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A Unique Informationally Efficient and Decentralized Mechanism with Fair Outcomes

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1 Introduction

Following the pioneering work of Hurwicz [15] and Mount and Reiter [29], there has been a significant amount of work on trying to determine the informational requirements of decentralized resource allocation mechanisms. By mechanism here we mean a system which communicates knowledge which is dispersed among agents and uses it to determine the allocation of resources. Agents send messages and these are translated into outcomes. In particular, the focus in the literature has been on the dimension of the space of messages used for communication between agents. These informational requirements depend upon two basic elements: the class of environments over which the mechanism is supposed to operate and the particular outcomes that the mechanism is required to achieve.

Most of the literature has centered on the informational requirements for obtaining Pareto optimal allocations in different environments. In this paper we discuss the information needed to obtain a more restricted class of outcomes: in the context of pure exchange classical environments, those which are fair, i.e., both efficient and envy free. The interest in such allocations is a longstanding one, and although the definition we use is that of Foley [12], the underlying notion goes back to the ancient Egyptians and has been formally investigated in another context by Dubins and Spanier [10][1]. Intuitively, it might seem that a great deal of information would be required to obtain such outcomes since establishing whether an allocation is fair would seem to involve each agent comparing his allocation with that of everyone else and this would require knowing the initial endowments of all agents. However, we will show that in fact very little more information is needed to obtain fairness in addition to efficiency and, in particular, we will prove rigorously the idea suggested by Thomson and Varian [37] that we can actually specify a mechanism which is the only one involving minimal informational requirements.

The design of a particular mechanism involves the specification of a message space (i.e. the set of signals used for communication), the response functions (i.e. how individuals choose the messages they send) and a rule to recognize the so called equilibrium messages, which are messages that indicate that the decisions of the independent agents have been coordinated. These equilibrium messages are then mapped by an outcome function into the allocations. The message space, response functions, equilibrium rules and outcome functions are regarded as chosen by the mechanism designer and not by the individuals. If all the outcomes of the mechanism are the ones prescribed by a social choice rule (also chosen by the designer) then it is said that the mechanism realizes that social choice rule. The problem of why the agent should be motivated to send the messages and behave in the way prescribed by the mechanism, that is the problem of incentive compatibility, is not directly dealt with, though useful discussions may be found in Hurwicz [16], Reichelstein [33] and Reichelstein and Reiter [32]. When a mechanism realizing a given social choice rule provides the right incentives, we say that it implements it. The problem we address in

[1] A survey of the literature on the subject is given by Crawford (1987).

this paper is of the following nature: given a social choice rule (fair allocations in our case), find a decentralized mechanism that realizes it with minimal message spaces. The question of its implementation is not considered. It is clear that the competitive mechanism which receives particular attention here has the same incentive compatibility problem as in the standard general equilibrium model. Agents' messages cannot be regarded as best responses in finite economies in a game theoretic sense. However, given the instructions received from the mechanism designer, they reflect the best choice.

There has been considerable work done on the problem of the informational requirements of an allocation mechanism which ensures efficient outcomes in classical environments[2]. This has been extended to environments with public goods (Sato [34]), and those which are stochastic (Jordan [21]), non-convex (Calsamiglia [2][3][4]), discrete (Hurwicz and Marschak [20]) and intertemporal (Hurwicz and Majumdar [19], Brock and Majumdar [1] and Dasgupta and Mitra [9]).

The outcomes that are selected as socially desirable are chosen by the designer, and in our case his two criteria are efficiency and fairness. He does not require that the agents should be able to verify these properties but merely affirms that these are the desired properties of acceptable outcomes. Indeed, since the process is decentralized and informationally efficient, it is not possible for the individuals themselves, with the information at their disposal, to check that the outcome is fair any more than they can check on efficiency.

In the context of classical environments the competitive mechanism plays a special role and indeed Jordan [22] has shown that, under certain assumptions, it is the unique informationally efficient way of obtaining Pareto optimality. This will be important in what follows. We know that, in classical environments, if all agents have the same consumption sets, there always exist fair and efficient allocations. [3]. This was shown by considering the Walrasian outcomes obtained after dividing income equally between all agents[4]. This indicates the route to follow. Since the competitive mechanism is informationally efficient in obtaining Pareto outcomes, all that remains is to distribute income equally. The question is how much additional information it is necessary to convey in order to perform the required redistribution.

The equal income Walrasian mechanism has received considerable attention in the literature. Apart from the papers mentioned above, results by Maskin [28] and Thomson [36][38] show that any Nash implementable social choice correspondence is closely related to the equal income Walrasian correspondence. Furthermore, as Varian [40], Hammond [14], Kleinberg [24], Champsaur and Laroque [5], and Mas Colell [26] [27] have shown, in economies with a large number of agents with sufficiently diverse characteristics the only fair outcomes are equal income Walrasian allocations. However, if there is not enough diver-

[2]Mount and Reiter [29], Hurwicz [17], Osana [30], Chander [6] and Calsamiglia [4].

[3]If consumption sets are not identical, i.e., there are non transferable commodities, then fair allocations may not exist. In particular, Pazner and Schmeidler [31]and Tillmann [39] have shown that, in a productive economy with agents of differing ability, there will be no such allocations.

[4]See Kolm [25] and Feldman and Kirman [11].

sity or not enough agents it is known that there may exist many other fair and efficient allocations.

The three basic results of this paper also indicate the central position of the Walrasian mechanism from a different perspective. They can be summarized as follows:

a) Any informationally decentralized mechanism that realizes fair allocations over the class of classical pure exchange environments has a message space of dimension greater than or equal to $n\ell$, that is the number of agents times the number of commodities.

b) The equal income Walrasian mechanism, in which all agents take prices parametrically and maximize utility subject to the average income constraint, realizes fair outcomes over the class of classical pure exchange environments and has a message space of dimension $n\ell$. Besides the typical competitive message, every agent has to send a real number expressing the value of his initial endowments at going prices. Thus, the equal income Walrasian mechanism is informationally efficient.

c) Athough in the class of environments considered there exist many fair allocations which are not equal income Walrasian allocations, we show that a mechanism that selects any of these necessarily has strictly larger informational requirements. In other words, if we insist on mechanisms with message spaces of minimal dimension, then the Walrasian mechanism from equal incomes is in fact the unique candidate.

2 Structure of the argument

To establish the framework let us look at the classical problem of obtaining Pareto optimal outcomes in an informationally efficient way. In the Walrasian competitive mechanism only net trades and prices must be known to check that a given outcome is Pareto efficient. But this information is compatible with infinitely many different economies (represented by different Edgeworth boxes and indifference curves) as shown in figure 1. The fact that there is no need to distinguish between all of these is the basis for the strong result that a finite dimensional message space (of dimension $n(\ell - 1)$) is sufficient to select Pareto efficient allocations over an infinite dimensional class of economies.

The Walrasian mechanism achieves this in an informationally decentralized way. In a decentralized mechanism, the decision process is decomposed into two phases. In the first phase there is a communication process. Because of the assumed initial dispersion of information, messages sent by agents depend only on messages of other agents and on their own characteristics. This important feature of the communication process implies that the so called "crossing condition" has to be satisfied: if two economies have the same equilibrium message, any "crossed economy" in which one agent from one of the two initial economies is "switched" with an agent from the other, must have the same equilibrium message [5]. For a given mechanism the translation of the equilibrium message into an action (net trade) is precisely prescribed by an outcome function, $z = h(m)$. Hence, if two economies have the same equilibrium message m, then the mechanism leads to the same action z for both. These two characteristics of any informationally decentralized mechanism have important implications that we will discuss.

Consider a mechanism that selects Pareto optimal outcomes and two economies which have the same equilibrium message. Then two facts are necessarily true. First, the common outcome z of the mechanism leads to an allocation which is Pareto efficient for both economies. Second, this very same trade z must also be the outcome of the mechanism for any of the "crossed" economies because of the "crossing condition". Therefore the trade z must lead to final allocations which are Pareto efficient not only in the two initial economies, but also in all the "crossed" economies. This is illustrated in figure 1, where an economy is represented by the Edgeworth box ABCD and the continuous indifference curves shown there. The point ω represents the initial endowments and z the trade leading to the final allocation x. A second economy is represented by the Edgeworth box EFGH and the dotted indifference curves respectively. It is easily seen that both economies have the same equilibrium message (p, z) and that the trade z leads to an allocation x which is Pareto efficient for both. Now consider the crossed economy in which we take the first agent from the first economy and the second agent from the second economy. The Edgeworth box for this economy is given by AIGK and the relevant indifference curves are one continuous and the other dotted. It is immediately seen that for this "crossed economy" the trade z still leads to a Pareto efficient allocation, as was to be expected from our previous argument. It is clear that the same competitive message is compatible with Edgeworth boxes of completely different sizes. This means that the equilibrium message does not reveal the "size" of the economy since no agent has any idea about the aggregate initial endowments.

Now, suppose that we are interested in a mechanism whose outcomes are not only Pareto optimal, but also fair. The competitive mechanism does not guarantee such outcomes. Notice that the allocation x happens to be fair for the economy EFGH, but not for ABCD. It seems clear that, to check whether an allocation is fair as well as being efficient, some information concerning the "size" of the Edgeworth box is needed. Indeed, the information needed turns out to be more than that of the competitive mechanism, but not very much.

At given prices let agents simply add their incomes to their information on net trades and prices, that is one real number per individual. Then let them maximize their utilities subject to the average income constraint. This mechanism guarantees fair and Pareto efficient allocations and has an $n\ell$-dimensional message space. Next we show that these are the minimal informational requirements of any decentralized mechanism that selects fair and efficient outcomes. The basic argument is fairly simple. Think of a class of economies in which all consumers have the same utility function, a Cobb Douglas with unit coefficients for example. This very specific class $E^{\star}$, which we shall refer to as the class of

[5] See section 3 for a formal statement of this property.

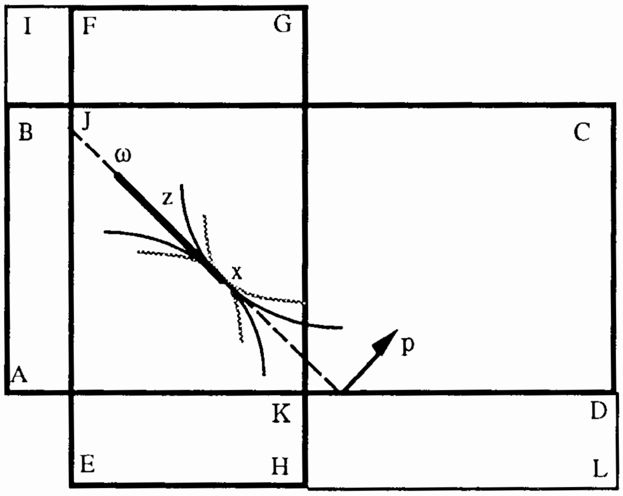

Figure 1: Illustration of the crossing condition

"canonical" economies, with its associated "canonical" utility function, will be particularly useful in what follows. It is clear that the only fair efficient allocation in such an economy is to give each agent the same bundle, i.e., to choose the allocation at the center of the Edgeworth box. To identify an economy in this class requires only the complete description of the initial bundles of all n agents , i.e., n bundles of ℓ goods. Thus its dimension is $n\ell$.

Now we claim that, in any informationally decentralized mechanism, two different economies in that subclass $E^\star$ must use different messages. Indeed, suppose that we have a mechanism for which two different economies share the same equilibrium message m. Consequently, the outcome of the mechanism in both economies must also be the same trade z. Consider the situation depicted in figure 2. We have two different economies, represented by the two Edgeworth boxes ABCD and EFGH, which have the same trade z as the outcome[6]. This outcome is fair because the final allocation point x obtained with the trade z from the initial endowment point ω is the center of the box for both economies. However, by the crossing condition, the "crossed economies" will have the same equilibrium message and consequently they must have the same outcome z. It is easily seen that, with the same trade, the final outcome x for the crossed economy given by the Edgeworth box EJCK is not fair because it is not in the center. The same argument holds for the other crossed economy: x is not at the center of AIGL. Therefore every economy in the subclass considered must use different messages. Hence the message space has to be at least as "big" as the $n\ell$-dimensional class of environments. If some regularity conditions are satisfied the dimension of the message space cannot be smaller than $n\ell$. This establishes the informational efficiency of the equal income Walrasian mechanism.

In the preceding argument we have been considering a very specific class of

[6]Since utility functions are assumed to be Cobb-Douglas, the flatter, (solid), indifference curves correspond to the bigger Edgeworth box. However along a ray given by the diagonal of the boxes the normalized utility gradients are always the same.

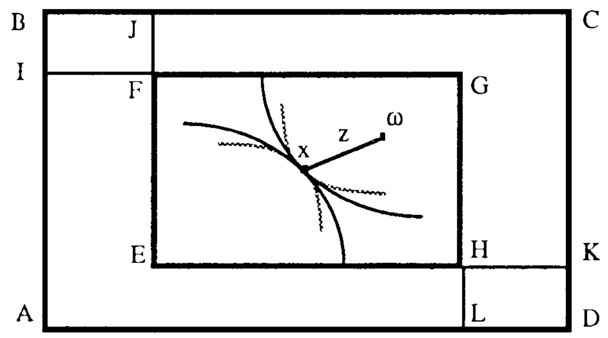

Figure 2: A subclass of economies with the uniqueness property

environments in which all agents have identical preferences. The information needed to attain fair efficient allocations within that restricted class is the same as that of the equal income Walrasian mechanism. Of course, for this class there are other mechanisms which use the same amount of information. Think, for example, of a mechanism in which every agent simply announces his own bundle. The average bundle can then be computed and the appropriate trades assigned to every agent. In this particular class of economies, this mechanism is essentially equivalent to the equal income Walrasian one because it yields precisely the same outcomes. The latter mechanism, however, has a satisfactory performance, i.e. gives both efficiency and fairness, over a much larger class of environments in which agents can have different utility functions, whilst the "bundle announcing" mechanism generally yields outcomes which are not even Pareto efficient.

It is important to note that the equilibrium message (prices, trades and individual incomes) for the equal income Walrasian mechanism does not reveal the particular Edgeworth box describing the economy. This must be so since, as we have seen, to specify the box completely requires $n\ell$-dimensional messages. However, to ensure efficiency in the general case, one needs to communicate supporting prices and in order to keep the message space to the same dimension, some information about the Edgeworth box must be sacrificed. This is illustrated in figure 3, where a whole class of Edgeworth boxes compatible with the same equal income Walrasian equilibrium message is shown : the upper-right corner of the box can be located at any point on a segment of the aggregate income budget line, 2A2B, which is clearly twice that facing each of the two individuals. The only condition is that the final outcome x should be non-negative for both inndividuals.

Finally, in section 6, we show that the only mechanism with minimal informational requirements achieving fair efficient outcomes is the equal income Walrasian one. Here we will sketch the argument used in the formal proof . Firstly, recall that if Pareto efficient outcomes are to be obtained in classical environments at least the information concerning supporting prices and competitive net trades must be conveyed. Furthermore, if fairness is required, as we have seen, some information about the total resources (the size of the Edgeworth box) is needed. This can be obtained by eliciting agents' incomes, for

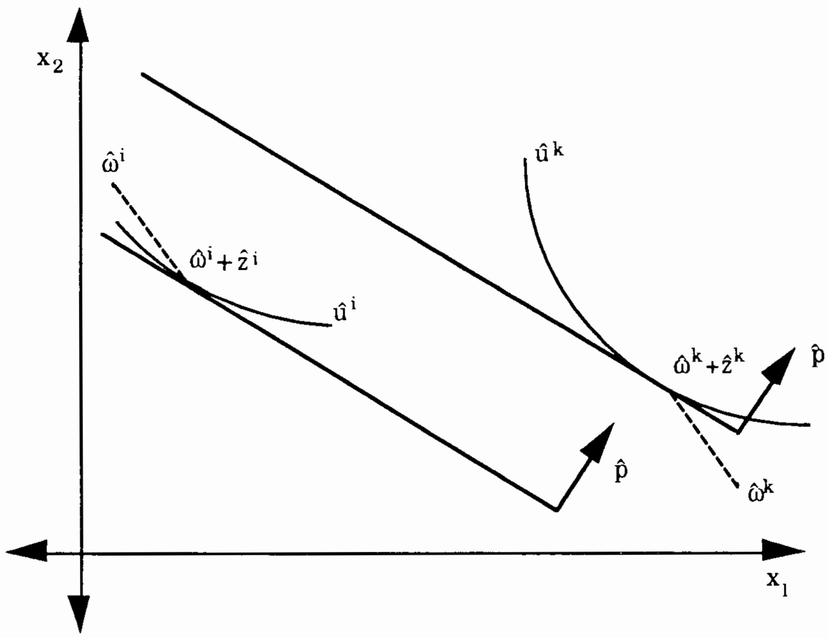

Figure 4: A mechanism which is not equal income Walrasian

"canonical" function. The initial endowments are then modified accordingly so as to preserve the same trade. The transition from the economy $\bar{e}$ to $e^\star$ is illustrated in figure 5 for agent k.

Now the essential point is that, in the similar economy $e^\star$ we construct, there is envy. Consider the situation in figure 6.

Each of the agents constructed in this way, has endowments $\omega^\star$, the canonical utility function $u^\star$, and the same income as before. Clearly, however, agent i is now envious of agent k. Thus $\hat{z}$ is not an equilibrium trade for $e^\star$ which is a contradiction with the way in which it was chosen. Thus unequal income final allocations cannot be equilibrium outcomes. The problem is clear. Any mechanism which would permit the sort of situation in figure 4 as an outcome must be able to distinguish between that and the situation in figure 6. This requires more information about the utility function and would therefore be more informationally demanding than the equal income Walrasian mechanism. This means that it cannot have been informationally efficient in the first place.

3 Informationally decentralized mechanisms

Consider an exchange economy with ℓ commodities, and a set of agents $\Im = \{1, 2, \ldots, n\}$. Every agent $i \in \Im$ is characterized by a utility function u^i, and an initial endowment ω^i. The i-th agent's characteristics are denoted by $e^i = \langle u^i, \omega^i \rangle$. An economy is denoted by the n-tuple $e = (e^1, e^2, \ldots, e^n)$. The class of possible economies, denoted by E, reflects the a priori knowledge available on the agents' characteristics. We shall specify a class of environments by putting some restrictions on utility functions. In order to ensure that equilibrium prices are strictly positive, we shall postulate a special type of strict monotonicity, in

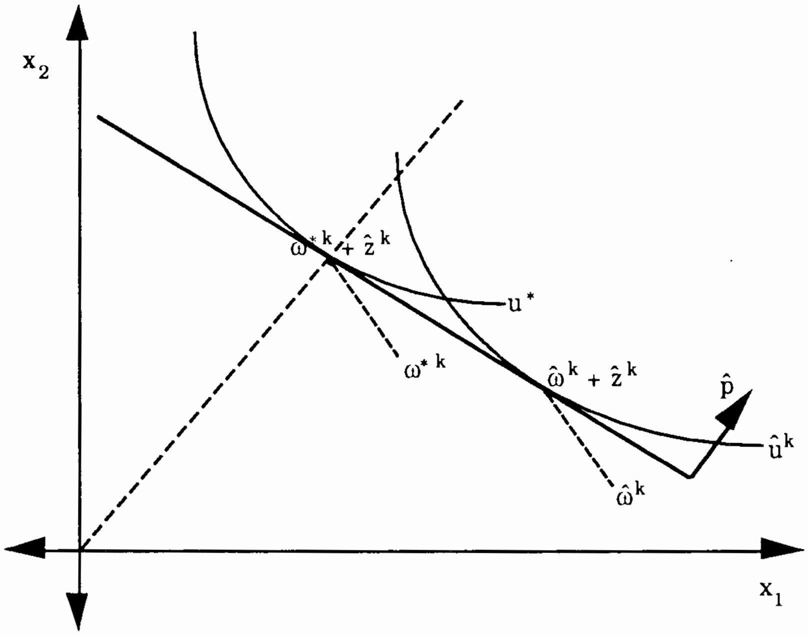

Figure 5: Construction of a similar economy

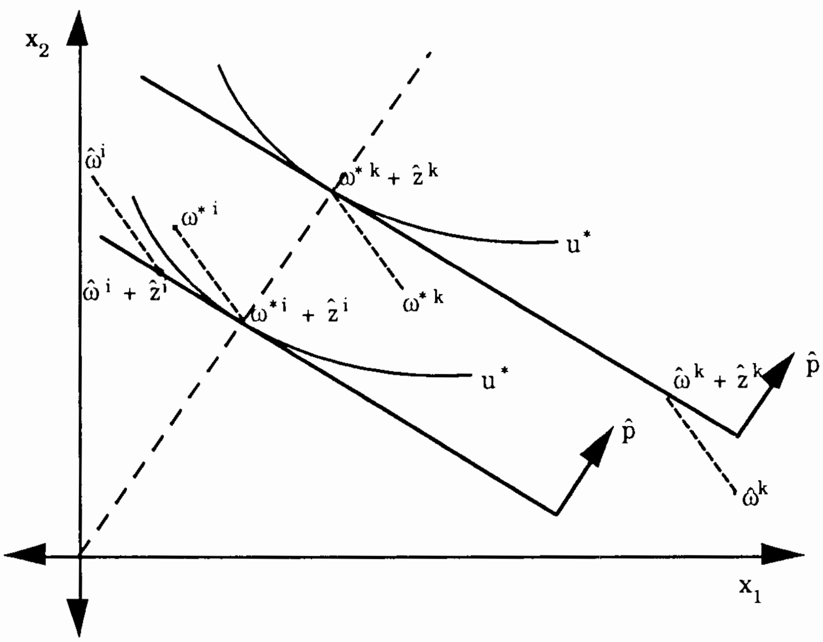

Figure 6: Envy in the similar economy

$\Pi = \langle M, \mu, h \rangle$ is privacy preserving over the class of economies E if for every $i \in \Im$ and every $\tilde{e}$ and $\bar{e}$ in E, $\mu(\bar{e}) \cap \mu(\tilde{e}) \neq \emptyset$ implies $\mu(\bar{e}) \cap \mu(\tilde{e}) = \mu(\tilde{e} \otimes_i \bar{e}) \cap \mu(\bar{e} \otimes_i \tilde{e})$ whenever $\tilde{e} \otimes_i \bar{e}$ and $\bar{e} \otimes_i \tilde{e}$ belong to E. This means that if m is an equilibrium message for both $\tilde{e}$ and $\bar{e}$ in E than it must also be an equilibrium message for any "crossed" economy. If the class of economies E is a cartesian product $E = E^1 \times E^2 \times \ldots \times E^n$, then the privacy property can be characterized in terms of "coordinate" correspondences[8], as established by the following proposition due to Mount and Reiter.

Proposition 3.1 (Mount and Reiter[29],Lemma 5, p.171) *Let the class of economies E be the cartesian product $E = E^1 \times E^2 \times \ldots \times E^n$. Then a mechanism $\Pi = \langle M, \mu, h \rangle$ is privacy preserving on E if and only if for every $i \in \Im$ there exists a correspondence $\mu^i : E^i \twoheadrightarrow M$ such that for every $e \in E$, $\mu(e) = \bigcap_{i=1}^{n} \mu^i(e^i)$.*

In this case, in a privacy preserving mechanism every agent can check independently whether a given message is an equilibrium message by looking at their own characteristics, $m \in \mu^i(e^i)$. The initial dispersion of information (every agent is assumed to know his own characteristics) and the so-called privacy property (according to which the knowledge of other agents' characteristics is only conveyed through formal messages) are the basic ingredients of the concept of informational decentralization. In what follows, a mechanism satisfying the privacy property will be said to be *informationally decentralized.*

The problem that we want to address is the minimal amount of information contained in M which is sufficient to guarantee that the mechanism yields fair outcomes over the class of classical economies[9]. Furthermore, we would like to know which are the properties that an informationally efficient (i.e., using minimal message spaces) mechanism must necessarily have.

4 The equal income Walrasian mechanism

Let $D = \{p \in R^{\ell}_{++} : \sum_{j=1}^{\ell} p_j = 1\}$ denote the $\ell - 1$-dimensional simplex and define the message space

$$M_w = \{(p, z, r) \in \Delta \times R^{\ell n} \times R^n : \sum_{i=1}^{n} z^i = 0 \text{ and } pz^k = \frac{1}{n} \sum_{i=1}^{n} r^i - r^k\}$$

where p is a vector of normalized prices, z is a n-tuple of net trades and $r = (r^1, r^2, \ldots, r^n)$ is the vector of initial incomes (or values of initial endowments). Now, for every agent, define the correspondence $\mu^i_w : E^i \twoheadrightarrow M_w$ as follows: a message $(\bar{p}, \bar{z}, \bar{r})$ is an equilibrium message from the point of view of agent i, that is $(\bar{p}, \bar{z}, \bar{r}) \in \mu^i_w(e^i)$, if the following two conditions are satisfied:

[8] See Chander [7] for a discussion of this more general definition of the privacy property and its relation to the original definition by Mount and Reiter[29].

[9] By classical economies, we understand the class of economies satisfying the conditions sufficient to guarantee the existence and optimality of a competitive equilibrium.

which monotonicity is not necessarily strict on the boundary (as is the case with Cobb-Douglas preferences). Hence, the set of possible utility functions is defined as follows.

To every real valued continuous utility function u on R^ℓ_+ we associate a set $\chi(u)$ defined as :

$$\chi(u) = \begin{cases} R^\ell_{++} & \text{if for any } \bar{x}, \{x \in R^l_{++} : u(x) \geq u(\bar{x})\} \subseteq R^\ell_{++} \\ R^\ell_+ & \text{otherwise.} \end{cases}$$

We can define the general class of environments E^g as follows:

$$\begin{aligned} \mathfrak{U}^g &= \{u : R^\ell_+ \to R : u is \text{ continuous,strictly monotone on } \chi(u) \text{ and strictly quasiconcave}\} \quad (1) \\ E^g &= \{(e^1, e^2, \ldots, e^n) : \text{ for all } i \in \Im, u^i \in \mathfrak{U}^g, \omega^i \in R^\ell \text{ and } \sum_{i=1}^n \omega^i > 0\} \end{aligned}$$

We shall also consider a more restricted class of utility functions which satisfy a boundary condition, namely,

$$\begin{aligned} \mathfrak{U}^b &= \{u \in \mathfrak{U}^g : \forall \bar{x}\{x \in R^\ell_{++} : u(x) \geq u(\bar{x})\} \subseteq R^\ell_{++} \quad (2) \\ E^b &= \{e \in E^g : \forall i, u^i \in \mathfrak{U}^b\} \end{aligned}$$

Let x^i denote the consumption vector of the i-th agent and let $z^i = x^i - \omega^i$ denote the net trade vector. Let $x = \langle x^1, x^2, \ldots, x^n \rangle$ and $z = \langle z^1, z^2, \ldots, z^n \rangle$ denote respectively the n-tuples of consumption and net trades. The set of possible outcomes of a resource allocation mechanism is given by the set of feasible net trades $Z = \{z \in R^{\ell n} : \sum_{i=1}^n z^i = 0\}$. The Pareto optimality correspondence $P : E \twoheadrightarrow Z$ assigns to every economy $e \in E$ the set of Pareto optimal trades, i.e. those which when added to the initial endowments give a Pareto optimal allocation. For brevity we call a trade fair if the resulting allocation is both Pareto optimal and envy-free [7]. Hence a net trade z is fair for the economy $e \in E$ if $z \in P(e)$ and for all agents i and k, $u^i(x^i) \geq u^i(x^k)$, where $x^i = \omega^i + z^i$, that is, there is no agent who envies other agent. Let $F : E \twoheadrightarrow Z$ denote the correspondence that assigns to every economy $e \in E$ the set of fair trades.

Following Mount and Reiter[29], an allocation mechanism is a triple $\Pi = \langle M, \mu, h \rangle$, where M is a set of abstract messages, $\mu : E \twoheadrightarrow M$ is a message correspondence that assigns to every economy the set of equilibrium messages, and $h : M \to Z$ is the outcome function, that assigns to every equilibrium message the corresponding net trade. An allocation mechanism is decisive over the class of economies E if for every $e \in E$, $\mu(e) \neq \emptyset$. An allocation mechanism is fair over the class of economies E if it is decisive and for every $e \in E$, and every $z \in h[\mu(e)]$, $z \in F(e)$.

Given $\bar{e}$ and $\tilde{e}$ in E, the "crossed" economy $(\tilde{e}^i, \tilde{e}^2, \ldots, \tilde{e}^{i-1}, \bar{e}^i, \tilde{e}^{i+1}, \ldots, \tilde{e}^n)$ is denoted by $\bar{e} \otimes_i \tilde{e}$, for $i \in \Im$, while $\tilde{e} = \bar{e} \otimes_0 \tilde{e}$. A resource allocation mechanism

[7] Thus we do not follow the terminology used by Schmeidler and Vind[35].

$$\begin{cases} u^i(\omega^i + \bar{z}^i) \geq u^i(\omega^i + z^i) & \text{for all } z^i \in R^\ell \text{ such that } \bar{p}(\omega^i + z^i) = \frac{1}{n}\sum_{k=1}^n \bar{r}^k \\ \bar{p}\omega^i = \bar{r}^i \end{cases} \tag{3}$$

According to the second condition,every agent checks whether the proposed initial income $\bar{r}^i$ corresponds to the value of his initial endowments at the going prices. According to the first condition, the agent maximizes utility subject to the average income constraint. This last magnitude can be computed by the agent from the messages sent by others. The message correspondence and outcome function are then defined as

$$\mu_w(e) = \bigcap_{i=1}^{n} \mu_w^i(e^i) \text{ and } h_w(p,z,r) = z$$

Hence, the equal income Walrasian mechanism is given by $\Pi_w = \langle M_w, \mu_w, h_w \rangle$. It is easily verified that M_w is a smooth $n\ell$-dimensional manifold.

5 Informational requirements for fair allocations

In this section we study the minimal amount of information - as measured by the dimension of the message space - that is required to guarantee that an informationally decentralized mechanism leads to fair outcomes over the general class of environments E^g. Let us define the canonical subclass of economies $E^\star$. An economy is a member of $E^\star$ if all agents have identical preferences which can be represented by a Cobb-Douglas utility function with unit coefficients. More specifically,

$$E^\star = \{e \in E^g : u^i(x^i) = \prod_{j=1}^{\ell} x_j^i\} \tag{4}$$

Thus, in an economy with n agents and ℓ commodities, every environment in the subclass $E^\star$ is completely specified by $n\ell$-dimensional vectors of initial endowments. Hence $\dim E^\star = n\ell$. Now we shall show that the fair correspondence F restricted to $E^\star$ is a single-valued function. Given a utility function $u \in \mathfrak{U}^g$ let us denote by $E(u) \subseteq E^g$ the class of economies in which initial endowments are arbitrary but all agents have the same utility function u.

Lemma 5.1 *Let $\bar{e} \in E(u)$ be an economy where all agents have identical utilities so that $\bar{u}^i = u$ for all i. Suppose further that $z \in F(\bar{e})$. Then for all agents $k \in \mathfrak{I}$ we must have $\omega^k + z^k = \frac{1}{n}\sum_{i=1}^n \omega^i$.*

PROOF:It suffices to show that for all i and k in $\mathfrak{I}$, $\omega^i + z^i = \omega^k + z^k$. It is clear that, since all agents have the same utility function u, fairness implies that $u(\omega^i + z^i) = u(\omega^k + z^k)$ for all i and k in $\mathfrak{I}$. Suppose that there exist two agents i and k such that $\omega^i + z^i \neq \omega^k + z^k$. Then we can construct a new trade $\hat{z}$ which is Pareto superior to z. Indeed, define

$$\hat{z}^i = \frac{\omega^k + z^k - \omega^i + z^i}{2} \text{ and } \hat{z}^k = \frac{\omega^i + z^k - \omega^k + z^i}{2}$$

while $\hat{z}^j = z^j$ for all other agents. Then, the new final consumption vectors satisfy $\hat{x}^i = \hat{x}^k = \frac{1}{2}(x^i + x^k)$. By strict quasiconcavity of the common utility function u it follows that $u(\omega^i + \hat{z}^i) > u(\omega^i + z^i)$ and $u(\omega^k + \hat{z}^k) > u(\omega^k + z^k)$, while all other agents are indifferent since they receive the same consumption vector. Moreover, since $\hat{z}^i + \hat{z}^k = z^i + z^k$ it follows that $\hat{z}$ is a feasible trade. This contradiction completes the proof.

Lemma 5.1 implies that in an economy with identical individuals with strictly quasiconcave utillity functions, the only fair allocation is the equal division of total endowments. In particular, this is true for our canonical subclass $E^\star$. Now we shall show the basic theorem of this section.

Proposition 5.2 (Informational Efficiency Theorem) *Suppose that* $\Pi = \langle M, \mu, h\rangle$ *is a resource allocation mechanism on* E^g *such that:*

a) it is fair on E^g*,*

b) it is informationally decentralized,

c) the message space M is a manifold,

d) when restricted to $E^\star$ *the message correspondence is locally threaded*[10] *at some point* $\bar{e}$.

Then the dimension of the message space is at least as large as that of the equal income Walrasian mechanism defined in section 4, that is, $\dim M \geq \dim M_w$.

PROOF: We first show that the restriction of μ to $E^\star$ is an injective correspondence. Suppose that $\bar{m} \in \mu(\bar{e}) \otimes_i (\tilde{e})$. We have to show that this implies $\bar{e} = \tilde{e}$. Since the mechanism is assumed to be informationally decentralized, it follows from the privacy property that $\bar{m} \in \mu(\tilde{e} \otimes_i \bar{e}) \cap \mu(\bar{e} \otimes_i \tilde{e})$ for all i. But if the mechanism is fair this implies that the outcome $\bar{z} = h(\bar{m})$ satisfies

$$\bar{z} \in F(\bar{e} \otimes_i \tilde{e}) \quad \Rightarrow \quad \bar{\omega}^i + \bar{z}^i = \tilde{\omega}^k + \bar{z}^k \text{ for all } k \in \mathfrak{I}$$
$$\bar{z} \in F(\bar{e} \otimes_0 \tilde{e}) \quad \Rightarrow \quad \tilde{\omega}^i + \bar{z}^i = \tilde{\omega}^k + \bar{z}^k \text{ for all } k \in \mathfrak{I}$$

Hence $\bar{\omega}^i = \tilde{\omega}^i$. Since the argument can be replicated for any other agent,it is clear that the initial endowments are the same in both economies and therefore $\bar{e} = \tilde{e}$. Using assumption d, let U be an open neighborhood of $\bar{e}$ and $f : U \to M$ a continuous function such that $f(e) \in \mu(e)$ for all $e \in U$. Then f is a continuous injection from U to $f[U]$. Since U and M are manifolds, we can use Theorem

[10] *Let X and Y be topological spaces. A correspondence* $\Psi : X \to Y$ *is said to be locally threaded at* $x \in X$ *if there exists a neighborhood U of x and a continuous function* $f : U \to Y$ *such that* $f(x) \in \Psi(x)$ *for all* $x \in U$

18 in Kelley [23]to conclude that f is a homeomorphism between U and $f[U]$. Then $\dim M_w = n\ell = \dim U \leq \dim M$.

The intuition behind this result is clear: the message space has to contain at least as much information as the subclass of environments $E^\star$ because every environment in $E^\star$ must have a different message. Therefore the message space has to contain enough information so as to distinguish between members of $E^\star$. Notice that in order to guarantee Pareto optimal allocations the minimal informational requirements are those of the Walrasian competitive process, i.e., $n(\ell - 1)$. The preceding result suggests the possibility of realizing a stronger optimality correspondence through mechanisms which require every agent to send just an additional real number. In the following proposition we show that the equal income Walrasian mechanism defined in section 4 is an informationally decentralized process satisfying all the assumptions of the informational efficiency theorem and whose message space is of dimension $n\ell$.

Corollary 5.3 *The equal income Walrasian mechanism is informationally efficient on E^g.*

PROOF: It is clear from the construction of $\Pi_w = \langle M_w, \mu_w, h_w \rangle$ that it is informationally decentralized and that M_w is a manifold of dimension $n\ell$. It is also clear that, from the conditions defining the class E^g, it can be shown that the competitive equilibrium from average income exists, is Pareto optimal and envy-free. Finally, the correspondence μ_w when restricted to $E^\star$ is a continuous function and thus it is locally threaded at any $\bar{e} \in E^\star$.

6 The uniqueness theorem in the special class of economies

In this section we follow the basic strategy of the proof given by Jordan for the uniqueness of the competitive mechanism in achieving efficiency in classical environments. Our proof differs from his in several essential respects, and it is worth indicating the differences between our problem and his, and the way in which our proof is adapted to handle them. Firstly, in his problem the equality of supporting hyperplanes is necessary for efficiency. In our case the analogous condition, equality of incomes and of supporting hyperplanes, is not necessary for achieving fairness and efficiency. Secondly, we construct a class of economies, those with what we have called "canonical" preferences for all individuals, where the fair outcome is unique and is also efficient. There is no equivalent in the ordinary efficiency problem. Thirdly, when considering the problem of how much information is necessary to discriminate between different economies, Jordan was free to modify total income in order to change the curvature of indifference curves of a given Cobb-Douglas utility function. We do not have this possibility, and so have to use a different construction.

The theorem is proved first for a class of economies $\tilde{E}$ in which all agents have the same utility function and then extended to more general economies. We define several binary relations on the class of economies $\tilde{E}$: the similarity relation S_π induced by a mechanism Π, the message relation T_π induced by a mechanism Π, and the message relation T_w induced by the equal income Walrasian mechanism. Two economies are similar relative to a mechanism Π if they have the same equilibrium net trades, the same slopes of their indifference curves at these trades and the same value of initial endowments. Two economies are "message related" by a mechanism Π if both have the same equilibrium message under Π. The basic strategy of the proof is to show that if Π is a fair mechanism with minimal informational requirements the three binary relations induce the same partitions on the class of economies $\tilde{E}$. The proof hinges on the fact that every equivalence class contains one and only one member of the canonical class $E^\star$ so that in an informationally efficient mechanism, where the message partitions should not be finer, what happens in $E^\star$ must be essentially the same as what happens everywhere. Now, it turns out that in the canonical class the equal income Walrasian outcome is the only fair outcome. This fact is later used to show that any informationally efficient mechanism is essentially the same as the equal income Walrasian because the outcomes are the same and the message spaces homeomorphic.

Let us start by defining rigorously the similarity relation. Given a mechanism $\Pi = \langle M, \mu, h \rangle$ define the correspondence $\Psi_\pi : E^g \twoheadrightarrow \Delta \times R^{\ell n} \times R^n$ that assigns a price-trade-income-tuple to every environment. Let $S(u^i, \omega^i + z^i)$ denote the set of all normalized prices supporting the utility u^i at the allocation $\omega^i + z^i$ For any given $e \in E^g$, we say that $(p, z, r) \in \Psi_\pi(e)$ if and only if for all i

$$
\begin{aligned}
z &\in h[\mu(e)] \\
p &\in S(u^i, \omega^i + z^i) \\
r^i &= p\omega^i
\end{aligned}
\tag{5}
$$

Hence, the image under Ψ_π of a given environment is given by the outcomes of the given mechanism, the normalized supporting pricevector at the corresponding final consumption points and the vector of values of initial endowments of the n agents. Notice that Ψ_π depends upon the mechanism.

Two economies $\bar{e}$ and $\hat{e}$ in E^g are said to be similar relative to mechanism $\Pi = \langle M, \mu, h \rangle$ if their images under Ψ_π have a nonempty intersection, that is, $\Psi_\pi(\bar{e}) \cap \Psi_\pi(\hat{e}) \neq \emptyset$. In this case we write $\bar{e} S_\pi \hat{e}$. It is easily verified that if the message correspondence of the mechanism, μ, is a function on E and the allocations are interior,then Ψ_π is a function given by:

$$
\begin{aligned}
z &= h[\mu(e)] \\
p &= \frac{Du^i(\omega^i + z^i)}{\|Du^i(\omega^i + z^i)\|} \\
r^i &= p\omega^i
\end{aligned}
\tag{6}
$$

Moreover, S_π is an equivalence relation that induces a partition of E into equivalence classes. The k-th agent of two similar economies, $e^\star$ and $\hat{e}$ was represented in figure 5.

Now, we define a special class of utility functions which, in a sense, are similar to Cobb-Douglas, but whose indifference curves can be made arbitrarily flat[11]. We define it as follows. Let $\bar{u}$ be a Cobb-Douglas utility function and consider the indifference curve going through $\bar{\omega}^k + \bar{z}^k$. For any given ray, let y^2 be the intersection of the indifference curve and thhis ray, and y^1 the intersection of the supporting hyperplane at $\bar{\omega}^k + \bar{z}^k$ and the ray, as shown in figure 7. Then the indifference curve of the new utility function $\bar{\bar{u}}$ cuts the given ray at a point $y^\star = \lambda y^1 + (1-\lambda) y^2$ for $0 \le \lambda < 1$. By choosing the appropriate weight λ, the new indifference curve can be made as flat as desired. Construct the utility function by just blowing up (or down) this indifference curve. The class $\mathfrak{U}^s$ of utility functions defined in this way is nicely behaved: all functions are smooth, homothetic, indifference curves are asymtotic to the coordinate axes and can be made arbitrarily flat.

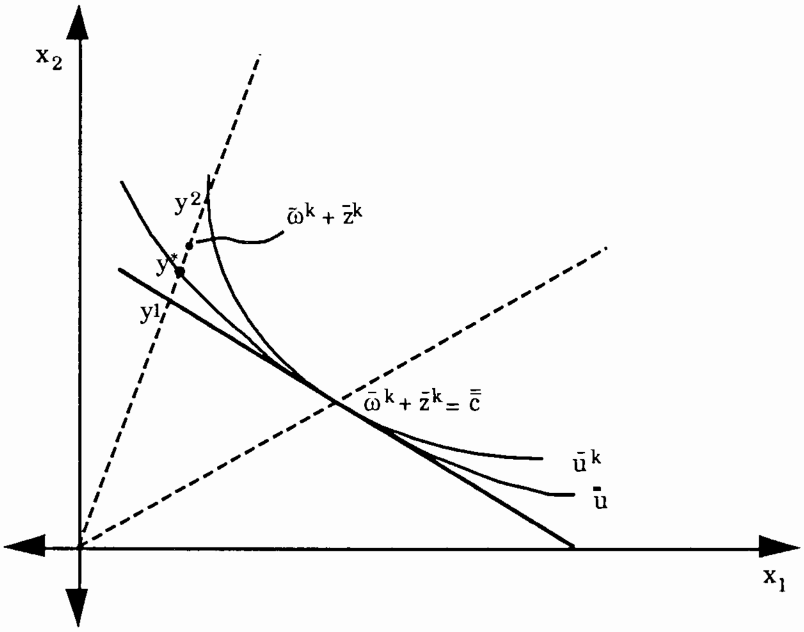

Figure 7: Construction of a flatter indifference map

Lt us define the following classes of economies that will be of some interest.

$$
\begin{aligned}
E^s &= \{e \in E^g : \forall i, u^i \in \mathfrak{U}^s\} \\
E(u) &= \{e \in E^g : \forall i, u^i = u\} \\
\tilde{E} &= \bigcup \{E(u) : u \in \mathfrak{U}^s\}
\end{aligned}
$$

The class E^s includes all economies with heterogeneous agents whose utilities are flattened" Cobb-Douglas functions. $E(u)$ contains economies in which

[11]That is, locally as close to a given supporting hyperplane as desired.

all agents have the same given utility function u. Finally the class $\tilde{E}$ includes all economies in wich all agents have the same utility function.

Now let us define some properties that the mechanisms might have:

F : The mechanism is fair, that is, leads always to final allocations which are Pareto optimal and envy-free.

I : The mechanism is informationally decentralized, that is, the message correspondence has the privacy property.

C : For every utility function $u \in \mathfrak{U}^s$, the restriction of the message correspondence μ to the subclass of economies $E(u)$ in which all agents have the same utility function u, is a continuous function.

O : For every utility function in the special class $u \in \mathfrak{U}^s$, the image of the subclass, $\mu[E(u)]$, is a closed subset of M.

M : The message space is of minimal dimension, that is, M is a connected manifold of dimension $n\ell$.

All these conditions, but O, are standard in the literature. In Jordan [22], condition O is assumed to hold for a class of economies with heterogeneous agents with Cobb-Douglas utility functions. Here we assume that it holds in a class of economies in which all agents have the same utility function $u \in \mathfrak{U}^s$. Notice that, by Lemma 5.1, in this class the performance function satisfies these properties since it is a nicely behaved single-valued function.

We start by showing that the subclass of environments $E(u)$, where all agents have the same utility function, generates the whole message space.

Proposition 6.1 *Let* Π *be a mechanism on* E^g *satisfying C,O,M. Then* $\mu[E(u)] = M$ *for all* $u \in \mathfrak{U}^s$.

PROOF: Take $e \in E(u)$ and let U_e be any open neighborhood of e. Consider the mapping $\varphi : E(u) \to R^{n\ell}$ given by the projection $\varphi(u,\omega) = \omega$. This is clearly a homeomorphism of $E(u)$ into an open subset of $R^{n\ell}$ so that we shall consider $E(u)$ as an open subset of $R^{n\ell}$. Using theorem A.1 in Greenberg [13] we conclude that $\mu[U_e]$ is an open set in M which is homeomorphic to U_e. Then $\mu[E(u)] = \bigcup\{\mu[U_e] : e \in E(u)\}$ is the union of open sets and so it is open. But it is also closed by assumption. Since M is connected, it is the only open and closed set[12], so that $\mu[E(u)] = M$ q.e.d.

Now we shall turn to a result whose intuition is the following. Given any economy $\bar{e}$, let $\bar{m}$ be an equilibrium message and $(\bar{p},\bar{z},\bar{r}) \in \Psi(\bar{e})$ an associated price-trade-income vector (which is not necessarily the equal income Walrasian message) where $\bar{z} = h(\bar{m})$. Then given any equilibrium message $\bar{m}$ and any consumer k we can construct an economy $\hat{e}$ such that:

a) The equilibrium message, and consequently the equilibrium trade, are the same as in $\bar{e}$.

[12] See page 53 in Kelley [23]

b) Agent k in the second economy is very similar to agent k in the original economy because the value of initial endowments has not changed and the indifference surface at the consumption point $\bar{x}^k$ in the first economy is very similar in the sense that, although it has been bent to make it flatter, the supporting hyperplane at the final consumption point is still the same.

c) All other agents change a lot and are modelled after agent k: they all have the same utility function similar to k's original utility function.

d) The new utility function is strictly quasi-concave, homothetic and can be locally as close to its supporting hyperplane as desired.

Proposition 6.2 *Let $\Pi = \langle M, \mu, h \rangle$ be an allocation mechanism on E^g satisfying F,I,C,O,M. Then given any economy $\bar{e} \in E^g$, any $\bar{m} \in \mu(\bar{e})$ and any agent k, there exists a cannonical economy $\hat{e}$ such that:*

a) All agents have the same homothetic utility function $\hat{u} = \in \mathfrak{U}^s$ so that $\hat{e} \in E(\hat{u})$.

b) The cannonical economy has the same equilibrium message, $\bar{m} = \mu(\hat{e})$

c) $\frac{D\hat{u}(\hat{\omega}^k + \bar{z}^k)}{\|D\hat{u}(\hat{\omega}^k + \bar{z}^k)\|} = \bar{p}$, where $\bar{p}$ is the comon supporting hyperplane at the Pareto optimal allocation $\bar{\omega} + h(\bar{m})$.

d) The value of the initial endowments of the k-th agent has not changed, i.e., $\bar{p}\bar{\omega}^k = \bar{p}\hat{\omega}^k$ and $\hat{\omega}^k + \bar{z}^k > 0$.

Moreover, the utility function $\hat{u}$ can be chosen to have indifference curves strictly convex and as flat as desired.

PROOF: Let $\bar{z} = h(\bar{m})$ be the equilibrium outcome of the mechanism and consider the point chosen by the k-th consumer, $\bar{\omega}^k + \bar{z}^k$. Since $\bar{p}(\bar{\omega}^k + \bar{z}^k) > 0$, we can choose an initial endowment for the k-th agent such that

a) $\bar{p}\tilde{\omega}^k = \bar{p}\bar{\omega}^k$

b) $\tilde{\omega}^k + \bar{z}^k > 0$

Now, since $\tilde{\omega}^k + \bar{z}^k$ is an interior allocation, there exists a Cobb-Douglas utility function $\tilde{u}$ whose supporting hyperplane at $\bar{\omega}^k + \bar{z}^k$ is the same as that of the original utility function $\bar{u}^k$. By choosing the parameter λ as close to one as desired, we can generate a utility function $\hat{u}$ in the special class $\mathfrak{U}^s$ as flat as desired and such that the normalized utility gradient equals the given price vector,

$$\frac{D\hat{u}(\tilde{\omega}^k + \bar{z}^k)}{\|D\hat{u}(\tilde{\omega}^k + \bar{z}^k)\|} = \bar{p}$$

So we get a situation such as that represented in figure 7. Consider now the class of economies $E(\hat{u})$ in which all agents have the same utility function $\hat{u}$.

By Proposition 6.1 there exists an economy $\hat{e} \in E(\hat{u})$ such that $\mu(\hat{e}) = \bar{m}$. This new economy satisfies assertions a and b of the proposition. Moreover, it has the same equilibrium trade as before, $\bar{z} = h(\bar{m})$.

Now we establish assertion c). By the crossing condition $\bar{m} \in \mu(\hat{e} \otimes_k \bar{e})$, so that $\bar{m}$ is also an equilibrium message for the old economy with the new k-th agent. That means that the allocation $(\hat{\omega} + \bar{z}) \otimes_k (\bar{\omega} + \bar{z})$ is a Pareto optimal allocation for the mixed economy $\hat{e} \otimes_k \bar{e}$. Assertion c) follows because it is a necessary condition for optimality.

Finally, we have to show statement d), namely, $\bar{\omega}^k = \tilde{\omega}^k$. First, note that by Lemma 5.1 in the subclass of economies $E(\hat{u})$ the only fair allocation is the center of the Edgeworth box, so that the final consumption of all agents equals $\hat{c} = \hat{\omega}^i + \bar{z}^i$ for all i. But then $\bar{p}$ supports the utility surface of $\hat{u}$ at both $\hat{c} = \hat{\omega}^k + \bar{z}^k$ and $\tilde{\omega}^k + \bar{z}^k$. Then, by homotheticity of $\hat{u}$, we know that $\hat{c}$ must be proportional to $\tilde{\omega}^k + \bar{z}^k$. But this implies that they must be equal. Otherwise we can construct the economy $\tilde{e} \otimes_k \hat{e}$ which by the crossing condition has the same equilibrium message and therefore the same equilibirum trade. In this economy all final consumptions would also be on the same ray, as shown in figure 8. If agent's k final consumption does not coincide with the common consumption $\hat{c}$ then there is envy and this contradicts the assumption that Π is a fair mechanism. q.e.d.

Next Proposition establishes that if two economies have the same equilibrium message, then they are similar, that is, the message binary relation is finer than the similarity relation.

Proposition 6.3 *Let $\Pi = \langle M, \mu, h \rangle$ be an allocation mechanism on $\tilde{E}$ satisfying F,I,C,O,M. Given any two economies $\bar{e}$ and $\tilde{e}$ in $\tilde{E}$, $\bar{m} = \mu(\bar{e}) = \mu(\tilde{e})$ implies that $\bar{e}$ and $\tilde{e}$ are similar,that is, $\Psi(\bar{e}) = \Psi(\tilde{e})$*

PROOF: The net trade $\bar{z} = h(\bar{m})$ is an equilibrium trade for both economies $\bar{e}$ and $\tilde{e}$. Then the allocation $\bar{\omega} + \bar{z}$ is Pareto optimal in $\bar{e}$ and the allocation $\tilde{\omega} + \bar{z}$ is Pareto optimal in $\tilde{e}$. By the crossing condition $\bar{m} \in \mu(\tilde{e}) \otimes_k \mu(\bar{e})$so, since the mechanism is fair, the allocation $(\tilde{\omega} + \bar{z}) \otimes_k (\bar{\omega} + \bar{z})$ is a Pareto optimal allocation for the mixed economy $\tilde{e} \otimes_k \bar{e}$. Then

$$\bar{p} = \frac{D\tilde{u}(\tilde{\omega}^k + \bar{z}^k)}{\|D\tilde{u}(\tilde{\omega}^k + \bar{z}^k)\|} = \frac{D\bar{u}(\bar{\omega}^i + \bar{z}^i)}{\|D\bar{u}(\bar{\omega}^i + \bar{z}^i)\|} \text{ for all } i.$$

because it is a necessary condition for optimality. So in order to show that the two economies are similar it remains to be proved that $\bar{p}\bar{\omega}^i = \bar{p}\tilde{\omega}^i$ so that agents have the same values of initial endowments in both economies.

Suppose that for some agent k, $\bar{p}\bar{\omega}^k < \bar{p}\tilde{\omega}^k$.As shown in figure 7, by Proposition 6.2 we can construct an economy $\hat{e} \in E(\hat{u})$ with a identical homothetic utility function $\hat{u}$ for all agents flat enough so that $\hat{u}(\hat{c}) < \hat{u}(\tilde{\omega}^k + \bar{z}^k)$, where $\hat{c} = \bar{\omega}^k + \bar{z}^k = \hat{\omega}^i + \bar{z}^i$ is the common final consumption obtained by all agents in the economy $\hat{e}$. Now take the economy $\hat{e}$ and substitute the k-th original agent so that we get the economy $\tilde{e} \otimes_k \hat{e}$. By the crossing condition we still have the same equilibrium message and net trade, so that in this new economy all agents, except k, get $\hat{c} = \bar{\omega}^i + \bar{z}^i$, while agent k gets $\tilde{\omega}^k + \bar{z}^k$. As can be seen

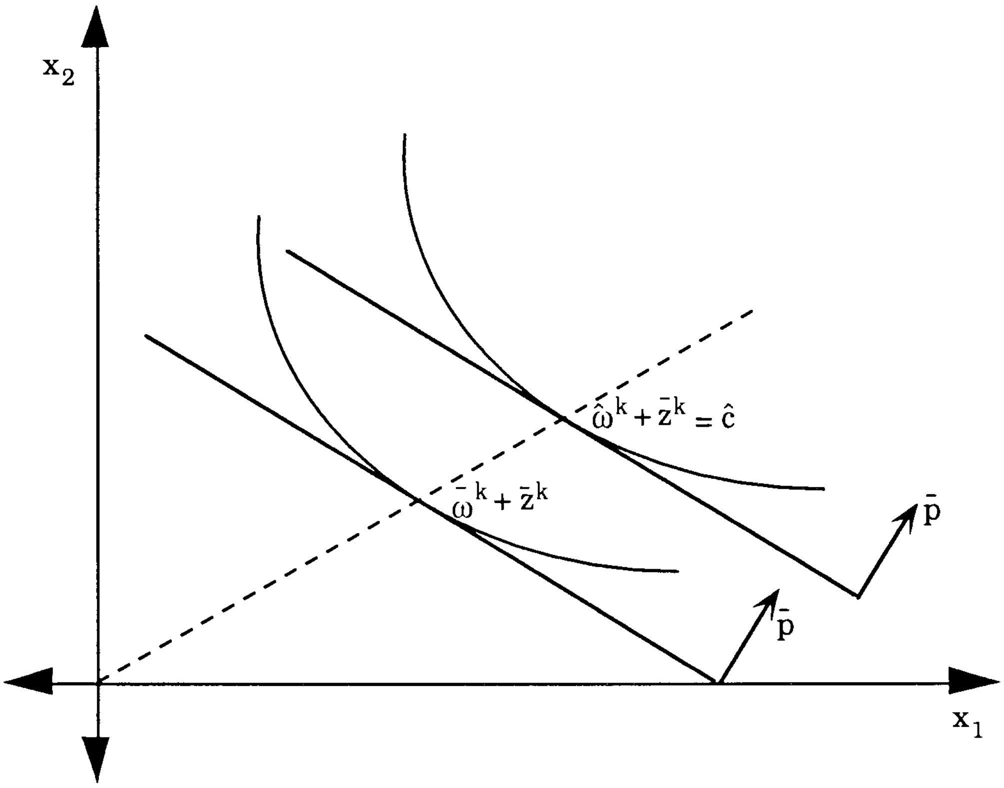

Figure 8: Unequal incomes generate envy

in figure 8 agent k is envied by all others and the mechanism is not fair. This contradiction completes the proof.

When restricted to $\tilde{E}$, the single-valuedness of the message correspondence implies that the message relation T_π and the similarity relation S_π are equivalence relations. The next Lemma establishes that, if S is the similarity relation, there exists a bijection between the quotient space $\tilde{E}/S$ and the subclass of environments $E^\star$, defined in (4). This means that in every equivalence class there is one and only one element of $E^\star$, which can be considered its canonical representation.

Lemma 6.4 *Suppose that* $\Pi = \langle M, \mu, h\rangle$ *is an allocation mechanism on* E^g *satisfying F,I,C,O,M. Given any economy* $\bar{e} \in \tilde{E}$ *there exists a unique* $e^\star \in E^\star$ *which is similar to it,* $\bar{e} S_\pi e^\star$*. Moreover* $\bar{m} = \mu(e^\star) = \mu(\bar{e})$.

PROOF: By proposition 6.1, given $\bar{m} = \mu(\bar{e})$, there exists $e^\star \in E^\star$ such that $\bar{m} = \mu(e^\star)$. By proposition 6.3, $\bar{e}$ and $e^\star$ are similar. Furthermore $e^\star$ is uniquely determined. Since the utility function of all the consumers is a Cobb- Douglas with unit coefficients, the price vector $\bar{p}$ uniquely determines the ray along which all final allocations must lie. The fact that in this economy the only fair allocation is the center of the Edgeworth box implies that this ray is its diagonal. The fact that total income is given uniquely determines the box. Finally, initial endowments of every agent are determined in such a way that the given equilibrium net trade leads to the center of the box. Hence the economy $e^\star$ is completely specified. Formally, if $\bar{e}$ is any economy and $(\bar{p}, \bar{z}, \bar{r}) = \Psi(\bar{e})$, the unique similar economy in $E^\star$ is given by

$$\omega_j^{\star i} = \frac{\bar{r}^i + \bar{p}\bar{z}^i}{\bar{p}_j \ell} - \bar{z}_j^i$$

As shown in proposition 5.2, when restricted to $E^\star$, the equal income walrasian message correspondence is an injective continuous function and the uniqueness of $e^\star$ follows.

Notice that Lemma 6.4 implies in particular that if two economies in $E^\star$ are similar, they are the same. We get as an immediate corollary that in an informationally efficient mechanism two similar economies in the class $\tilde{E}$ must use the same message.

Corollary 6.5 *Let* $\Pi = \langle M, \mu, h\rangle$ *be an allocation mechanism on* E^g *satisfying F,I,C,O,M. If two economies* $\bar{e}$ *and* $\tilde{e}$ *in* $\tilde{E}$ *are similar they have the same equilibrium message,* $\bar{m} = \mu(\bar{e}) = \mu(\tilde{e})$.

Proposition 6.3 and Corollary 6.5 imply that in the class $\tilde{E}$ the message relation and the similarity relation are exactly the same so that $\mu(\bar{e}) = \mu(\tilde{e})$ if and only if $\Psi(\bar{e}) = \Psi(\tilde{e})$. Therefore, as long as we preserve similarity, we can alter agents' characteristics without changing the equilibrium outcomes of the mechanism. The following proposition establishes that if a mechanism Π is informationally efficient, the partition induced by the similarity relation S_π is the same as that induced by the equal income Walrasian message relation T_w . In particular, the outcome of the mechanism must be the equal income Walrasian one.

Proposition 6.6 *Let $\Pi = \langle M, \mu, h \rangle$ be an allocation mechanism on E^g satisfying F,I,C,O,M. Then $\mu_w(\tilde{e}) = \Psi(\tilde{e})$ for all $\tilde{e} \in \tilde{E}$.*

PROOF: If $\tilde{e} \in \tilde{E}$ there exists some $\tilde{u} \in \mathfrak{U}^s$ such that $\tilde{u}^i = \tilde{u}$ for all agents $i \in \{1, 2, ..., n\}$. By Lemma 5.1 we know that there is a unique fair allocation for this economy, namely,

$$\tilde{\omega}^k + \tilde{z}^k = \frac{1}{n} \sum_{i=1}^{n} \tilde{\omega}^i$$

Hence, we must have

$$\tilde{z} = h_w[\mu_w(\tilde{e})] = h[\mu(\tilde{e})]$$

and this implies $\mu_w(\tilde{e}) = \Psi(\tilde{e})$ q.e.d.

Hence, if Π is an informationally efficient mechanism, the partitions of the class $\tilde{E}$ induced by the similarity relation S_π, the mechanism's message relation T_π and the equal income Walrasian message relation T_w are identical. Now we shall use this fact to show that every decentralized mechanism with a minimal message space is essentially the same as the equal income Walrasian mechanism $\Pi_w = \langle M_w, \mu_w, h_w \rangle$ in the sense that it can be transformed into it by a homeomorphism φ as shown in figure 9.

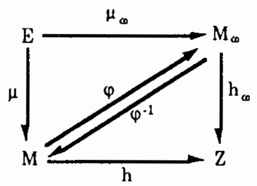

Figure 9: The basic homeomorphism

Proposition 6.7 *Let $\Pi = \langle M, \mu, h \rangle$ be an allocation mechanism on $\tilde{E}$ satisfying F,I,C,O,M. Then there exists a homeomorphism $\varphi : M \rightarrow M_w$ such that :*

a) $\mu^i_w(e^i) = \varphi[\mu^i(e^i)]$ *for all* $e^i = (u^i, \omega^i) \in \mathfrak{U}^s \times R^\ell$

b) $\mu_w = \varphi \circ \mu$

c) $h_w \circ \varphi = h$

PROOF: By Proposition 5.2, for any $u \in \mathfrak{U}^s$, the restriction of μ and μ_w to $E(u)$ is an injective function. By Proposition 6.1 $\mu[E(u)] = \mu_w[E(u)] = M$. Then, for any $m \in M$ there exists a unique economy $e \in E(u)$ such that $\mu(e) = m$. Call it $\gamma(m)$.

Define $\varphi : M \rightarrow M_w$ as the function given by $\varphi(m) = \Psi[\gamma(m)]$. This is indeed a function because, in view of Proposition 6.3 and Corollary 6.5 it follows that

$$\mu(\bar{e}) = \mu(\hat{e}) \Leftrightarrow \Psi(\bar{e}) = \Psi(\hat{e}) \tag{7}$$

so that the image of a message m is independent of the utility $u \in \mathfrak{U}^s$ chosen as reference. Using the definition of φ, Proposition 6.6, and Proposition 6.1 as applied to the equal income Walrasian mechanism we get

$$\varphi[M] = \Psi[E(u)] = \mu_w[E(u)] = M_w$$

so that φ maps M onto M_w. Now we show that it is injective. By assumption C, μ is a single-valued function on $E(u)$. Then γ is injective, that is, if $\bar{m} \neq \tilde{m}$ then $\gamma(\bar{m}) \neq \gamma(\tilde{m})$. As shown in Proposition 5.2, the message correspondence μ restricted to $E^\star$ is injective, so that $\mu(\gamma(\bar{m})) \neq \mu(\gamma(\tilde{m}))$. From 7 it follows that $\Psi(\gamma(\bar{m})) \neq \Psi(\gamma(\tilde{m}))$ and by definition of φ we get $\varphi(\bar{m}) \neq \varphi(\tilde{m})$ proving injectiveness. Hence, φ is a bijection.

In order to prove statement a) of the theorem given any agent $\bar{e}^i$ with utility $\bar{u}$ let $(\bar{p}, \bar{z}, \bar{r}) \in \varphi[\mu^i(e^i)]$ and $\bar{m} \in \mu^i(e^i)$. Let $\bar{e}$ be the unique economy in $E(\bar{u})$ such that $\mu(\bar{e}) = \bar{m}$. Then by definition of φ and Proposition 6.6

$$(\bar{p}, \bar{z}, \bar{r}) = \varphi(\bar{m}) = \Psi(\bar{e}) = \mu_w(\bar{e}) \subset \mu_w^i(\bar{e}^i)$$

Conversely, given any $(\bar{p}, \bar{z}, \bar{r}) \in \mu_w^i(\bar{e}^i)$, let $\bar{e}$ be the unique economy in $E(\bar{u})$ such that $(\bar{p}, \bar{z}, \bar{r}) = \mu_w(\bar{e})$. Then

$$(\bar{p}, \bar{z}, \bar{r}) = \mu_w(\bar{e}) = \Psi(\bar{e}) = \varphi[\mu(\bar{e})] \subset \varphi[\mu^i(\bar{e}^i)]$$

and statement a) of the Proposition is established. Since φ is a bijection, statement b) follows directly.

To prove statement c), note that $\varphi(m)$ is a vector (p, z, r) such that $z = h(m)$. Since h_w is just a projection, $h_w(\varphi(m)) = h(m)$ for all m.

Finally, it remains to show that φ and φ^{-1} are continuous. In order to show that φ^{-1} is continuous let $\{(p^\nu, z^\nu, r^\nu)\}$ be a sequence in M_w converging to $(\bar{p}, \bar{z}, \bar{r})$. We have to demonstrate that the sequence m^ν, where $m^\nu = \varphi^{-1}(p^\nu, z^\nu, r^\nu)$ converges to $\bar{m} = \varphi^{-1}(\bar{p}, \bar{z}, \bar{r})$. By taking

$$\begin{aligned}
\omega_j^{i\nu} &= \frac{r^{i\nu} + p^\nu z^{i\nu}}{p_j^\nu \ell} - z_j^{i\nu} \\
\alpha_j^{i\nu} &= 1
\end{aligned}$$

for all i, j and n, we define a sequence of economies $e^\nu = (\omega^\nu, \alpha^\nu)$ in the class $E^\star$ which converges to $\bar{e} = (\bar{\omega}, \bar{\alpha})$ given by

$$\begin{aligned}
\bar{\omega}_j^i &= \frac{\bar{r}^i + \bar{p}\bar{z}^i}{\bar{p}_j \ell} - \bar{z}_j^i \\
\bar{\alpha}_j^i &= 1
\end{aligned}$$

for all i, j and n. But by statement 2 of the present proposition $m^\nu = \varphi^{-1}(p^\nu, z^\nu, r^\nu) = \varphi^{-1}(\mu_w(e^\nu) = \mu(e^\nu)$ for all ν and $m = \varphi^{-1}(\bar{p}, \bar{z}, \bar{r}) = \varphi^{-1}(\mu_w(\bar{e})) = \mu(\bar{e})$. Since μ is continuous, $\{e^\nu\}$ converges to $\bar{e}$ implies $\{m^\nu\}$ converges to $\bar{m} = \mu(\bar{e})$. Hence φ^{-1} is continuous. Since M and M_w are manifolds of the same dimension, φ^{-1} is in fact a homeomorphism [13] from M_w onto M and the proof is complete.

[13] See page 82 in Greenberg [13]

7 Extension of the uniqueness theorem

In this section we extend the uniqueness theorem to a general class of economies in which agents may have different utilities.

Proposition 7.1 *Let $\Pi = \langle M, \mu, h\rangle$ be an allocation mechanism on E^g satisfying F,I,C,O,M. Then there exists a homeomorphism $\varphi : M \longrightarrow M_w$ such that for all $e \in E^g$:*

a) $\varphi[\mu^i(e^i)] \subset \mu^i_w(e^i)$ *for all* $e^i = \rangle u^i, \omega^i \in \mathfrak{U}^g \times R^\ell$

b) $\varphi[\mu(e)] \subset \mu_w(e)$ *for all* $e \in E^g$

c) $h_w \circ \varphi = h$

The homeomorphism φ given in Proposition 6.7 satisfies statement c). To show $\varphi[\mu^i(e^i)] \subset \mu^i_w(e^i)$, let $\bar{m} \in \mu^i(\bar{e}^i)$ and let $(\bar{p}, \bar{z}, \bar{r}) = \varphi(\bar{m})$. We have to show that $(\bar{p}, \bar{z}, \bar{r})$ is an equal income Walrasian equilibirum message for the i-th agent, that is, we have to establish i) $\bar{p}$ is a supporting price of $\bar{u}^i$ at $(\bar{\omega}^i + \bar{z}^i$, and ii) the equal income property is satisfied.

To show i) take any $\hat{u} \in \mathfrak{U}^s$. By Proposition 6.1 there exists $\hat{e} \in E(\hat{u})$ such that $\bar{m} = \mu(\hat{e})$. Then $\Psi(\hat{e}) = (\bar{p}, \bar{z}, \bar{r}) = \mu_w(\hat{e})$. Now, by the crossing property $\bar{m} \in \mu(\bar{e} \otimes_i \hat{e})$ so that the allocation $(\bar{\omega} + \bar{z}) \otimes_k (\hat{\omega} + \bar{z})$ is a Pareto optimal allocation for the mixed economy. Assertion i) follows because it is a necessary condition for optimality.

Suppose that the equal income property is not satisfied, so that for some agent k, $\bar{p}(\bar{\omega}^k + \bar{z}^k) < \bar{p}(\bar{\omega}^i + \bar{z}^i)$. By Proposition 6.2 we can construct an economy $\hat{e} \in E(\hat{u})$ with a identical homothetic utility function $\hat{u} \in \mathfrak{U}^s$ for all agents such that:

a) the unique fair final consumption is given by $\hat{c} = \bar{\omega}^k + \bar{z}^k$

b) the utility is flat enough so that $\hat{u}(\hat{c}) < \hat{u}(\bar{\omega}^i + \bar{z}^i)$

c) $\bar{m} = \mu(\hat{e})$

Now take the economy $\hat{e}$ and substitute the i-th original agent so that we get the economy $\bar{e} \otimes_i \hat{e}$. By the crossing condition we still have the same equilibrium message and net trade, so that in this new economy all agents, but k, get $\hat{c} = \bar{\omega}^k + \bar{z}^k$, while agent i gets $\bar{\omega}^i + \bar{z}^i$. Agent i is envied by all others and the mechanism is not fair. Statement 2 follows directly.

If preferences satisfy a boundary condition, the we get the following uniqueness theorem

Proposition 7.2 *Let $\Pi = \langle M, \mu, h\rangle$ be an allocation mechanism on E^g satisfying F,I,C,O,M,B. Then there exists a homeomorphism $\varphi : M \longrightarrow M_w$ such that for all $e \in E^b$:*

a) $\varphi[\mu^i(e^i)] = \mu^i_w(e^i)$ *for all agents*

b) $\varphi[\mu(e)] = \mu_w(e)$

c) $h_w \circ \varphi = h$

PROOF: In view of proposition 7.1 it remains to be shown that $\mu^i_w(e^i) \subset \varphi[\mu^i(e^i)]$. Take any $(\bar{p}, \bar{z}, \bar{r}) \in \mu^k_w(e^k)$. Define an economy $\hat{e} \in E(\hat{u})$ such that all agents have the same utility function $\hat{u} = u^k$ and $\hat{\omega}^i = \bar{\omega}^k + \bar{z}^k - \bar{z}^i$. Notice that the k-th agents' characteristic has not changed, $\bar{e}^k = \hat{e}^k$. Now we claim that $(\bar{p}, \bar{z}, \bar{r})$ is the unique equal income Walrasian equilibrium message, $(\bar{p}, \bar{z}, \bar{r}) = \mu_w(\hat{e})$. This follows immediately from Proposition 5.1, according to which the only fair allocation in this economy is the center of the Edgeworth box and this is in fact the allocation obtained since $\hat{\omega}^i + \bar{z}^i = \hat{\omega}^k + \bar{z}^k = \bar{\omega}^k + \bar{z}^k$ for all i. By the preceeding proposition we know that $\varphi[\mu(\hat{e})] \subset \mu_w(\hat{e}) = (\bar{p}, \bar{z}, \bar{r})$. Hence $\varphi[\mu(\hat{e})] = (\bar{p}, \bar{z}, \bar{r})$. On the other hand, $\mu(\hat{e}) \subset \mu^k(\hat{e}^k) = \mu^k(\bar{e}^k)$. Hence $\varphi[\mu(\hat{e}] \subset \varphi[\mu^k(\bar{e}^k)]$. Then we finally get the desired result $(\bar{p}, \bar{z}, \bar{r}) = \mu_w(\hat{e}) \subset \varphi[\mu^k(e^k)]$.

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