix of appropriate dimension, and other standard results of matrix calculus (e.g., Magnus & Neudecker, 1988), will be used extensively in the proof of the lemma.

First note that under (1):

$$E\, zz' = E\left(\sum_j B_j \delta_j\right)\left(\sum_i B_i \delta_i\right)' = \sum_i B_i E(\delta_i \delta_i') B_i'$$

since $E\, \delta_i \delta_j = 0$ when $i \neq j$. Hence,

(A1) $$(I+K)\ Ezz' \otimes Ezz' =$$

$$\sum_i (I+K)(B_i \otimes B_i)(E(\delta_i \delta_i') \otimes E(\delta_i \delta_i'))(B_i \otimes B_i)' +$$

$$\sum_{i \neq j} (I+K)(B_i \otimes B_j)(E(\delta_i \delta_i') \otimes E(\delta_j \delta_j'))(B_i \otimes B_j)'.$$

and

$$\text{var}(\text{vec}\, zz') =$$

$$\sum_i \sum_j \sum_k \sum_t (B_j \otimes B_i)[E(\delta_j \otimes \delta_i)(\delta_t \otimes \delta_k)' - E(\delta_j \otimes \delta_i) E(\delta_t \otimes \delta_k)'](B_t \otimes B_k)' =$$

$$\sum_i \sum_j \sum_k \sum_t (B_j \otimes B_i) X_{ijkt} (B_t \otimes B_k)',$$

where

$$X_{ijkt} := E(\delta_j \otimes \delta_i)(\delta_t \otimes \delta_k)' - E(\delta_j \otimes \delta_i) E(\delta_t \otimes \delta_k)'.$$

Now, computing $X_{ijkt}$ under different combinations of subscripts, we deduce

(A2) $$\text{Var}(\text{vec}\, zz') =$$

$$\sum_{i \neq j} (B_j \otimes B_i)[E(\delta_j \delta_j') \otimes E(\delta_i \delta_i')](B_j \otimes B_i)'(I + K).$$

$$\sum_{i \neq L} (I+K)(B_i \otimes B_L)[E(\delta_i \otimes \delta_L)(\delta_i \otimes \delta_i)'](B_i \otimes B_i)' +$$

$$\sum_{i \neq L} (B_i \otimes B_i)[E(\delta_i \otimes \delta_i)(\delta_L \otimes \delta_i)'](B_i \otimes B_L)'(I + K) +$$

$$\sum_i (B_i \otimes B_i)[\text{var vec}\, \delta_i \delta_i'](B_i \otimes B_i)'.$$

Consequently, we can write

(A3) $D^{+}\,(I+K)\;\; Ezz' \;\otimes Ezz'\; D^{+'} \;=\; 2\, D^{+}\, Ezz' \;\otimes Ezz'\; D^{+'} =$

$$\sum_{i} 2D^{+}(B_i \otimes B_i\,)\,[\,E(\delta_i\delta_i') \otimes \; E(\delta_i\delta_i')]\;\;(B_i \otimes B_i\,)'D^{+'} \;+$$

$$\sum_{i \neq j} 2D^{+}\,(B_i \otimes B_j\,)\,[E(\delta_i\delta_i') \otimes \; E(\delta_j\delta_j')]\;\;(B_i \otimes B_j\,)'D^{+'} \;;$$

and

(A4) $D^{+}\,\mathrm{Var}\,(\,\mathrm{vec}\; zz'\;\,)\; D^{+'} \;=$

$$\sum_{i \neq j} 2D^{+}\,(B_j \otimes B_i\,)\,[E(\delta_j\delta_j'\,) \otimes E(\delta_i\delta_i')\,]\;\;(B_j \otimes B_i\,)'\, D^{+'}\,.$$

$$\sum_{i \neq L} 2D^{+}\,(B_i \otimes B_L\,)[E(\delta_i \otimes \delta_L)(\delta_i \otimes \delta_i)'\,]\,(B_i \otimes B_i\,)'\, D^{+'} \;+$$

$$\sum_{i \neq L} 2D^{+}(B_i \otimes B_i\,)[E(\delta_i \otimes \delta_i)(\delta_L \otimes \delta_i)'\,]\,(B_i \otimes B_L\,)'\, D^{+'} \;+$$

$$\sum_{i} D^{+}\,(B_i \otimes B_i\,)\,[\,\mathrm{var\,vec}\;\delta_i\delta_i']\;\;(B_i \otimes B_i\,)'D^{+'}\,.$$

The proof concludes by combining results (A3) and (A4). Q.E.D.

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