Economics Working Paper 77

# Lecture Notes on Economic Growth (I): Introduction to the Literature and Neoclassical Models*

Xavier Sala-i-Martin

Yale University and UPF

May 1994

## Abstract

This is a survey of the economic growth literature growth. In the first two sections we analyze the main differences between exogenous and endogenous growth models using a fixed savings rate analysis. We argue that in order to have endogenous growth there must be constant returns to the factors that can be accumulated. A graphical tool is developed to show that changes in the saving rate have different effects on long-run growth in the two types of models. We show how different growth models can be generated by simply changing different features of the production function. We then explore Ramsey's optimal-saving neoclassical model. Finally, we discuss the role of technology in the neoclassical model of growth.

*"The consequences for human welfare involved in questions like these are simply staggering: once one starts to think about them, it is hard to think about anything else"*. Lucas (1988), p. 5.

## (1) INTRODUCTION

Most economic growth models discussed in the literature have a simple general equilibrium structure. First, there are households who own the assets and inputs in the economy and choose the fraction of income they will consume and the fraction they will save. Second, there are firms who hire the different inputs in order to produce the final output they sell to consumers. And finally, there are markets where households sell their inputs to firms and where firms sell their output to households.

In the initial section of these notes, however, we will use a setup with no markets or firms. We will think of household-producers who own the inputs and the technology which transforms them into output. These household-producers then choose how much of the output to consume and how much to save and invest. This setup is often known as a *Robinson Crusoe Economy*. The only asset in this (closed) economy is something we call $K_t$. One may wish to think of K as physical *capital* but it may also include other inputs that *can be accumulated*, such as knowledge or skills. The other input in the economy, $L_t$, cannot be accumulated in that it is assumed to grow at a rate which is independent of individual choices. We may want to think of L as labor but it may also include other non-reproducible resources such as land or energy. We assume that the technology available transforms combinations of these two inputs into output according to the following function:

$$(1.1) \qquad Y_t = F(K_t, L_t),$$

where $Y_t$ stands for aggregate output.

To simplify matters we assume that output is a homogeneous good which can be either *consumed* or *saved*. The reason why households may choose to save is that unconsumed output can be transformed into capital through a

1

process that we will call *investment*. If we let s() be the fraction of income that is saved, the increase in the capital stock is given by

$$(1.2) \qquad \dot{K}_t = s() \, F(K_t, L_t) - \delta K_t,$$

where $\dot{K}=dK/dt$ is the time derivative of K (this notation will be used throughout these notes) and $\delta$ is the constant rate of depreciation.[1] Equation (1.2) says that in a closed economy, gross investment (the sum of net investment, $\dot{K}$, and depreciation) is equal to gross saving.[2]

In most of the recent economic growth literature, households are assumed to choose a consumption path by maximizing a utility function subject to some intertemporal budget constraint.[3] In Section 3 we will find that the optimal saving rate s() is a complicated function for which there are in general no closed form solutions. The complicated mechanics of dynamic optimization, however, obscure some of the important points and issues. Hence, before studying optimal-savings models it will be convenient to follow Solow (1956) and Swan (1956) and start with the assumption that the saving rate, s(), is an exogenous constant which we denote by 's'.[4] We

----

[1] This assumption implies that the depreciation rate is independent of economic conditions. More realistically, firms choose the intensity at which they use their capital, and when capital is used more intensively, it depreciates faster. Hence, in the real world, $\delta$ is a function of economic conditions. We will, however, abstract from the the choice of capacity utilization here, and assume that $\delta$ is constant. Nonetheless we will see later that the rate of depreciation is an important determinant of the rate of economic growth. Thus, endogeneizing the depreciation rate is potentially a productive area of research.

[2] More generally, in an open economy, the difference between saving and investment is equal to the current account balance.

[3] Early economists used to confine the intertemporal optimization analysis to normative issues. The celebrated Ramsey (1928) paper starts with the sentence *"The first problem I propose to tackle is this: how much of its income should a nation save?" (p.543).* Contemporaneous economists, on the other hand, use intertemporal optimizing models for descriptive or positive analysis as well. Following Barro (1974), the representative agent is assumed to be a family or group of individuals linked to each other through bequests.

[4] In Section 3 we will show that a constant saving rate is optimal under

assume that the production function is Cobb Douglas so

$$(1.3) \qquad Y_t = AK_t^{\beta}L_t^{\alpha},$$

where A is the level of the technology. From a macroeconomic perspective, we should think of 'technology' in a broad sense that includes government distortions, protection of property rights, and things of this sort. In other words, for the same amount of K and L, economy 1 may get more output than economy 2 because it is less distorted, its government is more efficient, or its institutions favor private production more effectively. The 'technology' parameter A, therefore, should capture all these concepts. Using the Cobb Douglas production function and the assumption of constant saving rates, we can write the net increase in capital as

$$(1.4) \qquad \dot{K} = s \, AK^{\beta}L^{\alpha} - \delta K,$$

where time subscripts have been omitted (we will keep omitting them when no ambiguity arises).

We imagine that population is equal to employment, and thereby abstract from unemployment and labor force participation issues. We further assume that population grows at an exogenously determined constant rate, $\dot{L}/L=n$.[5]

---

certain conditions.

[5] In the real world, people choose how many children to have and whether to migrate or not. For example, if the production of children requires that parents spend time with their kids, then high wages will tend to deter reproduction. Similarly, high future wages induce higher fertility rates because children will become grown-ups who will earn those future wages. Interest rates will also affect fertility rates if the parent's utility function exhibits diminishing returns to children. This is true because, in this case, parents would like to smooth children over time (much in the same way they want to smooth consumption). Wages and interest rates should also affect mortality rates by affecting the amount of time people spend working rather than taking care of themselves or their children. In terms of migration, high wages in a country tend to attract immigrants. Hence, economic conditions (such as wages, interest rates and so on) should, in principle, affect the rate of population growth. However, we simplify the analysis here by abstracting from these important, interesting, and largely unresolved issues, which should be the subject of future research (see Barro and Sala-i-Martin (1994), Chapter 9, for examples of growth models with endogenous fertility and migration. I know of no growth models that

3

Define lower case k as the capital-labor ratio (or capital per worker), K/L. By taking the time derivative of $k_t$, we can rewrite (1.4) in per capita terms as:[6]

$$(1.5) \qquad \dot{k} = sAk^{\beta}L^{\alpha+\beta-1} - (\delta+n)k.$$

The growth rate of capital per worker is given by $\dot{k}_t/k_t \equiv \gamma_k$. We can compute this growth rate by dividing both sides of (1.5) by k. Define *steady state* as the state where all variables grow at a constant (possibly zero) rate. Thus, the steady-state growth rate, $\gamma_k^*$, is constant[7] by definition. Hence, we can write $[\gamma_k^* + \delta +n]/sA = k^{\beta-1}L^{\alpha+\beta-1}$, where all the variables in the left-hand-side are constants. Take logarithms and time derivatives of both sides and get

$$(1.6) \qquad 0 = (\beta-1)\gamma_k^* + n(\alpha+\beta-1).$$

This *key* equality deserves some attention. Consider first the neoclassical production function where output is assumed to exhibit *constant returns to scale* (CRS) and positive but *diminishing returns to each input*. In the Cobb Douglas setup, these two assumptions respectively require that $\alpha+\beta=1$ and $0<\beta<1$. Given $\alpha+\beta=1$, the second term in the right-hand-side of (1.6) vanishes so we are left with

$$(1.7) \qquad 0 = (\beta-1)\gamma_k^*.$$

---

incorporate endogenous mortality rates)

[6]    Notice that the difference between expressing the accumulation equation in levels or in per capita terms is the addition of the term nk to δk. We can in fact think of nk as some extra depreciation since it represents the loss of *capital per person* due to the addition of population. Note in particular that in the hypothetical case where people do not save anything (s=0), capital per person would fall both because capital falls (depreciation) and because the number of people increases (population growth).

[7]    From now on we denote steady-state values of the various variables with stars.

4

The assumption of diminishing returns to capital, $\beta<1$, implies that the only sustainable steady-state growth rate is $\gamma_k^*=0$. In other words, *the only steady-state growth rate consistent with the neoclassical model is zero.*

An interesting question arises. If the only steady-state growth rate is zero, how did the neoclassical theorists of the 1950s and 1960s explain the fact that most industrialized countries have experienced centuries-long positive growth rates? Their answer was that the technology used by these countries had improved over time. In order to capture this idea, they allowed the term A in (1.3) to grow at an exogenously given rate, g (in other words, $\dot{A}/A = g$).[8] When technology grows at a constant rate, the rest of the variables follow. Hence, the steady-state growth rates of income per capita, capital per capita, and consumption per capita in a neoclassical model with *exogenous productivity growth* are all equal to g.

A second (and possibly more interesting) way to read equation (1.6) is the following: "If we want to have *positive steady-state growth rates* ($\gamma_k>0$) in a model *with constant returns to scale* ($\alpha+\beta=1$), *then the production function must exhibit constant returns to the inputs that can be accumulated,* $\beta=1$." This implies that $\alpha = 0$, and the production function takes the form:

$$(1.8) \qquad Y_t = AK_t,$$

where A is a constant. This technology, known as the "AK technology", yields the simplest model of endogenous growth. One of the main differences between endogenous and neoclassical models of growth is that the steady-state growth rate, $\gamma_k^*$, derived from endogenous growth models can be positive, *even when no variable is assumed to grow at an exogenous rate.* The steady-state growth rate depends on different *choice or endogenous variables* like the saving rate or the tax structure, rather than on the

---

[8]    Productivity growth in the neoclassical model had to be exogenous because in a world of competitive markets and CRS technologies, the rewards to all private inputs (given by the marginal products) exhaust the value of the final product. Since technology is a public good (in the sense of being a non-rival and non-excludable good), there are no resources left to finance activities such as research and development.

exogenously given rate of productivity growth. This is why they are called *Models of Endogenous Growth*.

Despite its simplicity, it is important to study the AK model in detail because all endogenous growth models embed a linearity that makes them look like the AK model. We will study the AK model in Section 5.

There are various ways to motivate the AK technology. The most obvious one is to take (1.1) and to think of labor as capital: what really matters for production is not the number of bodies (raw labor), but rather the amount of quality-adjusted labor. This quality, in turn, can be accumulated by foregoing consumption, just like physical capital can. In other words, people invest in their *human capital* much in the same way that they invest in their physical capital. If the production functions for human and physical capital are similar, then we can combine the two concepts into a broad measure of capital to get a production function that resembles AK. This idea, which we will derive more formally in Section 5 of these notes, underlies the work of Rebelo (1991).

Another way to motivate the AK technology is to think in terms of private capital along with publicly provided inputs (such as roads, infrastructure, or law enforcement). The production function could be written as $Y=AK^{\beta}g^{1-\beta}$, where K is the private capital good and g is a publicly-provided good. If the government increases the supply of public inputs in proportion to the supply of private capital (perhaps because increases in private capital generate increases in tax collection), then the setup resembles an AK technology (see Barro (1990)). In Section 6 we will examine the Barro model in more detail.

Equation (1.6) allows for a third reading: "Assuming no population growth, n=0, we can have non-reproducible inputs, $\alpha>0$, together with positive steady-state growth, $\gamma_k^*>0$, if there are constant returns to the inputs that can be accumulated, $\beta=1$. But notice that this implies $\alpha+\beta>1$; that is, it implies *increasing returns to scale* (IRS)".[9]

---

[9] Note that when the rate of population growth is positive, n>0, thetechnology exhibits IRS, so that $\alpha+\beta>1$ applies, then there is no steady-state growth rate $\gamma_k^*$ which satisfies the key equality (1.6). What occurs in this case is that the growth rate is never constant, but rather,

The problem is that if we plainly postulate an (IRS) production function we may have trouble finding a set of prices to support a general competitive equilibrium. Furthermore, the usual optimization techniques cannot be used because the usual concavity requirements for the first order necessary conditions to be sufficient are not met.

There are at least two ways to get around this problem. The first one is to follow Alfred Marshall and introduce IRS at the aggregate level but CRS at the firm level. This can be formulated through production *externalities or spillovers*: each producer's decision affects all other producers' output, but no one takes this into account. Hence, all producers face a concave problem so the usual optimizing tools can be applied. The economy as a whole, however, faces an IRS production function which, (under some conditions that we will outline below) generates endogenous growth. The Cobb Douglas version of this production function is

$$Y = AK^{\beta}L^{1-\beta}\kappa^{\psi},$$

where K is private capital and $\kappa$ is the aggregate capital stock in the economy. Individual firms do not realize that their own investment decisions affect $\kappa$ so they take it as given. In the aggregate, however, total capital will equal the sum of the capital stocks of individual firms. Therefore, $\kappa=K$. It follows that, effectively, aggregate output is given by

$$(1.9) \qquad Y = AK^{\beta+\psi}L^{1-\beta}.$$

Note that if the size of the externality is such that $\beta+\psi=1$, then we have *constant returns to capital* in an IRS world. Thus, by modeling IRS through externalities we get around the problem of the existence of competitive general equilibrium. As it is well known, however, competitive equilibrium models with externalities tend to generate non-optimal outcomes. In Section 7 we show how Romer (1986), following Arrow (1962) and Sheshinski

---

increases over time. This phenomenon, also known as *scale effect*, explains why all the models of endogenous growth with IRS always assume no population growth.

(1967), uses capital externalities in the aggregate production function to generate endogenous growth.

A second way to get around the problem of the non-existence of competitive equilibrium is to drop the assumption of competitive behavior. This is sometimes called the Chamberlinian approach to increasing returns. Among other things, this approach is interesting because, under imperfect competition the rewards to all inputs of production do not exhaust total output. Hence, there are rents that can be assigned to activities such as research and development (R&D), which are not directly productive but which may contribute to the expansion of the frontiers of knowledge (and knowledge benefits all firms in the economy). Not surprisingly, therefore, this approach has been extensively used by economists who think that R&D is an important source of economic growth. In Section 9 we explore a model of R&D and growth based on Romer (1987, 1990) and Grossman and Helpman (1991, Chapter 3) where firms invest in R&D in search of new capital goods. In this model, there are no diminishing returns to the introduction of new varieties so the incentive to undertake R&D never diminishes. This keeps the economy growing.

Before working through the mechanics of all these models, let us introduce a graphical device that will further clarify the basic differences between exogenous and endogenous growth models. This device will also help us understand why the saving (or investment) rate does not affect the long-run growth rate in the former models but does affect it in the latter.

## (2) MODELS WITH CONSTANT SAVING R1TE: A GRAPHICAL EXPOSITION

We often hear economic advisors to Third World countries argue that one of the necessary conditions for economic growth and development is an increase in national saving rate. The suggested mechanism is that higher savings lead to higher investment (since they must be equal in a closed economy), and higher investment leads to more rapid economic growth. In this section, we analyze the conditions under which this policy recommendation is valid. For this analysis, we continue to assume a constant saving rate.

8

## Neoclassical Growth

Let us first imagine that the production function is neoclassical in that exhibits constant returns to scale, $\alpha+\beta=1$, and diminishing returns to each input, $0<\beta,\alpha<1$. Capital per person accumulates according to (1.5). If we divide both sides of (1.5) by k we get

$$(2.1) \qquad \gamma_k = \dot{k}/k = sAk^{-(1-\beta)} - (\delta+n).$$

The left-hand-side of this equation is the instantaneous growth rate of the stock of capital per person. The right-hand-side says that this growth rate is given by the difference between two functions, $sAk^{-(1-\beta)}$ and $(\delta+n)$. We depict these two functions in Figure 1. The function $\delta+n$ is independent of k so it is a flat line. Since we assume $\beta<1$, the function $sAk^{-(1-\beta)}$ is downward sloping in k, it approaches infinity as k approaches zero, and it approaches zero as k approaches infinity. Since $\delta+n$ is strictly positive, the two curves must cross once and only once in the positive quadrant. The value of k at which they cross, denoted by k*, is the *steady-state capital* per worker given by

$$k^* = [sA/(\delta+n)]^{1/(1-\beta)}.$$

We can use Figure 1 to study the behavior of an economy over time. Equation (2.1) says that the growth rate of k is given by the vertical difference between the two curves. Hence, the growth rate is positive for k<k* and negative for k>k*. Moreover, the growth rate is larger the further below the steady state an economy is. Consider an economy with an initial level of capital $k_0$ below k*. The growth rate of capital is initially large but falls over time as the economy grows towards its steady-state position. When the economy reaches the steady state, the economy stops growing. The behavior is symmetric if the initial capital stock is above steady state. We can take logarithms and derivatives of the production function to see that the growth rate of output per capita is proportional to the growth rate of capital per person, $\gamma_y=\beta\gamma_k$. It follows that the dynamic behavior of y parallels the behavior of k.

The intuitive reason behind the lack of steady-state growth is the

9

assumption that the returns to capital diminish and approach zero; when the capital stock is low, each addition to the capital stock generates a large increase in output (that is, the marginal product of capital is high). Since, by assumption, agents save and invest a constant fraction of this additional output, the increase in the capital stock is large. As the capital stock grows however, each additional unit generates fewer and fewer units of output. Since agents still save a constant fraction of it, the new additions to the capital stock are smaller and smaller. In fact, they would approach zero if the capital stock were arbitrarily large. Before reaching zero, however, the economy reaches a point where the new additions to the capital stock are just sufficient to replace the depreciated stock and makeι up for population growth (at rate n). This is just enough to keep the capital per person at a constant level. Once the economy reaches this (steady-state) point, it remains there forever.

Imagine that, starting from a steady-state position, the saving rate, s, experiences a sudden and permanent increase (maybe because the government changes the tax structure). The curve $sAk_t^{-(1-\beta)}$ will shift to the right while the line $(\delta+n)$ will be unaffected. We can see in Figure 1 that the following things are true:

(a) the growth rate experiences an immediate increase.

(b) the growth rate falls over time and eventually returns to zero.

(c) the new steady-state stock of capital per worker is higher.

The main point is that, even though the permanent increase in the saving rate leads to a short-term increase in the growth rate and an increase in the steady state *level* of capital per worker, the steady-state *growth rate* remains unaffected. The transitional dynamics following an exogenous and permanent increase in A, $\delta$ or n are very similar: there is a short-run effect on the growth rate, but in the long run, only the levels of capital and output change. Incidentally, when the level of technology A increases continuously at a constant rate g (as is the case of the neoclassical models with exogenous productivity growth), then the curve $sA(t)f(k)/k$ shifts to the right continuously. It follows that the steady-state capital stock $k^*$ also shifts to the right at the same rate, g. Hence, the steady-state per-capita growth rate of the economy is positive

10

and equal to g.

### A Quantitative Measure of the Length of the Transition

An important question is how quickly the economy reaches the new steady state. To answer this question we can log-linearize equation (2.1) around the steady state to get[10]

$$(2.2) \qquad \gamma_k = -(1-\beta)(\delta+n)[\log(k)-\log(k^*)].$$

The speed of convergence is given by $(1-\beta)(\delta+n)$. To get a quantitative measure of this speed of convergence we note that the rate of population growth in industrialized nations is between 0.01 and 0.02. The depreciation rate is somewhere between 0.05 and 0.1, depending on how we treat residential capital and other durable goods. The physical capital share in industrialized countries lies between 0.25 and 0.30. Hence, the speed of convergence predicted by the model is somewhere between 0.042 and 0.09. In other words, between 4.2 and 9 percent of the gap between k(0) and k* is closed every year. These numbers imply half lives of 7.7 and 16 years respectively (that is, half of the distance between k(0) and k* disappears in 7.7 years). Hence, the speed of convergence towards the steady state is quite large, implying that the transition takes a short period of time.

The predicted speed of convergence would be much smaller if we took a broad view of capital (so as to include things like human capital). For instance, if the broad capital share was 0.75, then the predicted speed of convergence would lie somewhere 0.015 and 0.03 (with implied half lives of 23 to 47 years). Barro and Sala-i-Martin (1991, 1992) and Mankiw, Romer and Weil (1992) show that these smaller speeds of convergence accord better with the data.

### The Convergence Hypothesis.

---

[10] To log-linearize the equation, we rewrite (2.1) in terms of $\log(k)$. Note that $\gamma_k$ is the time derivative of $\log(k)$. The term $Ak^{-(1-\beta)}$ can be written as $e^{-(1-\beta)\log(k)}$. The steady-state value of $sAe^{-(1-\beta)\log(k^*)}$ is equal to $\delta+n$. Take a first-order Taylor approximation of (2.1) around $\log(k^*)$ to get (2.2).

11

Figure 1 suggests that the growth rate for an economy which starts below the steady state is high and decreasing. This implies that if economies differ ONLY in their initial capital-labor ratios, then we should observe that poor economies grow faster than rich ones (in Figure 1, different economies would be represented by different stocks of $k_0$ but all of them would have the same steady state $k^*$). Since the growth rate of income per capita is proportional to the growth rate of capital per person, the model also predicts a negative relation between the initial level of income and its growth rate. This inverse relation between the level of income and its growth rate is known as the *convergence hypothesis*. This hypothesis is interesting because it can be easily tested using data for a cross-section of countries by simply plotting growth rates and levels of income. If the correlation is negative, then the economies tend to converge.

But note that the neoclassical model just outlined predicts a negative relation between income and growth rates if the only difference across countries is their initial capital stocks. However, if economies also differ in the level of technology, A, the savings rate, s, the depreciation rate, $\delta$, or the rate of population growth, n, then the model does not predict that poor countries should grow faster. As an example, consider Figure 2 where two economies (called P for poor and R for rich) have the capital stocks $k_{OP}$ and $k_{OR}$ respectively (with $k_{OP} < k_{OR}$). Imagine that the saving rate in the poor economy is also lower so it converges to a smaller steady-state capital ratio, $k_P^* < k_R^*$. Note that in this particular example, it happens that the poor economy grows less than the rich one so there is no convergence in the absolute sense. Yet there is *conditional convergence* in the sense that the growth rate of a country is inversely related to the distance from its steady state. In other words, the model predicts convergence only after controlling for the determinants of the steady state. This can also be seen in equation (2.2), where the growth rate is negatively related to (the log of) k relative to $k^*$. Hence, from an empirical point of view, we need to hold $k^*$ constant. Barro and Sala-i-Martin (1991, 1992) and Mankiw, Romer and Weil (1991) find empirical support for the conditional convergence hypothesis and, therefore, for the neoclassical model. The

12

capital share needed by the model to fit the data is substantially larger than 0.3 and close to 0.75.

## An Extended Solow Model

The empirical evidence on the convergence hypothesis suggests that the neoclassical model is consistent with the data if the capital share is close to 0.75. Empirical estimates of the physical capital share in industrialized economies suggest that it is closer to 0.3 than to 0.75. Hence, we need to think of K in a broad sense to include other forms of non-physical capital.

To incorporate this idea, Mankiw, Romer and Weil (1992) constructed what they call an 'extended Solow model'. The model includes three inputs: capital, raw labor and human capital (denoted by H) in a Cobb Douglas technology

$$(2.3) \qquad Y = BK^{\lambda}H^{\alpha}L^{1-\lambda-\alpha}.$$

They assume that both human and physical capital can be accumulated out of the output stream so

$$\dot{K} + \dot{H} = BK^{\lambda}H^{\alpha}L^{1-\lambda-\alpha} - C - \delta_K K - \delta_H H,$$

where $\delta_K$ and $\delta_H$ are the depreciation rates for physical and human capital respectively. For simplicity, they assume $\delta_K = \delta_H$. The marginal product of physical and human capital in this model must be equalized at all points in time so $\lambda Y/K = \alpha Y/H$. This can be rewritten as $H = (\alpha/\lambda)K$ so human capital must be proportional to physical capital for all t. If we plug this equality into the output equation we get $Y = AK^{\beta}L^{1-\beta}$, where the effective capital share, $\beta$, is the sum of the physical and human capital shares, $\beta=\lambda+\alpha$, and where $A=B(\alpha/\lambda)^{\alpha}$ is a constant. Hence, the extension of the Solow model to include human capital is just a way to argue that the relevant capital share is larger than the physical capital share. In other words, it is a way to argue that the relevant capital share is closer to 0.75 than to 0.3. Note that the speed of convergence derived in equation (2.2) would depend on the 'broad capital share' $\beta = \lambda+\alpha$, rather than the physical

13

capital share $\lambda$ (so the speed would be equal to $(1-\lambda-\alpha)(\delta+n)$.) If the share of physical capital is $\lambda=0.3$ and the share of human capital is $\alpha=0.45$, then the relevant capital share is 0.75 and the predicted speed of convergence lies somewhere 0.015 and 0.03. This is much closer to the findings of Barro and Sala-i-Martin (1991, 1992) and Mankiw, Romer and Weil (1992).

## Open Economy Considerations

The growth models described up to this point assume that the economy is closed, in that there is no trade of goods, assets or labor across economies. The empirical evidence mentioned above deals with economies, such as states within the United States, prefectures within Japan, or even OECD economies, that are not obviously closed. Barro, Mankiw and Sala-i-Martin (1992) present an open-economy model where economies can borrow in international capital markets but cannot use all their capital as collateral. Using the production (2.3), imagine that physical capital can freely move across borders but human capital cannot. Imagine that the world capital market faces a constant world real interest rate $r^*$. The assumption of perfect physical mobility would equalize the marginal product of physical capital to the world real interest rate so $\lambda Y/K = r^* + \delta$. Using this equality, we can write K as a function of Y as $K = \lambda Y/(r^*+\delta)$. Plug this in the production function (2.3) to get a reduced form production function

$$Y = A \ H^{\beta} \ L^{1-\beta},$$

where $\beta \equiv \alpha/(1-\lambda)$ is the relevant capital share and $A \equiv B^{1/(1-\lambda)}[\lambda/(r^*+\delta)]^{\lambda/(1-\lambda)}$ is a constant. Note that the reduced form production function of this open economy model is identical to the production function of the neoclassical model. Moreover, the relevant capital share is numerically very close. If we continue to assume that $\lambda$ is close to 0.3 and $\alpha$ is 0.45, then the relevant capital share is $\beta = 0.45/0.7 = 0.65$. The implied speed of convergence lies between 0.21 and 0.42 (recall that the closed economy speed for similar capital shares lied between 0.15 and 0.31). Hence, allowing for capital mobility in the neoclassical model does not change substantially the qualitative or quantitative predictions about the speed of the transition, as long as the fraction of the capital

stock that can be used as collateral is not very large. The implication is that, for most practical purposes, assuming a closed economy setup may not be a bad idea.

## Endogenous Growth

Imagine now that the capital share of the previous model is equal to one, $\beta=1$. This corresponds to the AK technology described in (1.8). The growth rate of the economy is still given by equation (2.1). The difference is that the $sAk^{-(1-\beta)}$ curve is now a flat line at sA, as displayed in Figure 3. If we assume that the economy is productive enough so that $sA>\delta+n$, then the growth rate (the difference between the two lines) is constant and positive.

There are four important differences between this and the neoclassical model. First, the economy has no transitional dynamics in that it grows at a constant rate equal to $sA-(\delta+n)$, independently of the capital stock.

Second, an exogenous increase in the saving rate increases *both* the short-run and the steady-state growth rates. Hence, contrary to the neoclassical predictions, policies directed to increase the saving (and investment) rate affect the long-run growth rate of the economy. The same thing is true for policies that affect the level of technology, A, the rate of population growth, n, or the depreciation rate, $\delta$.

Third, this model predicts no relation between the growth rate of an economy and its level of income. In other words, this model does not predict convergence (conditional or absolute). This explains why the convergence hypothesis has received so much attention in the modern growth literature: it is one of the features that distinguishes the new endogenous growth models from the old neoclassical models and, as a consequence, it is a way to test the validity of the two approaches.

Finally, the AK model predicts that a temporary recession will have permanent effects. That is, if the capital stock temporarily falls for some exogenous reason (an earthquake, a natural tragedy or a war that destroys part of the capital stock), the economy will not grow temporarily faster so as to go back to the prior path of capital accumulation. The AK model predicts that after such a temporary reduction in the capital stock, the growth rate will still be the same so the loss will tend to be permanent.

15

For completeness, Figure 4 depicts the case where $\beta>1$ (increasing returns to the inputs that can be accumulated).[11] The curve $sAk^{-(1-\beta)}$ is upward sloping (and if $\beta>2$ its slope is increasing!). Notice that this implies growth rates that increase over time. The prediction of ever-increasing growth rates does not seem empirically attractive.

### The Harrod-Domar Model

Long before the neoclassical theory came to life in the mid 1950s, the most popular model of economic growth was the Harrod-Domar model (developed by Harrod (1939) and Domar (1946)). We can use the graphical tool developed in the last subsection to learn about this older growth model.

Harrod and Domar tried to combine two of the key features of Keynesian economics -the multiplier and the accelerator- in a model that explained long run economic growth. We have been using the multiplier assumption (savings are a fixed proportion of income) all along so let us describe the distinguishing feature of the Harrod-Domar model: the accelerator. The increase in capital required to produce a given increase in output is assumed to be a constant number. In particular, it is independent of the capital-labor ratio. That is

$$(2.4) \qquad \Delta Y_t = A\Delta K_t,$$

where $A$ is constant. Notice that one production function that satisfies this relationship is the AK function used by the endogenous growth literature. Thus, one could be tempted to identify the the new endogenous growth models with the old Harrod-Domar model. Yet that would be a mistake. The reason is that Harrod and Domar were very concerned about the effects of growth on long-run employment and unemployment[12] (their study could be thought of as an explanation for the then-existing long-run unemployment of

---

[11] In this case the assumption of CRS, $\alpha+\beta=1$, must be dropped since a negative labor share makes little economic sense. Think of this case as one where $\alpha=0$ (so all inputs can be accumulated) and $\beta>1$ (so there are both IRS and increasing returns to capital.)

[12] In fact, Domar's paper is called "Capital Expansion, Rate of Growth, and Employment".

16

the Great Depression). Although they never introduced a specific production function, the fact that they worried so much about employment seems to indicate that they were not thinking of a function such as AK, where there is no role for inputs like labor.

Another production function which satisfies the accelerator principle and which is closer to the spirit of what Harrod and Domar had in mind is Leontief's fixed coefficients function. Output is here assumed to be produced by a fixed proportion of capital and labor. Given this proportion, an increase in the level of one of the inputs without a corresponding increase in the other leaves output unchanged. Thus, we should replace the production function (1.1) by

$$(2.5) \qquad Y_t = \min(AK_t, BL_t),$$

where A and B are exogenous production parameters. After rewriting this function in per capita terms, $y = \min(Ak, B)$, we graph it in Figure 5. We see that there is a capital-labor ratio $\tilde{k} = B/A$ that has the following property: for capital-labor ratios smaller than $\tilde{k}$, Ak is smaller than B so output is determined by Ak. For capital-labor ratios larger than $\tilde{k}$, Ak is larger than B so output is determined by B. In other words, this production function can be expressed as

$$(2.6) \qquad y = \begin{cases} Ak & \text{for all } k < \tilde{k} = B/A \\ B & \text{for all } k > \tilde{k} = B/A \end{cases}$$

Note that this technology is similar to the Ak model, but only for small capital-labor ratios. For large k, the production function is flat so the marginal product of capital is equal to zero. We can now apply our basic growth equation (2.1) to this technology to get

$$(2.7) \qquad \dot{k}/k = \begin{cases} sA - (\delta+n) & \text{for } k < \tilde{k} = B/A \\ sB/k - (\delta+n) & \text{for } k > \tilde{k} = B/A. \end{cases}$$

Harrod and Domar pointed out that there are three possible configurations of parameters, each of which will yield different implications for growth and employment.

17

CASE 1: $sA < \delta+n$

When the savings rate and/or the marginal productivity of capital are very small compared to the aggregate depreciation rate (which includes population growth), there is no possible steady state. This is pictured in Figure 6a. Notice that the economy converges to a point where the logarithm of the capital-labor ratio is negative infinity (so the capital-labor ratio converges to zero). In this case not only will there be unemployment (because $Y=AK<BL$), but also it will grow over time. Harrod and Domar thought that this was a good description of the observed large and growing unemployment rates of the 1930s.

CASE 2: $sA = \delta+n$

When, by chance, the exogenously-given saving rate and marginal product of capital are such that $sA = \delta+n$ (see Figure 6b), the economy will reach a steady state where all per capita variables are constant, $\gamma^*=0$, and $k^*=\tilde{k}$.

CASE 3: $sA>\delta+n$

The third case, depicted in Figure 6c, is one where the marginal product of capital or/and the savings rate are large relative to the aggregate depreciation rate, $\delta+n$. We see in Figure 6c that, for small capital-labor ratios, this case looks very much like the AK model. The growth rate is positive and constant. There is a moment, however, when the capital stock reaches the level $\tilde{k}=B/A$. At this point the marginal product of capital is zero, but because of the constant saving rate assumption, people keep saving (and investing) a constant fraction of their income. We can see in Figure 6c that the growth rate starts falling towards zero. The steady-state capital stock is such that $k^*>\tilde{k}$. This inequality implies that, in the steady state, $AK>BL$, so that there are idle machines. Since the stock of capital per person is constant at $k^*$ and population grows at the rate n, there is a perpetual growth in excess capacity. Again, this is an undesirable outcome.

Two out of the three configurations of parameters yield long-run equilibria where there are idle resources. The only one that does not, can only be achieved by chance because all the relevant parameters (A, s, $\delta$ and n) are given exogenously. Hence, in all probability the economy will be stuck in one of the bad equilibria.

18

In the 1950s, the neoclassical approach led by Solow and Swan was seen as a way of solving this knife-edge property of the Harrod-Domar model. That is, the neoclassical production function achieves the equality between sA and $\delta$+n by allowing for the marginal product of capital to vary in k.[13]

Another way to avoid the knife-edge property of the Harrod-Domar model is to endogenize the saving rate. The old Cambridge School in England, for example, argued that the savings rate was endogenous because workers had a different marginal propensity to save from capitalists. In the process of economic growth, their argument went, the distribution of income would change, and with it, so would the aggregate saving rate.[14]

The saving rate can also be endogenized by allowing agents to make optimal intertemporal decisions. In fact, one could argue that the main reason behind the instability of the Harrod-Domar model were that the household-producers were assumed to keep saving and investing (purchasing machines) a constant fraction of their income, even if there was a large and growing number of idle machines! Optimizing agents would never choose to behave in this manner.[15]

### 'Sobelow' and CES Production Functions

With this graphical approach we can visualize the behavior of the economy for more complicated production functions. Consider Figure 7 for

---

[13] We know that there will be a level of capital k such that the marginal product of capital is equal to $(\delta$+n$)$/s since the marginal product is assumed to range from zero $(f'(\infty)=0)$ to infinity $(f'(0)=\infty)$ in a continuous fashion.

[14] This was one of the main differences between the Cambridge (U.S.) and the Cambridge (U.K.) schools of thought. The other main difference was that the British rejected the neoclassical production function and, in particular, they rejected the notion of an aggregate capital stock. They thought of capital as a number of different machines which, combined with different types of workers, yielded different types of output. Such a heterogeneous set of objects, they argued, is impossible to aggregate into a single variable called aggregate capital stock. See Robinson (1954).

[15] This is another reason against those who argue that the endogenous growth literature is just a new version of the Harrod-Domar model: the endogenous growth literature always uses optimizing models, whereas all the results derived by Harrod and Domar disappear as soon as agents are allowed to choose saving and investment in an optimal manner.

example. The steady state is similar to the one described by the AK model but the transitional dynamics are different. One production function that exhibits such dynamics is the following:

$$(2.8) \qquad Y = AK + BK^{\beta}L^{1-\beta}$$

This production function was first proposed by Kurz (1968) later reintroduced in the endogenous growth literature by Jones and Manuelli (1990). Notice that this function is half way between Solow $(BK^{\beta}L^{1-\beta})$ and Rebelo (AK).[16] It has all the nice concavity properties required by the usual optimization theorems so we can apply straightforward optimization techniques to find solutions.

In per capita terms, the Solow production function is concave, and as k tends to infinity, the marginal product of capital approaches zero. The Rebelo production function in per capita terms is linear with slope equal to A for all values of k. The Sobelow production function is also concave for all capital-labor ratios. As k goes to infinity, however, the slope of the production function does not go to zero but rather to A. Hence, *the only difference between the Solow and the Sobelow functions is that the latter does not satisfy the Inada condition.*

We observe in Figure 7 that sf(k)/k now does not approach zero asymptotically; instead, it approaches A>0. If A is sufficiently large (in this case if sA>δ+n), then the steady state growth rate is positive, even though there is a transition period where growth rates are decreasing monotonically.

It is worth noticing that if the economy has been going on for a while, the decreasing returns part of the production function will be almost irrelevant. Hence, in the long run, this model is essentially AK.

A production function that exhibits similar behavior is the Constant Elasticity of Substitution (CES) production function given by:

$$(2.9) \qquad Y = A \left[ \beta(bK)^{\psi} + (1-\beta)[(1-b)L]^{\psi} \right]^{1/\psi},$$

where A, $\beta$, b, $\psi$ are constant parameters with $0<\beta<1$, $0<b<1$, and $-\infty<\psi<1$. The

---

[16] Hence, the label "Sobelow".

(constant) elasticity of substitution between capital and labor is given by
$\varepsilon=1/(1-\psi)$. As $\psi$ tends to $-\infty$, the production function approaches Leontief's
fixed proportions function $Y=\min[bK,(1-b)L]$ with a zero elasticity, $\varepsilon=0$. As
$\psi$ approaches 0, (2.9) becomes Cobb Douglas with a capital share equal to $\beta$
with a unit elasticity, $\varepsilon=1$. Finally, for $\psi=1$, the production function is
linear $Y = A\left[\beta(bK) + (1-\beta)[(1-b)L]\right]$, with an infinite elasticity, $\varepsilon=+\infty$.

The average product of capital in per capita terms, $f(k)/k$, is given by

$$(2.10) \qquad f(k)/k = A\left[\beta b^{\psi} + (1-\beta)(1-b)^{\psi}k^{-\psi}\right]^{1/\psi}.$$

The reader can check that when $0<\psi<1$ (that is, when the elasticity between
capital and labor is relatively large) this average product approaches
$Ab\beta^{1/\psi}>0$ as k goes to infinity. Hence, the function $sf(k)/k$ in the growth
equation remains bounded above zero, just as in the Sobelow model. The
dynamics of the model closely resemble those depicted in Figure 7. The main
point is that conventional CES production functions can generate perpetually
positive growth rates if the elasticity of substitution is sufficiently
large.[17]

### Poverty Traps

Another possibility is depicted in Figure 8. Here we see the function
$sf(k)/k$ crossing the horizontal line $(\delta+n)$ twice so there are two steady
states. The lower crossing represents a *stable poverty trap*. That is,
countries whose initial capital is very low will tend to this zero-growth,
low-income trap. In fact, all countries whose initial capital lies to the
left of $k_2^*$ will fall into this trap. Countries that start to the right of
$k_2^*$ will enjoy positive growth rates forever.

Consider a country stuck at $k_1^*$. Imagine that the government manages to
generate a small increase in the saving rate so that the $sf(k)/k$ curve

---

[17] Again, the only difference between the CES and the 'normal'
neoclassical production function is that when $0<\psi<1$, the CES does not
satisfy the Inada condition $\lim_{k\to\infty} f'(k)=0$. Instead, the marginal product of
capital approaches $Ab\beta^{1/\psi}$, which is a positive constant. In words, the
marginal product of capital does fall as k increases, but it does not
converge towards zero.

shifts up slightly. Imagine that, after the shift, the sf(k)/k line still crosses the (δ+n) line. The country will experience a period of positive growth rates but it will quickly converge to another zero-growth steady state. However, if the increase in s was so large that the sf(k)/k curve no longer crossed the (δ+n) line the country would escape the poverty trap forever as the growth rate would become permanently positive. Similar predictions would arise with reductions in fertility rates, n, or improvements in A (which, as argued before, include measures of aggregate distortions and government inefficiencies): small changes in policy would have no effect on the long-run growth rate.

For modern versions of models with poverty traps, see Murphy, Shleifer and Vishny (1989), and Azariadis and Drazen (1990). Durlauf and Johnson (1992) propose and use econometric techniques to look for evidence of poverty traps using large cross-sections of countries.

### (3)    NEOCLASSICAL GROWTH:   THE RAMSEY MODEL
#### The Model of Household-Production

Up to now we have assumed that household-producers save a constant fraction of their income, without worrying about whether this behavior is rational or not. In this section we describe the behavior of the same household-producers[18], when they are allowed to choose their consumption path in an optimal manner. The original model is due to Ramsey (1928) and it was later refined by Cass (1965) and Koopmans (1965).

In this setup, agents are assumed to maximize a utility function of the form

$$(3.1) \qquad U(0) = \int_0^\infty e^{-\rho t} u(c_t) L_t dt = \int_0^\infty e^{-\rho t} \left( \frac{c_t^{1-\sigma}-1}{1-\sigma} \right) L_t dt,$$

where $\rho$ is the discount rate, $c_t$ is consumption per capita at time t, and $L_t$ is population. Equation (3.1) says that utility is the sum (or integral) of instantaneous utility functions, $u(c_t)$, between times 0 and infinity.

---

[18]    In the next section we show that the results are identical to those found in a competitive model with separate firms and households.

Instantaneous utility functions are sometimes called *felicity functions*. Each felicity function is discounted at rate $\rho$. Note that the planning horizon is infinite. This may seem to be an unreasonable assumption given that lifetimes are obviously finite. Following Barro (1974), however, we may think that individuals care about their utility and about their children's utility. In this sense, we must think of agents as dynasties or families where the number of individuals belonging to each dynasty is $L_t$. Under this interpretation, the discount rate (which was described by Ramsey (1928) as "*ethically indefensible and arises only from the weakness of the imagination*"[19], (p. 543) at the individual level) represents the fact that individuals care more for their own utility than for that of their children so they discount the future. Since $c_t$ is consumption per capita, $u(c_t)$ is the instantaneous per capita felicity. Hence, the instantaneous felicity for the whole dynasty or family is equal to the individual times the number of people in the family (this explains the term $L_t$ in (3.1)).

The felicity function, $u(c)$, is assumed to take the form $u(c) = \frac{c^{1-\sigma}-1}{1-\sigma}$. The parameter $\sigma$ measures how concave this utility function is. We assume that people's preferences are concave, so $\sigma > 0$, which reflects their desire to enjoy smooth consumption paths over time (that is, people prefer to consume a little bit every day rather than starve to death throughout the month and have a big party at the end of the month). The larger the parameter $\sigma$, the larger the desire to smooth consumption. If $\sigma = 0$, utility is linear so individuals do not particularly like to smooth consumption. As $\sigma$ approaches 1, the utility function becomes logarithmic.[20]

---

[19] Ramsey was considering the optimal choice from a government's point of view. He thought that introducing a discount rate was ethically indefensible because that meant the government was giving a larger weight to current as opposed to future generations.

[20] This can be seen by taking the limit $\lim_{\sigma \to 1} \frac{c^{1-\sigma}-1}{1-\sigma} = \frac{0}{0}$, which is an indeterminate number. We can apply l'Hopital's rule and take derivatives of numerator and denominator with respect to $\sigma$ to get that the limit is equal to $\log(c)$ (recall that the derivative of $c^{1-\sigma}$ with respect to $\sigma$ is $c^{1-\sigma}\log(c)$). The reason for the term '-1' in the felicity function is that,

23

We still consider a one-good closed economy, where total output must be either consumed or invested. Capital depreciates at the constant rate $\delta$ and population grows at rate n. The budget constraint faced by this household-producer is similar to (1.2). If we write total savings as total output per capita, F(K,L), minus total consumption, C, the budget constraint can be written as

$$(3.2) \qquad \dot{K} = F(K,L) - C - \delta K$$

We assume now that the production function is neoclassical in the sense that it satisfies the following three properties:

(i) The production function exhibits constant returns to scale [so F() is homogeneous of degree one: $F(\lambda K, \lambda L) = \lambda F(K,L)$].

(ii) The marginal products of all inputs are positive and diminishing [that is $\partial F/\partial K > 0$, $\partial F/\partial L > 0$, $\partial^2 F/\partial K^2 < 0$ and $\partial^2 F/\partial L^2 < 0$]

(iii) F() satisfies the Inada conditions. The Inada conditions require that the marginal product of capital go to zero as capital tends to infinity and go to infinity as capital tends to zero; a similar condition applies for labor.

Note that the utility function depends on consumption per capita (lower case c) while the budget constraint depends on aggregate consumption. We can divide both sides of (3.2) and use the assumption of CRS to write

$$(3.3) \qquad \dot{K}/L = f(k) - c - \delta k$$

The right-hand-side of (3.3) is in per capita terms but the left-hand-side is not. We can write $\dot{K}/L$ as a function of k as follows

$$(3.4) \qquad \dot{k} = \frac{\dot{K}L - \dot{L}K}{L^2} = \dot{K}/L - nk.$$

Plug (3.4) in (3.3) and rearrange to get

_____

in its absence, the limit as $\sigma$ tends to 1 would be infinity rather than an indeterminate number so l'Hopital's rule could have not been applied.

$$(3.5) \qquad \dot{k} = f(k) - c - (n+\delta)k.$$

Assumption (ii) implies $f'(k)>0$ and $f''(k)<0$ while assumption (iii) requires $\lim_{k \to 0} f'(k) = \infty$ and $\lim_{k \to \infty} f'(k) = 0$. As mentioned in Sections 1 and 2, a simple production function that satisfies the neoclassical properties is the Cobb Douglas function $F = AK^{\beta}L^{1-\beta}$ with $0<\beta<1$. This function can be expressed in per-capita terms as $f(k)=Ak^{\beta}$.

Agents maximize (3.1) subject to (3.5), given the initial stock of capital $k_0>0$. In other words, the neoclassical growth problem can be written as

$$\max \qquad U(0) = \int_0^{\infty} e^{-(\rho-n)t}\left(\frac{c_t^{1-\sigma}-1}{1-\sigma}\right)dt,$$

$$(3.6) \qquad \text{subject to} \qquad \dot{k} = f(k) - c - (\delta+n)k,$$

$$\text{where} \qquad k_0 > 0 \text{ is given}$$

In order to have a bounded or finite utility (so as to have a meaningful economic problem), we must require that the term inside the integral to go to zero as t goes to infinity. This requires

$$(3.7) \quad \lim_{t \to \infty} e^{-(\rho-n)t}\left(\frac{c_t^{1-\sigma}-1}{1-\sigma}\right) = \lim_{t \to \infty} e^{-(\rho-n)t}\left(\frac{c_t^{1-\sigma}}{1-\sigma}\right) - \lim_{t \to \infty} e^{-(\rho-n)t}\left(1/(1-\sigma)\right) = 0.$$

As we will see later, the steady-state level of consumption is constant. It follows that, if the limit in (3.7) has to be zero, it must be the case that

$$(3.8) \qquad \rho > n.$$

To solve the model, we set up the Hamiltonian:[21]

---

[21] See Barro and Sala-i-Martin (1994, mathematical appendix) for a detailed discussion of the mathematical techniques used to solve this type of dynamic problems.

$$(3.9) \qquad H() = e^{-(\rho-n)t}\left[\frac{c_t^{1-\sigma}-1}{1-\sigma}\right] + \nu\left[f(k) - c - nk - \delta k\right],$$

where $\nu$ is the dynamic Lagrange multiplier (or shadow price of investment). The first order conditions are:

$$(3.10) \qquad H_c = 0 \Leftrightarrow e^{-(\rho-n)t}c^{-\sigma} - \nu = 0$$

$$(3.11) \qquad H_k = -\dot{\nu} \quad \rightarrow \quad -\dot{\nu} = \nu(f'(k)-n-\delta)$$

$$(3.12) \qquad \lim_{t\to\infty} (k_t\nu_t) = 0.$$

Equation (3.10) says that the marginal value of consumption must equal the marginal value of investment. Take logarithms of (3.10) to get $-(\rho-n)t-\sigma\log(c_t) = \log(\nu)$. Now take the derivative with respect to time to get $-(\rho-n)-\sigma(\dot{c}/c) = \dot{\nu}/\nu$. We can now plug this in (3.11) to get the traditional condition for consumption growth:

$$(3.13) \qquad \gamma_c \equiv \dot{c}/c = \sigma^{-1}[f'(k)-\rho-\delta].$$

Equation (3.13) is often called the *Euler Equation*. To interpret this Euler equation, it will be convenient to rewrite it as

$$\rho + \sigma(\dot{c}/c) = f'(k) - \delta.$$

The left-hand-side is the return to consumption. The discount rate represents the gain in utility from consuming today since we prefer consumption for ourselves rather than for our children. The return to consumption also includes the term $\sigma(\dot{c}/c)$. If we like a smooth consumption path over time (that is if $\sigma>0$), then whenever we expect consumption to be higher in the future (ie, when $\dot{c}/c>0$), we will want to increase consumption today. In other words, positive consumption growth rates imply non-smooth consumption paths. Hence, we will want to shift part of the future consumption to today and that is we put a premium on $\rho$ to compensate for

26

'early consumption'.[22]    The right-hand-side is the return to saving (and investment), which is equal to the marginal product of capital minus the depreciation rate, $\delta$.  Optimizing individuals should, at the margin, be indifferent between consuming and investing.  This indifference is the one represented by equality (3.13).  Using the Cobb-Douglas technology, $y = Ak^{\beta}$, equation (3.13) can be written as

$$(3.14) \qquad \dot{c}/c = \sigma^{-1}[\beta Ak^{-(1-\beta)} - \rho - \delta]$$

Equation (3.12) is the transversality condition.  One interpretation of this condition is the following:  agents who are optimizing do not want to leave anything of value after they die.  If there were something valuable left at the end of their planning horizons, then they could have consumed it earlier thereby increasing their utility.  It follows that in this case they would not be optimizing in the first place.  Since the agent in this economy is assumed to 'die' at infinity, equation (3.12) says that the value of the capital stock in the last moment of the planning horizon (that is, at infinity) must be zero.  The value of the capital stock, in turn, is equal to the stock of capital, $k_t$, times its shadow price $\nu_t$.[23]

Equations (3.13) (or (3.14) in the case of Cobb Douglas production) and (3.5) along with the initial condition $k_0$, and the transversality condition (3.12) fully determine the dynamics of the economy.  Before studying such

---

[22]    The opposite is also true:  if $\gamma_c$ is negative, then we know that we will have less consumption tomorrow than today.  Since we want to have a smooth consumption path, we will want to shift some of today's consumption to the future.  Hence, the term $\sigma(\dot{c}/c) < 0$ would in this case represent a negative return to consumption today.

[23]    We will discuss in a later sub-section that, in a finite horizon problem, the transversality condition would state $K_T \nu_T = 0$, where T is the last moment of the planner horizon.  The interpretation here would be that if the agents are optimizing, then they would not want to leave any capital when they died, unless its price was zero.  Strictly speaking, this intuition does not carry over to the infinite horizon case because there are counter examples where optimal behavior of infinite horizon economic agents violates the transversality conditions.  See the mathematical appendix in Barro and Sala-i-Martin (1994) for further details on the validity of the transversality condition in infinite horizon problems.

dynamics, we want to show that these dynamic equations also apply to two different setups: a competitive market economy and a planner economy. The planner economy is immediate because a planner would choose to maximize utility subject to the economy-wide budget constraint. Since (3.5) represents the economy-wide constraint, the planner would use the same utility function and the same constraint as our household-producer. The dynamic equations describing the solution would be the same.

### Competitive Market Solution

Consider a setup where households own assets, B, and labor. The two sources of income are labor and asset income. The competitive wage rate is denoted by w so total wage income is wL. The rate of return to assets is the rate of interest, r, so total asset income is rB. Total income is spent on consumption and in the accumulation of assets. Hence, the budget constraint (in per capita terms) is

$$(3.15) \qquad \dot{b} = w + rb - c - nb,$$

where b are assets per person. Agents are assumed to maximize (3.1) subject to (3.15), given the initial stock of assets, $b_0$. Following the same procedure as before, we find that the first order conditions yield a rate of growth for consumption equal to

$$(3.16) \qquad \dot{c}/c = (1/\sigma)[r - \rho],$$

and a transversality condition $\lim_{t \to \infty} \nu_t b_t = 0$. Firms hire labor and capital at competitive rates and sell output. We assume that capital depreciates at rate $\delta$. If R is the rental rate of capital, the net rate of return is given by R–$\delta$. Since there is no risk or uncertainty, the rate of return to assets and capital must be the same so r=R–$\delta$. Firms maximize profits which are given by

$$(3.17) \qquad \pi = F(K,L) - (r+\delta)K - wL.$$

The first order conditions require the equalization of rental rates to

28

marginal products:

$$(3.18) \qquad r + \delta = f'(k)$$

$$(3.19) \qquad w = f(k) - k \, f'(k).[24]$$

Since this is a closed economy with no government, the only asset in non-zero net supply is capital so b=k. Plug (3.18) into (3.16) to get (3.13). Plug (3.18), (3.19), and $\dot{b} = \dot{k}$ into (3.15) to get (3.5). Plug the equality b=k in the transversality condition to get (3.12). Hence, we find that the dynamic equations and terminal conditions describing the behavior of the competitive market economy over time coincide with those derived for the household-producer and planner economy. It follows that the three setups are equivalent.

### Steady-State Analysis

In the steady-state, the growth rates of all per-capita variables are zero.[25] By setting (3.13) to zero we get that the steady capital stock is given by $f'(k^*) = \rho + \delta$. In the case of Cobb Douglas output, $A\beta k^{*-(1-\beta)} = \rho + \delta$, we see

$$(3.20) \qquad k^* = \left[ \beta A / (\delta + \rho) \right]^{1/(1-\beta)}.$$

Plug this into (3.5) to get the steady-state value of consumption:

---

[24] Write Y=Lf(k). Differentiate Y with respect to K, holding constant L, to get (3.18). Differentiate Y with respect to L, holding constant K, to get (3.19).

[25] We can show that the only sustainable growth rate is zero: take the constraint $\dot{k} = k^\beta - c - nk - \delta k$ and divide it by k. Define $\dot{k}/k = \gamma_k$, which in steady state will, by definition, be a constant. Noticing that $k^{(\beta-1)} = (\gamma_c^* \sigma + \rho + \delta)/\beta$, rearrange to get $c/k = (\gamma_c^* \sigma + \rho + \delta)/\beta - \gamma_k^* - n - \delta$, which is a constant. Take logs and time derivatives to conclude that $\dot{c}/c = \dot{k}/k = \gamma_k^* = \gamma^*$. Now consider again the equality, $k^{(\beta-1)} = (\gamma^* \sigma + \rho + \delta)/\beta$. The right-hand-side of this expression is a constant. Take logs and time derivatives of both sides to conclude that $(\beta-1)\gamma_k^* = 0$. Since $\beta < 1$, it follows that $\gamma_k^* = \gamma^* = 0$.

$$(3.21) \qquad c^* = Ak^{*\beta} - (\delta+n)k^*,$$

where $k^*$ is given by (3.20).

### Transitional Dynamics and the Shape of the Stable Arm

The neoclassical model just outlined is NOT a very interesting model of steady-state growth (because steady state growth is zero). It is, however, an interesting model of the transition towards the steady state. This transition is shown in Figure 9. The vertical line is the $\dot{c}=0$ locus. This locus determines the steady-state value of $k^*$. The inverse-u-shaped curve is the $\dot{k}=0$ locus. The intersection of the two loci is the steady state.

These two loci divide the space into four regions. In the region closest to the origin, the dynamics of the system are such that the economy moves in the northeastern direction (as depicted by the arrows in Figure 9). Moving across regions in a counter clock manner, the arrows point south east, south west and north west, respectively (as shown in Figure 9). Since we can get to the steady state from only two of the four regions, the system exhibits saddle-path stability. We will argue later that, in this infinite horizon setup, the economy will always find itself on this unique stable path. The exact functional form for this stable arm cannot be found in general. Its shape, however, depends on the different parameters in interesting and intuitive ways. For example, if the parameter $\sigma$ is large (that is, if agents really want to smooth consumption), then the stable arm for $k<k^*$ will be very close to the $\dot{k}=0$ locus. Agents will try to consume as much as possible early on in order to have a relatively smooth consumption path (note that consuming as much as possible means investing as little as possible; this is why the stable arm is close to the no investment, $\dot{k}=0$, locus). Similarly, when $\sigma$ is very low agents do not mind non-smooth consumption paths. The stable arm in this case will be close to the horizontal axis for $k<k$. Close to the steady state, the stable arm will be very vertical. Hence, agents will choose low consumption and rapid accumulation of capital early on. As the economy reaches the steady state, consumption rises rapidly. Hence, this consumption path is very uneven, but

30

this does not make household unhappy because $\sigma$ is close to zero.

Notice that the economy can converge to the steady state from below or from above depending on $k_0$ being larger or smaller than $k^*$. The interesting case is the one where we converge from below so the economy actually grows. Along this path, per capita capital grows, but it does so at a decreasing rate (which ends up being zero in steady state). As capital increases, the marginal product of capital falls, and therefore, so do the interest and growth rates. Since output is proportional to capital, the qualitative behavior of output mimics the behavior of capital. The initial level of consumption is below the steady-state level, $c^*$, and it grows at positive rates until it converges to $c^*$. Hence, along the transition, the growth rate of consumption, capital, and output per capita are positive.

### Golden Rule, and Dynamic Efficiency

It is worth noticing that, in Figure 9, there is a stock of capital called $k_{gold}$ (for Golden Rule). This is the capital stock that maximizes steady-state consumption. From the budget constraint we see that when $\dot{k}=0$, steady state consumption is equal to $c^*=f(k)-(\delta+n)k$. The capital stock that maximizes $c^*$ is the one that satisfies $f'(k_{gold})=(n+\delta)$. This level of capital divides the set of capital labor ratios in two. Capital stocks above the $k_{gold}$ have the property that in order to achieve higher steady-state consumption, the economy needs to get rid of some capital. In other words, in order to achieve higher consumption in the future, the economy would need to dissave (which of course means higher consumption today). Therefore, if the economy were to find itself with such excess capital, everybody could increase consumption at all points in time. The points above $k_{gold}$ are called the *dynamically inefficient region* because some generations could be made better off without making any generation worse off. Notice that for stocks of capital below the Golden Rule, if the economy wants to increase the steady state consumption, it needs to save: higher consumption tomorrow would have to be traded for lower consumption today. This may be good or bad depending on exact preferences and discount rates. This region is called *dynamically efficient region.*

We can ask whether our economy will ever be in the dynamic inefficient

31

region. To answer this question, we can integrate (3.11) forward between 0 and t and get

$$(3.22) \qquad \nu_t = \nu_0 \, e^{-\int_0^t (f'(k_s) - \delta - n)ds} \qquad ,$$

which, after substituting in (3.12) yields

$$(3.23) \qquad \lim_{t \to \infty} \nu_0 e^{-\int_0^t (f'(k_s) - \delta - n)ds} k_t = 0.$$

Since $\nu_0$ is positive (and equal to $c_0^{-\sigma}$), it must be the case that the second term in (3.23) is equal to zero. This implies that in the steady state, the marginal product of capital must be larger than $\delta+n$. This condition is always satisfied in steady state since we assumed $\rho > n$ (recall that we assumed this inequality in order to get a bounded utility function). To see this, recall that in the steady state, the marginal product of capital is equal to $\rho+\delta$ so that $f'(k^*) > n+\delta$ applies. Since $k_{gold}$ satisfies $f'(k_{gold})$ and since $f''(k) < 0$, it follows that $k^* < k_{gold}$. Hence our economy will never be in the dynamically inefficient region.

### Ruling out explosive paths

We now want to show that, if the transversality condition (3.12) is to be satisfied, *then the economy will find itself on the unique stable arm.* To show this we must rule out all other possible paths. Suppose that we start with the capital stock $k_0$ in Figure 9. Let $c_0$ be the consumption level that corresponds to that on the saddle path. Let us imagine first that the initial consumption level is $c_0' > c_0$. If this is the case, the economy will follow the path depicted in Figure 9: at first both c and k will be growing. At some finite time, the economy will hit the $\dot{k}=0$ schedule, and after that, consumption will keep growing while the stock of capital is falling. The economy will hit the zero capital axis in finite time. At this point, there will be a jump in c (because with zero capital there is zero output, and therefore, zero consumption) which will violate the first order condition (3.14) (see Barro and Sala-i-Martin [1994, chapter

32

2] for a detailed analysis of why the economy hits the vertical axis in finite time.) If the initial level of c were below $c_0$ (as $c_0''$ in Figure 9), then the economy would display increasing c and k for a while. After crossing the $\dot{c}=0$ line, the economy would converge to $k^{**}$. Since $k^{**}>k_{gold}$, this path would not be optimal (as shown in the previous sub-section, points above the golden rule are not optimal). Hence, the only $c_0$ that satisfies the optimality conditions is the $c_0$ on the stable arm.

### The Importance of the Transversality Condition: A Finite Horizon Example

To emphasize the importance of the transversality condition in choosing the initial level of $c_0$, it is useful to compare the previous example with a finite horizon case. That is, let us consider the problem (3.6) with one change: the terminal date is not infinity but rather T<∞:

$$\max \quad U(0) = \int_0^T e^{-(\rho-n)t}\left(\frac{c_t^{1-\sigma}-1}{1-\sigma}\right)dt,$$

(3.24)
$$\text{subject to} \quad \dot{k} = f(k) - c - (\delta+n)k,$$
$$\text{where} \quad k_0 > 0 \text{ is given}$$

Note that the only difference between (3.6) and (3.24) is that the little number on top of the integral is not infinity but T. The first order conditions (3.10) and (3.11) still apply. Therefore, the dynamic equations characterizing the solution (3.5) and (3.14) also apply. The transversality condition, on the other hand, is no longer given by (3.12) but, rather by

$$(3.25) \qquad \nu_T k_T = 0.$$

In words, (3.25) says that the capital stock the agents choose to leave at the moment of 'death', must have no value. Equation (3.10) says that $\nu_T = e^{-(\rho-n)T}c_T^{-\sigma}$, which is a positive number for all finite values of consumption. Hence, the transversality condition (3.25) implies that the stock of capital left at the moment of death must be zero:

$$(3.26) \qquad k_T = 0.$$

33

We can analyze the transitional dynamics implied by this new terminal condition. Note that, because (3.14) and (3.5) still hold, the phase diagram which applies to this case is *the same* as in Figure 9. The question is, given the initial value of $k_0$, what is the optimal choice of consumption at time zero? Is it still the one on the stable arm?

The transversality condition says that at time T, it is optimal for the agent to have zero capital stock. Hence, *the optimal strategy will involve choosing a consumption path such that, at T, the system sits exactly on the vertical axis, where k=0 applies*. Note that this immediately rules out the stable arm: if we follow the dynamics implied by the stable arm, it will be impossible for the economy to be on the vertical axis at time T. The same is true for any initial c below the stable arm. It follows that the initial choice of consumption must be above the stable arm.

In fact, there is a unique value of $c_0$ with the property that, if we follow the dynamics implied by (3.5) and (3.14), the system lies on the vertical axis at exactly time T. In Figure 10, this point is denoted by $c_0$. If we follow the dynamics after $c_0$, then consumption and capital increase over time as the economy gets closer to the steady state. Since we are riding along a path above the stable arm, however, we must eventually hit the $\dot{k}=0$ locus. At that time, capital starts falling while consumption keeps rising. Hence, along this path we have a rising consumption profile and an inverse u-shaped capital profile. Note the similarity between the predictions of this model and the life-cycle model. Here, we get the inverse u-shape for k, however, without assuming that people retire during the last period of their lives.

If the agent had chosen an initial level of consumption which was 'too low' (like point $A_0$ in Figure 10), then the qualitative path for consumption and capital would be very similar. A key difference would be that the new path is closer to the steady state so $\dot{c}$ and $\dot{k}$ are closer to zero. This means that the economy would spend longer time in that position because when $\dot{c}$ and $\dot{k}$ are close to zero, c and k do not move much. Eventually, however, the capital stock would start falling. But, because we spent so much time around the steady state, time T would arrive and the capital stock would not

34

be zero (in Figure 10, at time T the economy finds itself at $A_T$ with a positive capital stock). This, of course, would violate the transversality condition.

Similarly, if the initial choice of consumption were too high (like point $B_0$ in Figure 10), then the economy would reach the vertical axis before time T. As it was the case in the infinite horizon problem, when the system crashes into the vertical axis it must jump to the origin (there is no capital left so there must be no consumption). Such a jump violates the Euler equation (3.14).

Hence, there is a unique optimal choice of consumption, $c_0$. The main lesson is that this initial $c_0$ is different from the one choose in the case of infinite horizon. In other words, by changing the transversality condition we arrive at a different choice of $c_0$.

### The Turnpike Theorem

We can use the analysis of the finite horizon case to describe the so called 'turnpike theorem' of Dorfman, Samuelson and Solow (1958). The turnpike theorem says that if the horizon T is large, then the optimal way to go from the initial stock of capital $k_0$ to the final $k_T=0$ is to get very close to the steady-state for a long period of time, and then diverge towards zero (as seen in Figure 10). We can see that this is true by looking at the optimal path $c_0$ in Figure 10 and asking ourselves: what if the horizon was a little bit larger? In this case, we would choose a $c_0$ closer to the stable arm. The dynamics of the system would move closer to the steady state. But note that being closer to the steady state means being closer to the $\dot{c}=0$ and $\dot{k}=0$ loci. In this region, the economy does not move very fast (that is why these are the $\dot{c}=0$ and $\dot{k}=0$ schedules!) so it remains close to the steady state for a long time before eventually departing towards the $k_T=0$ point. Hence, if T is sufficiently large, the optimal path will involve *being around the steady state for a long period of time*. An example of such a path for capital is given in Figure 11.

### Can a Constant Saving Rate be Optimal?

In Sections 1 and 2 we used a constant saving rate rule without

worrying about whether this rule was optimal.  We can show that there is a set of parameters for which a constant saving rate is optimal.  To do so, we construct a phase diagram in c/y and k (the previous phase diagram in Figures 9 and 10 were in c and k).  Note that $\dot{y}/y = \beta \dot{k}/k$ and $\dot{(c/y)}/(c/y) = \dot{c}/c - \dot{y}/y = \dot{c}/c - \beta \dot{k}/k$.  We can use (3.5) and (3.14) to get

$$(3.27) \qquad [\dot{c/y}]/[c/y] = (1/\sigma)\left[\beta A k^{-(1-\beta)} - \rho - \delta\right] - \beta\left[A k^{-(1-\beta)}(1-c/y) - \delta - n\right].$$

The $\dot{c/y}=0$ schedule is given by

$$(3.28) \qquad c/y = -(1-\sigma)/\sigma + (1/A\beta)k^{1-\beta}\left[(\rho+\delta)/\sigma - \beta(n+\delta)\right].$$

Note that the $\dot{c/y}=0$ schedule is upward sloping if $(\rho+\delta)/\sigma > \beta(n+\delta)$, it is downward sloping if $(\rho+\delta)/\sigma < \beta(n+\delta)$, and it is horizontal if $(\rho+\delta)/\sigma = \beta(n+\delta)$.  The three cases are depicted in Figures 12a, b, and c, respectively.  The $\dot{k}=0$ schedule requires $c/y = 1-(n+\delta)k^{1-\beta}/A$, which is an unambiguously downward sloping curve.  The dynamics depicted in Figure 12 suggest that the system is saddle-path stable in all three parameterizations.  The key difference among the three cases is in the slope of the stable arm.  Note that it is upward-sloping when $(\rho+\delta)/\sigma > \beta(n+\delta)$, it is downward-sloping when $(\rho+\delta)/\sigma < \beta(n+\delta)$, and it is horizontal when $(\rho+\delta)/\sigma = \beta(n+\delta)$.  In this last case, the ratio c/y is constant at the level $(\sigma-1)/\sigma$ along the transition.  Since the saving rate is equal to s=1-c/y, it follows that the saving rate is constant at the value $s^*=1/\sigma$.  In other words, the saving rate is optimally chosen to be constant when the parameters are such that $(\rho+\delta)/\sigma = \beta(n+\delta)$.

When $(\rho+\delta)/\sigma > \beta(n+\delta)$, then c/y monotonically rises along a transition from low capital stocks, and as a result, the saving rate unambiguously falls.  The opposite is true when $(\rho+\delta)/\sigma < \beta(n+\delta)$.

Even though the optimizing Ramsey model is consistent with a constant saving rate, there is an important difference between this case and the exogenously-given saving rate of the Solow-Swan model.  The level of the saving rate in the Ramsey model is dictated by the parameters of the model and cannot be chosen arbitrarily.  In particular, it cannot be chosen to be

in the dynamically inefficient region. The same is not true in the Solow-Swan model where saving rates can be chosen to be arbitrarily large so that they can generate steady-state capital stocks above the golden rule.

## Convergence and Convergence Regressions

Like the constant saving-rate neoclassical model described in Sections 1 and 2, the Ramsey model with optimal consumption predicts that, if all countries share the same production and utility parameters, then poor countries tend to grow at a faster rate than rich ones. In other words, income or output levels will converge over time. Following Sala-i-Martin (1990), we can show this important implication by log-linearizing the two key differential equations (3.14) and (3.5) around the steady state. In the appendix we show that, by doing so, we can express the growth rate of output per capita as a negative function of the initial level of output per capita:

$$(3.29) \qquad [\ln(y_t)-\ln(y_0)]/t = [(1-e^{-\lambda_1 t})/t] \, [\ln(y^*)-\ln(y_0)]\ln(y_0),$$

where $-\lambda_1=(1/2)\left[\rho-n-\left((\rho-n)^2+4\mu(\rho+\delta(1-\beta)-\beta n)/\beta\right)^{(1/2)}\right]$, and $\mu\equiv(1-\beta)(\rho+\delta)/\sigma>0$. Equation (3.29) says that if a set of economies have the same deep parameters (discount rate, coefficient of intertemporal elasticity of substitution, capital share, depreciation and population growth rates, etc) so that they converge to the same steady state, then the cross section regression of growth on the log of initial income should display a negative coefficient. In other words, poor countries should tend to grow faster. The reason for this is that countries with low initial capital stocks would have high initial marginal product of capital. That would lead them to save, invest, and therefore, grow fast.

If countries converge to different steady states, however, there should be no relation between growth and initial income, unless we hold constant the determinants of the steady state. Sala-i-Martin (1990) and Barro and Sala-i-Martin (1992) use a slightly more complicated[26] version of (3.29) to show that the states of the U.S. (which we may think of as described by

---

[26] It is a slightly more complicated version because they include exogenous productivity growth.

similar production and utility parameters) converge to each other exactly the way equation (3.29) predicts. They also show that, once they hold constant the determinants of the steady state, a large sample of countries also converges to each other the way equation (3.29) predicts.

## (4) EXOGENOUS PRODUCTIVITY GROWTH
### Classification of Technological Innovations

We just mentioned that the simple neoclassical model predicts that the long-run rate of growth is zero. In order to explain observed long-run growth neoclassical economists amended the model and incorporated exogenous productivity growth. In Section 1 we saw that, in the context of a fixed saving rate, the introduction of productivity growth led to long-run economic growth. The question is *what kind* of technological progress should we introduce in our models. Some inventions *save* capital relative to labor (capital saving technological progress), some *save* labor relative to capital (labor saving technological progress), and some do not save either input relative to the other (neutral or unbiased technological progress).

Notice that the definition of neutral innovations depends on what we mean by "saving". The two most popular definitions of unbiased or neutral technological progress are due to Hicks and Harrod respectively.

Hicks argues that a technological innovation is neutral (Hicks-neutral) with respect to capital and labor if and only if the ratio of marginal products remains unchanged for a given capital-labor ratio. Consequently, a technological innovation is labor (capital) saving if the marginal product of capital (labor) increases by more than the marginal product of labor (capital) at a given capital-labor ratio. Notice that Hicks neutrality amounts to renumbering the isoquants. Production functions with Hicks-neutral technological progress can be written as:

$$(4.1) \qquad Y_t = A(t)F(K_t, L_t),$$

where $A(t)$ is an index of the state of technology at moment t, evolving according to $A_t = A_0 e^{gt}$ (that is $\dot{A}/A = g$), and where $F()$ is still homogeneous of degree one.

The second definition of technological unbiasedness is due to Harrod.

He says that a technical innovation is neutral (Harrod-neutral) if the relative shares ($KF_k/LF_L$) remain unchanged for a given capital-*output* ratio. Robinson (1938) and Uzawa (1961) showed that this implies a production function of the form

$$(4.2) \qquad Y_t = F(K_t, A(t)L_t),$$

where, again, $A(t)$ is an index of technology at time t, $\dot{A}/A=g$ and $F()$ is homogeneous of degree one. Notice that this production function says that, with the same amount of capital, we need less and less labor to produce the same amount of output. Therefore, this function is also known as labor-augmenting technological progress. By symmetry we could have thought of technological change as being "capital augmenting", ie $Y=F(B_tK_t,L_t)$. This would mean that, for a given number of hours of work ($L_t$), we need decreasing amounts of capital to achieve the same isoquant.

The reason why we care about what kind of technological progress we should postulate is that, as Phelps (1962, 1966) showed, *a necessary and sufficient condition for the existence of a steady state in an economy with exogenous technological progress is for this technological progress to be Harrod-neutral or labor-augmenting* (see Barro and Sala-i-Martin (1994, Chapter 2) for a detailed and easy derivation of this result).

Note, however, that when we work with Cobb Douglas utility functions the two types of technological progress are identical since

$$Y(K,AL) = K^\beta(AL)^{1-\beta} = K^\beta e^{g(1-\beta)t}L^{1-\beta} = e^{g(1-\beta)t}\left(K^\beta L^{1-\beta}\right) = BY(K,L)$$

### The Irrelevance of Embodiment

All types of technological change we have discussed up to now are disembodied in the sense that, when a technological innovation occurs, all existing machines become more productive. An example of this would be improvements in computer software: it makes all existing computers better. There are many inventions, however, that are not of this type. When one invention occurs, only the NEW machines are more productive (as is the case with computer hardware). Economists call this *embodied technological progress*.

In the 1960s, when the neoclassical model of exogenous productivity growth was being developed, there was a debate on the importance of embodiment in economic growth. Proponents of what at the time was called "New Investment Theory" (embodied technological progress) said that investment in new machines had the usual effect of increasing the capital stock and the additional effect of modernizing the average capital stock. Proponents of the "unimportance of the embodiment question" argued that this new effect was a level effect but that it did not affect the steady state rate of growth. In a couple of important papers Solow (1969) and Phelps (1962) showed the following:

(i) The neoclassical model with embodied technological progress and perfect competition (so the marginal product of labor is equal for all workers no matter what the vintage of the machine they are using is) can be rewritten in a way that is equivalent to the neoclassical model with disembodied progress (Solow [1969]).

(ii) The steady-state rate of growth is independent of the fraction of progress that is embodied although it is dependent on the total rate of technical progress (Phelps [1962]).

(iii) The speed of convergence is larger the larger is the fraction of embodied progress, (Phelps [1962]).

Thus, the distinction between embodied and disembodied progress seems unimportant when studying long-run issues but might be crucial when studying short-run dynamics.[27] The modeling of embodied technological progress is quite complicated because one has to keep track of all old vintages of capital and associated labor. Yet a simple way to think about it is to postulate a technology-free production function $Y=F(K,L)$, and an accumulation function of the form $\dot{K}=A(t)(Y_t-C_t)$, where $\dot{A(t)}/A(t)=g$ and $K(t)$ is a measure of aggregate capital. This function reflects the fact that a unit of saving $(Y-C)$ in a later period generates a larger increase in

---

[27] The importance of embodiment in modeling business cycles can be seen from the fact that an embodied shock affects the marginal product of capital but does NOT affect the marginal product of labor or current output supply. This is a key difference with respect to a disembodied shock, especially in as far as the implications for the procyclicality of real wages and real interest rates is concerned.

capital than a unit of saving in an earlier period. This is like saying that later vintages of capital are more productive.

### The Neoclassical Model with Technological Progress

Let us go back to the labor augmenting form as depicted in equation (4.2). Note that output depends on K and on the factor $A(t)L$. This factor is sometimes called the effective labor input. Agents maximize utility (3.1) subject to a constraint like (3.5), except that output is now given by (4.2). In order to solve the model, we will express it in units of effective labor, much in the same way we solved the Ramsey model in Section 3 by expressing all equations in per capita terms. Let variables with 'hats' be in effective units (so $\hat{c}=c/(LA)$, $\hat{k}=K/(LA)$, and $\hat{y}=Y/(LA)$). We can write the utility function as

$$(4.3) \qquad U(0) = \int_0^\infty e^{-(\rho-n)t}\left(\frac{(\hat{c}_t e^{gt})^{1-\sigma}-1}{1-\sigma}\right)L_0 dt,$$

and the budget constraint as

$$(4.4) \qquad \dot{\hat{k}} = f(\hat{k}) - (n+g+\delta)\hat{k} - \hat{c}.$$

Agents maximize (4.3) subject to (4.4), given $k_0>0$. The model is exactly the same as the Ramsey model of the previous section with two small differences. First, the effective discount rate in the utility function is $\rho-n-(1-\sigma)g$ rather than $\rho-n$. Second, the effective depreciation rate is $\delta+n+g$ rather than $\delta+n$. Leaving these two trivial differences aside, the model exactly parallels that of the previous section so we will not reproduce its solution here. Let us just mention that the bounded utility condition is now

$$(4.5) \qquad \rho > n + (1-\sigma)g,$$

while the differential equation describing consumption growth is now

$$(4.6) \qquad \dot{\hat{c}}/\hat{c} = (1/\sigma)\left(f'(\hat{k}) - \delta - \rho - \sigma g\right).$$

41

The transversality condition requires $\lim_{k \to \infty} \hat{k}_t \nu_t = 0$, where $\nu$ is the shadow price of capital. Equations (4.4) and (4.6) determine the dynamics of $\hat{c}$ and $\hat{k}$. Their behavior parallels that of $c$ and $k$ in Figure 9. The steady-state growth rate for all per capita variables is g.

---------------------------------------------------------------------------

### APPENDIX: Derivation of the Convergence Regression (3.29)

If we express all variables in logarithms, the two key differential equations (3.5) and (3.14) can be written as

(A.1)

$$\ln(\dot{c}_t) = (1/\sigma)(\beta e^{-(1-\beta)\ln(k_t)} - (\rho+\delta))$$

$$\ln(\dot{k}_t) = e^{-(1-\beta)\ln(k_t)} - e^{(\ln(c_t)-\ln(k_t))} - (n+\delta).$$

In steady state the two equations are equal to zero so:

(A.2)

$$e^{-(1-\beta)\ln(k^*)} = (\rho+\delta)/\beta$$

$$e^{(\ln(c^*)-\ln(k^*))} = e^{-(1-\beta)\ln(k^*)} - (n+\delta) = h > 0,$$

where $h=(\rho+\delta(1-\beta)-\beta n)/\beta$. We can now Taylor-expand the system (A.1) around (A.2) and get

(A.3)

$$\ln(\dot{c}_t) = -\mu[\ln(k_t)-\ln(k^*)]$$

$$\ln(\dot{k}_t) = -h[\ln(c_t)-\ln(c^*)] + (\rho-n)[\ln(k_t)-\ln(k^*)],$$

where $\mu \equiv (1-\beta)(\rho+\delta)/\sigma > 0$. Alternatively, we have:

$$(A.4) \quad \begin{bmatrix} \ln(\dot{c}_t) \\ \ln(\dot{k}_t) \end{bmatrix} = \begin{bmatrix} 0 & -\mu \\ -h(\rho-n) \end{bmatrix} \begin{bmatrix} \ln(c_t)-\ln(c^*) \\ \ln(k_t)-\ln(k^*) \end{bmatrix}.$$

Notice that the determinant of the matrix is detA=$-h\mu<0$, which implies that

42

the system is saddle-path stable.  The eigenvalues of the system are

$$-\lambda_1 = (1/2)(\rho-n - \left[(\rho-n)^2+4\mu h\right]^{(1/2)}) < 0$$

(A.5)

$$\lambda_2 = (1/2)(\rho-n + \left[(\rho-n)^2+4\mu h\right]^{(1/2)}) > 0.$$

The solution for $\ln(k_t)$ has the usual form

$$(A.6) \qquad \ln(k_t) - \ln(k^*) = \psi_1 e^{-\lambda_1 t} + \psi_2 e^{\lambda_2 t},$$

where $\psi_1$ and $\psi_2$ are two arbitrary constants.  To determine them, we notice that since $\lambda_2$ is positive, the capital stock will violate the transversality condition unless $\psi_2=0$.  The initial conditions help us determine the other constant since at time 0 the solution implies

$$(A.7) \qquad \ln(k_0) - \ln(k^*) = \psi_1 e^{0}$$

Hence, the final solution for the log of the capital stock has the form

$$(A.8) \qquad \ln(k_t) - \ln(k^*) = [\ln(k_0) - \ln(k^*)]e^{-\lambda_1 t}.$$

If we realize that $\ln(k_t)=\ln(y_t)/\alpha$, and we subtract $\ln(y_0)$ from both sides of equation (A.8) we get equation (3.26) in the text:

$$(3.24) \qquad [\ln(y_t)-\ln(y_0)]/t = [(1-e^{-\lambda_1 t})/t] [\ln(y^*)-\ln(y_0)]\ln(y_0).$$

43

# References

Arrow, K. J. (1962), "The Economic Implications of Learning by Doing", *Review of Economic Studies,* 24, June, 155-173.

Azariadis, C. and A. Drazen, (1990), "Threshold Externalities in Economic Development", *Quarterly Journal of Economics,* 90, 501-526.

Barro, R. (1974), "Are Government Bonds Net Wealth?", *Journal of Political Economy* 82: 1095-1117, November/December.

Barro, R. J. (1990), "Government Spending in a Simple Model of Endogenous Growth", *Journal of Political Economy,* October, 98, S103-125.

Barro, R. J., N. G. Mankiw and X. Sala-i-Martin, (1992) "Capital Mobility in Neoclassical Models of Economic Growth", *NBER working paper* 4206, November.

Barro, R.J and X. Sala-i-Martin, (1991) "Convergence Across States and Regions", *Brookings Papers of Economic Activity,* I.

Barro, R.J and X. Sala-i-Martin, (1992) "Convergence", *Journal of Political Economy,* April, 100, 223-251.

Barro, R.J and X. Sala-i-Martin, (1994), *"Economic Growth",* McGraw Hill.

Cass, D. (1965), "Optimum Growth in an Aggregative Model of Capital Accumulation", R*eview of Economic Studies,* 32, 233-240.

Domar, E. D. (1946), "Capital Expansion, Rate of Growth, and Employment", *Econometrica,* April.

Dorfman, R., P. Samuelson and R. Solow, (1958), *"Linear Programming and Economic Analysis",* McGraw Hill, New York.

Durlauf, S. and P. Johnson (1992), "Local versus Global Convergence across National Economies", NBER working paper 3996, February.

Grossman, G. and Helpman, E. (1991), *"Innovation and Growth in the Global Economy",* MIT Press, Cambridge, MA.

Harrod, R. F. (1937), "Review of Joan Robinson's *Essays in the Theory of Employment",* The Economic Journal vol 47.

Harrod, R.F. (1939), "An Essay in Dynamic Theory", *The Economic Journal,* March.

Hicks, J. R., "The Theory of Wages", Macmillan, London.

Jones, L and Manuelli, R. (1990), "A model of optimal equilibrium growth", *Journal of Political Economy,* 98, 1008-37.

Koopmans, T. C. (1965), "On the Concept of Optimal Growth", in *"The Econometric Approach to Development Planning",* North Holland, Amsterdam.

Kurz, M., (1968), "The General Instability of a Class of Competitive Growth Processes", *Review of Economic Studies,* march, 155-165.

Lucas, R. E. (1988), "On the mechanics of economic development", *Journal of Monetary Economics,* 22, 3-42.

Mankiw, N.G., D. Romer, D., and D. N. Weil, (1992), "A Contribution to the Empirics of Economic Growth", *Quarterly Journal of Economics,* 107, 407-38.

Murphy, K., A. Shleifer and R. Vishny, (1989), "Industrialization and the Big Push", *Journal of Political Economy,* 97, 1007-26.

Phelps, E., (1961) "The Golden Rule of Capital Accumulation: A Fable for Growthmen", *American Economic Review.*

Phelps, E. S., (1962) "The New View of Investment: A Neoclassical Analysis", *Quarterly Journal of Economics,* November.

Phelps, E. (1966), "Golden Rules of Economic Growth", *Norton*, New York, 1966.

Ramsey, F. P. (1928), "A Mathematical Theory of Saving", *Economic Journal*, 38, p. 543-59.

Rebelo, S. (1991) "Long Run Policy Analysis and Long Run Growth", *Journal of Political Economy,* 99, 500-21.

Robinson, J. (1938), "The Classification of Inventions", *Review of Economic Studies,* February.

Robinson, J. (1954), "The Production Function and the Theory of Savings", *Review of Economic Studies*.

Romer, P., (1986), "Increasing Returns and Long Run Growth", *Journal of Political Economy,* 94, 1002-1037.

Romer, P. (1987), "Growth Based on Increasing Returns Due to Specialization", *American Economic Review,* May.

Romer, P. (1990), "Endogenous Technological Change", *Journal of Political Economy,* October, 98, S71-102.

Sala-i-Martin, X. (1990), "On Growth and States", Unpublished PhD Dissertation, Harvard University, May.

Sheshinski, E. (1967), "Optimal Accumulation with Learning by Doing", in Karl Shell ed. "Essays on the Theory of Optimal Economic Growth", M.I.T. Press.

Solow, R. (1956), "A Contribution to the Theory of Economic Growth", The *Quarterly Journal of Economic,* 70, February, 65-94.

Solow, R. M., (1969), "Investment and Technical Progress", in Arrow et al., eds, *"Mathematical Methods in the Social Sciences",* Stanford University Press.

Swan, T. W. (1956), "Economic growth and capital accumulation", *Economic Record,* 32, 334-61.

Uzawa, H. (1961), "Neutral Inventions and the Stability of Growth Equilibrium", *Review of Economic Studies,* February.

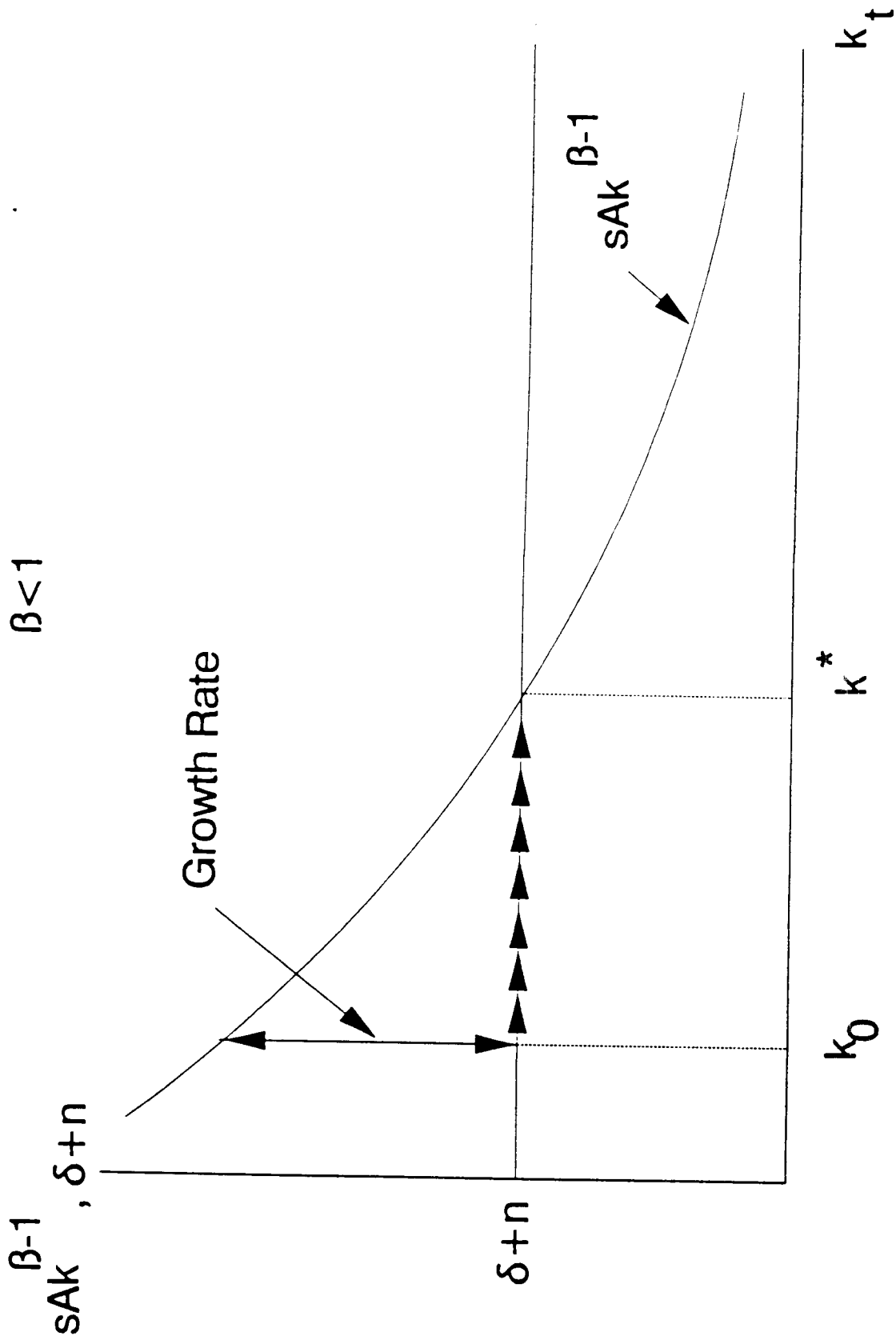Figure 1: The Neoclassical Model

$\beta < 1$

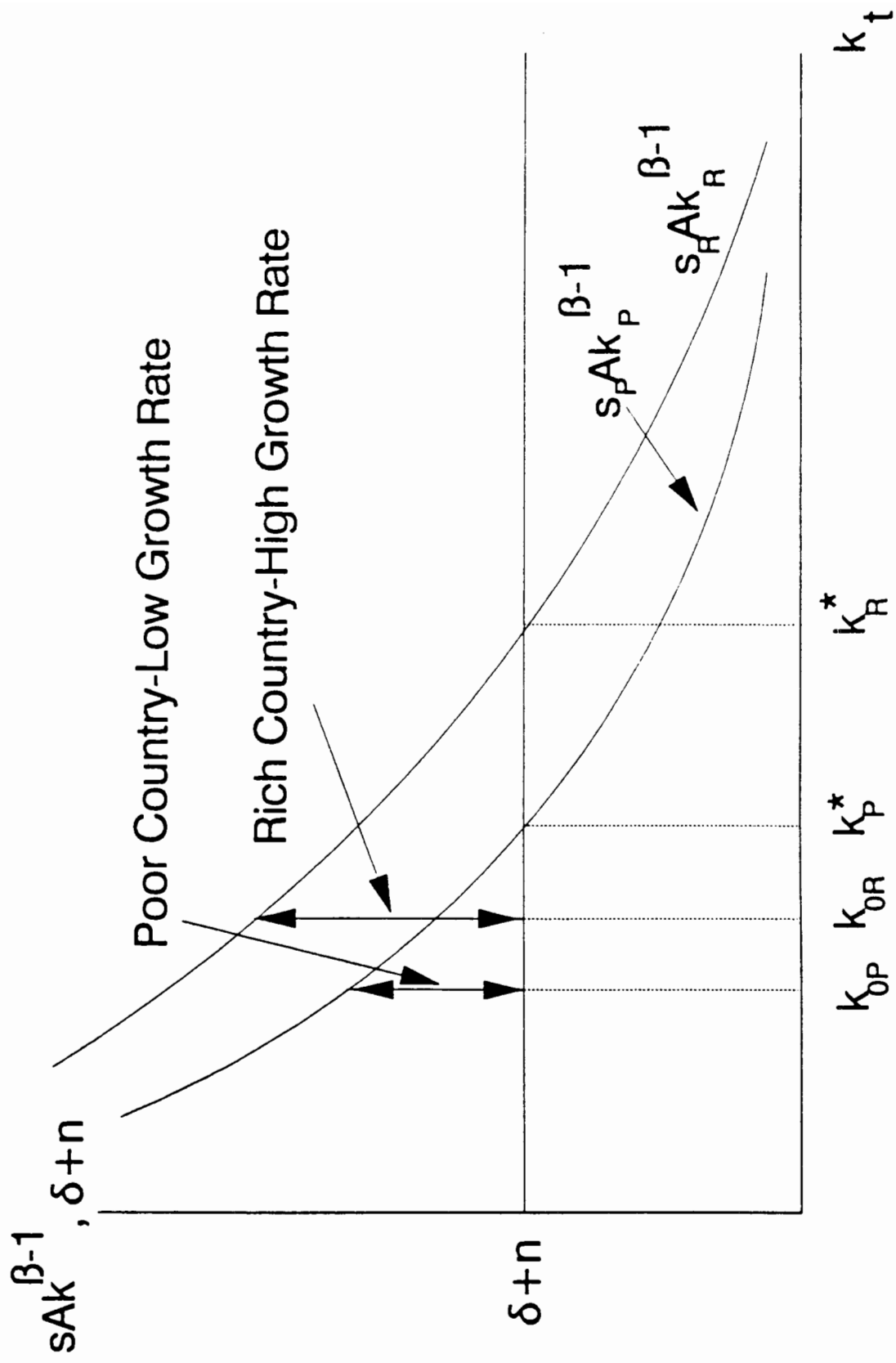Figure 2: Conditional Convergence in the Neoclassical Model
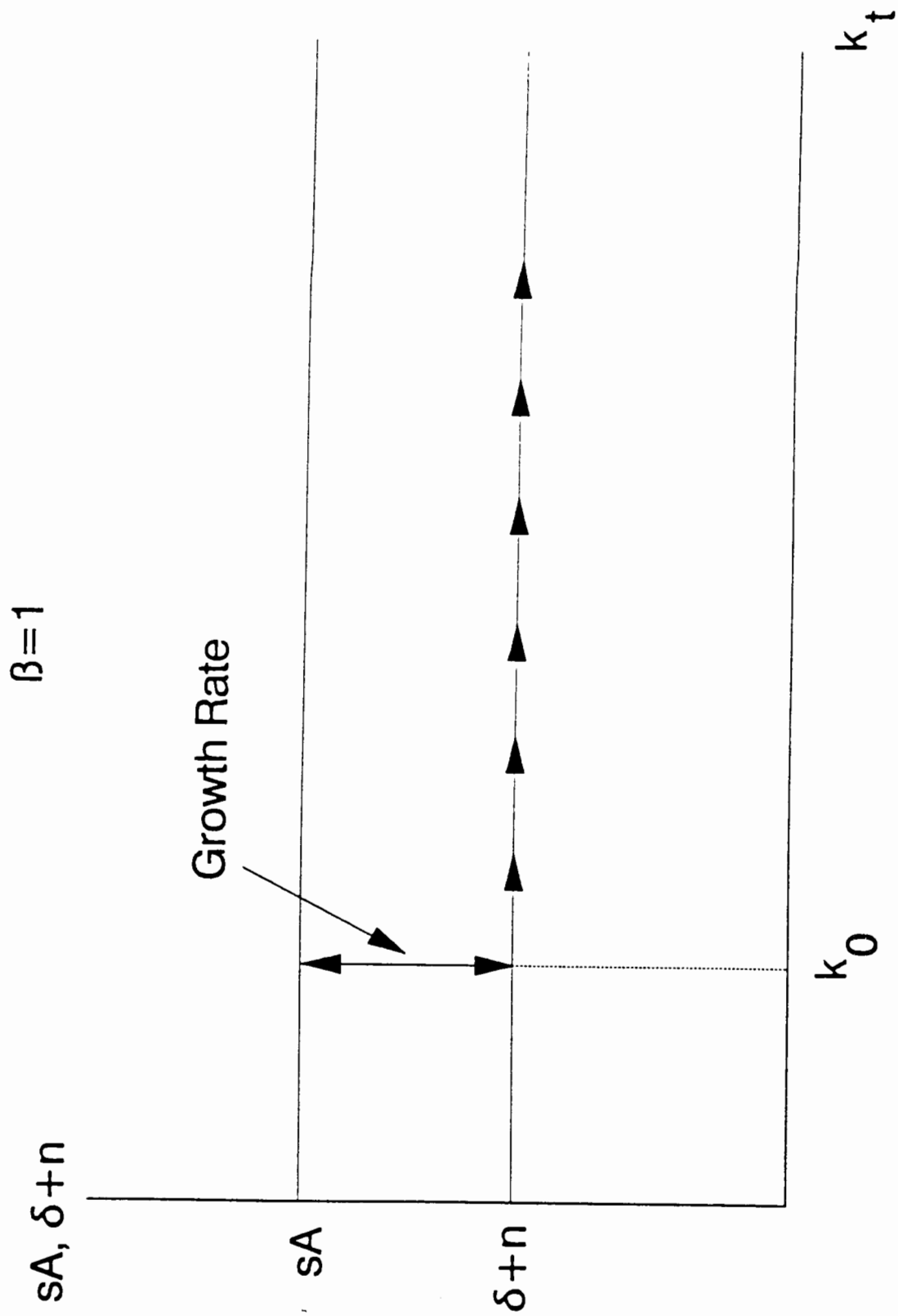
Figure 3: The Rebelo-Ak model

$\beta = 1$
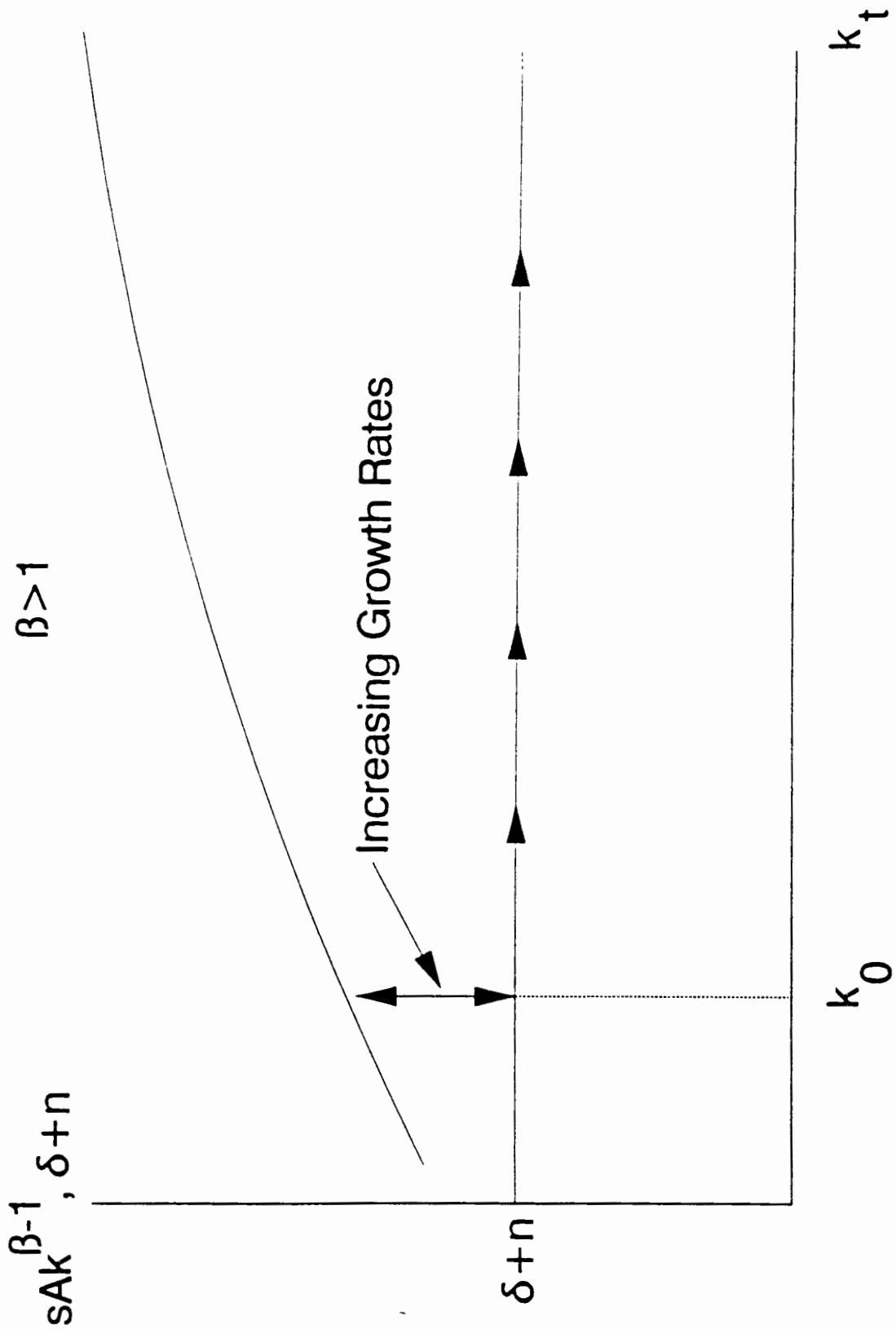
Figure 4: Increasing Returns and Increasing Growth Rates

$\beta > 1$

$sAk^{\beta-1}, \delta+n$

Increasing Growth Rates

$\delta+n$

$k_0$

$k_t$

# Figure 5: The Harrod-Domar Production Function

f(K/L)

B

$f(k) = B$ →

← $f(k) = Ak$

0

0          k=B/A          K/L
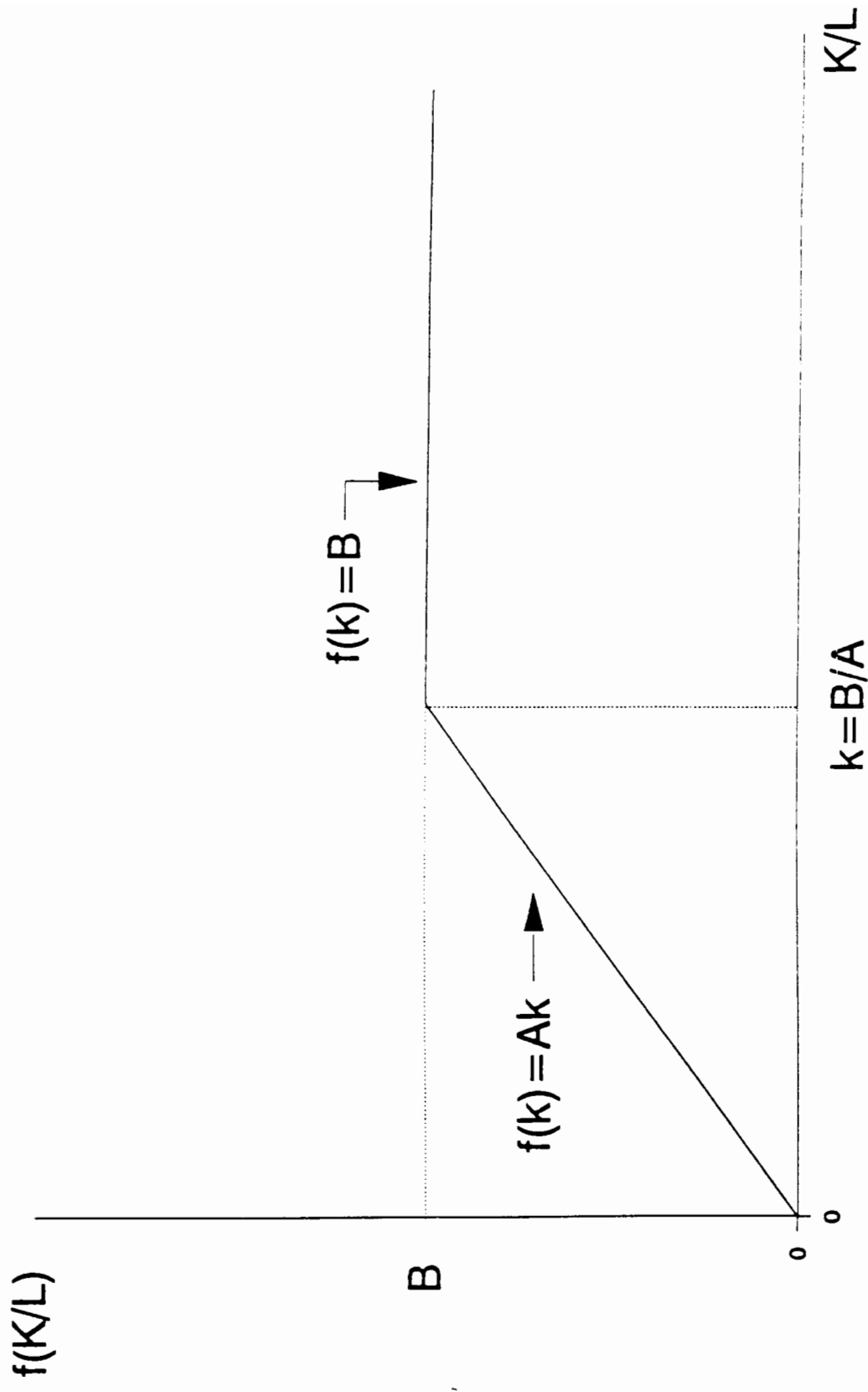
# Figure 6: The Harrod-Domar Model

## (A) Case 1: $sA < (\delta + n)$



## (B) Case 2: $sA = (\delta + n)$



## (C) Case 3: $sA > (\delta + n)$
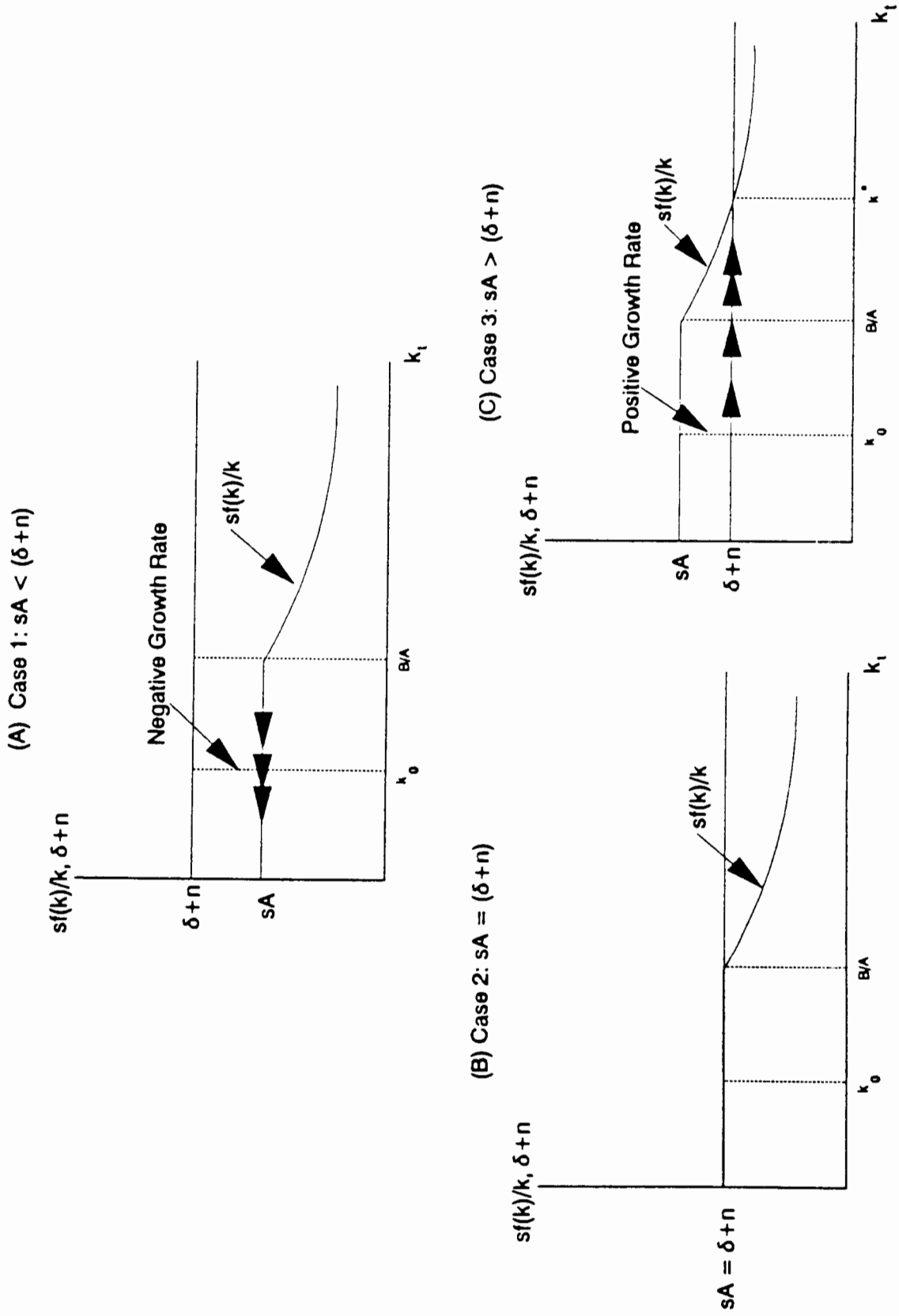
Figure 7: The "Sobelow" and CES Models

sf(k)/k, $\delta$+n

Current Growth Rate

$s(A+Bk^{\beta-1})$

Assymptotic Growth Rate

sA

$\delta$+n

$k_0$

$k_t$
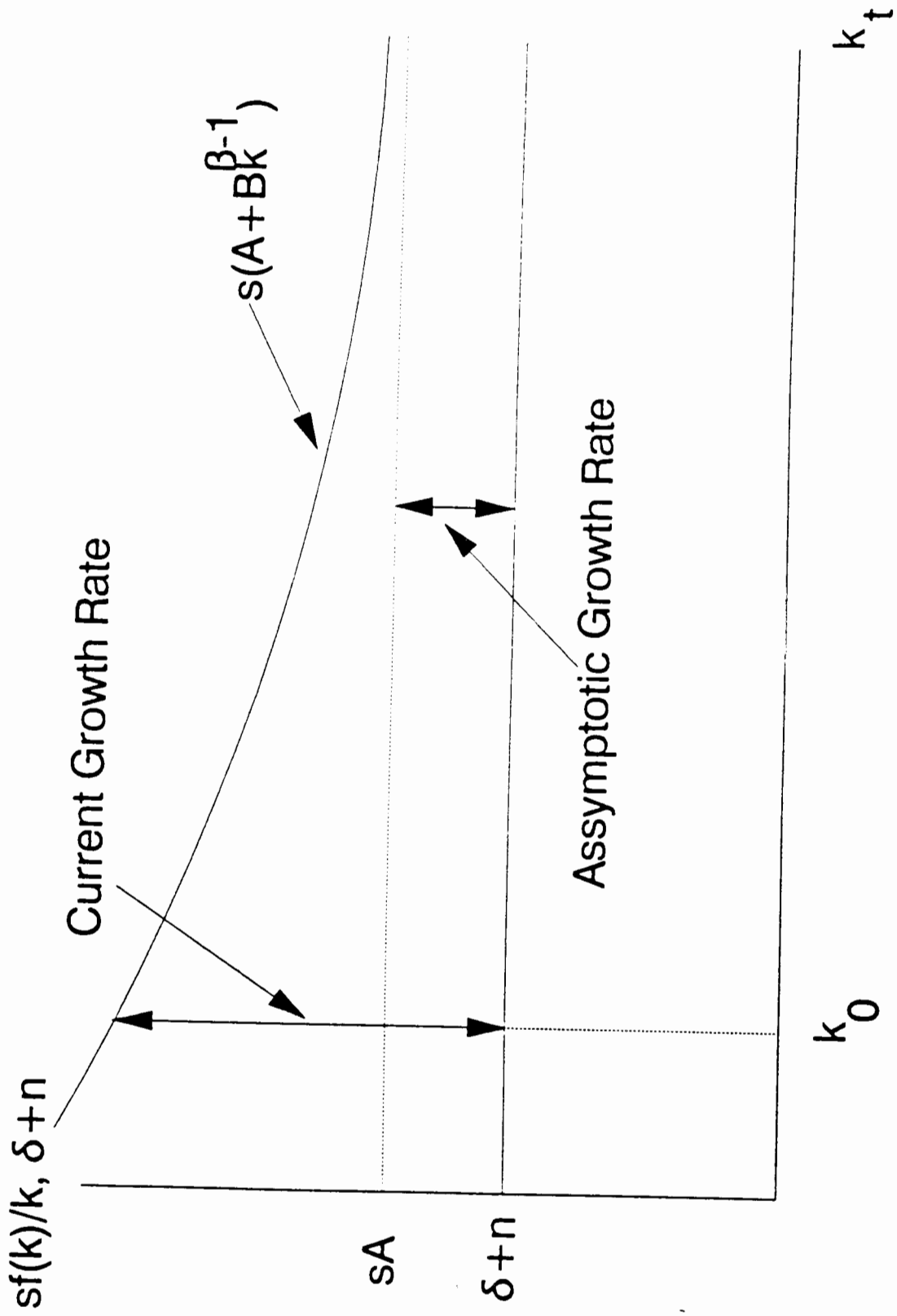
Figure 8: Stable Poverty Trap
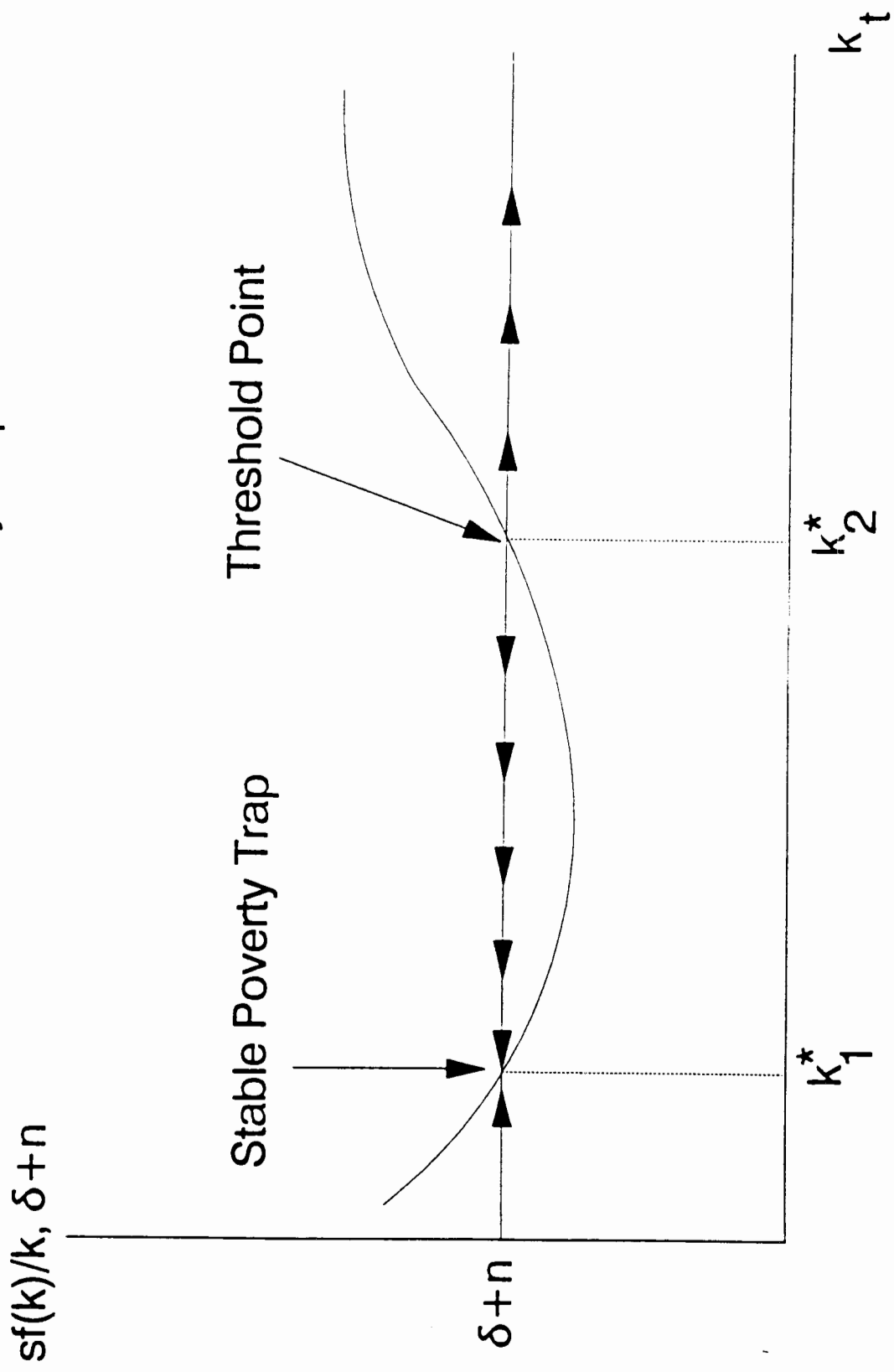
Figure 9: The Phase Diagram of the Ramsey Model
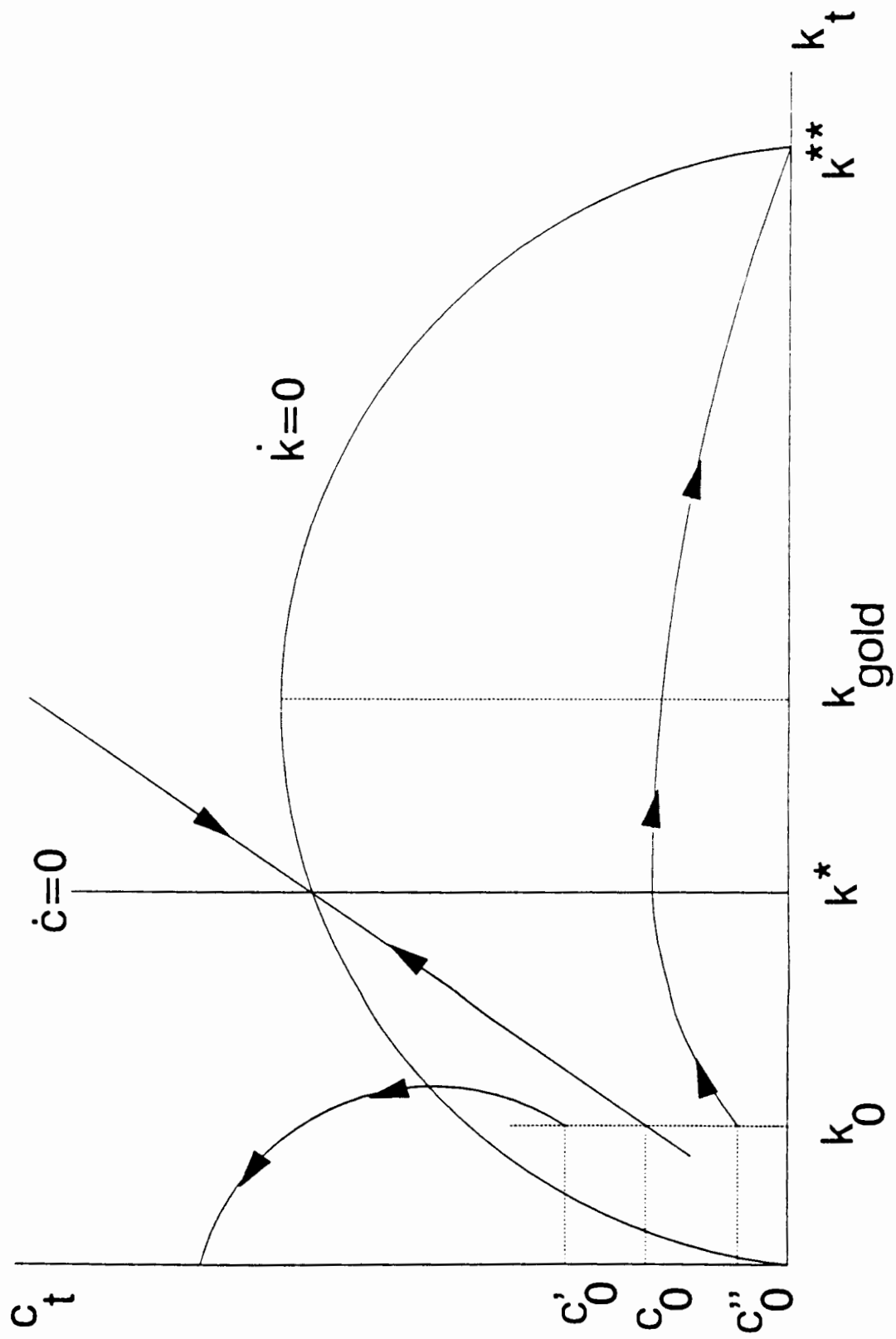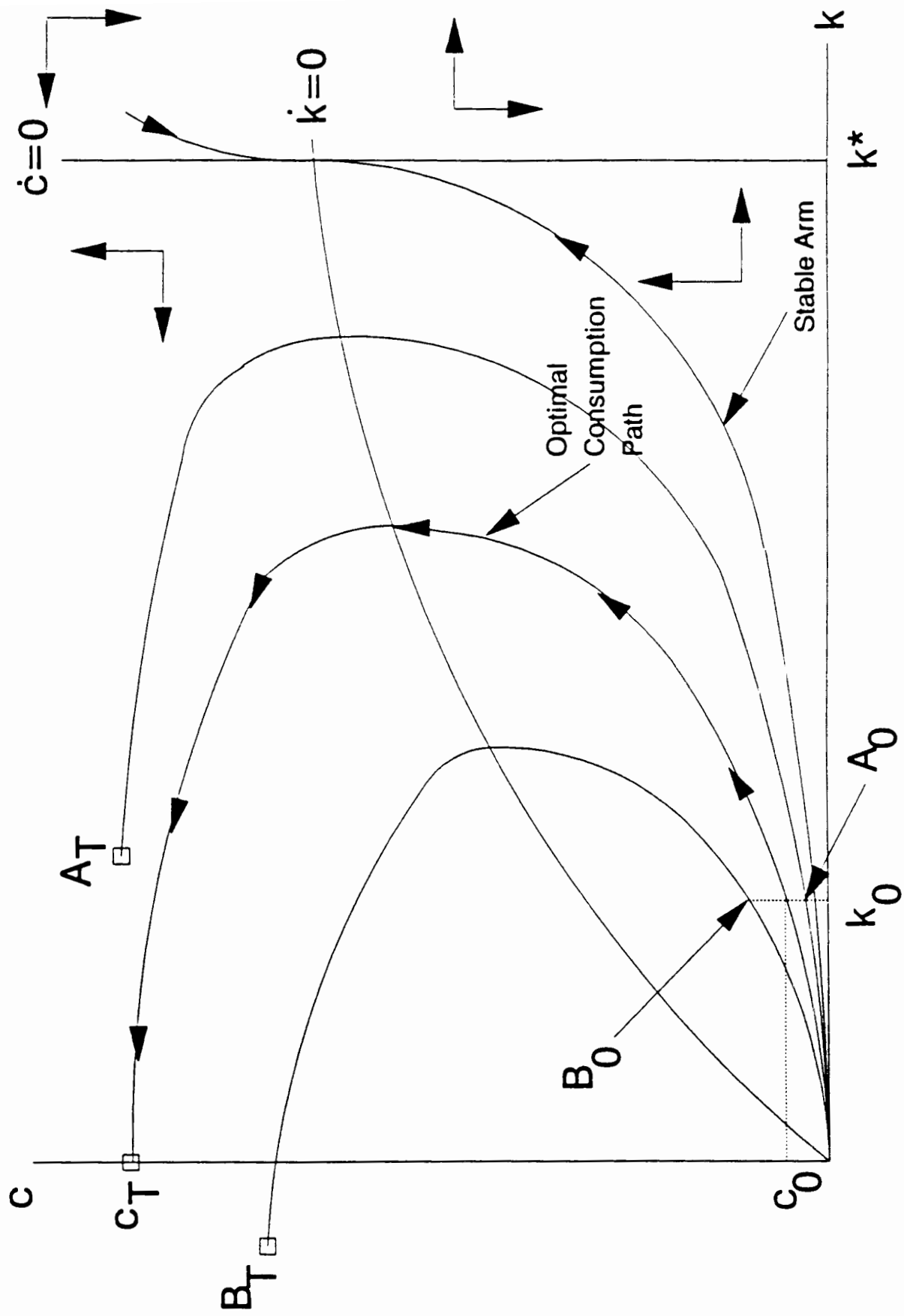
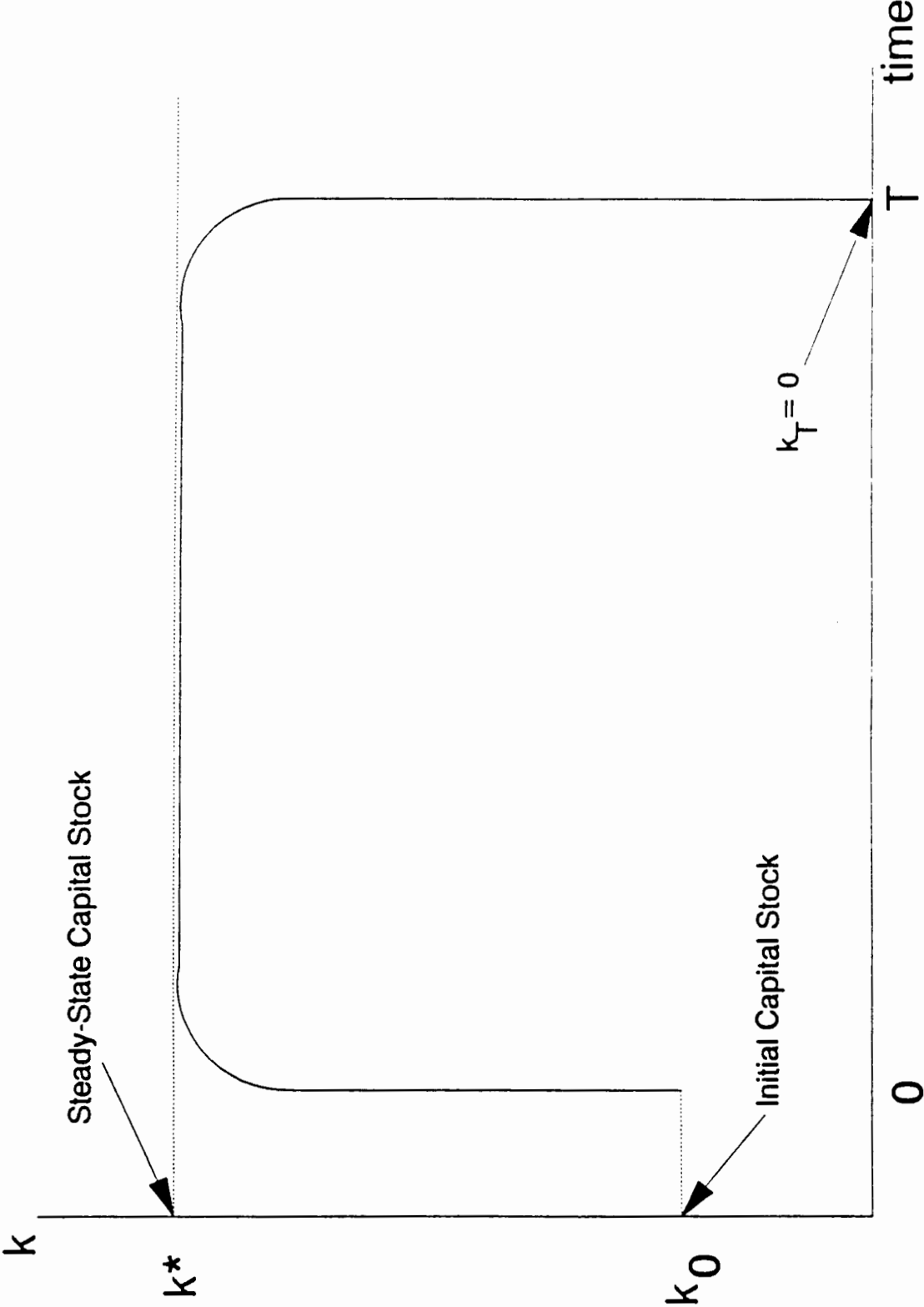Figure 10: The Ramsey Model with Finite Horizons

Figure 11: The Turnpike Theorem

Figure 12: Behavior of the Saving Rate along the Transition