Biased Representation of Homothetic Preferences on Homogeneous Sets

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Abstract

In the homogeneous case of one-dimensional objects, we show that any preference relation that is positive and homothetic can be represented by a quantitative utility function and a unique bias. This bias may favor or disfavor the preference for an object. In the first case, preferences are complete but not transitive and an object may be preferred even when its utility is lower. In the second case, preferences are asymmetric and transitive but not negatively transitive and it may not be sufficient for an object to have a greater utility to be preferred. In this manner, the bias reflects the extent to which preferences depart from the maximization of a utility function.

Keywords: Intransitive preferences, Incomplete preferences, Irrational Behavior, Bias, Procedural concerns, Process of Choice.

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1 Introduction

Biased representation of preferences consists in representing preferences with a utility function and a multiplicative bias outside this utility function. The utility function provides for a measurement of the objects of preferences while the bias explains why preferences may be inconsistent with the maximization of the utility function. In this manner, the bias allows for a larger class of preferences to be represented and captures the extent to which preferences may depart from the maximization of a utility function.

In the *homogeneous* case of *one-dimensional* objects, we prove in this paper that there exist a positively-valued function u unique up to multiplication by a positive constant and a unique bias $0 \leq \alpha \leq 1$ such that

$$x \succ y \Leftrightarrow \alpha u(x) > (1-\alpha)u(y). \qquad (i)$$

If $\alpha = \frac{1}{2}$, preferences are represented by the maximization of u. If $\alpha > \frac{1}{2}$ then the bias "favors" the preference over x, and x may be preferred even if its utility is lower. If $\alpha < \frac{1}{2}$ then the factor "disfavors" the preference over x, and it is not sufficient for x to have a greater utility to be preferred. If $\alpha = 0$ then preferences are empty and if $\alpha = 1$, they hold for any pair of objects. Note that these values of 0 and 1 for α are representing two limit cases where the bias determines preferences independently of the utility of objects. Also, when $\alpha \neq 0$, we may like to write the inequality as $\frac{u(x)}{u(y)} \geqslant \frac{(1-\alpha)}{\alpha}$ and make more explicit the proportional threshold of preferences (for threshold models, see Suppes et Al. 1989).

In the homogeneous case presented here, two main axioms are required to obtain this result. Firstly, preferences are assumed to be *positive*: if an object x is preferred to an object y, then a relative increase of the quantity of x does not change its preference. Secondly, preferences verify a form of *scale invariance* called *homotheticity*: a preference for x over y remains invariant if x and y are replicated an identical number of times.

As no specific property for the binary relation $\succ$ alone is required, this model covers a broad class of preferences over homogeneous sets. If $\alpha > \frac{1}{2}$, the relation $\succ$ is complete but not transitive. If $\alpha < \frac{1}{2}$, the relation $\succ$ is asymmetric and transitive but its negation is not transitive, leading to intransitivity of indifference. Of course, $\succ$ is asymmetric and negatively transitive, i.e. it is a weak order, if and only if $\alpha = \frac{1}{2}$.

We obtain the representation (i) assuming that the primitive relation, noted $\succ$, verifies an Archimedean axiom. When the primitive relation is non-Archimedean, we note it $\succsim$ and we obtain a representation of the form

$$x \succsim y \Leftrightarrow \alpha u(x) \geq (1-\alpha)u(y). \qquad (i')$$

In that case, we define two relations $\succ$ and $\sim$ such that $\succ$ is represented by (i) and $\sim$ is represented by

$$x \sim y \Leftrightarrow \alpha u(x) = (1-\alpha)u(y). \qquad (i'')$$

In this manner, we naturally have $x \succsim y \Leftrightarrow x \succ y$ or $x \sim y$ and the relation $\sim$ defines a particular type of "indifference". In terms of interpretation, x is indifferent to y if and only if x is preferred to y but any "increase" in the quantity of y negates this preference (note that in general, such "indifference" is neither symmetric nor transitive). This later representation should be useful to experimentally elicit the value of the bias.

While this model requires weak assumptions on preferences to prove the existence of the function u, the biased representation obtained has strong uniqueness properties. Indeed, the function u is unique up to *multiplication by a positive constant*, hence allowing the measurement of both differences and ratios of utility, which corresponds to a full *quantitative* measurement (also called a ratio-scale, see Krantz et Al. 1971). As to the factor α, it is *uniquely determined* by the preferences, thus allowing a precise measurement of the departure from utility maximization. In this manner, relaxing the maximization hypothesis does not impede the measurement of utility while offering a way to model preferences that would have been considered "irrational".

The result presented in this paper is an extension of the biased representations introduced in Le Menestrel and Lemaire (2004). There, the relation of preferences $\succ$ was assumed to be asymmetric and transitive, and corresponds, in the present paper, to the case when the bias disfavors the preferred object, i.e. when $0 < \alpha \leqslant \frac{1}{2}$. An Archimedean condition was also assumed and the interpretation was developed around the idea of a proportional lack of discrimination (also known as Weber's law). In the present paper, the bias can also favor preferences and may be used to interpret situations in which individuals forgo part of their utility because of considerations that cannot be easily included in the utility function. Typical examples are considerations proper to the process of choice (i.e. procedural concerns in Sen, 1997) or ethical considerations. Future work shall develop such interpretations.

Finally, we have generalized Le Menestrel and Lemaire (2004) to the non-homogeneous case in Lemaire and Le Menestrel (2004). Then, the biasing factor is not necessarily constant. Similarly, it is probably possible to generalize the result presented here to non-homogeneous sets.

2 Extending the homogeneous case of biased representation

Let A be a nonempty set of elements $x, y, z... \in A$. Denote $\mathbb{N}^*$ the set of positive integers, and assume A to be endowed with a map $\mathbb{N}^* \times A \rightarrow A, (m, x) \longmapsto mx$ such that $(mm')x = m(m'x)$ and $1x = x$. Such a A is called a $\mathbb{N}^* - set$. In this manner, nx can be naturally interpreted as the quantity n of an object x. Denoting $\mathbb{R}_{>0}$ the set of positive real numbers, note that a $\mathbb{R}_{>0} - set$ is a special case of a $\mathbb{N}^* - set$. Hence, the main results presented here (Theorems 1 and 2) would remain valid if one wants to let n be a positive real number instead of a positive natural number. Here, we use natural numbers because we do not need quantities to be non-denumerable to obtain our result.

We say that A is *homogeneous* if, for all $x, y \in A$, there exists $(m, n) \in \mathbb{N}^* \times \mathbb{N}^*$ such that $mx = ny$. A homogeneous set can hence be readily interpreted as consisting of quantities of one-dimensional objects. We note $\mathbb{Q}_{>0}$ the set of positive rational numbers.

Let $\succ$ be a binary relation on A and consider the three following axioms ($x, y \in A; m, n \in \mathbb{N}^*$):

A1 *(Positivity):* $\forall(x, y, m, n)$ such that $m > n$, we have $x \succ y \Rightarrow mx \succ ny$.

A2 *(Homotheticity):* $\forall(x, y, m)$ we have $x \succ y \Leftrightarrow mx \succ my$;

A3 *(Archimedean):* $\forall(x, y)$ such that $x \succ y$, $\exists(m, n)$ such that $m < n$ and $mx \succ ny$.

Theorem 1 *Let A be a $\mathbb{N}^* - set$ endowed with a binary relation $\succ$ that verifies A1, A2 and A3. Suppose A is homogeneous. Then there exist a function $u : A \rightarrow \mathbb{R}_{>0}$ and a number $0 \leq \alpha \leq 1$ such that, for all $x, y \in A$ and $m \in \mathbb{N}^*$, we have*

$$x \succ y \Longleftrightarrow \alpha u(x) > (1 - \alpha)u(y), \tag{i}$$

$$u(mx) = mu(x). \tag{ii}$$

Moreover, the pair (u, α) of (i) is unique up to replacing u by λu for $\lambda > 0$.

Proof. We can always choose a function $u : A \rightarrow \mathbb{R}_{>0}$ such that $u(mx) = mu(x)$ for all $(x, m) \in A \times \mathbb{N}^*$. Since A is homogeneous, such a function exists and is unique up to multiplication by a positive scalar; in other words, given an element $a \in A$, u is uniquely determined by its value at a. If the relation $\succ$ (respectively $\not\succ$) is empty, we take $\alpha = 0$ (resp. $\alpha = 1$). So in both cases, we have

$$x \succ y \Longleftrightarrow \alpha u(x) > (1-\alpha)u(y).$$

From now on, suppose that both the relation $\succ$ and the relation $\not\succ$ are nonempty. For $x, y \in A$, we define the subset of $\mathbb{Q}_{>0}$

$$\mathcal{P}_{x,y} = \{\frac{m}{n} : mx \succ ny, \exists (m,n) \in \mathbb{N}^* \times \mathbb{N}^*\}.$$

Let $x, y \in A$. By A1 and A2, if $q \in \mathcal{P}_{x,y}$ then $\mathbb{Q}_{\geq q} \subset \mathcal{P}_{x,y}$. And since $\succ$ is nonempty and A is homogeneous, we have $\mathcal{P}_{x,y} \neq \emptyset$. Put $t_{x,y} = \inf_{\mathbb{R}_{\geq 0}} \mathcal{P}_{x,y}$. By A1 and A2, we have $\mathbb{Q}_{>t} \subset \mathcal{P}_{x,y}$ with $t = t_{x,y}$.

Now, if $q \in \mathcal{P}_{x,y}$, then by A2 and A3, there exists $q' \in \mathbb{Q}_{<q} \cap \mathcal{P}_{x,y}$, which implies $q > t_{x,y}$. Hence, we have $\mathcal{P}_{x,y} \subset \mathbb{Q}_{>t}$ and then $\mathcal{P}_{x,y} = \mathbb{Q}_{>t}$.

We thus have

$$x \succ y \Longleftrightarrow 1 \in \mathcal{P}_{x,y} \Longleftrightarrow t_{x,y} < 1.$$

Since $\mathcal{P}_{mx,ny} = \frac{n}{m}\mathcal{P}_{x,y}$, we have $t_{mx,ny} = \frac{n}{m}t_{x,y}$. Using the homogeneity of A, we obtain $t_{x,y} > 0$ (recall the relation $\not\succ$ is supposed to be nonempty). Now, choose $a \in A$. Since we can always replace u by λu with $\lambda = u(a)^{-1}t_{a,a}$, we can suppose $u(a) = t_{a,a}$. Since $t_{a,nx} = nt_{a,x}$, we thus have $u(x) = t_{a,x} \in \mathbb{R}_{>0}$. Also put

$$\sigma(x,y) = t_{a,x}^{-1}t_{x,y}^{-1}t_{a,y} \in \mathbb{R}_{>0}.$$

Since $\mathcal{P}_{mx,ny} = \frac{n}{m}\mathcal{P}_{x,y}$, we have $t_{mx,ny} = \frac{n}{m}t_{x,y}$. Hence $u(mx) = mu(x)$ and

$$\sigma(mx,ny) = (mt_{a,y})^{-1}(\frac{n}{m}t_{x,y})^{-1}nt_{a,y} = \sigma(x,y).$$

Since A is homogeneous, σ is constant on $A \times A$; let $\alpha = \frac{\sigma(A\times A)}{\sigma(A\times A)+1}$. We have $0 < \alpha < 1$ (both $\succ$ and $\not\succ$ being nonempty). Also

$$\begin{aligned} x \succ y \quad &\Leftrightarrow t_{x,y}^{-1} > 1 \\ &\Leftrightarrow (t_{a,x}^{-1}t_{x,y}^{-1}t_{a,y})t_{a,x} > t_{a,y} \\ &\Leftrightarrow \alpha u(x) > (1-\alpha)u(y). \end{aligned}$$

As for the uniqueness property (in the general case, i.e. without hypothesis on the emptiness of $\succ$ and $\not\succ$), let (v,β) be a pair such that $v : A \to \mathbb{R}_{>0}, v(mx) = mv(x), 0 \leq \beta \leq 1,$ *and* $x \succ y \Leftrightarrow \beta v(x) > (1-\beta)v(y)$. Since A is homogeneous, we necessarily have $v = \lambda u$ for some $\lambda \in \mathbb{R}_{>0}$. It is then easy to deduce that $\beta = \alpha$. ■

We can summarize the properties of $\succ$ in the following corollary:

Corollary 1 *Let $\succ$ be a binary relation on a homogeneous $\mathbb{N}^* -$ set A that verifies A1, A2 and A3, and let (u, α) be a pair that verifies conditions (i) and (ii) of. Theorem 1. The relation $\succ$ is*

- *nonempty if and only if* $\alpha > 0$,
- *asymmetric and transitive if and only if* $\alpha \leq \frac{1}{2}$,
- *complete if and only if* $\alpha > \frac{1}{2}$.

Note also (with the notation of Corollary 1) that the relation $\not\succ$ is given by

$$x \not\succ y \Longleftrightarrow \alpha u(x) \leq (1-\alpha)u(y).$$

In particular, $\not\succ$ is nonempty if and only if $\alpha < 1$.

In Theorem 1, we assume that the primitive relation $\succ$ is Archimedean. Now, starting with a binary relation $\succsim$ on A that verifies A1 and A2, we define two binary relations $\succ$ and $\sim$ on A as follows:

- $x \succ y \Leftrightarrow (x \succsim y$ and $\exists(m,n)$ such that $m < n$ and $mx \succsim ny)$;
- $x \sim y \Leftrightarrow (x \succsim y$ and $x \not\succ y)$.

As suggested by the notation, we have $x \succsim y \Leftrightarrow (x \succ y$ *or* $x \sim y)$. Since $\succsim$ is homothetic and positive, then $\succ$ and $\sim$ are homothetic, and $\succ$ is positive and Archimedean. Note that $\sim$ may not be symmetric (i.e. it may not verify $x \sim y \Rightarrow y \sim x$); and $\succ$ may not be asymmetric (i.e. it may not verify $x \succ y \Rightarrow y \not\succ x$). Note also that if $x \not\succ y \Leftrightarrow y \succsim x$ for all $x, y \in A$, then the relation $\sim$ is given by

$$x \sim y \Leftrightarrow (x \succsim y \text{ and } y \succsim x) \Leftrightarrow (x \not\succ y \text{ and } y \not\succ x);$$

in which case it is clearly symmetric.

The relation $\sim$ is empty if and only if the relation $\succsim$ verifies A3, in which case we have $\succ = \succsim$. Hence assuming that $\sim$ is not empty amounts to assume that $\succsim$ verifies the following axiom $(x, y \in A; m, n \in \mathbb{N}^*)$:

A3' *(Non-Archimedean):* $\exists(x,y)$ such that $x \succsim y$ and, $\forall(m,n)$ such that $m < n$, we have $mx \not\succsim ny$.

This leads us to a slightly different formulation of Theorem 1:

Theorem 2 *Let A be a $\mathbb{N}^*-$ set endowed with a binary relation $\succsim$ that verifies A1, A2 and A3'. Suppose A is homogeneous. Then there exist a function $u : A \to \mathbb{R}_{>0}$ and a number $0 < \alpha \leq 1$ such that, for all $x, y \in A$ and $m \in \mathbb{N}^*$, we have*

$$x \succsim y \Longleftrightarrow \alpha u(x) \geq (1-\alpha)u(y), \qquad (i')$$

$$u(mx) = mu(x). \qquad (ii)$$

Moreover, the pair (u, α) of (i) is unique up to replacing u by λu for $\lambda > 0$.

Proof. Choose a function $u : A \to \mathbb{R}_{>0}$ such that $u(mx) = mu(x)$ for all $(x, m) \in A \times \mathbb{N}^*$ (Cf. the proof of Theorem 1). If $\not\succsim$ is empty, we take $\alpha = 1$; hence the pair (α, u) verifies (i').

We now suppose $\not\succsim$ is not empty. Since $\succsim$ verifies A3', it is not empty. Also, $\sim$ is not empty. Because A is homogeneous, for all $(x, y) \in A \times A$, there exists $(m_0, n_0) \in \mathbb{N}^* \times \mathbb{N}^*$ such that $m_0 x \sim n_0 y$. We define $\mathcal{P}_{x,y}$ and $t_{x,y}$ as in the proof of Theorem 1. We also define the subset of $\mathbb{Q}_{>0}$

$$\mathcal{Q}_{x,y} = \{\frac{m}{n} : mx \succsim ny, \exists (m, n) \in \mathbb{N}^* \times \mathbb{N}^*\}.$$

So we have the inclusion $\mathcal{P}_{x,y} \subset \mathcal{Q}_{x,y}$. If $q \in \mathcal{Q}_{x,y}$, then by A1 and A2, we have $\mathbb{Q}_{\geq q} \subset \mathcal{Q}_{x,y}$; and by the definition of $\succ$, we have $\mathbb{Q}_{>q} \subset \mathcal{P}_{x,y} = \mathbb{Q}_{>t_{x,y}}$, and $q \geq t_{x,y}$. Since $\succ \neq \succsim$, the inclusion $\mathcal{P}_{x,y} \subset \mathcal{Q}_{x,y}$ is strict. Hence we have $\mathcal{Q}_{x,y} = \mathbb{Q}_{\geq t_{x,y}}$, and

$$x \succsim y \Longleftrightarrow 1 \in \mathcal{Q}_{x,y} \Longleftrightarrow t_{x,y} \leq 1.$$

Since $\not\succsim$ is supposed to be nonempty, we also have $t_{x,y} > 0$. From Theorem 1, there exists a $0 \leq \alpha \leq 1$ such that $x \succ y \Longleftrightarrow \alpha u(x) > (1-\alpha)u(y)$. We then have

$$x \succsim y \Longleftrightarrow \alpha u(x) \geq (1-\alpha)u(y)$$

and

$$x \sim y \Longleftrightarrow \alpha u(x) = (1-\alpha)u(y).$$

Since the relation $\succsim$ is nonempty, the relation $\succ$ is also nonempty, and we have $\alpha > 0$.

Finally, if (v, β) is another pair verifying conditions (i') and (ii), then it verifies condition (i) of Theorem 1, which implies $(v, \beta) = (\lambda u, \alpha)$ for a positive scalar λ. ∎

We can summarize the properties of $\succsim$ in the following corollary:

Corollary 2 *Let $\succsim$ be a binary relation on a homogeneous $\mathbb{N}^* - set$ A that verifies A1, A2 and A3', and let (u, α) be a pair that verifies conditions (i') and (ii) of. Theorem 2. The relation $\succsim$ is*

- *asymmetric if and only if* $\alpha < \frac{1}{2}$,
- *transitive if and only if* $\alpha \leq \frac{1}{2}$,
- *complete if and only if* $\alpha \geq \frac{1}{2}$.

Note also (with the notation of Corollary 2) that the relation $\not\succsim$ is given by

$$x \not\succsim y \Longleftrightarrow \alpha u(x) < (1 - \alpha)u(y).$$

In particular, $\not\succsim$ is nonempty if and only if $\alpha < 1$.

The following two corollaries help to further understand the link between Theorem 1 and Theorem 2. A key condition is whether the biasing factor is a rational number.

Corollary 3 *Let $\succsim$ be a binary relation on a homogeneous $\mathbb{N}^* - set$ A that verifies A1, A2 and A3', and let (u, α) be a pair that verifies conditions (i') and (ii) of. Theorem 2. Then $\alpha \in \mathbb{Q}_{>0}$.*

Proof. For all $(x, y) \in A \times A$, there exists $(m_0, n_0) \in \mathbb{N}^* \times \mathbb{N}^*$ such that $m_0 x \sim n_0 y$ and we have $t_{x,y} = \frac{m_0}{n_0}$ (Cf. the proof of Theorem 2). From the definition of α (Cf. the proof of Theorem 1), we conclude that $\alpha \in \mathbb{Q}_{>0}$. ■

Corollary 4 *Let $\succ$ be a nonempty binary relation on a homogeneous $\mathbb{N}^* - set$ A that verifies A1, A2 and A3, and let (u, α) be a pair that verifies conditions (i) and (ii) of. Theorem 1. Let $\succsim$ be the binary relation defined by $x \succsim y \Longleftrightarrow \alpha u(x) \geq (1 - \alpha)u(y)$. Then $\succsim$ verifies A1 and A2, and it verifies A3' if and only if $\alpha \in \mathbb{Q}_{>0}$.*

Proof. Clearly, the relation $\succsim$ verifies A1 and A2. And we have

$$x \succ y \Leftrightarrow (x \succsim y \text{ and } \exists (m, n) \text{ with } m < n \text{ such that } mx \succsim ny.$$

Hence if the relation $\succsim$ verifies A3', from Corollary 3, we have $\alpha \in \mathbb{Q}_{>0}$. Now suppose $\alpha \in \mathbb{Q}_{>0}$. Since $\alpha \in \mathbb{Q}_{>0}$, there exists $(m, n) \in \mathbb{N}^* \times \mathbb{N}^*$ such that $\alpha m = (1 - \alpha)n$. Hence for all $x \in A$, we have $mx \succsim nx$ but $mx \not\succ nx$. So $\succ \neq \succsim$, and $\succsim$ verifies A3'. ■

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