On moments and tail behaviors of storage processes

Arturo Kohatsu-Higa*
Universitat Pompeu Fabra, Department of Economics
and Business, Ramón Trias Fargas 25-27, 08005 Barcelona, Spain
Makoto Yamazato
University of the Ryukyus, Department of Mathematics, Faculty of Science,
Senbaru1, Nishihara-cho, Okinawa, Japan 903-0213.

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Abstract

We study the existence of moments and the tail behaviour of the densities of storage processes. We give sufficient conditions for existence and non-existence of moments using the integrability conditions of submultiplicative functions with respect to Lévy measures. Then, we study the asymptotical behavior of the tails of these processes using the concave or convex envelope of the release rate function.

Keywords: dam process, storage process, subordinator, submultiplicative functions, subexponentiality.
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1 Introduction

Storage processes are stochastic processes $\{X(t)\}$ defined through a stochastic differential equation of the type

$$X(t) = x - \int_0^t r(X(s))ds + A(t). \tag{1}$$

Here, A is an increasing stochastic process called input process and r is a non-negative function, usually called the release rate. The solution of this type of stochastic differential equation has applications in storage systems theory, economics and insurance risk theory. For example, A can represent the amount of items arriving at a storage, the amount of rain that a dam receives or the amount of stochastic interest accrued by an account. The function r can represent the way and/or rate the stored items are sold or delivered, how the water is released or the amount of money that is being used. Some references to possible applications can be found as early as Kendall (1957) and in Desmukh and Pliska (1980).

In this article we are interested in a mathematical property related to storage processes with increasing Lévy processes as inputs. That is, first we study the existence of moments and second, we study asymptotics of tail probabilities of X.

We will study the nonexistence (in Section 3) and the existence (in Section 4) of $E(g(X(t)))$ for a positive function g. In various cases g will be a submultiplicative function. Power functions are particular cases of submultiplicative functions. In fact, our results applied to these particular examples give the following table:

$r(y) = y^{\alpha}$	$g(y) = y^{\beta}$	Criteria
$0 \le \alpha \le 1$	$\beta > 0$	$\int_1^{+\infty} y^{\beta} \nu(dy)$
$\alpha > 1$	$\beta < \alpha - 1$	always finite
$\alpha > 1$	$\beta = \alpha - 1$	$\int_1^{+\infty} \log y \, \nu(dy)$
$\alpha > 1$	$\beta > \alpha - 1$	$\int_1^{+\infty} y^{\beta - \alpha + 1} \nu(dy)$

The above criteria determines which moments of $X(t)$ are finite or infinite.

Although these results obviously give some information on the tail probabilities of these processes, we give more precise asymptotics (exact order) of tail probabilities of storage processes via their Lévy measures in Section 6. To do this, the concept subexponentiality ([3]) plays an important role. Our results show that process of Ornstein-Uhlenbeck type ($r(y) = ay$ case) occupies a critical position on the tail behavior (Theorem 6.2). A result similar in part to one of our results (Theorem 6.3) has been obtained by Grigoriu and Samorodnitsky [8] under a different assumption.

Articles related with the properties we study here are Asmussen (1998) where the tail behaviour of the stationary distribution of storage processes and the distribution of the ruin time of risk processes is investigated. In Sigman and Yao (1994) the existence of moments for storage processes is studied although this study does not cover stable processes while ours does.

Possible applications of these results are in statistical properties of parameter estimators, simulation and numerical analysis of these systems such as weak or strong approximation results.

2 Preliminaries

In this section we describe how to construct a solution to (1). For further details, we refer to [2]. We assume the following hypotheses for r throughout the article.

(H0) $r : [0, \infty) \to [0, \infty)$ such that $r(0) = 0$, $r(x) > 0$, for $x > 0$, left continuous and $\lim_{y \to x+} r(y) > 0$ for all $x > 0$.

We call $r(x)$ a release rate. Let $\{A(t)\}$ be an increasing cadlag Lévy process such that $A(0) = 0$ and

$$E[e^{-\theta A(t)}] = \exp\{\int_0^{\infty} t(e^{-\theta y} - 1)\nu(dy)\}; \quad \theta \ge 0$$

where ν is a measure on $(0,\infty)$ satisfying

$$0 < \int_0^\infty (x \wedge 1)\nu(dx) < \infty.$$

This measure ν is called the Lévy measure of $\{A(t)\}$.

The idea of the construction is to first treat the case of finite number of jumps. This can be solved explicitly pathwise. Finally one takes limits in the number of jumps in order to define a solution to (1).

Consider first the simple case of $\lambda = \nu(0,+\infty) < +\infty$. Then the number of jumps of A is finite in any compact interval. Denote the jump times and the jump sizes by T_n and Y_n, $n = 1,\dots$, respectively. The interarrival times are denoted by $\tau_n = T_n - T_{n-1}$. In between two jumps X is a solution of an ordinary differential equation that can be written using some auxiliary function q which we define now. Set

$$R(x,y) = \int_x^y \frac{1}{r(z)}dz$$

for $0 < x \le y$. Define $R(0,y) := R(0+,y) \le \infty$. Since the function $R(x,y)$; for $0 < x \le y$; is continuous and strictly decreasing in x, it has a continuous inverse $R_y^{-1}(t)$ for $t \in [0,R(0,y))$. Define $q(t,y)$ by

$$q(t,y) = \begin{cases} R_y^{-1}(t) & \text{for } 0 \le t < R(0,y), \\ 0 & \text{for } R(0,y) \le t \quad (\text{if } R(0,y) < \infty). \end{cases}$$

Then, q satisfies the following properties:

1. $q(R(x,y),y) = x$ for $0 < x \le y < \infty$ and $R(q(t,y),y) = t$ for $y > 0$ and $0 \le t < R(0,y)$.
2. $q(t,y)$ is continuous, decreasing in t and increasing in y.
3. Since $R(x,y)$ is nondecreasing and left differentiable in x, $q(t,y)$ is right differentiable in t and it satisfies

$$\begin{cases} \frac{d^+}{dt}q(t,y) &= -r(q(t,y)); \\ q(0,y) &= y; \end{cases}$$

   for $x > 0$ and $0 \le t < R(x,0)$.

Under this situation the solution X of (1) is given by

$$\begin{aligned} X(0) &= x \\ X(T_m) &= q(\tau_m, X(T_{m-1})) + Y_m \quad (m \ge 1); \\ X(t) &= q(t - T_m, X(T_m)) \quad \text{for} \quad T_m < t < T_{m+1}. \end{aligned}$$

To solve the general situation with a general Lévy measure ν, we set $\{A_n(t)\}$ for $n \ge 1$ by

$$A_n(t) = \sum_{s \le t} (A(s) - A(s-))1_{\{A(s)-A(s-)>\frac{1}{n}\}}.$$

Then, $\{A_n(t)\}$ is an increasing Lévy process with Lévy measure $\nu_n(\cdot) = \nu(\cdot \cap (\frac{1}{n},\infty))$ with $\lambda_n = \nu_n(0,\infty) < +\infty$. For each $n \ge 1$, there is unique process $\{X_n(t)\}$ satisfying (1). Since $A_n(t)$ is nondecreasing in n, $\{X_n(t)\}$ is also nondecreasing in n. One can therefore define $X(t) = \lim_{n\to\infty} X_n(t)$. Then, $\{X(t)\}$ satisfies (1) with driving noise $\{A(t)\}$. This process $\{X(t)\}$ is called a storage process starting at x corresponding to r and ν. This process is a Hunt process. We call the Lévy process $\{A(t)\}$ input process of $\{X(t)\}$. We denote $X(t)$ starting at x by $X(t,x)$ if necessary. For uniqueness and any further details we refer the reader to [2].

We denote the distribution of the i.i.d. random variables Y_n by $F_n = \lambda_n^{-1}\nu_n$. The random variables $\tau_k = T_k - T_{k-1}$, $k \ge 1$ are i.i.d. with identical density $\lambda_n e^{-\lambda_n t}$. The sequences $\{Y_k\}$ and $\{\tau_k\}$ are mutually independent.

3 Non-existence of moments

In this and the next sections, we study relations between the tail behavior of the Lévy measure and the existence of the moments of $\{X(t)\}$. We show that there is a remarkable difference between the two cases $\int_1^\infty \frac{1}{r(y)}dy < \infty$ and $\int_1^\infty \frac{1}{r(y)}dy = \infty$ (Examples 3.1 ∼ 4.2. See also the table in the Introduction).

Lemma 3.1 Let g be a nonnegative and nondecreasing function on $[0,\infty)$. Let $\lambda = \nu((1,\infty))$. Then

$$E[g(X(t,x))] \geq g(q(t,x))e^{-\lambda t} + \int_1^\infty \left\{\int_0^t e^{-\lambda s} g(q(s,y))ds\right\}\nu(dy)$$

for all $x, y \geq 0$.

Proof Let $\{X_1(t)\}$ be a storage process corresponding to $\{A_1(t)\}$ defined in Section 2. Then $X(t,x) \geq X_1(t,x)$ and we have

$$\begin{aligned} E[g(X(t,x))] &\geq E[g(X_1(t,x))] \\ &= \sum_{n=0}^{\infty} E[g(X_1(t,x)) : T_n \leq t < T_{n+1}]. \end{aligned}$$

Since

$$X_1(t,x) \geq q(t-T_n, q(T_n,x)+Y_n) \quad \text{on } \{T_n \leq t < T_{n+1}\},$$

we have

$$E[g(X_1(t,x)) : t < T_1] = g(q(t,x))e^{-\lambda t}$$

and

$$\begin{aligned} &E[g(X_1(t,x)) : T_n \leq t < T_{n+1}] \\ \geq\ &E[g(q(t-T_n, q(T_n,x)+Y_n)) : T_n \leq t < T_{n+1}] \end{aligned}$$

for all $n \geq 1$ and for all $x, t \geq 0$. We have, for $n \geq 1$,

$$\begin{aligned} &E[g(q(t-T_n, q(T_n,x)+Y_n)) : T_n \leq t < T_{n+1}] \\ =\ &e^{-\lambda t}\int_1^\infty \left(\int_0^t \frac{(\lambda s)^{n-1}}{(n-1)!} g(q(t-s, q(s,x)+y))ds\right)\nu(dy). \end{aligned}$$

Hence we have

$$\begin{aligned} E[g(X(t,x))] &\geq g(q(t,x))e^{-\lambda t} + e^{-\lambda t}\int_1^\infty \left(\int_0^t e^{\lambda s} g(q(t-s,y))ds\right)\nu(dy) \\ &\geq g(q(t,x))e^{-\lambda t} + \int_1^\infty \left(\int_0^t e^{-\lambda s} g(q(s,y))ds\right)\nu(dy). \end{aligned}$$

Hence we have the lemma. □

First, we give sufficient conditions for non-existence of moments.

Theorem 3.1 Let g be a nonnegative and nondecreasing function defined on $[0,\infty)$. If there is $v > 0$ such that

$$\int_1^\infty \left(\int_{q(v,y)}^y \frac{g(z)}{r(z)}dz\right)\nu(dy) = \infty,$$

then

$$E[g(X(t,x))] = \infty \quad \text{for all } x \geq 0 \text{ and for all } t \geq v.$$

Proof We have, by Lemma 3.1, for any $x \geq 0$,

$$
\begin{aligned}
E[g(X(t;x))] &\geq E[g(X_1(t;x))] \\
&\geq e^{-\lambda t}\int_1^{\infty}\left(\int_{q(t;y)}^{y}\frac{g(z)}{r(z)}dz\right)\nu(dy) = \infty
\end{aligned}
$$

for $t \geq v$. Here we used the change of variable $z = q(s;y)$. $\Box$

Remark 3.1 In the conclusion of Therem 3.1, we can not substitute "for $t \geq v$" to "for all $t > 0$". We give a counter example. Let $r(y) \leq r$ for $y \geq 0$, $\nu(dy) = \frac{e^{-y}}{y}dy$ for $y > 0$ and

$$
g(y) = \begin{cases} e^y & \text{for } y < 0 \\ (y^{-1} \wedge 1)e^y & \text{for } y \geq 0. \end{cases}
$$

Note that

$$
x - rt + A(t) \leq X(t;x) \leq x + A(t). \tag{2}
$$

The distribution $P(t;dy)$ of $x - rt + A(t)$ is $\frac{1}{\Gamma(t)}(y - x + rt)^{t-1}e^{-(y-x+rt)}dy$. Hence

$$
\begin{aligned}
\int_0^{\infty} g(y)P(t;dy) &= \frac{1}{\Gamma(t)}\int_{x-rt}^{\infty} g(y)(y - x + rt)^{t-1}e^{-(y-x+rt)}dy \\
&= C(t;x)\int_0^{\infty}(y^{-1}\wedge 1)(y - x + rt)^{t-1}dy + D(t;x) \quad \begin{cases} < \infty & \text{for } t < 1, \\ = \infty & \text{for } t > 1, \end{cases}
\end{aligned}
$$

where $0 < C(t;x) < \infty$ and $0 \leq D(t;x) < \infty$ for every $t \geq 0$. Then by (2), we have

$$
E[g(X(t;x))] \quad \begin{cases} < \infty & \text{for } t < 1, \\ = \infty & \text{for } t > 1. \end{cases}
$$

Theorem 3.1 may not be suitable for direct application due to the necessity of computing $q(s;y)$ in order to check that the condition is valid. One can simplify the above restriction if more conditions are assumed like the following corollaries.

Corollary 3.2 Assume that $r(y) = O(y)$ as $y \to \infty$ and g is a nonnegative and nondecreasing function on $[0;\infty)$ such that, for any $0 < a \leq 1$ and $y > 0$, $g(ay) \geq c(a)g(y)$ with $c(a) > 0$. If for

$$
\int_1^{\infty} g(y)\nu(dy) = \infty
$$

then, $E[g(X(t;x))] = \infty$ for all $x \geq 0; t > 0$.

Proof Note that

$$
\int_1^{\infty}\frac{1}{r(y)}dy < \infty \qquad \text{if and only if } q(t;\infty) = \lim_{y\to\infty} q(t;y) < \infty \text{ for all } t > 0: \tag{3}
$$

By the assumption, there is $M > 0$ and y_0 such that

$$
\frac{1}{r(y)} \geq \frac{1}{My}
$$

for $y \geq y_0$. Hence $\lim_{y \to \infty} q(t;y) = \infty$ for any $t > 0$. Fix $t_0 > 0$ arbitrarily. Choose $y_1 > 0$ so that $q(t_0; y_1) \geq y_0$. Note that $q(t;y) \geq y_0$ for $y \geq y_1$ and $t \leq t_0$. By the definition of $q(t;y)$, we have

$$t = \int_{q(t;y)}^{y} \frac{1}{r(z)} dz \geq \frac{1}{M} \int_{q(t;y)}^{y} \frac{1}{z} dz = \frac{1}{M} \log \frac{y}{q(t;y)}$$

for $y \geq y_1$ and $t \leq t_0$. Hence, $q(t;y) \geq y e^{-Mt}$ for all $t \in [0; t_0]$ and all $y \geq y_1$. We have, by the assumption,

$$\int_{q(t;y)}^{y} \frac{g(z)}{r(z)} dz \geq g(q(t;y)) \int_{q(t;y)}^{y} \frac{1}{r(z)} dz \geq tc(e^{-Mt}) g(y)$$

for $y \geq y_1$ and $0 \leq t \leq t_0$. Hence

$$\int_{1}^{\infty} \left(\int_{q(t;y)}^{y} \frac{g(z)}{r(z)} dz \right) \nu(dy) \geq tc(e^{-Mt}) \int_{y_1}^{\infty} g(y) \nu(dy) = \infty$$

for $0 < t \leq t_0$. Then, by Theorem 3.1,

$$E[g(X(t;x))] = \infty \quad \text{for all } x \geq 0 \text{ and for all } t > 0. \; \square$$

Example 3.1 Let $r(y) = cy^{\alpha}$ $(\alpha \leq 1)$ and $g(y) = y^{\beta}$ $(\beta > 0)$. If $\int_1^{\infty} y^{\beta} \nu(dy) = \infty$, then $E(X(t;x)^{\beta}) = \infty$ for $x \geq 0; t > 0$ by Corollary 3.2.

Corollary 3.3 Assume that $\int_1^{\infty} \frac{1}{r(y)} dy < \infty$ and g is a nonnegative and nondecreasing function on $[0; \infty)$. If

$$\int_{1}^{\infty} \left(\int_{1}^{y} \frac{g(z)}{r(z)} dz \right) \nu(dy) = \infty;$$

then

$$E[g(X(t;x))] = \infty \quad \text{for all} \quad x \geq 0; t > 0.$$

Proof Note that due to (3), we have that $q(t; \infty) < \infty$ for any $t > 0$. Without loss of generality we assume that $q(t; \infty) > 1$. Then for $y > q(t; \infty)$

$$\int_{q(t;y)}^{y} \frac{g(z)}{r(z)} dz \geq \int_{q(t;\infty)}^{y} \frac{g(z)}{r(z)} dz.$$

Then conclusion holds by Theorem 3.1. $\square$

Example 3.2 Let $r(y) = y^{\alpha}$ $(\alpha > 1)$, $g(y) = y^{\beta}$ $(\beta > 0)$ and $\beta > \alpha - 1$. If

$$\int_{1}^{\infty} y^{\beta - \alpha + 1} \nu(dy) = \infty;$$

then

$$E[X(t;x)^{\beta}] = \infty \quad \text{for all} \quad x \geq 0; t > 0$$

by Corollary 3.3.

4 Existence of moments

So far, we have studied sufficient conditions for non-existence of moments of $X(t)$. Now, we give sufficient conditions for the existence of moments. Let $x_0 \geq 0$. We say that a function g on $[x_0, \infty)$ is submultiplicative on $[x_0, \infty)$ if it is nonnegative and there is a constant $a > 0$ such that

$$g(y + z) \leq a g(y) g(z) \quad \text{for} \quad y, z \geq x_0.$$

Lemma 4.1 Suppose that $g(y)$ is a nonnegative and nondecreasing function on $[0, \infty)$ which is submultiplicative on $[x_0, \infty)$, where $x_0 \geq 0$. Then, $\int_1^\infty g(z)\nu(dz) < \infty$ if and only if $E[g(A(t))] < \infty$ for all $t \geq 0$.

Proof Under the assumption of the lemma, necessary facts for the proof of Theorem 25.3 of [12] hold. $\Box$

Theorem 4.1 Let $x_0 \geq 0$ and let $g(y)$ be a nonnegative and nondecreasing function on $[0, \infty)$ which is submultiplicative on $[x_0, \infty)$. Then $\int_1^\infty g(z)\nu(dz) < \infty$ implies

$$E[g(X(t; x))] < \infty \quad \text{for all} \quad x \geq 0, t > 0.$$

Proof Note that $X(t; x) \leq x + A(t)$ for all $t \geq 0$. By Lemma 4.1,

$$E[g(A(t))] < \infty.$$

This yields the conclusion. $\Box$

Example 4.1 Let $g(y) = y^\beta$ with $\beta > 0$ and r satisfying (H0). If

$$\int_1^\infty y^\beta \nu(dy) < \infty, \tag{4}$$

then, by Theorem 4.1, $E[X(t; x)^\beta] < \infty$ for all $x \geq 0, t > 0$.

In the case $r(y) = y^\alpha$, $\alpha > 1$, conditions weaker than (4) are sufficient for the existence of moments of $\{X(t)\}$ of order β. Next three theorems treat this case.

Theorem 4.2 Assume that $\nu((0, \infty)) < \infty$. Let g be a nonnegative and nondecreasing function on $[0, \infty)$. Suppose that there is $x_0 > 0$ and $C \geq 0$ such that $G(y) = \int_{x_0}^y \frac{g(z)}{r(z)} dz + C$ is submultiplicative on $[x_0, \infty)$. Then

$$\int_{x_0}^\infty G(y)\nu(dy) < \infty$$

implies

$$E[g(X(t; x))] < \infty \quad \text{for all} \quad x \geq 0, t > 0.$$

Proof Let $\lambda = \nu((0, \infty))$ and $F(dy) = \frac{1}{\lambda}\nu(dy)$. We define $\{\tau_k\}$, $\{Y_k\}$, $\{T_k\}$ as Section 2 for $\{X(t)\}$. Choose x so that $x > x_0$. Note that

$$E[g(X(t; x)) : 0 \leq t < T_1] = g(q(t; x))e^{-\lambda t}$$

and, for $n \geq 1$,

$$
\begin{aligned}
& E[g(X(t,x)) : T_n \leq t < T_n + \tau_{n+1}] \\
= \; & E[g(q(t - T_n, X(T_n -, x) + Y_n)) : T_n \leq t < T_n + \tau_{n+1}] \\
\leq \; & E[g(q(t - T_n, x + Y_1 + \cdots + Y_n)) : T_n \leq t < T_n + \tau_{n+1}] \\
= \; & \int_0^\infty \cdots \int_0^\infty \left\{ \int_0^t e^{-\lambda(t-s)} g(q(t - s, x + y_1 + \cdots + y_n)) P(T_n \in ds) \right\} F(dy_1) \cdots F(dy_n) \\
= \; & \frac{\lambda^n e^{-\lambda t}}{(n-1)!} \int_0^\infty \cdots \int_0^\infty \left\{ \int_0^t g(q(t - s, x + y_1 + \cdots + y_n)) s^{n-1} ds \right\} F(dy_1) \cdots F(dy_n) \\
\leq \; & \frac{\lambda^n t^{n-1} e^{-\lambda t}}{(n-1)!} \int_0^\infty \cdots \int_0^\infty \left\{ \int_0^t g(q(s, x + y_1 + \cdots + y_n)) ds \right\} F(dy_1) \cdots F(dy_n) \\
= \; & \frac{\lambda^n t^{n-1} e^{-\lambda t}}{(n-1)!} \int_0^\infty \cdots \int_0^\infty \\
& \left\{ G(x + y_1 + \cdots + y_n) - G(q(t, x + y_1 + \cdots + y_n)) \right\} F(dy_1) \cdots F(dy_n) \\
\leq \; & \frac{\lambda^n t^{n-1} e^{-\lambda t}}{(n-1)!} \int_0^\infty \cdots \int_0^\infty \left\{ G(x + y_1 + \cdots + y_n) - G(q(t, x)) \right\} F(dy_1) \cdots F(dy_n).
\end{aligned}
$$

Since $G(y)$ is submultiplicative on $[x_0, \infty)$, there is $c > 0$ such that

$$G(x + y_1 + \cdots + y_n) \leq c^{n-1} \prod_{i=1}^{n} G(x + y_i).$$

Also by the submultiplicativity,

$$\int_0^\infty G(x + y)F(dy) \leq \int_0^{x_0} G(x + y)F(dy) + cG(x) \int_{x_0}^\infty G(y)F(dy) =: K < \infty.$$

Then, we have

$$E[g(X(t,x)) : T_n \leq t < T_n + \tau_{n+1}] \leq \frac{\lambda^n t^{n-1} e^{-\lambda t}}{(n-1)!} \{K(cK)^{n-1} - G(q(t,x))\}.$$

Hence,

$$
\begin{aligned}
E[g(X(t,x))] & \leq g(q(t,x))e^{-\lambda t} + \lambda \sum_{n=1}^{\infty} \frac{(\lambda t)^{n-1}}{(n-1)!} e^{-\lambda t} \{K(cK)^{n-1} - G(q(t,x))\} \\
& = g(q(t,x))e^{-\lambda t} + \lambda K e^{\lambda t(cK-1)} - \lambda G(q(t,x)).
\end{aligned}
$$

To prove the finiteness of the above expectation is enough to prove that $G(q(t,x))$ can not take the value $-\infty$. This is only possible if $q(t,x) = 0$. In such a case $\int_0^x \frac{1}{r(z)} dz < \infty$ which implies that $G(0) > -\infty$. Therefore $E[g(X(t,x))] < \infty$ for $x \geq x_0$. To conclude for $x < x_0$ is enough to note that $X(t,x) \leq X(t,x_0)$ for $0 \leq x < x_0$. □

In the previous result we had to assume that the Lévy measure was finite, in the next we exchange this condition with a restriction on r and further restriction on g.

Theorem 4.3 Assume that r is nondecreasing. Let g be a nonnegative and nondecreasing on $[0, \infty)$. Suppose that there is $x_0 > 0$ and $C \geq 0$ such that both $g(y)$ and $G(y) = \int_{x_0}^y \frac{g(z)}{r(z)} dz + C$ are submultiplicative on $[x_0, \infty)$. Then

$$\int_{x_0}^\infty G(y)\nu(dy) < \infty$$

implies

$$E[g(X(t;x))] < \infty \quad \text{for} \quad x \ge 0; t > 0.$$

Proof Let $A(t)$ be an input process with Lévy measure ν. Let

$$A_1(t) = \sum_{s \le t} \{A(s) - A(s-)\} 1_{\{A(s) - A(s-) > 1\}}$$

and let $X_1(t;x)$ be the storage process starting at x with input process $A_1(t)$. Then $\{X_1(t;x)\}$ satisfies

$$X_1(t;x) = x - \int_0^t r(X_1(s;x))ds + A_1(t).$$

Since $X(t;x) \ge X_1(t;x)$ and r is nondecreasing, we have

$$\begin{aligned} X(t;x) - X_1(t;x) &= -\int_0^t \{r(X(s;x)) - r(X_1(s;x))\}ds + A(t) - A_1(t) \\ &\le A(t) - A_1(t). \end{aligned}$$

By the submultiplicativity and nondecreasingness of g, we have

$$\begin{aligned} E[g(X(t;x))] &\le E[ag(x_0 + X(t;x) - X_1(t;x))g(x_0 + X_1(t;x))] \\ &\le E[ag(x_0 + A(t) - A_1(t))g(x_0 + X_1(t;x))] \\ &= aE[g(x_0 + A(t) - A_1(t))]E[g(x_0 + X_1(t;x))] \end{aligned}$$

where $a > 0$ is the constant of submultiplicativity of g. Here, we also used the mutual independence of $A - A_1$ and X_1. By Lemma 4.1 (note that the Lévy process $A - A_1$ has a Lévy measure with bounded support),

$$E[g(x_0 + A(t) - A_1(t))] < \infty \quad \text{for } t > 0.$$

Now we prove the finiteness of $E[g(x_0 + X_1(t;x))]$. If $X_1(t;x) \le x_0$, then $g(x_0 + X_1(t;x)) \le g(2x_0)$. If $X_1(t;x) > x_0$ then $g(x_0 + X_1(t;x)) \le ag(x_0)g(X_1(t;x))$. In any case,

$$g(x_0 + X_1(t;x)) \le g(2x_0) + ag(x_0)g(X_1(t;x))$$

By the preceding Theorem, we have

$$E[g(x_0 + X_1(t;x))] < \infty \quad \text{for all} \quad t > 0.$$

We get the conclusion. $\square$

If we assume convexity of r, condition on g for the existence of g-moment of X becomes quit simple. In order to show this, we prepare two lemmas.

Lemma 4.2 If r is convex on $[0, \infty)$, then $q(t; y + z) \le q(t; y) + q(t; z)$ for all $t, y, z \ge 0$.

Proof Since r is convex on $[0, \infty)$ and $r(0) = 0$, $\int_{0+}^{y} \frac{1}{r(z)}dz = \infty$ for all $y > 0$. Then

$$\int_{q(t;y)}^{y} \frac{1}{r(z)}dz = t$$

for all $y > 0$ and $t \geq 0$. Hence $q(t;y)$ is left differentiable in $y > 0$ and the left derivative $\frac{\partial^-}{\partial y} q(t;y)$ satisfies the equation

$$\frac{1}{r(y)} - \frac{1}{r(q(t;y))} \frac{\partial^-}{\partial y} q(t;y) = 0.$$

Hence

$$\frac{\partial^-}{\partial y} q(t;y) = \frac{r(q(t;y))}{r(y)}.$$

Since r is convex, r has a nondecresing left derivative r^*. Hence $\frac{\partial^-}{\partial y} q(t;y)$ has a left derivative which satisfies

$$\left(\frac{\partial^-}{\partial y} q(t;y)\right)^* = r(q(t;y)) \frac{r^*(q(t;y)) - r^*(y)}{r(y)^2} \leq 0 \quad \text{a.e. } y > 0.$$

Here, we used $q(t;y) \leq y$. Hence $q(t;y)$ is concave in y. Since

$$q(t;y+z) - q(t;y) = \int_0^z \frac{\partial^-}{\partial u} q(t;y+u)du,$$

we have

$$\begin{aligned} q(t;y+z) - q(t;y) - q(t;z) &= \int_0^z \left\{ \frac{\partial^-}{\partial u} q(t;y+z) - \frac{\partial^-}{\partial u} q(t;u) \right\} du \\ &\leq 0 \end{aligned}$$

for all $y, z, t \geq 0$, by concavity of q. $\Box$

Remark 4.1 Assume that r is convex on $[0, \infty)$. Let $z(t)$ be a nonnegative nondecreasing step function on $[0, \infty)$. Define $x(t;z)$ by

$$x(t;z) = -\int_0^t r(x(s;z))ds + z(t). \tag{5}$$

Let $z_0(t) = x$ and $z_1(t) = z_1 1_{[t_1, \infty)}(t)$ for $x, z_1 \geq 0$ and $t \in [0, \infty)$. Then, by Lemma 4.2,

$$x(t; z_0 + z_1(\cdot)) = q(t - t_1; q(t_1;x) + z_1) \leq q(t;x) + q(t - t_1; z_1) = x(t;z_0) + x(t;z_1)$$

for $t \geq t_1$. The above inequality also holds for $t < t_1$. In the same way, for nonnegative and nondecreasing step functions $z_1(t)$ and $z_2(t)$ with finite steps up to t, we have

$$x(t; z_1 + z_2) \leq x(t;z_1) + x(t;z_2).$$

Taking a limit, we have the above inequality for all nonnegative and nondecreasing step functions $z_1(t)$ and $z_2(t)$. This shows that the storage process $X(t;x)$ is a subadditive functional of $x + A$ provided that r is convex on $[0, \infty)$ ([11]).

Let $Y(t;x)$ be a nonnegative random variable with Laplace transform

$$E[\exp\{-\theta Y(t;x)\}] = \exp[-\theta q(t;x) + \int_0^\infty \{ \int_0^t (e^{-\theta q(s;y)} - 1) ds\} \nu(dy)]. \tag{6}$$

The integral of the right side of the above equality is well defined since $q(t;y) \leq y$. It may be interesting that $Y(t;x)$ is represented as

$$\begin{aligned} Y(t;x) &= q(t;x) + \int_{(0,t]} q(t-s; dA(s)) \\ &= q(t;x) + \int_{[0,t] \times (0,\infty)} q(s;y) N(dsdy) \end{aligned}$$

in law. Here,

$$\int_{(0,t]} q(t-s; dA(s)) := \lim_{n\to\infty} \sum_{0<s\le t;\, A(s)-A(s-)>1/n} q(t-s; A(s)-A(s-))$$

and N is a Poisson random measure on $[0,t]\times(0,\infty)$ with intensity measure $ds\,\nu(dy)$. The stochastic integral $\int_{(0,t]} q(t-s; dA(s))$ can be regarded as a natural extension of the stochastic integral representation $\int_{(0,t]} e^{-a(t-s)}dA(s)$ of process of Ornstein-Uhlenbeck type (that is when $r(y) = ay$, see [13]).

For two random variables X, Y, it is said that $X \ge Y$ in stochastic ordering sense if $P(X > y) \ge P(Y > y)$ for all y ([5]).

Lemma 4.3 If r is convex on $[0,\infty)$, then

$$X(t;x) \le Y(t;x)$$

for all $x, t \ge 0$ in stochastic ordering sense.

Proof First, we assume that $\lambda = \nu((0,\infty)) < \infty$. Let N_t be a Poisson process with intensity λ. Since X is a subadditive functional of $x + A$, we have that

$$\begin{aligned} X(t;x) &\le q(t;x) + q(t-T_1; Y_1) + \cdots + q(t-T_{N_t}; Y_{N_t}) \\ &= q(t;x) + \int_{[0,t]\times(0,\infty)} q(t-s;y)N(dsdy). \end{aligned}$$

Now, assume that $\lambda = \infty$. Let $X_n(t;x)$ be a storage process defined in Section 2 via $A_n(t)$. Then

$$X_n(t;x) \le q(t;x) + \int_{[0,t]\times(1/n,\infty)} q(s;y)N(dsdy)$$

in stochastic ordering sense, $X_n(t;x) \uparrow X(t;x)$ and

$$q(t;x) + \int_{[0,t]\times(1/n,\infty)} q(s;y)N(dsdy) \uparrow q(t;x) + \int_{[0,t]\times(0,\infty)} q(s;y)N(dsdy)$$

a.s. as $n\to\infty$. Hence $X(t;x) \le Y(t;x)$ in stochastic ordering sense. $\Box$

Theorem 4.4 Assume that r is convex on $[0,\infty)$. Let g be a nonnegative and nondecreasing function on $[0,\infty)$ such that g is submultipicative on an interval $[x_0,\infty)$, $(x_0 > 0)$. If

$$\int_{x_0}^{\infty}\left(\int_{x_0}^{y} \frac{g(z)}{r(z)}dz\right)\nu(dy) < \infty; \tag{7}$$

then

$$E[g(X(t;x))] < \infty \quad \text{for all} \quad x \ge 0, t > 0. \tag{8}$$

Proof We define $z(t;x)$ by

$$z(t;x) = \sup\{z > 0 : \int_x^z \frac{1}{r(u)}du \le t\} \tag{9}$$

for $x > 0$. If (7) holds, then

$$
\begin{aligned}
&\int_0^\infty \left(\int_0^t 1\{q(s;y) \geq x_0\} g(q(s;y))\,ds \right) \nu(dy) \\
= &\int_{z(t;x_0)}^{x_0} \left(\int_{x_0}^{y} \frac{g(z)}{r(z)} dz \right) \nu(dy) + \int_{z(t;x_0)}^{\infty} \left(\int_{q(t;y)}^{y} \frac{g(z)}{r(z)} dz \right) \nu(dy) \\
\leq &\int_{x_0}^{\infty} \left(\int_{x_0}^{y} \frac{g(z)}{r(z)} \right) \nu(dz) + t\nu((z(t;x_0);\infty)) < \infty :
\end{aligned}
$$

Hence by Lemma 4.1 and 4.3, we get (8). □

Example 4.2 Let $g(y) = y^\beta$ and $r(y) = y^\alpha$ ($\alpha > 1; \beta > 0$). Then by Theorem 4.3 or 4.4 we have that in the following cases $E[X(t;x)^\beta] < \infty$ for all $x \geq 0; t > 0$:

(a) If $\beta < \alpha - 1$.

(b) If $\beta = \alpha - 1$ and $\int_1^\infty \log y \nu(dy) < \infty$.

(c) If $\beta > \alpha - 1$ and $\int_1^\infty y^{\beta-\alpha+1} \nu(dy) < \infty$.

5 Examples

In all the previous examples we always used power functions. Here we exhibit functions g and G which are not power functions and satisfy the assumptions of previous Theorems and Corollaries.

Example 5.1 Let $g(y)$ be a function of the form

$$g(y) = c(y) \exp\left\{ \int_1^y \frac{\epsilon(u)}{u} du \right\};$$

where $0 < c_1 \leq c(y) \leq c_2$ and $0 < \epsilon(y) \leq c_3$ for all $y \geq 0$.

(a) For $0 < a < 1$ and $y > 0$,

$$\frac{g(ay)}{g(y)} = \frac{c(ay)}{c(y)} \exp\left\{ -\int_{ay}^y \frac{\epsilon(u)}{u} du \right\} \geq \frac{c_1}{c_2} a^{c_3}:$$

Hence $g(\cdot)$ satisfies the assumption of Corollary 3.2 provided that $c(\cdot)$ is nondecreasing.

(b) Suppose additionally that $\frac{\epsilon(u)}{u}$ is nonincreasing on $[1;\infty)$, then, for $y;z \geq 1$,

$$
\begin{aligned}
g(y+z) &= c(y+z) \exp\left\{ \int_1^{y+z} \frac{\epsilon(u)}{u} du \right\} \\
&= \frac{c(y+z)}{c(y)c(z)} g(y)g(z) \exp\left(\int_y^{y+z} - \int_1^z \right) \frac{\epsilon(u)}{u} du \\
&= \frac{c(y+z)}{c(y)c(z)} g(y)g(z) \exp\left\{ \int_1^{z+1} \frac{\epsilon(w+y-1)}{w+y-1} dw - \int_1^z \frac{\epsilon(u)}{u} du \right\} \\
&\leq \frac{c(y+z)}{c(y)c(z)} g(y)g(z) \exp\left\{ \int_z^{z+1} \frac{\epsilon(u)}{u} du \right\} \\
&\leq \frac{2^{c_3} c_2}{(c_1)^2} g(y)g(z);
\end{aligned}
$$

that is, g is submultiplicative on $[1;\infty)$. Hence $g(y)$ satisfies the assumption of Theorem 4.1 provided that $c(\cdot)$ is nondecreasing.

Example 5.2 Let $c > 0$ and $\alpha > 0$ be two fixed constants. Let $\epsilon(y)$ be a nonnegative and nondecreasing function on $[1;\infty)$ such that $\lim_{y\to\infty}\epsilon(y) = \beta > 0$, $\epsilon(1) + 1 - \alpha > 0$ and $\int_1^\infty \frac{\beta-\epsilon(u)}{u}du < \infty$. Define g and G by

$$g(y) = \begin{cases} c(\epsilon(y) + 1 - \alpha)\exp\{\int_1^y \frac{\epsilon(u)}{u}du\} & \text{for } y > 1, \\ c(\epsilon(1) + 1 - \alpha); & \text{for } 0 \le y \le 1 \end{cases}$$

and

$$G(y) = \int_1^y z^{-\alpha} g(z)dz \quad \text{for } y \ge 1,$$

respectively. Then,

$$G(y) = c\exp\{\int_1^y \frac{\epsilon(u) - \alpha + 1}{u}du\} - c \quad \text{for } y \ge 1:$$

The function g is nonnegative and nondecreasing on $[0;\infty)$. Let $y;z \ge 1$. Then,

$$\begin{aligned} \int_y^{y+z}\frac{\epsilon(u)}{u}du - \int_1^z \frac{\epsilon(u)}{u}du &= \int_0^z \frac{\epsilon(y+u)}{y+u}du - \int_1^z \frac{\epsilon(u)}{u}du \\ &\le \int_0^1 \frac{\beta}{y+u}du + \int_1^z \frac{\beta}{u}du - \int_1^z \frac{\epsilon(u)}{u}du \\ &= \beta\log\frac{y+1}{y} + \int_1^z \frac{\beta-\epsilon(u)}{u}du \\ &\le \beta\log 2 + \int_1^\infty \frac{\beta-\epsilon(u)}{u}du < \infty: \end{aligned}$$

Repeating similar calculations as in example 5.1 we obtain the submultiplicativity of $g(y)$ on $[1;\infty)$. In the same way, we have the submultiplicativity of $G(y)$ on $[1;\infty)$. Hence $g(y)$ satisfies the assumptions in Theorems 4.3 and 4.4 with $x_0 = 1$ and $r(y) = y^\alpha$.

Examples 5.1 and 5.2 are slight departures of functions g that are of polynomial type. The following introduces a similar study for exponential type functions.

Example 5.3 Let $r(y) = y^\alpha$ $(\alpha > 0)$. Then, $g(y) = y^\alpha e^{ay}$, $(a > 0)$, satisfies the assumption of Theorem 4.1 with $x_0 = 1$. Also, $G(y) = a^{-1}e^{ay}$ satisfies the assumption of Theorem 4.3 and Theorem 4.4 with $x_0 = 1$. Here one can also extend these examples to generate a similar class as in Examples 5.1 and 5.2. In fact, if $g(y) = c(y)\exp(\int_1^y f(u)du)$ for a submultiplicative function c and a positive function f nonincreasing and bounded then g is submultiplicative in $[1;\infty)$. Similarly, one has that if $g(y) = (f(y) - \frac{\alpha}{y})\exp(\int_1^y f(u)du)$ for $f(y) - \frac{\alpha}{y}$ bounded above and below by positive constants then g and G are submultiplicative functions in $[1;\infty)$.

6 Tail probability

In this section, we discuss tail probabilities of storage processes.

Lemma 6.1 For all $x;t \ge 0$,

$$P(X(t;x) > y) \le P(x + A(t) > y):$$

Proof It is obvious by the inequality $X(t;x) \le x + A(t)$. $\Box$

Lemma 6.2 If r is concave on $(0, \infty)$, then

$$q(t, y+z) \geq q(t, y) + q(t, z)$$

for all $y, z > 0$ and $t \geq 0$.

Proof We have

$$\int_{q(t,y)}^{y} \frac{1}{r(z)} dz = t \quad \text{for } y > z(t, 0+).$$

As in the proof of Lemma 4.2,

$$q(t, y+z) - q(t, y) - q(t, z) \geq 0$$

for $y, z > z(t, 0+)$ and $t \geq 0$. If $0 \leq y \leq z(t, 0+)$, then

$$q(t, y) + q(t, z) = q(t, z) \leq q(t, y+z)$$

by nondecreasingness of $q(t, y)$ in $y \geq 0$. □

Lemma 6.3 If r is concave on $(0, \infty)$, then

$$X(t, x) \geq Y(t, x)$$

in stochastic ordering sense for all $x, t \geq 0$, where $Y(t, x)$ is defined by (6).

Proof The proof is accomplished using the same argument as the proof of Lemma 4.3 using Lemma 6.2. □

Let $Q(y) = \int_0^\infty \left(\int_0^t 1_{(y,\infty)}(q(s, z)) ds\right) \nu(dz)$ for $y > 0$.

Lemma 6.4 (a) If r is bounded and $\nu((y+z, \infty))/\nu((y, \infty)) \to 1$ as $y \to \infty$ for every $z \geq 0$, then

$$Q(y) \sim t\nu((y, \infty))$$

(b) If $r(y) = o(y)$ and $\nu(((1+o(1))y, \infty)) \sim \nu((y, \infty))$ as $y \to \infty$, then

$$Q(y) \sim t\nu((y, \infty))$$

(c) If $r(y) = O(y)$ as $y \to \infty$ and $\nu((y, \infty))$ is slowly varying at infinity, then

$$Q(y) \sim t\nu((y, \infty)).$$

(d) If $r(y) \sim ay$ as $y \to \infty$ and $\nu((y, \infty)) = y^{-\beta} L(y)$, $(a, \beta > 0)$ where $L(y)$ is slowly varying at infinity, then

$$Q(y) \sim \frac{1 - e^{-a\beta t}}{a\beta} y^{-\beta} L(y).$$

(e) If $\int_1^\infty \frac{1}{r(y)} dy < \infty$, then

$$Q(y) \sim \int_y^\infty \frac{\nu((z, \infty))}{r(z)} dz.$$

Proof We have

$$
\begin{aligned}
Q(y) &= \int_0^\infty \left(\int_0^t 1\{q(s;z) > y\} ds \right) \nu(dz) \\
&= \int_{(y, z(t;y))} \left(\int_y^z \frac{1}{r(u)} du \right) \nu(dz) + t \int_{[z(t;y), \infty)} \nu(dz) \\
&= \int_y^{z(t;y)} \frac{1}{r(z)} \nu((z, \infty)) dz. \qquad (10)
\end{aligned}
$$

Hence

$$
t\nu((z(t;y), \infty)) \leq Q(y) \leq t\nu((y, \infty)). \qquad (11)
$$

Note that in cases (a) ∼ (d), $\int_1^\infty \frac{1}{r(u)} du = \infty$. (a) If there is $M > 0$ such that $r \leq M$ on $[0, \infty)$, then $y \leq z(t;y) \leq y + Mt$. By Lemma 1 (a) in [6], $\nu((z(t;y), \infty)) \sim \nu((y, \infty))$. By (11), we get the conclusion.
(b) For any $\epsilon > 0$, there is $z_0 > 0$ such that $r(u) < \epsilon u$ for all $u > z_0$. Then $\frac{1}{r(u)} > \frac{1}{\epsilon u}$ for all $u > z_0$. Hence $t \geq \int_y^{z(t;y)} \frac{1}{\epsilon u} du$. This yields $1 \leq \frac{z(t;y)}{y} \leq e^{\epsilon t}$. By the assumption on ν, $\nu((z(t;y), \infty)) \sim \nu((y, \infty))$. We get the conclusion by (11).
(c) There is $M > 0$ and $z_0 > 0$ such that

$$
\frac{1}{r(u)} \geq \frac{1}{Mu}
$$

for $u \geq z_0$. By the argument in the proof of (a), $\frac{z(t;y)}{y} \leq e^{Mt}$. Since ν is slowly varying, $\nu((z(t;y), \infty)) \sim \nu((y, \infty))$. We get the conclusion by (11).
(d) For any $\epsilon > 0$, there is $y_0 > 0$ such that $(1 - \epsilon)ay \leq r(y) \leq (1 + \epsilon)ay$ for $y > y_0$. By (10),

$$
\begin{aligned}
(1+\epsilon)^{-1} \int_y^{z(t;y)} \frac{\nu((z, \infty))}{az} dz &\leq Q(y) \\
&\leq (1-\epsilon)^{-1} \int_y^{z(t;y)} \frac{\nu((z, \infty))}{az} dz
\end{aligned}
$$

for $y > y_0$. As $\nu((z, \infty)) \sim z^{-\beta} L(z)$, we have

$$
Q(y) \sim \int_y^\infty \frac{1}{a} z^{-1-\beta} L(z) dz - \int_{z(t;y)}^\infty \frac{1}{a} z^{-1-\beta} L(z) dz
$$

as $y \to \infty$. Hence

$$
Q(y) \sim \frac{1}{a\beta} \left(y^{-\beta} L(y) - z(t;y)^{-\beta} L(z(t;y)) \right)
$$

as $y \to \infty$. Since $z(t;y) \sim ye^{at}$ as $y \to \infty$,

$$
Q(y) \sim \frac{1 - e^{-a\beta t}}{a\beta} y^{-\beta} L(y)
$$

as $y \to \infty$.
(e) Since $\int_1^\infty \frac{1}{r(z)} dz < \infty$, $q(t;\infty) < \infty$ for any $t > 0$ and $z(t;y) = \infty$ for $y \geq q(t;\infty)$. We have

$$
Q(y) = \int_y^\infty \frac{1}{r(z)} \nu((z, \infty)) dz.
$$

□

A probability measure P on $[0,\infty)$ is said to be subexponential ([3]) if

$$\lim_{y\to\infty}\frac{P^{2*}((y,\infty))}{P((y,\infty))}=2$$

holds. Note that if $P((y,\infty))$ is regularly varying at infinity with nonpositive exponent, then P is subexponential by Corollary on p. 279 of [7].

Theorem 6.1 Assume that r is concave and $\nu|_{(1,\infty)}/\nu(1,\infty)$ is subexponential. If r is bounded or, $r(y)=o(y)$ and $\nu(((1+o(1))y,\infty))\sim\nu((y,\infty))$ as $y\to\infty$, then

$$P(X(t,x)>y)\sim t\nu((y,\infty))$$

as $y\to\infty$, for all $x,t\geq 0$.

Proof Let $\{A(t)\}$ be an input process of $\{X(t,x)\}$. Since r is concave, we have

$$P(Y(t,x)>y)\leq P(X(t,x)>y)\leq P(A(t)>y-x). \tag{12}$$

By Lemma 6.4 (a), (b) and Theorem 1 ([6]), we have

$$P(Y(t,x)>y)\sim t\int_y^\infty \nu(dz).$$

By Theorem 1 ([6]) and Lemma 1 ([6]), we have

$$\begin{aligned}P(A(t)>y-x) &\sim t\int_{y-x}^\infty \nu(dz)\\ &\sim t\int_y^\infty \nu(dz).\end{aligned}$$

□

Theorem 6.2 (a) If r is concave on $(0,\infty)$, $r(y)\sim ay$ $(a>0)$ as $y\to\infty$ and $\nu((y,\infty))=L(y)$, then

$$P(X(t,x)>y)\sim tL(y)$$

and

(b) if r is convex on $[0,\infty)$, $r(y)\sim ay$ and $\nu((y,\infty))=y^{-\beta}L(y)$ $(\beta>0)$, then

$$P(X(t,x)>y)\sim\frac{1-e^{-a\beta t}}{a\beta}y^{-\beta}L(y)$$

as $y\to\infty$ for all $x,t\geq 0$. Here $L(y)$ is a function slowly varying at infinity.

Proof (a) Note that (12) holds. By Lemma 6.4 (c), we have

$$P(X(t,x))\sim tL(y) \text{ as } y\to\infty.$$

(b) By Lemma 6.4 (d),

$$Q(y)\sim\frac{1-e^{-a\beta t}}{a\beta}y^{-\beta}L(y) \text{ as } y\to\infty.$$

Let

$$A_y = \{(s;u) \in [0;t] \times (0;\infty) : x(t; u1_{[s;\infty)}(\cdot)) > y\};$$

where $x(t;z)$ is a functional of a nonnegative and nondecreasing step function z defined by (5). Note that

$$\int_{(0;\infty)} \int_0^t 1_{A_y}((s;u))ds\nu(du) = Q(y):$$

Since $X(t;x)$ is a subadditive functional of $x + A$ and a probability measure $-Q(dy)/Q(1)$ on $(1;\infty)$ is subexponential,

$$P(X(t;x) > y) \sim Q(y)$$

as $y \to \infty$ by Theorem 3.1 and Example (Lévy motion) in [11]. We get the conclusion. $\square$

Theorem 6.3 If r is convex and a probability measure $-Q(dy)/Q(1)$ on $(1;\infty)$ is subexponential, then

$$P(X(t;x) > y) \sim \int_y^{z(t;y)} \frac{\nu((z;\infty))}{r(z)}dz$$

as $y \to \infty$ for all $x \geq 0$ and $t > 0$. Under an additional assumption $\int_1^\infty \frac{1}{r(y)}dy < \infty$,

$$P(X(t;x) > y) \sim \int_y^{\infty} \frac{\nu((z;\infty))}{r(z)}dz$$

as $y \to \infty$ for all $x \geq 0$ and $t > 0$.

Proof As in the proof of Theorem 6.2 (b), we have

$$P(X(t;x) > y) \sim Q(y)$$

as $y \to \infty$. By (10),

$$P(X(t;x) > y) \sim \int_y^{z(t;y)} \frac{\nu((z;\infty))}{r(z)}dz$$

as $y \to \infty$. If $\int_1^\infty \frac{1}{r(y)}dy < \infty$, then we have by Lemma 6.4 (e), that

$$P(X(t;x) > y) \sim \int_y^{\infty} \frac{\nu((z;\infty))}{r(z)}dz$$

as $y \to \infty$. $\square$

Remarks

1a. In Theorems 4.2-4.4, assumptions for r can be relaxed as follows: There exists a function r_- such that $r_- \leq r$ and the assumptions on r are replaced by the same ones where r is replaced by r_-.

1b. Additionally, if one assumes that

$$\int_y^{\infty} \frac{\nu((z;\infty))}{r(z)}dz \sim \int_y^{\infty} \frac{\nu((z;\infty))}{r_-(z)}dz$$

as $y \to \infty$ then Theorem 6.3 is also satisfied.

2. In Theorem 6.1, assumptions for r can be relaxed as follows: There exists a function r_+ such that $r_+ \geq r$ and the assumptions on r are replaced by the same ones where r is replaced by r_+.

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