HOMOTHETIC INTERVAL ORDERS

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Abstract We give a characterization of the non-empty binary relations $\succ$ on a $\mathbb{N}^*$-set A such that there exist two morphisms of $\mathbb{N}^*$-sets $u_1, u_2 : A \to \mathbb{R}_+$ verifying $u_1 \leq u_2$ and $x \succ y \Leftrightarrow u_1(x) > u_2(y)$. They are called *homothetic interval orders*. If $\succ$ is a homothetic interval order, we also give a representation of $\succ$ in terms of one morphism of $\mathbb{N}^*$-sets $u : A \to \mathbb{R}_+$ and a map $\sigma : u^{-1}(\mathbb{R}_+^*) \times A \to \mathbb{R}_+^*$ such that $x \succ y \Leftrightarrow \sigma(x,y)u(x) > u(y)$. The pairs (u_1, u_2) and (u, σ) are "uniquely" determined by $\succ$, which allows us to recover one from each other. We prove that $\succ$ is a semiorder (resp. a weak order) if and only if σ is a constant map (resp. $\sigma = 1$). If moreover A is endowed with a structure of commutative semigroup, we give a characterization of the homothetic interval orders $\succ$ represented by a pair (u, σ) so that u is a morphism of semigroups.

Résumé On donne une caractérisation des relations binaires non vides $\succ$ sur un $\mathbb{N}^*$-ensemble A telles qu'il existe deux morphismes de $\mathbb{N}^*$-ensembles $u_1, u_2 : A \to \mathbb{R}_+$ vérifiant $u_1 \leq u_2$ et $x \succ y \Leftrightarrow u_1(x) > u_2(y)$. On les appelle des *ordres intervalles homothétiques*. Si $\succ$ est un ordre intervalle homothétique, on donne aussi une représentation de $\succ$ à l'aide d'un morphisme de $\mathbb{N}^*$-ensembles $u : A \to \mathbb{R}_+$ et d'une application $\sigma : u^{-1}(\mathbb{R}_+^*) \times A \to \mathbb{R}_+^*$ tels que $x \succ y \Leftrightarrow \sigma(x,y)u(x) > u(y)$. Les paires (u_1, u_2) et (u, σ) sont déterminées "de manière unique" par $\succ$, ce qui nous permet de retrouver l'une à partir de l'autre. On montre que $\succ$ est un semiordre (resp. un ordre faible) si et seulement si σ est une application constante (resp. $\sigma = 1$). Si de plus A est muni d'une structure de semigroupe commutatif, on donne une caractérisation des ordres intervalles homothétiques $\succ$ représentés par une paire (u, σ) telle que u soit un morphisme de semigroupes.

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Introduction Let us start with an example, which has been our main source of inspiration for this work. Consider a two-armed-balance, the two arms of which not necessarily being of the same length; such a balance is said to be *biased*. Let denote P_1 and P_2 its two pans. If the arms are not of the same length, we assume that P_1 is located at the end of the shortest arm. Suppose also we are given a set A of objects to put on P_1 and P_2. We define as follows a binary relation $\succ$ on A: $x \succ y$ if the balance tilts towards x when we put x on P_1 and y on P_2. This relation is always asymmetric and transitive, but it is negatively transitive if and only if the two arms are of the same length. However we can observe it is always *strongly transitive*: $x \succ y \succsim z \succ t \Rightarrow x \succ t$ with $y \succsim z \Leftrightarrow z \not\succ y$. In particular, $\succ$ is an *interval order* (cf. [F]). Furthemore, suppose that A is endowed with a structure of $\mathbb{N}^*$-set. Then the relation $\succ$ verifies the following property of *homothetic independence*: $x \succ y \Leftrightarrow (mx \succ my,\ \forall m \in \mathbb{N}^*)$. We can continue to identify the properties satisfied by $\succ$. That naturally brings us to introduce the notion of *homothetic structure* (cf. section 2). A homothetic structure is by definition a $\mathbb{N}^*$-set A endowed with a binary relation $\succ$ verifying five properties of compatibility, the most striking two being the homothetic independence

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introduced before and the following property : if $x \succ y$, then $\exists m \in \mathbb{N}^*$ such that $mx \succ (m+1)y$. A homothetic structure $(A, \succ)$ is called a *homothetic interval order* if the relation $\succ$ is assymmetric and strongly transitive. The main goal of this paper is to give a caracterization of the *homothetic interval orders* via their representations in $\mathbb{R}_+$.

So let $(A, \succ)$ be a non-empty homothetic interval order. If $(A, \succ)$ is obtained from a biased balance as above, then we know there exists a morphism of $\mathbb{N}^*$-sets $u : A \to \mathbb{R}_+$ (the mass) and a real number $\alpha \in]0,1]$ (the ratio of the shortest arm to the longest one) such that $x \succ y \Leftrightarrow \alpha u(x) > u(y)$. It is this kind of result we are looking for here. Let us begin with the simplest case: $\succ$ is a *homothetic weak order*; i.e., the relation $\succ$ is negatively transitive. Then we prove (proposition (4.1)) that there exists a morphism of $\mathbb{N}^*$-sets $u : A \to \mathbb{R}_+$, unique up to multiplication by a positive scalar, such that $x \succ y \Leftrightarrow u(x) > u(y)$. Let us point out that no countable hypothesis on the quotient-set $A/\!\sim$ is needed here; where $\sim$ denotes the *indifference relation* on A defined by $x \sim y \Leftrightarrow x \succsim y \succsim x$.

Now let us return to the general case. So as to simplify this introduction, we assume that $\forall (x,y) \in A \times A$, the set $\mathcal{P}_{x,y} = \{mn^{-1} : (m,n) \in \mathbb{N}^* \times \mathbb{N}^*,\ mx \succ ny\}$ is *non-empty*. Hence we can put $s_{x,y} = \inf_{\mathbb{R}} \mathcal{P}_{x,y} \in \mathbb{R}_+$. This invariant is one the most important tool of this work; we prove in particular that $x \succ y \Leftrightarrow s_{x,y} < 1$. Let $\mathcal{E}(A)$ be the set of pairs (u, σ) made up of a morphism of $\mathbb{N}^*$-sets $u : A \to \mathbb{R}_+^*$ and a map $\sigma : A/\mathbb{N}^* \times A/\mathbb{N}^* \to \mathbb{R}_+^*$ such that $\sigma(x,y)\sigma(z,t) = \sigma(x,t)\sigma(z,y)$ and $\sigma(x,x) \leq 1$. The main result of this paper (propositions (6.1) and (7.2)) is stated as follows.

Main result. — *The four following conditions are equivalent:*

(1) there exists a pair $(u,\sigma) \in \mathcal{E}(A)$ *such that* $x \succ y \Leftrightarrow \sigma(x,y)u(x) > u(y)$*;*

(2) there exists a morphism of $\mathbb{N}^*$*-sets* $u : A \to \mathbb{R}_+^*$ *and a map* $\gamma : A/\mathbb{N}^* \to]0,1]$ *such that* $x \succ y \Leftrightarrow \gamma(x)u(x) > \gamma(y)^{-1}u(y)$*;*

(3) there exists two morphisms of $\mathbb{N}^*$*-sets* $u_1, u_2 : A \to \mathbb{R}_+^*$ *such that* $u_1 \leq u_2$ *and* $x \succ y \Leftrightarrow u_1(x) > u_2(y)$*;*

(4) $\succ$ *is a homothetic interval order.*

Moreover, if $\succ$ *is a homothetic interval order, then the pair* (u,γ) *of (2) is* unique *up to multiplication of* u *by a positive scalar; and the pair* (u_1,u_2) *of (3) is unique up to multiplication by a positive scalar (i.e., up to replacing it by* $(\lambda u_1, \lambda u_2)$ *for a constant* $\lambda > 0$*).*

The link between the two characterizations (2) and (3) is precisely described (corollary (7.4)): if (u,γ) is a pair verifying (2), then the pair $(u_1,u_2) = (\gamma u, \gamma^{-1}u)$ clearly verifies (3). Conversely, if (u_1,u_2) is a pair verifying (3), then the pair $(u,\gamma) = ((u_1u_2)^{\frac{1}{2}}, (u_1\bar{u}_2)^{\frac{1}{2}})$ verifies (2); where $\bar{u}_2 : A \to \mathbb{R}_+^*$ denotes the map defined by $\bar{u}_2(x) = u_2(x)^{-1}$.

For $i = 0, 1, 2$, we define as follows a binary relation $\succ_i$ on A:

- $x \succ_0 y \Leftrightarrow s_{x,y} < s_{y,x}$,
- $x \succ_1 y \Leftrightarrow (mx \succsim z \succ my,\ \exists (z,m) \in A \times \mathbb{N}^*)$,
- $x \succ_2 y \Leftrightarrow (mx \succ z \succsim my,\ \exists (z,m) \in A \times \mathbb{N}^*)$.

Suppose $\succ$ is a homothetic interval order. Then we prove that for $i = 0, 1, 2$, $\succ_i$ is a *homothetic weak order*; i.e., a homothetic structure which is a weak order. Moreover, for any (i.e., for one) pair (u,γ) verifying (2), u represents $\succ_0$; and for any (i.e., for one) pair (u_1,u_2) verifying (3), u_i represents $\succ_i$ ($i = 1, 2$). Let denote $\gamma_\succ : A/\mathbb{N}^* \to]0,1]$ the map defined by $\gamma_\succ = \gamma$ for any (i.e., for one) pair (u,γ) verifying (2). We prove (proposition (7.5)) that the following conditions are equivalent:

- $\gamma_\succ$ is a constant map;
- $\succ_1 = \succ_2$;
- $\succ$ is a semiorder.

We are also interested in the case of a commutative semigroup A (sections 5 and 8). A binary relation $\succ$ on A is said to be $\circ$-*independent* if $x \succ y \Leftrightarrow (x \circ z \succ y \circ z,\ \forall z \in A)$. We introduce a weaker notion of compatibility between $\circ$ and $\succ$, called $\circ$-*pseudoindependence* (cf. section 5). We prove in particular (corollary (8.3)) that if $(A, \circ)$ is a commutative semigroup endowed with a non-empty homothetic interval order $\succ$, then the weak order $\succ_0$ is $\circ$-independent if and only if $\succ$ is a $\circ$-pseudoindependent semiorder; we also remark (proposition (8.2)) that $\succ$ is $\circ$-pseudoindependent if and only if for $i = 1, 2$, $\succ_i$ is $\circ$-independent.

Let us make a few remarks about the nature of the results explained here above. Characterization (3) with the help of two maps u_1 and u_2, is the usual way to represent interval orders ([F] theorem 2.7); in fact, the homothetic weak orders $\succ_1$ and $\succ_2$ are simple variants of the weak orders associated with $\succ$ by Fishburn ([F] theorem 2.6). Novelty resides in that the pair of morphisms (u_1, u_2) is unique up to mutiplication by a positive constant. The advantage provided by the characterization (2) is to put in a prominent position the twisting factor $\gamma_\succ : A/\mathbb{N}^* \to]0, 1]$, conveying explicitely the guiding line of our thinking: to consider a homothetic interval order $\succ$ as a deformation of its associated homothetic weak order $\succ_0$. This characterization leads us to contemplate a classification of homothetic interval orders in terms of their invariant $\gamma_\succ$, a task left to a future work. Finally let us mention that this paper is a generalization of [LL], in which we deal with the particular case of a $\mathbb{N}^*$-set A so that $\forall (x, y) \in A^2$, $\exists (m, n) \in \mathbb{N}^* \times \mathbb{N}^*$ such that $mx = ny$.

Notations, writing conventions. The symbols $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}$ denote respectively the field of real numbers, the field of rational numbers, and the ring of integers. For every part $X \subset \mathbb{R}$ and every $r \in \mathbb{R}$, we put $X_{>r} = \{x \in X : x > r\}$ and $X_{\geq r} = \{x \in X : x \geq r\}$. Let $\mathbb{R}_+ = \mathbb{R}_{\geq 0}$, $\mathbb{R}_+^* = \mathbb{R}_{>0}$, $\mathbb{N} = \mathbb{Z}_{\geq 0}$; and for every part $X \subset \mathbb{R}_+$, let $X^* = X \cap \mathbb{R}_+^*$.

Let $\mathbb{R}_+^\infty = \mathbb{R}_+ \coprod \{\infty\}$ where ∞ denotes an arbitrary element not belonging to $\mathbb{R}$. The standard strict order $>$ on $\mathbb{R}_+$ extends naturally to a strict order on $\mathbb{R}_+^\infty$, still denoted $>$: for $x \in \mathbb{R}_+$, we put $\infty > x$, $x \not> \infty$ and $\infty \not> \infty$. And for $x, y \in \mathbb{R}_+^\infty$, we put $x \geq y \Leftrightarrow y \not> x$. For every part $X \subset \mathbb{R}_+^\infty$, we put

$$\inf_{\mathbb{R}_+^\infty} X = \begin{cases} \inf_{\mathbb{R}_+}(X \cap \mathbb{R}_+) & \text{if } X \cap \mathbb{R}_+ \neq \emptyset \\ \infty & \text{if not} \end{cases}.$$

Let (writing conventions) $\infty^{-1} = 0$, $0^{-1} = \infty$ and $\emptyset^{-1} = \emptyset$. And for all non-empty parts $X \subset \mathbb{R}_+^\infty$ and $Y, Z \subset \mathbb{R}_+$, we put $X^{-1} = \{q^{-1} : q \in X\} \subset \mathbb{R}_+^\infty$ and $YZ = \{yz : y \in Y, z \in Z\} \subset \mathbb{R}_+$.

At last, if A is a set, for $n \in \mathbb{Z}_{\geq 1}$, we put $A^n = A \times \cdots \times A$ (n times).

1. Let A be a set endowed with a binary relation $\succ$. Let denote $\sim$ and $\succsim$ the binary relations on A defined as follows:

- $x \sim y \Leftrightarrow x \not\succ y \not\succ x$,
- $x \succsim y \Leftrightarrow (x \succ y$ or $x \sim y)$.

The relation $\succ$ is said to be:

(A) *asymmetric* if $\forall (x, y) \in A^2$, we have $x \succ y \Rightarrow y \not\succ x$;

(T) *transitive* if $\forall (x, y, z) \in A^3$, we have $x \succ y \succ z \Rightarrow x \succ z$;

(ST) *strongly transitive* if it satisfies (A) and $\forall (x, y, z, t) \in A^4$, we have $x \succ y \succsim z \succ t \Rightarrow x \succ t$;

(NT) *negatively transitive* if it satisfies (A) and the relation $\succsim$ is transitive;

(S) *strict* if $\forall (x, y) \in A^2$, we have $x \succsim y \succsim x \Rightarrow x = y$.

The relation $\succ$ satisfies (A) if and only if $\forall (x, y) \in A^2$, we have $x \not\succ y \Leftrightarrow y \succsim x$. Then we deduce that if $\succ$ satisfies (A), then it satisfies (NT) if and only if the two following equivalent properties are true (x, y, $z \in A$):

- $\forall(x,y,z) \in A^3$, we have $x \succ y \succsim z \Rightarrow x \succ z$;
- $\forall(x,y,z) \in A^3$, we have $x \succsim y \succ z \Rightarrow x \succ z$.

Thus we have the implications:

$$\text{(NT)} \Rightarrow \text{(ST)} \Rightarrow \text{(T) \& (A)}.$$

(1.1) Remarks. — Suppose the relation $\succ$ satisfies (A). Then we have:
- $\succ$ satisfies (ST) if and only if $\forall(x,y,z,t) \in A^4$, we have $(x \succ y \text{ and } z \succ t) \Rightarrow (x \succ t \text{ or } z \succ y)$;
- $\succ$ satisfies (NT) if and only if $\forall(x,y,z,t) \in A^4$, we have $x \succsim y \succ z \succsim t \Rightarrow x \succ t$;
- $\succ$ satifies (S) if and only if $\forall(x,y) \in A^2$, we have $x \neq y \Rightarrow (x \succ y \text{ or } y \succ x)$;
- if $\succ$ satisfies (T), then it satisfies (NT) if and only if $\sim$ is an equivalence relation. ⋆

Using the terminology of Fishburn [F], we will say that the relation $\succ$ is a:
- *interval order* if it satisfies (ST);
- *semiorder* if it is an interval order and $\forall(x,y,z,t) \in A^4$, we have $x \succ y \succ z \Rightarrow (t \succ z \text{ or } x \succ t)$;
- *weak order* if it satisfies (NT);
- *strict order* if it satisfies (NT) and (S).

It is easy to check that the definition of interval order given above coincides with the one of [F]. Thus we have the implications:

$$\text{strict order} \Rightarrow \text{weak order} \Rightarrow \text{semiorder} \Rightarrow \text{interval order}.$$

(1.2) Definition. — *Let A be a set endowed with a* non-empty *binary relation $\succ$ (i.e., satisfying: $\exists(x,y) \in A^2$ such that $x \succ y$; in particular, A est non-empty), and let u be a map $A \to \mathbb{R}_+$. We say that u* represents *$\succ$ if $\forall(x,y) \in A^2$, we have $x \succ y \Leftrightarrow u(x) > u(y)$.*

2. Let G be a commutative monoid (written multiplicatively); i.e., a set endowed with a map $G \times G \to G$, $(g,g') \mapsto gg'$ and an element $1 = 1_G \in G$, such that $\forall(g,g',g'') \in G^3$, we have $(gg')g'' = g(g'g'')$, $gg' = g'g$ and $1g = g$. We call *G-set* a set A endowed with a map $G \times A \to A$, $(g,x) \mapsto gx$ such that $\forall(g,g',x) \in G^2 \times A$, we have $g(g'x) = (gg')x$ and $1x = x$. If A is a G-set, we denote A/G the quotient-set of A by the equivalence relation $\sim_G$ on A defined by:
- $x \sim_G y$ if and only if $\exists(g,g') \in G^2$ such that $gx = g'y$.

Let G be a commutative monoid, and let A be a G-set endowed with a binary relation $\succ$. The relation $\succ$ is said to be :
- ($_G$I) *G-independent* if $\forall(x,y,g) \in A^2 \times G$, we have $x \succ y \Leftrightarrow gx \succ gy$;
- ($_G$SS) *G-strongly separable* if $\forall(x,y,z,t) \in A^4$ such that $x \succ y$ and $z \succ t$, $\exists(g,g',g'') \in G^3$ such that $gx \succ g'z \succsim g''z \succ gy$;
- ($_G$C) *G-coherent* if $\forall(x,y,z) \in A^3$ such that $x \succ y \succsim z$, $\exists(g,g') \in G^2$ such that $gx \succ g'z$.

From section 1, we know that if the relation $\succ$ satisfies (NT), then it satisfies ($_G$C). Suppose moreover that G is endowed with a *weak order* $>$. Then the relation $\succ$ is said to be:
- ($_G$A) *G-archimedean* if $\forall(x,y) \in A^2$ such that $x \succ y$, $\exists(g,g') \in G^2$ such that $g' > g$ and $gx \succ g'y$;
- ($_G$P) *G-positive* if $\forall(x,y,g,g') \in A^2 \times G^2$ such that $g > g'$, we have $x \succ y \Rightarrow gx \succ g'y$.

(2.1) Remark. — Let G be a commutative monoid endowed with a weak order $>$, and let A be a G-set endowed with a binary relation $\succ$. Let denote ($_G$NI) (resp. ($_G$NP)) the property obtained by replacing the symbol $\succ$ by the symbol $\succsim$ in ($_G$I) (resp. in ($_G$P)). It is easy to prove that if $\succ$ satisfies (A), ($_G$I), ($_G$A) and ($_G$P), then $\succsim$ satisfies ($_G$NI) and ($_G$NP). ⋆

(2.2) Definition. — *Let G be a commutative semigroup endowed with a weak order $>$. A binary relation $\succ$ on a G-set A is called a:*
- G-structure *if it satisfies* $({}_G\mathrm{I})$, $({}_G\mathrm{SS})$, $({}_G\mathrm{C})$, $({}_G\mathrm{A})$ *and* $({}_G\mathrm{P})$*;*
- G-strict order *if it is a G-structure and a strict order;*
- G-weak order *if it is a G-structure and a weak order;*
- G-semiorder *if it is a G-structure and a semiorder.*
- G-interval order *if it is a G-structure and an interval order.*

The set $\mathbb{N}^*$ is a monoid for the multiplication, and the standard strict order $>$ on $\mathbb{R}_+$ induces by restriction a strict order on $\mathbb{N}^*$. To ease the notation, we will replace the index $\mathbb{N}^*$ in $({}_{\mathbb{N}^*}\mathrm{I})$, $({}_{\mathbb{N}^*}\mathrm{SS})$ (etc.), by an index "h" for *homothetic*; and we will call *homothetic structure* (resp. *homothetic strict order*, etc.) a $\mathbb{N}^*$-structure (resp. a $\mathbb{N}^*$-strict order, etc.). In this paper, we intend to give a characterization — by means of their representations in $\mathbb{R}_+$ — of the $\mathbb{N}^*$-sets endowed with a non-empty homothetic interval order. We will also give a characterization of the $\mathbb{N}^*$-sets endowed with a non-empty homothetic semiorder (resp. a non-empty homothetic weak order, a non-empty homothetic strict order).

3. Let A be a $\mathbb{N}^*$-set endowed with a binary relation $\succ$. For x, $y \in A$, we denote $\mathcal{P}_{x,y} = \mathcal{P}^{\succ}_{x,y}$ and $\mathcal{Q}_{x,y} = \mathcal{Q}^{\succ}_{x,y}$ the subsets of $\mathbb{Q}_{>0}$ defined by

$$\mathcal{P}_{x,y} = \left\{mn^{-1} : (m,n) \in (\mathbb{N}^*)^2,\, mx \succ ny\right\},$$
$$\mathcal{Q}_{x,y} = \left\{mn^{-1} : (m,n) \in (\mathbb{N}^*)^2,\, mx \succsim ny\right\};$$

and we put $s_{x,y} = \inf_{\mathbb{R}_+^\infty} \mathcal{P}_{x,y}$ and $r_{x,y} = \inf_{\mathbb{R}_+^\infty} \mathcal{Q}_{x,y}$. If $\succ$ satisfies (A), then $\forall (x,y) \in A^2$, we have the partitions of $\mathbb{Q}_{>0}$:

$$\mathbb{Q}_{>0} = \mathcal{P}_{x,y} \coprod \mathcal{Q}^{-1}_{y,x} = \mathcal{P}^{-1}_{y,x} \coprod \mathcal{Q}_{x,y}. \tag{3.1}$$

(3.2) Lemma. — *Let A be $\mathbb{N}^*$-set endowed with a non-empty binary relation $\succ$ satisfying* $({}_\mathrm{h}\mathrm{A})$ *and* $({}_\mathrm{h}\mathrm{P})$*. Then $\forall (x,y) \in A^2$, we have $\mathcal{P}_{x,y} = \mathbb{Q}_{>s_{x,y}}$.*

Proof : Let x, $y \in A$, and put $s = s_{x,y}$. If $\mathcal{P}_{x,y} = \emptyset$, then there is nothing to prove. Thus we may (and do) assume that $\mathcal{P}_{x,y} \neq \emptyset$. From $({}_\mathrm{h}\mathrm{P})$, if $q \in \mathcal{P}_{x,y}$, then $\mathbb{Q}_{\geq q} \subset \mathcal{P}_{x,y}$. If $q \in \mathbb{Q}_{>s}$, then by definition of s, $\exists q' \in \mathcal{P}_{x,y}$ such that $s \leq q' < q$. Thus we have $\mathbb{Q}_{>s} \subset \mathcal{P}_{x,y}$. From $({}_\mathrm{h}\mathrm{A})$, we have $s \in \mathbb{Q}_{>0} \Rightarrow s \notin \mathcal{P}_{x,y}$. From which we deduce that $\mathcal{P}_{x,y} = \mathbb{Q}_{>s}$. ∎

If A is a $\mathbb{N}^*$-set endowed with a binary relation $\succ$, we denote $A^* = A^*_{\succ}$ and $A^{**} = A^{**}_{\succ}$ the subsets of A defined as follows:

$$A^* = \{x \in A : \mathcal{P}_{x,y} \neq \emptyset,\, \exists y \in A\},$$
$$A^{**} = \{x \in A : \mathcal{P}_{x,y} \neq \emptyset,\, \forall y \in A\}.$$

(3.3) Remarks. — Suppose the relation $\succ$ satisfies $({}_\mathrm{h}\mathrm{I})$. Then A^* is a sub-$\mathbb{N}^*$-set of A, and we have:
- $\succ$ satisfies $({}_\mathrm{h}\mathrm{SS})$ if and only if $\forall (x,y,z) \in A^2 \times A^*$ such that $x \succ y$, $\exists (p,m,n) \in (\mathbb{N}^*)^3$ such that $px \succ mz \succsim nz \succ py$;
- if $\succ$ satisfies $({}_\mathrm{h}\mathrm{SS})$, then $\succ$ satisfies $({}_\mathrm{h}\mathrm{C})$ if and only if $A^{**} = A^*$. ⋆

(3.4) Lemma. — *Let A be a $\mathbb{N}^*$-set endowed with a non-empty interval order $\succ$ satisfying* ($_{\rm h}$I), ($_{\rm h}$SS) *and* ($_{\rm h}$C), *and let $(x,a) \in (A^*)^2$. Then $\forall y \in A$, we have $\mathcal{P}_{x,y} = \mathcal{P}_{x,a}\mathcal{Q}_{a,a}\mathcal{P}_{a,y}$.*

Proof : Since $A^{**} = A^*$, we have $\mathcal{P}_{x,a} \neq \emptyset$ and $\mathcal{P}_{a,y} \neq \emptyset$. From (FT) and ($_{\rm h}$I), we have $\mathcal{P}_{x,a}\mathcal{Q}_{a,a}\mathcal{P}_{a,y} \subset \mathcal{P}_{x,y}$. And from ($_{\rm h}$SS) and ($_{\rm h}$I), we have $\mathcal{P}_{x,y} \subset \mathcal{P}_{x,a}\mathcal{Q}_{a,a}\mathcal{P}_{a,y}$. ∎

4. The following proposition characterizes the $\mathbb{N}^*$-sets endowed with a *homothetic weak order* (resp. a *homothetic strict order*).

(4.1) Proposition. — *Let A be a $\mathbb{N}^*$-set endowed with a non-empty binary relation $\succ$. The two following conditions are equivalent:*

(1) there exists a morphism of $\mathbb{N}^*$-sets $u : A \to \mathbb{R}_+$ which represents $\succ$;

(2) $\succ$ is a homothetic weak order.

Moreover, if $\succ$ is a homothetic weak order, then the morphism u of (1) is unique *up to mutiplication by a positive scalar. And $\succ$ is a* homothetic strict order *if and only if there exists an* injective *morphism of $\mathbb{N}^*$-sets $u : A \to \mathbb{R}_+$ which represents $\succ$.*

Proof : Suppose there exists a morphism of $\mathbb{N}^*$-sets $u : A \to \mathbb{R}_+$ which represents $\succ$. Clearly we have $u^{-1}(u(A)^*) = A^*$, and the relation $\succsim$ is given by: $x \succsim y \Leftrightarrow u(x) \geq u(y)$. Then it is easy to check (and left to the reader) that $\succ$ is a homothetic weak order.

Conversely, suppose $\succ$ is a homothetic weak order. Let $(x,y) \in A^2$. From ($_{\rm h}$I) and (3.2), we have $x \succ y \Leftrightarrow s_{x,y} < 1$. And from (3.1) and (3.2), we have $\mathcal{Q}_{y,x} = \mathbb{Q}_{\geq r_{y,x}}$ with $r_{y,x} = s_{x,y}^{-1}$.

Let us prove that $\mathcal{P}_{x,x} \neq \emptyset \Leftrightarrow s_{x,x} = 1$. The implication $\Leftarrow$ is clear. Conversely, if $s_{x,x} \neq 1$, then $r_{x,x} < 1$. Hence $\exists (m,n) \in (\mathbb{N}^*)^2$ such that $m < n$ and $mx \succsim nx$. From ($_{\rm h}$NI) and ($_{\rm h}$NP) (cf. remark (2.1)), we have $m^2x \succsim mnx \succsim n^2x$, from which we obtain (using (NT)) $m^2x \succsim n^2x$. Therefore $\forall k \in \mathbb{N}^*$, we have $m^kx \succsim n^kx$. Since $\lim_{k\to+\infty}(\frac{m}{n})^k = 0$, we obtain $r_{x,x} = 0$; i.e., $\mathcal{P}_{x,x} = \emptyset$.

Since the relation $\succ$ is non-empty, we have $A^* \neq \emptyset$. Choose an element $a \in A^*$. We have $\mathcal{P}_{a,a} \neq \emptyset$; i.e., $s_{a,a} = 1$.

Suppose $x \succ y$. From (3.3), we have $\mathcal{P}_{x,a} \neq \emptyset$, hence $r_{a,x} \in \mathbb{R}_{>0}$. Let us prove that $s_{a,x} = r_{a,x}$. From (3.4), we have $\mathcal{P}_{x,y} = \mathcal{P}_{x,a}\mathcal{Q}_{a,a}\mathcal{P}_{a,y} = \mathcal{P}_{x,a}\mathcal{P}_{a,y}$, which implies the equality $s_{x,y} = s_{x,a}s_{a,y} = r_{a,x}^{-1}s_{a,y}$. Hence we have $s_{a,y} < r_{a,x}$ because $s_{x,y} < 1$. Seing that $r_{a,x} \in \mathbb{R}_+$, we have $\mathcal{Q}_{a,x} \neq \emptyset$. Let $(m,n) \in (\mathbb{N}^*)^2$ such that $ma \succsim nx$. Since $s_{a,a} = 1 = s_{x,x}$, from ($_{\rm h}$P) and ($_{\rm h}$NI), $\forall p \in \mathbb{N}^* \smallsetminus \{1\}$, we have $(p+1)ma \succ pma \succsim pnx \succ (p-1)nx$; therefore (using (ST)), we have $(p+1)ma \succ (p-1)nx$. Tending towards the limit, we obtain the inclusion $\mathbb{Q}_{>\frac{m}{n}} \subset \mathcal{P}_{a,x}$. So we have $r_{a,x} \geq s_{a,x}$, which is an equality because $\mathcal{P}_{a,x} \subset \mathcal{Q}_{a,x}$. Finally we obtain $s_{a,x} > s_{a,y}$.

We don't suppose any more that $x \succ y$.

Let us prove that $r_{a,x} \in \mathbb{R}_+$ by reducing it to the absurd: suppose $r_{a,x} = \infty$; i.e., suppose $\mathcal{P}_{x,a} = \mathbb{Q}_{>0}$. Then ($_{\rm h}$I) we have $x \succ a$; therefore ($_{\rm h}$SS), $\exists (p,m,n) \in (\mathbb{N}^*)^3$ such that $pa \succ mx \geq nx \succ pb$. In particular, $\frac{p}{m} \in \mathcal{P}_{a,x}$; contradiction. Hence $r_{a,x} \in \mathbb{R}_+$.

Let $u = u_a : A \to \mathbb{R}_+$ be the map defined by $u(x) = r_{a,x}$. From ($_{\rm h}$NI), $\forall (z,t,m) \in A^2 \times \mathbb{N}^*$, we have $\mathcal{Q}_{z,mt} = m\mathcal{Q}_{z,t}$. Hence u is a morphism of $\mathbb{N}^*$-sets. Let us prove that $x \succ y \Leftrightarrow u(x) > u(y)$. We have seen that if $x \succ y$, then $r_{a,x} = s_{a,x} > s_{a,y}$. But we have the inclusion $\mathcal{P}_{a,y} \subset \mathcal{Q}_{a,y}$, from which we deduce the implication: $x \succ y \Rightarrow u(x) > u(y)$. Conversely, suppose $u(x) > u(y)$. Then $\exists (m,n) \in (\mathbb{N}^*)^2$ such that $ma \succsim ny$ and $ma \not\succsim nx$. But $ma \not\succsim nx \Leftrightarrow nx \succ ma$, from which we obtain $nx \succ ma \succsim ny$. From (NT) we have $nx \succ ny$; hence ($_{\rm h}$I) we have $x \succ y$. We thus proved that u represents $\succ$. And clearly, $\succ$ satisfies (S) if and only if u is injective.

We still have to prove the uniqueness property. Let $v : A \to \mathbb{R}_+$ be another morphism of $\mathbb{N}^*$-sets such that $\forall(x,y)\in A^2$, we have $x\succ y \Leftrightarrow v(x)>v(y)$. Since $u^{-1}(u(A)^*)=A^*=v^{-1}(v(A)^*)$, $\forall x\in A$, we have $u(x)\neq 0\Leftrightarrow v(x)\neq 0$. Let $\lambda : A\to\mathbb{R}_{>0}$ be the map defined by

$$\lambda(x)=\begin{cases} u(x)^{-1}v(x) & \text{if } u(x)\neq 0\\ u(a)^{-1}v(a) & \text{if not}\end{cases}.$$

Since u and v are morphisms of $\mathbb{N}^*$-sets, λ factorizes through the quotient-set $A/\mathbb{N}^*$. Suppose $\exists x\in A$ such that $\lambda(x)\neq\lambda(a)$. Put $\alpha=\lambda(a)\lambda(x)^{-1}$. First of all suppose $\alpha<1$. Then $\exists q\in\mathbb{Q}_{>0}$ such that $\alpha u(a)u(x)^{-1}<q<u(a)u(x)^{-1}$. In other words, we have $v(a)<qv(x)$ and $qu(x)<u(a)$, contradiction. Now if $\alpha>0$, then $\exists q'\in\mathbb{Q}_{>0}$ such that $u(a)u(x)^{-1}<q'<\alpha u(a)u(x)^{-1}$; i.e., $u(a)<q'u(x)$ and $q'v(x)<v(a)$, contradiction. Hence $\alpha=1$, and λ is a constant map. This completes the proof of the proposition. ∎

(4.2) Corollary. — *Let A be a $\mathbb{N}^*$-set endowed with a non-empty homothetic weak order $\succ$, and let $a\in A^*$. Then the map $A\to\mathbb{R}_+$, $x\mapsto r_{a,x}$ is a morphism of $\mathbb{N}^*$-sets which represents $\succ$.*

5. Let $(A,\circ)$ be a *commutative semigroup*; i.e., a set A endowed with a map $A\times A\to A$, $(x,y)\mapsto x\circ y$ such that $\forall(x,y,z)\in A^3$, we have
- $x\circ(y\circ z)=(x\circ y)\circ z$ (associativity),
- $x\circ y=y\circ x$ (commutativity).

Let remark that A is a fortiori a $\mathbb{N}^*$-set, for the operation of $\mathbb{N}^*$ on A defined by the map $\mathbb{N}^*\times A\to A$, $(m,x)\mapsto mx=x\circ\cdots\circ x$ (m times). For all parts $X,Y\subset A$, we put $X\circ Y=\{x\circ y : x\in X,\, y\in Y\}\subset A$

A binary relation $\succ$ on A is said to be:
($_\circ$I) $\circ$-*independent* if $\forall(x,y,z)\in A^3$, we have $x\succ y\Leftrightarrow x\circ z\succ y\circ z$;
($_\circ$PI) $\circ$-*pseudoindependent* if $A^*\circ(A\smallsetminus A^*)\subset A^*$ and $\forall(x,y,z,t)\in A^4$, we have

$$\begin{cases}(x\succ y,\ z\succ t)\Rightarrow x\circ z\succ y\circ t\\ (x\succsim y,\ z\succsim t)\Rightarrow x\circ z\succsim y\circ t\end{cases}.$$

(5.1) Proposition (variant of (4.1)). — *Let $(A,\circ)$ be a commutative semigroup endowed with a non-empty binary relation $\succ$. The three following conditions are equivalent:*
(1) there exists a morphism of semigroups $u : A\to\mathbb{R}_+$ which represents $\succ$;
(2) $\succ$ is a $\circ$-independent homothetic weak order*;*
(3) $\succ$ is a $\circ$-pseudoindependent homothetic weak order.
Moreover, if $\succ$ is a homothetic weak order, then the morphism u of (1) is unique *up to multiplication by a positive scalar.*

Proof : The implication (1) ⇒ (2) is clear.

Let us prove the implication (2) ⇒ (3). Supposose $\succ$ is a $\circ$-independent homothetic weak order. Let $(x,y)\in A^*\times(A\smallsetminus A^*)$ such that $x\circ y\in A\smallsetminus A^*$. Thus we have $x\succ x\circ y$. From ($_\circ$I), we have $x\circ y\succ(x\circ y)\circ y=x\circ(2y)$ and $y\succ 2y$, hence $y\in A^*$; contradiction. Therefore $A^*\circ(A\smallsetminus A^*)\subset A^*$. Then using (T) and (NT), we easily deduce that the relation $\succ$ is $\circ$-pseudoindependent. So we have (2) ⇒ (3).

Let us prove the implication (3) ⇒ (1). Suppose $\succ$ is a $\circ$-pseudoindependent homothetic weak order. Choose an element $a\in A^*$, and let $u=u_a : A\to\mathbb{R}_+$ be the morphism of $\mathbb{N}^*$-sets defined by $u(x)=r_{a,x}$. From (4.3), u represents $\succ$. Let $(x,y)\in A^2$. If $(m,n,m',n')\in(\mathbb{N}^*)^4$ satisfies

$ma \succsim nx$ and $m'a \succsim n'y$, then from ($_\circ$PI), we have $(nm' + n'm)a \succsim nn'(x \circ y)$. Therefore we have $r_{a,x\circ y} \leq \frac{m}{n} + \frac{m'}{n'}$. From which we deduce that $r_{a,x\circ y} \leq r_{a,x} + r_{a,y}$; i.e., that $u(x \circ y) \leq u(x) + u(y)$.

First of all suppose $(x, y) \in (A^*)^2$. If $(m, n, m', n') \in (\mathbb{N}^*)^4$ is such that $mx \succ na$ et $m'y \succ n'a$, then from ($_\circ$PI), we have $mm'(x \circ y) \succ (m'n + mn')a$. Hence we have $s_{x\circ y,a} \leq \frac{mm'}{m'n+mn'} = (\frac{n}{m} + \frac{n'}{m'})^{-1}$. From which we deduce that $r_{a,x\circ y} = s^{-1}_{x\circ y,a} \geq s^{-1}_{x,a} + s^{-1}_{y,a} = r_{a,x} + r_{a,y}$; i.e., that $u(x \circ y) \geq u(x) + u(y)$. Hence we have $u(x \circ y) = u(x) = u(y)$.

Now suppose $(x, y) \in (A \smallsetminus A^*)^2$. Then the inequality $u(x \circ y) \leq u(x) + u(y) = 0$ implies $u(x \circ y) = 0$. So we have $u(x \circ y) = 0 = u(x) + u(y)$.

Last of all suppose $(x, y) \in A^* \times (A \smallsetminus A^*)$. Assume $u(x \circ y) < u(x) + u(y)$. Since $u(y) = 0$, we have $x \succ x \circ y$. Hence ($_\mathrm{h}$P), $\exists(m, n) \in (\mathbb{N}^*)^2$ such that $m > n$ and $nx \succ m(x \circ y) = nx \circ z$ with $z = (m - n)x \circ my$. But $(m - n)x \in A^*$ and $my \in A \smallsetminus A^*$. Thus from ($_\circ$PI), we have $z \in A^*$. Because $(nx, z) \in (A^*)^2$, we have (cf. above) $u(nx \circ z) = u(nx) + u(z)$. But since $nx \succ nx \circ z$, we also have $u(nx) > u(nx \circ z)$; contradiction. Hence we have $u(x \circ y) = u(x) + u(y)$.

Since $x \circ y = y \circ x$, the case $(x, y) \in (A \smallsetminus A^*) \times A^*$ is already done.

So we proved that u is a morphism of semigroups. This completes the proof of the implication (3) ⇒ (1).

At last, the uniqueness property is a consequence of (4.1). □

6. Let E be a set, and $E' \subset E$ be a subset. Let denote $\mathcal{G}(E' \times E)$ the set of maps $f : E' \times E \to \mathbb{R}^*_+$ such that $\forall(x', y', x, y) \in (E')^2 \times E^2$, we have $f(x', x') \leq 1$ and $f(x', x)f(y', y) = f(x', y)f(y', x)$. And let denote $\mathcal{G}_0(E' \times E) \subset \mathcal{G}(E' \times E)$ the subset made up of maps f such that $\forall(x', y') \in (E')^2$, we have $f(x', y') = f(y', x')$. Let remark that if $f \in \mathcal{G}_0(E' \times E)$, then $\forall(x', y') \in (E')^2$, we have $f(x, y) = f(x, x)^{\frac{1}{2}} f(y, y)^{\frac{1}{2}} \leq 1$.

Let A be a $\mathbb{N}^*$-set endowed with a binary relation $\succ$ satisfying ($_\mathrm{h}$I). Put $\bar{A} = A/\mathbb{N}^*$ and let denote $\bar{A}^* = \bar{A}^*_\succ$ the subset of $\bar{A}$ defined by $\bar{A}^* = A^*_\succ/\mathbb{N}^*$. We denote $\mathcal{E}(A, \succ)$ the set of pairs (u, σ) made up of a morphism of $\mathbb{N}^*$-sets $u : A \to \mathbb{R}_+$ and a map $\sigma \in \mathcal{G}(\bar{A}^* \times \bar{A})$; i.e., a map $\sigma \in \mathcal{G}(A^* \times A)$ such that $\forall(x, y, m, n) \in A^* \times A \times (\mathbb{N}^*)^2$, we have $\sigma(mx, ny) = \sigma(x, y)$. We denote $\mathcal{E}_0(A, \succ) \subset \mathcal{E}(A, \succ)$ the subset made up of pairs (u, σ) such that $\sigma \in \mathcal{G}_0(\bar{A}^* \times \bar{A})$. At last, for $(u, \sigma) \in \mathcal{E}(A, \succ)$, we denote σ^* the restriction $\sigma|_{\bar{A}^* \times \bar{A}^*}$.

The following proposition characterizes the homothetic interval orders.

(6.1) Proposition. — *Let A be a $\mathbb{N}^*$-set endowed with a non-empty binary relation $\succ$. The two following conditions are equivalent:*

(1) there exists a pair $(u, \sigma) \in \mathcal{E}(A, \succ)$ such that $\forall(x, y) \in A^2$, we have $x \succ y \Leftrightarrow \sigma(x, y)u(x) > u(y)$;

(2) $\succ$ is a homothetic interval order.

Moreover, if $\succ$ is a homothetic interval order, then there exists a pair $(u, \sigma) \in \mathcal{E}_0(A, \succ)$ verifying (1); and if $(u_1, \sigma_1), (u_2, \sigma_2) \in \mathcal{E}_0(A, \succ)$ are two pairs verifying (1), then $\sigma^*_2 = \sigma^*_1$ and there exists a (unique) constant $\lambda > 0$ such that $u_2 = \lambda u_1$.

Proof: Suppose there exists a pair $(u, \sigma) \in \mathcal{E}(A, \succ)$ verifying (1). Clearly we have $u^{-1}(u(A)^*) = A^*$. For $x \in A$, put $\overline{x} = u(x)$. Let $(x, y) \in A^2$ such that $x \succ y$, and suppose $y \succ x$. Then we have $\sigma(y, x)\sigma(x, y)\overline{x} > \sigma(y, x)\overline{y} > \overline{x}$. But since $\sigma \in \mathcal{G}(A^* \times A)$, we also have $\sigma(y, x)\sigma(x, y) = \sigma(y, y)\sigma(x, x) \leq 1$, which contradicts the inequality $\sigma(y, x)\sigma(x, y)\overline{x} > \overline{x}$. Therefore $\succ$ satisfies (A).

Since $\succ$ satisfies (A), for $(x, y) \in A \times A^*$, we have $x \succsim y \Leftrightarrow \overline{x} \geq \sigma(y, x)\overline{y}$. Let $(x, y, z, t) \in A^4$

such that $x \succ y \succsim z \succ t$. Thus we have

$$\begin{cases} \sigma(x,y)\overline{x} > \overline{y} \geq \sigma(z,y)\overline{z} \\ \sigma(z,t)\overline{z} > \overline{t} \end{cases},$$

hence $\frac{\sigma(x,y)\sigma(z,t)}{\sigma(z,y)}\overline{x} > \overline{t}$. But $\sigma(x,y)\sigma(z,t) = \sigma(x,t)\sigma(z,y)$, hence $\sigma(x,t)\overline{x} > \overline{t}$; i.e., $x \succ t$. Therefore $\succ$ satisfies (ST); so it is an interval order.

It remains to prove that $\succ$ is a homothetic structure. The conditions ($_{\mathrm{h}}$I), ($_{\mathrm{h}}$A) and ($_{\mathrm{h}}$P) are clearly satisfied. Let $(x,y,z) \in A^3$ such that $x \succ y \succsim z$. We have $\sigma(x,y)\overline{x} > \overline{y}$, hence $\overline{x} > 0$ and $\exists m \in \mathbb{N}^*$ such that $m\sigma(x,z)\overline{x} > \overline{z}$; i.e., such that $mx \succ z$. Therefore $\succ$ satisfies ($_{\mathrm{h}}$C). Concerning the condition ($_{\mathrm{h}}$SS), let $(x,y,z,t) \in A^4$ such that $x \succ y$ and $z \succ t$. We have $\sigma(x,y)\overline{x} > \overline{y}$ and $r = \sigma(z,y)\overline{z} > 0$. Hence $\exists (p,m,n) \in (\mathbb{N}^*)^3$ such that

$$\sigma(x,y)\overline{x} > \frac{m}{p\sigma(z,z)} r \geq \frac{n}{p} r > \overline{y}.$$

Since $\sigma(x,y)\frac{\sigma(z,z)}{\sigma(z,y)} = \sigma(x,z)$, multiplying by $p\frac{\sigma(z,z)}{\sigma(z,y)}$, we obtain

$$p\sigma(x,z)\overline{x} > m\overline{z} \geq n\sigma(z,z)\overline{z} > p\frac{\sigma(z,z)}{\sigma(z,y)}\overline{y};$$

i.e., $px \succ mz \succsim nz \succ py$. Therefore $\succ$ satisfies ($_{\mathrm{h}}$SS).

Conversely, suppose $\succ$ is a homothetic interval order. Then $\forall (x,y) \in A^2$, we have (cf. the proof of (4.1)) $x \succ y \Leftrightarrow s_{x,y} < 1$ and $\mathcal{Q}_{y,x} = \mathbb{Q}_{\geq r_{y,x}}$ with $r_{y,x} = s_{x,y}^{-1}$.

Let denote $>$ the binary relation on A defined by $x > y \Leftrightarrow s_{x,y} < s_{y,x}$; i.e., by $x > y \Leftrightarrow \mathcal{P}_{x,y} \supsetneqq \mathcal{P}_{y,x}$. In particular, we have $x > y \Rightarrow x \in A^*$. Clearly, $>$ satisfies (A). Let $(x,y,z) \in A^3$ such that $x > y > z$. If $z \in A \smallsetminus A^*$, then $\emptyset = \mathcal{P}_{z,x} \subsetneqq \mathcal{P}_{x,z}$. And if $z \in A^*$, then from (3.4), we have $\mathcal{P}_{z,x} = \mathcal{P}_{z,y}\mathcal{Q}_{y,y}\mathcal{P}_{y,z} \subsetneqq \mathcal{P}_{x,z}$. Therefore $>$ satisfies (T).

Let denote $\approx$ the binary relation on A defined by $x \approx y \Leftrightarrow x \not> y \not> x$. Thus we have

$$x \approx y \Leftrightarrow s_{x,y} = s_{y,x} \Leftrightarrow \mathcal{P}_{x,y} = \mathcal{P}_{y,x}.$$

We clearly have $x \approx y \Leftrightarrow y \approx x$. Let us prove that $\approx$ is transitive. Let $(x,y,z) \in A^3$ such that $x \approx y \approx z$. Since $\mathcal{P}_{x,y} = \mathcal{P}_{y,x}$, we have $(x,y) \in (A^*)^2 \cup (A \smallsetminus A^*)^2$. If $(x,y) \in (A^*)^2$, then from (3.4), we have $\mathcal{P}_{x,z} = \mathcal{P}_{x,y}\mathcal{Q}_{y,y}\mathcal{P}_{y,z} = \mathcal{P}_{y,x}\mathcal{Q}_{y,y}\mathcal{P}_{z,y} = \mathcal{P}_{z,y}$; i.e., $x \approx z$. Suppose $(x,y) \in (A \smallsetminus A^*)^2$. Since $A^{**} = A^*$, we have $\mathcal{P}_{x,z} = \mathcal{P}_{y,z} = \emptyset = \mathcal{P}_{z,y}$; i.e., $z \in A \smallsetminus A^*$, which implies $\mathcal{P}_{z,x} = \emptyset$. Hence $x \approx z$.

Since $\approx$ is transitive, it is an equivalence relation. Hence $>$ is a weak order. Let remark that $\forall (x,y) \in A^2$, we have $x \succ y \Rightarrow x > y$, therefore $x \geq y \Rightarrow x \succsim y$.

Let us prove that $>$ is a homothetic structure. For $(x,y,m,n) \in A^2 \times (\mathbb{N}^*)^2$, we have $\mathcal{P}_{mx,ny} = \frac{n}{m}\mathcal{P}_{x,y}$. From which we deduce that $>$ satisfies ($_{\mathrm{h}}$I), ($_{\mathrm{h}}$A) and ($_{\mathrm{h}}$P). Since $>$ satisfies (NT), $>$ satisfies ($_{\mathrm{h}}$C). Concerning the condition ($_{\mathrm{h}}$SS), let $(x,y,z,t) \in A^4$ such that $x > y$ and $z > t$. Since $(x,z) \in (A^*)^2$, we have (3.4) $\mathcal{P}_{x,y} = \mathcal{P}_{x,z}\mathcal{Q}_{z,z}\mathcal{P}_{z,y}$. And if $y \in A^*$, we also have $\mathcal{P}_{y,x} = \mathcal{P}_{y,z}\mathcal{Q}_{z,z}\mathcal{P}_{z,x}$. Since $s_{x,y} < s_{y,x}$ with $s_{y,x} = \infty$ if $y \in A \smallsetminus A^*$, $\exists (p,m,n) \in (\mathbb{N}^*)^3$ such that $n < m$, $(\frac{m}{p})^2 s_{x,z} < s_{z,x}$ and $(\frac{p}{n})^2 s_{z,y} < s_{y,z}$; i.e., such that $px > mz \geq nz > py$. Thus $>$ satisfies ($_{\mathrm{h}}$SS), and $>$ is a homothetic structure.

Since $>$ is a homothetic weak order, from (4.1), there exists a morphism of $\mathbb{N}^*$-set $u : A \to \mathbb{R}_+$ such that $\forall (x,y) \in A^2$, we have $x > y \Leftrightarrow u(x) > u(y)$. For $x \in A$, we have $u(x) = 0$ if and only if $\forall y \in A$, we have $r_{y,x} = 0$; i.e., if and only if $x \in A \setminus A^*$. Thus we have $u^{-1}(u(A)^*) = A^*$. Let $\sigma^* : A^* \times A^* \to \mathbb{R}_+^*$ be the map defined by $\sigma^*(x,y) = r_{y,x}u(x)^{-1}u(y)$. We extend σ^* to $A^* \times A$ in the following way: let choose an element $a \in A^*$, and for $(x,y) \in A^* \times (A \setminus A^*)$, put $\sigma(x,y) = \sigma^*(x,a)$. For $(x,y,m,n) \in (A^*)^2 \times (\mathbb{N}^*)^2$, we have $r_{my,nx} = \frac{n}{m} r_{y,x}$. Therefore σ factorizes through $\bar{A}^* \times \bar{A}$. For $(x,y,z,t) \in (A^*)^4$, we have $\sigma^*(x,x) = r_{x,x} \leq 1$ and $\mathcal{P}_{x,y} = \mathcal{P}_{x,t}\mathcal{Q}_{t,t}\mathcal{P}_{t,y}$, from which we deduce that $s_{x,y} = s_{x,t}r_{t,t}s_{t,y}$ and (switching to the inverse) that $r_{y,x} = r_{t,x}s_{t,t}r_{y,t}$; hence $r_{y,x}r_{t,z} = r_{t,x}(r_{t,z}s_{t,t}r_{y,t}) = r_{t,x}r_{y,z}$ and $\sigma(x,y)\sigma(z,t) = \sigma(x,t)\sigma(z,y)$. From the definition of σ, this last equality remains true for $(y,t) \in A^2$. Hence $(\sigma,u) \in \mathcal{E}(A,\succ)$, and by construction $\forall (x,y) \in A^2$, we have $x \succ y \Leftrightarrow \sigma(x,y)u(x) > u(y)$.

It remains to prove the last two assertions of the proposition. For $(x,y) \in (A^*)^2$, we have $r_{y,x} = \sigma(x,y)u(x)u(y)^{-1}$, hence

$$\begin{aligned} u(x) > u(y) &\Leftrightarrow \sigma(x,y)u(x)u(y)^{-1} > \sigma(y,x)u(y)u(x)^{-1} \\ &\Leftrightarrow \sigma(x,y)^{\frac{1}{2}}u(x) > \sigma(y,x)^{\frac{1}{2}}u(y); \end{aligned}$$

which is possible only if $\sigma(x,y) = \sigma(y,x)$. Hence $(u,\sigma) \in \mathcal{E}_0(A,\succ)$. Concerning the uniqueness property, for $i = 1, 2$, let $(u_i,\sigma_i) \in \mathcal{E}_0(A,\succ)$ such that $\forall (x,y) \in A^2$, we have $x \succ y \Leftrightarrow \sigma_i(x,y)u_i(x) > u_i(y)$. Let recall that $u_1^{-1}(u_1(A)^*) = A^* = u_2^{-1}(u_2(A)^*)$. For $x \in A$, let write $u_2(x) = \lambda_x u_1(x)$ with $\lambda_x > 0$ and $\lambda_x = 1$ if $u_1(x) = 0$. Let remark that the map $x \mapsto \lambda_x$ factorizes through $\bar{A}$. For $(x,y) \in (A^*)^2$, we have (easy checking left to the reader) $\sigma_2(x,y) = \lambda_x^{-1}\lambda_y\sigma_1(x,y)$, therefore

$$\begin{aligned} & \sigma_2(x,y) = \sigma_2(y,x) \\ \Leftrightarrow\ & \lambda_x^{-1}\lambda_y\sigma_1(x,y) = \lambda_y^{-1}\lambda_x\sigma_1(y,x) \\ \Leftrightarrow\ & \lambda_y^2 = \lambda_x^2; \end{aligned}$$

i.e., $\lambda_x = \lambda_y$. So $x \mapsto \lambda_x$ is a constant map on A^*. This completes the proof of the proposition. ∎

(6.2) Remark. — Let A be $\mathbb{N}^*$-set endowed with a non-empty binary relation $\succ$. If $(u,\sigma) \in \mathcal{E}(A,\succ)$ is a pair verifying (6.1)-(1), then we have $u^{-1}(u(A)^*) = A^*$; and the relation $\succ$ is completely determined by the pair $(u|_{A^*},\sigma^*)$. But for $\sigma \in \mathcal{G}_0(A^* \times A)$ and $(x,y) \in (A^*)^2$, we have $\sigma(x,y) = \gamma(x)\gamma(y)$ with $\gamma(x) = \sigma(x,x)^{\frac{1}{2}}$. Therefore, the condition (1) of (6.1) is equivalent to the following condition (1'):

(1') there exists a morphism of $\mathbb{N}^*$-sets $u^* : A^* \to \mathbb{R}_+$ and a map $\gamma : \bar{A}^* \to]0,1]$, such that $\forall (x,y) \in (A^*)^2$, we have $x \succ y \Leftrightarrow \gamma(x)u(x) > \gamma(y)^{-1}u(y)$

Moreover, if $\succ$ is a homothetic interval order, then the pair (u^*,γ) of (1') is unique up to multiplication of u^* by a positive scalar. ⋆

(6.3) Corollary/definition. — *Let A be a $\mathbb{N}^*$-set endowed with a non-empty interval homothetic order $\succ$, and let $(u,\sigma) \in \mathcal{E}_0(A,\succ)$ be a pair verifying (6.1)-(1). Then u represents the homothetic weak order $\succ_0$ (denoted $>$ in the proof of (6.1)) on A defined by $x \succ_0 y \Leftrightarrow r_{y,x} > r_{x,y}$; and $\forall (x,y) \in (A^*)^2$, we have $\sigma^*(x,y) = r_{y,x}u(y)u(x)^{-1}$. The invariant $\sigma^* \in \mathcal{G}_0(\bar{A}^* \times \bar{A}^*)$ does not depend on u; we denote it $\sigma^*_\succ$. At last, let denote $\gamma^*_\succ : \bar{A}^* \to \mathbb{R}_+^*$ the map defined by $\gamma^*_\succ(x) = \sigma^*_\succ(x,x)^{\frac{1}{2}}$; so we have $\sigma^*_\succ(x,y) = \gamma^*_\succ(x)\gamma^*_\succ(y)$.*

(6.4) Corollary. — *Let A be a $\mathbb{N}^*$-set endowed with a non-empty homothetic interval order $\succ$, and let $u : A \to \mathbb{R}_+$ be a morphism of $\mathbb{N}^*$-sets which represents $\succ_0$. Then $\forall (x,y) \in (A^*)^2$, we have $u(x)u(y)^{-1} = (r_{y,x}s_{y,x})^{\frac{1}{2}}$.*

Proof: For $(x,y)\in(A^*)^2$, we have $\sigma^*_{\succ}(x,y)=r_{y,x}u(y)u(x)^{-1}$ and $\sigma(x,y)=\sigma(y,x)$; from which we deduce that $u(x)u(y)^{-1}=(r_{y,x}r_{x,y}^{-1})^{\frac{1}{2}}=(r_{y,x}s_{y,x})^{\frac{1}{2}}$. □

(6.5) Remark. — Let A be a $\mathbb{N}^*$-set endowed with a non-empty homothetic interval order $\succ$, and let $u:A\to\mathbb{R}_+^*$ be a morphism of $\mathbb{N}^*$-sets which represents $\succ_0$. One may wonder if the map $A\times A\to\mathbb{R}_+^\infty$, $(x,y)\mapsto r_{y,x}=s_{x,y}^{-1}$ factorizes through the product-map $u\times u$; i.e., if $\forall(x,y,x',y')\in A^4$ such that $u(x)=u(x')$ and $u(y)=u(y')$, we have $r_{x,y}=r_{x',y'}$. In general the answer is negative: cf. the example (7.5) below. ⋆

Let A be a $\mathbb{N}^*$-set endowed with a non-empty homothetic interval order $\succ$, and let $u:A\to\mathbb{R}_+$ be a morphism of $\mathbb{N}^*$-sets $u:A\to\mathbb{R}_+$ which represents $\succ_0$. Choose an element $a\in A^*$ and let denote $\sigma^a_{\succ}:A^*\times A\to\mathbb{R}_+^*$ the map extending $\sigma^*_{\succ}$ defined by $\sigma^a_{\succ}(x,y)=\sigma^*_{\succ}(x,a)$ for $(x,y)\in A^*\times(A\smallsetminus A^*)$. Then $(u,\sigma^a_{\succ})\in\mathcal{E}_0(A,\succ)$ and $\forall(x,y)\in A^2$, we have $x\succ y\Leftrightarrow\sigma^a_{\succ}(x,y)u(x)>u(y)$. The map $\sigma^a_{\succ}$ is *split*: there exist two maps $\sigma_1:\bar{A}^*\to\mathbb{R}_+^*$ and $\sigma_2:\bar{A}\to\mathbb{R}_+^*$ such that $\sigma^a_{\succ}=\sigma_1\times\overline{\sigma}_2$ with $\overline{\sigma}_2(x)=\sigma_2(x)^{-1}$ $(x\in A)$. In fact, for $(x,y)\in(A^*)^2$, put $\sigma_1(x)=s_{a,a}r_{a,x}u(x)^{-1}$ and $\sigma_2^*(y)=s_{a,y}u(y)^{-1}$; since $r_{y,x}=r_{y,a}s_{a,a}r_{a,x}$ (3.4), we have $\sigma_1(x)\sigma_2^*(y)^{-1}=\sigma^*_{\succ}(x,y)$. Let $\sigma_2:A\to\mathbb{R}_+^*$ be the map extending σ_2^* defined by $\sigma_2(y)=\sigma_2(a)$ for $y\in A\smallsetminus A^*$. The maps $\sigma_1:A^*\to\mathbb{R}_+^*$ and $\sigma_2:A\to\mathbb{R}_+^*$ defined in this way factorize through $\bar{A}^*$ and $\bar{A}$ respectively. And by construction, we have $\sigma^a_{\succ}=\sigma_1\times\overline{\sigma}_2$. In other words, $\forall(x,y)\in A^2$, we have $x\succ y\Leftrightarrow u_1(x)>u_2(y)$ with $u_i(x)=\sigma_i(x)u$. For $i=1,\,2$, the map $u_i:A\to\mathbb{R}_+$ is a morphism of $\mathbb{N}^*$-sets. This formulation by means of a pair of maps (u_1,u_2) is the one usually employed to represent interval orders; cf. [F] theorem 2.7. Let remark that in the general (i.e., not necessarily homothetic) theory of interval orders, there is a priori no possible uniqueness result for the pair (u_1,u_2). As we will see in section 7 below, for homothetic interval orders the result is quite different.

7. Let A be a $\mathbb{N}^*$-set endowed with a binary relation $\succ$. We denote $\succ_1$ and $\succ_2$ the binary relations on A defined by:

- $x\succ_1 y\Leftrightarrow(mx\succ z\succsim my,\ \exists(z,m)\in A\times\mathbb{N}^*)$,
- $x\succ_2 y\Leftrightarrow(mx\succsim z\succ my,\ \exists(z,m)\in A\times\mathbb{N}^*)$.

(7.1) Lemma. — *Let A be a $\mathbb{N}^*$-set endowed with a non-empty homothetic interval order $\succ$. Then for $i=1,\,2$, $\succ_i$ is a non-empty homothetic weak order.*

Proof: Let a pair $(u,\sigma)\in\mathcal{E}_0(A,\succ)$ satisfying (6.1)-(1). We may (and do) suppose $\sigma=\sigma^a_{\succ}$ for an element $a\in A^*$. For $(x,y)\in A^2$, we have $x\succ y\Rightarrow x\succ_i y$ $(i=1,\,2)$. Therefore the relations $\succ_1$ and $\succ_2$ are non-empty. Let us prove that $\succ_1$ is a homothetic weak order. Let $(x,y)\in A^2$ such that $x\succ_1 y$, and let $(z,m)\in A\times\mathbb{N}^*$ such that $mx\succ z\succsim my$. Thus we have $x\in A^*$. First of all suppose $(y,z)\in(A^*)^2$. Hence we have $\sigma(x,z)u(mx)>u(z)\geq\sigma(x,y)u(my)$. We obtain

$$r_{z,x}\frac{u(z)}{u(x)}u(mx)>u(z)\geq r_{z,y}\frac{u(z)}{u(y)}u(my),$$

hence $r_{z,x}>r_{z,y}$. But from (3.4), we have $r_{z,x}=r_{z,a}s_{a,a}r_{a,x}$ and $r_{z,y}=r_{z,a}s_{a,a}r_{a,y}$. From which we deduce that $r_{a,x}>r_{a,y}$. Now if $(y,z)\in(A\smallsetminus A^*)\times A$, then this last inequality remains true: we have $r_{a,x}>0$ and $r_{a,y}=0$. At last, if $(y,z)\in A^*\times(A\smallsetminus A^*)$, then replacing z by a in the calculation above, we still obtain $r_{a,x}>r_{a,y}$.

Conversely, let $(x,y)\in A^2$ such that $r_{a,x}>r_{a,y}$. Then $x\in A^*$, and $\exists(m,n)\in(\mathbb{N}^*)^2$ such that $r_{a,x}>\frac{n}{m}\geq r_{a,y}$. Since $\frac{1}{n}r_{a,t}=r_{na,t}$ $(t\in A)$, we have $mr_{na,x}>1\geq mr_{na,y}$. First of all suppose $y\in A^*$. Then we obtain $\sigma(x,a)u(mx)>u(na)\geq\sigma(y,a)u(my)$; i.e., $mx\succ na\succsim my$. Thus we have $x \succ_1 y$. Now if $y \in A \smallsetminus A^*$, then $\forall m \in \mathbb{N}^*$ such that $m > s_{x,a}$, we have $mx \succ a \succ my$; therefore $x \succ_1 y$.

So we proved that the morphism of $\mathbb{N}^*$-sets $u_1 : A \to \mathbb{R}_+$, $x \mapsto r_{a,x}$ represents the relation $\succ_1$. Then it is easy to check (and left to the reader) that $\succ_1$ is a homothetic weak order.

Let $(x,y) \in A^2$ such that $x \succ_2 y$, and let $(z,m) \in A \times \mathbb{N}^*$ such that $mx \succsim z \succ my$. Then $z \in A^*$, $u(mx) \geq \sigma(z,x)u(z)$ and $\sigma(z,y)u(z) > u(my)$, from which we obtain $\sigma(z,x)^{-1}u(mx) \geq u(z) > \sigma(z,y)^{-1}u(my)$. In particular, we have $x \in A^*$. First of all suppose $y \in A^*$. Like for $\succ_1$, we obtain $s_{a,x} > s_{a,y}$; and this inequality remains true for $y \in A \smallsetminus A^*$. Conversely, like for $\succ_1$ we prove that if $(x,y) \in A^2$ is such that $s_{a,x} > s_{a,y}$, then $x \succ_2 y$. Hence the morphism of $\mathbb{N}^*$-sets $u_2 : A \to \mathbb{R}_+$, $x \mapsto s_{a,x}$ represents $\succ_2$. And like for $\succ_1$, it is easy to check that $\succ_2$ is a homothetic weak order. □

(7.2) Proposition. — *Let A be a $\mathbb{N}^*$-set endowed with a non-empty binary relation $\succ$. The two following conditions are equivalent:*

(1) there exists two morphisms of $\mathbb{N}^*$-sets u_1, $u_2 : A \to \mathbb{R}_+$ such that $u_1 \leq u_2$ and $\forall (x,y) \in A^2$, we have $x \succ y \Leftrightarrow u_1(x) > u_2(y)$;

(2) $\succ$ is a homothetic interval order.

Moreover, if $\succ$ is a homothetic interval order, then the pair (u_1, u_2) of (1) is unique up to multiplication by a positive scalar (i.e., up to replacing it by $(\lambda u_1, \lambda u_2)$ for a $\lambda > 0$); and for $i = 1$, 2, u_i represents $\succ_i$.

Proof : Let u_1, $u_2 : A \to \mathbb{R}_+$ be two morphisms of $\mathbb{N}^*$-sets verifying (1). Since $u_1 \leq u_2$, $\succ$ satisfies (A); and $\forall (x,y) \in A^2$, we have $x \succsim y \Leftrightarrow u_2(x) \geq u_1(y)$. It is easy to check (and left to the reader) that $\succ$ is a homothetic interval order.

Conversely, suppose $\succ$ is a homothetic interval order. Choose an element $a \in A^*$, and let u_1^*, $u_2^* : A^* \to \mathbb{R}_+^*$ be the morphisms of $\mathbb{N}^*$-sets defined by $u_1^*(x) = s_{a,a}r_{a,x}$ and $u_2^*(x) = s_{a,x}$. For $i = 1$, 2, let $u_i : A \to \mathbb{R}_+$ be the morphism of $\mathbb{N}^*$-sets obtained extending u_i^* by zero on $A \smallsetminus A^*$. For $(x,y) \in (A^*)^2$, we have

$$\begin{aligned} x \succ y &\Leftrightarrow r_{y,x} > 1 \\ &\Leftrightarrow r_{y,a}s_{a,a}r_{a,x} > 1 \\ &\Leftrightarrow u_1(x) > u_2(y). \end{aligned}$$

By construction, we have $u_i^{-1}(u_i(A)^*) = A^*$ ($i = 1$, 2), therefore the equivalence above remains true for $y \in A \smallsetminus A^*$. Since $\succ$ satisfies (A), we have $u_1 \leq u_2$. From the proof of (7.1), we already know that for $i = 1$, 2, u_i represents $\succ_i$.

Concerning the uniqueness property, let u_1', $u_2' : A \to \mathbb{R}_+$ be two others morphisms of $\mathbb{N}^*$-sets verifying (1). For $(m,n,p) \in (\mathbb{N}^*)^3$, we have $mu_1(x) > nu_2(x) > pu_1(x)$ if and only if $mu_1'(x) > nu_2'(x) > pu_1'(x)$. Thus for $i = 1$, 2, we have $u_i'(x) = 0 \Leftrightarrow u_i(x) = 0$ ($x \in A$). For $i = 1$, 2, let $\lambda_i : A^* \to \mathbb{R}_+^*$ be the map defined by $\lambda_i(x) = u_i(x)^{-1}u_i'(x)$; since u_i and u_i' are morphisms of $\mathbb{N}^*$-sets, λ_i factorizes through the quotient-set $\bar{A}^*$. Let $f : \bar{A}^* \times \bar{A}^* \to \mathbb{R}_+^*$ be the map defined by $f(x,y) = \lambda_2(y)^{-1}\lambda_1(x)$. Let $(x,y) \in (A^*)^2$, and put $\mu = u_1(x)^{-1}u_2(y)$ and $\alpha = f(x,y)$. For $(m,n) \in (\mathbb{N}^*)^2$, we have $mx \succ ny \Leftrightarrow \frac{m}{n} > \mu$; but we also have $mx \succ ny \Leftrightarrow u_1'(mx) > u_2'(ny) \Leftrightarrow \alpha\frac{m}{n} > \mu$. If $\alpha > 1$, let choose $(m,n) \in (\mathbb{N}^*)^2$ such that $\alpha\frac{m}{n} > \mu \geq \frac{m}{n}$; then we have $mx \succ nx \succsim mx$, contradiction. If $\alpha < 1$, let choose $(m,n) \in (\mathbb{N}^*)^2$ such that $\frac{m}{n} > \mu \geq \alpha\frac{m}{n}$; then we have $mx \succ nx \succsim mx$, contradiction. Hence $\alpha = 1$. So we proved that $f = 1$. This implies there exists a constant $\lambda > 0$ such that $\lambda_1 = \lambda_2 = \lambda$. This completes the proof of the proposition. □

(7.3) Corollary. — *Let A be a $\mathbb{N}^*$-set endowed with a non-empty homothetic interval order $\succ$. Let $a \in A^*$ and u_1, $u_2 : A \to \mathbb{R}_+$ be the morphisms of $\mathbb{N}^*$-sets defined by $u_1(x) = s_{a,a}r_{a,x}$ and $u_2(x) = s_{a,x}$. Then the pair (u_1, u_2) verifies (7.2)-(1).*

(7.4) Corollary. — *Let A be a $\mathbb{N}^*$-set endowed with a non-empty homothetic interval order $\succ$.*

(1) Let $(u, \sigma) \in \mathcal{E}(A, \succ)$ be a pair verifying (6.1)-(1). Let u_1, $u_2 : A \to \mathbb{R}_+$ be the morphisms of $\mathbb{N}^*$-sets defined by $u_i(A \smallsetminus A^*) = 0$ ($i = 1, 2$), $u_1(x) = \gamma^*_\succ(x)u(x)$ and $u_2(x) = \gamma^*_\succ(x)^{-1}u(x)$ ($x \in A^*$). Then the pair (u_1, u_2) verifies (7.2)-(1).

(2) Let u_1, $u_2 : A \to \mathbb{R}_+$ be two morphisms of $\mathbb{N}^*$-sets verifying (7.2)-(1). Let $u : A \to \mathbb{R}_+$ be the morphism of $\mathbb{N}^*$-sets defined by $u = (u_1u_2)^{\frac{1}{2}}$, and let $v^* : \bar{A}^* \to \mathbb{R}^*_+$ be the map defined by $v^* = (u_1\overline{u}_2)^{\frac{1}{2}}$ with $\overline{u}_2(x) = u_2(x)^{-1}$. Then u represents $\succ_0$ and $\gamma^*_\succ = v^*$.

Proof : Let choose an element $a \in A^*$ and let u'_1, $u'_2 : A \to \mathbb{R}_+$ be the morphisms of $\mathbb{N}^*$-sets defined by $u'_1(x) = s_{a,a}r_{a,x}$ and $u'_2(x) = s_{a,x}$. For $(x, y) \in (A^*)^2$, we have

$$\begin{aligned} r_{x,y} < r_{y,x} &\Leftrightarrow r_{x,a}s_{a,a}r_{a,y} < r_{y,a}s_{a,a}r_{a,x} \\ &\Leftrightarrow s_{a,a}r_{a,x}s_{a,x} > s_{a,a}r_{a,y}s_{a,y} \\ &\Leftrightarrow (u'_1u'_2)(x) > (u'_1u'_2)(y). \end{aligned}$$

Since for $i = 1, 2$, we have $u'^{-1}_i(u'_i(A)^*) = A^*$, the equivalence above remains true for $(x, y) \in A^2$. Hence $u'_1u'_2$ represents $\succ_0$. Therefore $u' = (u'_1u'_2)^{\frac{1}{2}}$ represents $\succ_0$, and u' is a morphism of $\mathbb{N}^*$-sets. Moreover, it is easy to check (and left to the reader) that the map $\gamma^*_\succ : \bar{A}^* \to \mathbb{R}^*_+$ is given by $\gamma^*_\succ(x) = u'_1(x)^{\frac{1}{2}}u'_2(x)^{-\frac{1}{2}}$. By construction, for $x \in A^*$, we have $u'_1(x) = \gamma^*_\succ(x)u'(x)$ and $u'_2(x) = \gamma^*_\succ(x)^{-1}u'(x)$. Finally the uniqueness properties in (6.1) and (7.2) implie the corollary. □

The following proposition characterizes the homothetic semiorders.

(7.5) Proposition. — *Let A be a $\mathbb{N}^*$-set endowed with a non-empty binary relation $\succ$. The three following conditions are equivalent:*

(1) there exists a morphism of $\mathbb{N}^*$-sets $u : A \to \mathbb{R}_+$ and a constant $\alpha \in\,]0, 1]$ such that $\forall (x, y) \in A^2$, we have $x \succ y \Leftrightarrow \alpha u(x) > u(y)$;

(2) $\succ$ is a homothetic interval order such that $\succ_1 = \succ_2$ (in that case, we have $\succ_1 = \succ_0 = \succ_2$);

(3) $\succ$ is a homothetic semiorder.

Moreover, if $\succ$ is a homothetic semiorder, then the pair (u, α) of (1) is unique *up to multiplication of u by a positive scalar.*

Proof : Suppose there exists a morphism of $\mathbb{N}^*$-sets $u : A \to \mathbb{R}_+$ and a constant $\alpha \in\,]0, 1]$ verifying (1). Let $(x, y) \in A^2$. We have $x \succ_1 y$ if and only if $\exists (z, m) \in A \times \mathbb{N}^*$ such that $\alpha u(mx) > u(z) \geq \alpha u(my)$; i.e. (cf. the proof of (7.1)), if and only if $u(x) > u(y)$. And we have $x \succ_2 y$ if and only if $\exists (z, m) \in A \times \mathbb{N}^*$ such that $\alpha u(mx) \geq \alpha u(z) \geq \alpha u(my)$; i.e., if and only if $u(x) > u(y)$. Thus we have $\succ_1 = \succ_0 = \succ_2$. Now let $(x, y, z, t) \in A^4$ such that $x \succ y \succ z$. Since $\alpha u(x) > u(y) > \alpha^{-1}u(z)$, we have $\alpha^2 u(x) > u(z)$. If $t \succsim x$, we have $u(t) \geq \alpha u(x)$ and $\alpha u(t) \geq \alpha^2 u(x) > u(z)$, hence $t \succ z$. And if $z \succsim t$, we have $\alpha^{-1}u(z) \geq u(t)$ and $\alpha u(x) > \alpha^{-1}u(z) > u(t)$, hence $x \succ t$. Therefore $\succ$ is a semiorder.

Conversely, suppose $\succ_1 = \succ_2$. Let $a \in A^*$. From the uniqueness property in (4.1), there exists a (unique) $\beta > 0$ such that $\forall x \in A$, we have $r_{a,x} = \beta s_{a,x}$; taking $x = a$, we obtain $r_{a,a} = \beta s_{a,a}$. From (7.3) and (7.4), we have $\succ_0 = \succ_1$, and $\forall (x, y) \in (A^*)^2$, we have $\sigma^*_\succ(x, y) = \sigma^*_\succ(a, a) = r_{a,a}$. Put

$\alpha = r_{a,a} \in]0,1]$. If $u : A \to \mathbb{R}_+^*$ is a morphism of $\mathbb{N}^*$-sets which represents $\succ_0$, then $\forall (x,y) \in A^2$, we have $x \succ y \Leftrightarrow \alpha u(x) > u(y)$.

The implication (1) $\Rightarrow$ (3) and the equivalence (1) $\Leftrightarrow$ (2) are proved. Let us prove the implication (3) $\Rightarrow$ (1). Suppose $\succ$ is a homothetic semiorder. Let a pair $(u,\sigma) \in \mathcal{E}_0(A,\succ)$ verifying (6.1)(1). We have to prove that $\sigma^* = \sigma^*_{\succsim}$ is a constant map. Let $(x,y,z,t) \in (A^*)^4$ such that $x \succ y \succ z$. We have $\sigma(x,y)u(x) > u(y)$ and $\sigma(y,z)u(y) > u(z)$. Mutiplying the first inequality by $\sigma(y,t)$ and the second one by $\sigma(z,t)$, we obtain $\sigma(y,y)\sigma(x,t)u(x) > \sigma(y,t)u(y)$ and $\sigma(z,z)\sigma(y,t)u(y) > \sigma(z,t)u(z)$. From which we deduce that

$$\frac{\sigma(y,y)\sigma(x,t)\sigma(z,z)}{\sigma(z,t)} u(x) > u(z);$$

i.e., that $\sigma(y,y)\sigma(x,z)u(x) > u(z)$. Suppose σ^* is not a constant map. Then we may (and do) assume $\sigma(t,t) \neq \sigma(y,y)$. Up to permuting t and y, and replacing x, t, z par some multiples of themselves (in order to have $x \succ t \succ z$), we may (and do) assume $\sigma(t,t) < \sigma(y,y)$. Put $\mu = \frac{\sigma(y,y)}{\sigma(t,t)} > 1$. Since $\mathcal{P}_{x,y} = \mathbb{Q}_{>s_{x,y}}$, $\mathcal{P}_{y,z} = \mathbb{Q}_{>s_{y,z}}$ and $s_{x,y}s_{y,z} = s_{x,y}r_{y,y}^{-1} = s_{x,y}s_{y,y}$, we have $\mathcal{P}_{x,y}\mathcal{P}_{y,z} = \mathbb{Q}_{>s_{x,z}s_{y,y}}$. Thus we deduce that for every $\epsilon > 0$, there exists $(m,n,p) \in (\mathbb{N}^*)^3$ such that $mx \succ py \succ nz$ and $s_{x,z}s_{y,y} < \frac{m}{n} < s_{x,z}s_{y,y} + \epsilon$. So let $(m,n,p) \in (\mathbb{N}^*)^3$ such that $s_{x,z}s_{y,y} < \frac{m}{n} < \mu\, s_{x,z}s_{y,y}$. Since $\sigma(x,z) = s_{x,z}^{-1}u(x)^{-1}u(z)$, multiplying by $u(x)u(z)^{-1}$, we obtain

$$\frac{1}{\sigma(y,y)\sigma(x,z)} < \frac{u(mx)}{u(nz)} < \frac{\mu}{\sigma(y,y)\sigma(x,z)}.$$

Therefore, up to replacing (x,y,z) by (mx,py,nz), we may (and do) suppose that we have $\sigma(y,y)\sigma(x,z)u(x) > u(z) > \sigma(t,t)\sigma(x,z)u(x)$. Then $\exists (a,b) \in (\mathbb{N}^*)^2$ such that

$$u(z) \geq \frac{a}{b}\sigma(t,z)u(t) \geq \sigma(t,t)\sigma(x,z)u(x).$$

Again, up to replacing (x,y,z,t) by (bx,by,bz,at), we may (and do) suppose $a = b = 1$. Thus we have $z \succsim x$; and $u(t) \geq \sigma(t,z)^{-1}\sigma(t,t)\sigma(x,z)u(x) = \sigma(x,t)u(x)$, that is $t \succsim x$. Therefore $\succ$ is not a semiorder, contradiction. So we proved that σ^* is a constant map, which implies (1). ∎

Let A be a $\mathbb{N}^*$-set endowed with a non-empty homothetic interval order $\succ$. From (7.5), $\succ$ is a semiorder if and only if its invariant $\sigma^*_{\succsim}$ is a constant map. And $\succ$ is a weak order if and only if $\sigma^*_{\succsim} = 1$. We can see the homothetic interval order $\succ$ as a deformation of its associated homothetic weak order $\succ_0$; the invariant $\sigma^*_{\succsim}$ being the expression of this deformation. So the homothetic semiorders are the homothetic interval orders for which the deformation is as simple as possible, that is expressed by a constant invariant.

(7.6) Example. — Let $A = \mathbb{N}^*x \coprod \mathbb{N}^*y$ be the union of two copies of $\mathbb{N}^*$. Let α, β be two real numbers such that $0 < \alpha, \beta \leq 1$, and let $\sigma : \bar{A} \times \bar{A} \to \mathbb{R}_+^*$ be the map defined by $\sigma(x,x) = \alpha$, $\sigma(y,y) = \beta$ and $\sigma(x,y) = \sigma(y,x) = (\alpha\beta)^{\frac{1}{2}}$. Let $u : A \to \mathbb{R}_+$ be the morphism of $\mathbb{N}^*$-sets defined by $u(x) = u(y) = 1$. From (6.1), the binary relation $\succ$ on A defined by $z \succ t \Leftrightarrow \sigma(z,t)u(z) > u(t)$, is a *homothetic interval order*. Let remark that we have $A^*_{\succsim} = A$. Moreover, $\succ$ is a *semiorder* if and only if $\alpha = \beta$; in which case we have $\sigma^*_{\succsim} = \alpha$.

Otherwise, we have $r_{x,x} = \sigma(x,x)$ and $r_{y,y} = \sigma(y,y)$. So if $\alpha \neq \beta$, then the map $A \times A \to \mathbb{R}_+^*$, $(z,t) \mapsto r_{z,t}$ *do not factorizes* through the product-map $u \times u$; which answers the question asked in (6.5). ⋆

8. In this section, we generalize proposition (5.1) to the homothetic interval orders.

(8.1) Lemma. — *Let* $(A, \circ)$ *be a commutative semigroup endowed with a non-empty homothetic interval order* $\succ$. *If* $\succ_0$ *is* $\circ$*-independent, then* $\succ$ *est un semiorder.*

Proof : Suppose $\succ_0$ is $\circ$-independent. In particular, we have $A^* \circ A \subset A^*$. Let $a \in A^*$. For $(x, y, z) \in A^3$, we have $x \circ z \succ_1 y \circ z \Leftrightarrow r_{a,x\circ z} > r_{a,y\circ z}$. Replacing a by $a \circ z \in A^*$, we obtain

$$x \circ z \succ_1 y \circ z \Leftrightarrow r_{a\circ z,x\circ z} > r_{a\circ z,y\circ z} \Leftrightarrow r_{a,x} > r_{a,y} \Leftrightarrow x \succ_1 y.$$

Thus $\succ_1$ is $\circ$-independent. In the same way, we prove that $\succ_2$ est $\circ$-independent. Let u_0, u_1, $u_2 : A \to \mathbb{R}_+$ be the morphisms of $\mathbb{N}^*$-sets defined by $u_1(x) = s_{a,a}r_{a,x}$, $u_2(x) = s_{a,x}$ and $u_0 = (u_1u_2)^{\frac{1}{2}}$. From (7.3), for $i = 0$, 1, 2, u_i represents $\succ_i$; and from (5.1), u_i is a morphism of semigroups. For $(x, y)^2 \in A$, we have (easy calculation)

$$\begin{aligned} u_0(x \circ y)^2 &= u_0(x)^2 + u_0(y)^2 + u_1(x)u_2(y) + u_1(y)u_2(x) \\ &= [u_0(x) + u_0(y)]^2 + ([u_1(x)u_2(y)]^{\frac{1}{2}} - [u_1(y)u_2(x)]^{\frac{1}{2}})^2, \end{aligned}$$

from which we deduce that $([u_1(x)u_2(y)]^{\frac{1}{2}} - [u_1(y)u_2(x)]^{\frac{1}{2}})^2 = 0$; i.e., that $u_1(x)u_2(y) = u_1(y)u_2(x)$. That is possible only if $u_2 = \lambda u_1$ for a constant $\lambda > 0$. Hence $\succ$ is a semiorder (7.5). ∎

(8.2) Proposition. — *Let* $(A, \circ)$ *be a commutative semigroup endowed with a non-empty homothetic interval order* $\succ$. *The two following conditions are equivalent:*
(1) $\succ$ *is* $\circ$*-pseudoindependent;*
(2) for $i = 1$, 2, $\succ_i$ *est* $\circ$*-independent.*

Proof : Suppose $\succ$ is $\circ$-pseudoindependent. Let $a \in A^*$. From the proof of (5.1), for x, $y \in A$, we have $r_{a,x\circ y} = r_{a,x} + r_{a,y}$; and in the same way, we obtain $s_{a,x\circ y} = s_{a,x} + s_{a,y}$. So the implication (1) $\Rightarrow$ (2) is proved.

Conversely, suppose for $i = 1$, 2, $\succ_i$ est $\circ$-independent. Let u_1, $u_2 : A \to \mathbb{R}_+$ be two morphisms of $\mathbb{N}^*$-sets verifying (7.2)-(1). For $i = 1$, 2, since u_i represents $\succ_i$ (7.2), it is a morphism of semigroups (5.1). From this we deduce that for $(x, y, z, t) \in A^4$, we have

$$\begin{cases} (x \succ y,\ z \succ t) \Rightarrow x \circ z \succ y \circ t \\ (x \succsim y,\ z \succsim t) \Rightarrow x \circ z \succsim y \circ t \end{cases}.$$

Let $(x, y) \in A^* \times (A \setminus A^*)$. If $x \circ y \in A \setminus A^*$, then we have $x \succ x \circ y$, that is $u_1(x) > u_2(x \circ y) = u_2(x) + u_2(y) = u_2(x)$, which is impossible because $u_1 \leq u_2$. Hence $\succ$ is $\circ$-pseudoindependent. ∎

(8.3) Corollary. — *Let* $(A, \circ)$ *be a commutative semigroup endowed with a non-empty homothetic interval order* $\succ$. *The two following conditions are equivalent:*
(1) $\succ_0$ *is* $\circ$*-independent;*
(2) $\succ$ *is a* $\circ$*-pseudoindependent semiorder.*

Proof: If $\succ_0$ is $\circ$-independent, then $\succ$ is a semiorder (8.1), therefore $\succ_1 = \succ_0 = \succ_2$ (7.5) and $\succ$ is $\circ$-pseudoindependent. So we have (1) $\Rightarrow$ (2). Conversely, if $\succ$ is a $\circ$-pseudoindependent semiorder, then we have $\succ_1 = \succ_0 = \succ_2$ (7.5) and $\succ_0$ is $\circ$-independent (8.2). ∎

(8.4) Example. — Let $A = \mathbb{N}^*x \times \mathbb{N}^*y$ be the product of two copies of $\mathbb{N}^*$, endowed with the structure of commutative semigroup $\circ$ defined by $(mx, ny) \circ (m'x, n'y) = ((m+m')x, (n+n')y)$. Let λ, μ be two real numbers such that $0 < \lambda \leq \mu$, and let u_1, $u_2 : A \to \mathbb{R}_+$ be the morphisms of semigroups defined by $u_1(mx, ny) = \lambda m + n$ and $u_2(mx, ny) = \mu m + n$. Then from (7.2) and (8.2), the binary relation $\succ$ on A defined by $z \succ t \Leftrightarrow u_1(z) > u_2(t)$, is a $\circ$-*pseudoindependent homothetic interval order*. But the homothetic weak order $\succ_0$ is $\circ$-*independent* (i.e., $\succ_1 = \succ_2$) if and only if we have $\lambda = \mu$; in which case $\succ$ is a *homothetic weak order*. ⋆

For once, let us conclude with a definition.

(8.5) Definition. — *We call* biased balance *a commutative semigroup* $(A, \circ)$ *endowed with a* $\circ$-*pseudoindependent homothetic semiorder* $\succ$.

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