PRICE INCREASE AND STABILITY WITH NEW ENTRIES IN COURNOT MARKETS

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Abstract. It is widely accepted in the literature about the classical Cournot oligopoly model that the loss of quasi–competitiveness is linked, in the long run as new firms enter the market, to instability of the equilibrium. In this paper, though, we present a model in which a stable unique symmetric equilibrium is reached for any number of oligopolists as industry price increases with each new entry. Consequently, the suspicion that non–quasi–competitiveness implies, in the long run, instability is proved false.

JEL: Primary D43; Secondary C62.
Keywords: Cournot equilibrium, non-cooperative oligopoly, quasi–competitiveness, stability.

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Date: January 2012.

1. Introduction

Cournot's oligopoly classical model raises four kind of issues in the literature: the existence and uniqueness of the equilibrium, the stability of such an equilibrium, the quasi–competitiveness of the model and, lastly, perfect competition in the limit as the number of oligopolists increase. Under diverse general assumptions, the four questions have been taken care of. Stability was first considered in Theocharis (1960), where it was proved that Cournot's equilibrium solution was stable if there are two sellers, oscillatory if the number of sellers is three, and unstable if the number is greater than three. These results were immediately "corrected and appraised" in McManus & Quandt (1961) where it was shown that Theocharis' results were very restrictive as they depended profoundly on the adjustment system chosen and also because a discrete approach had been used whose dynamics were those of a system of difference equations. Consequently, the stability depended strongly on the coefficients. McManus & Quandt (1961) considered a continuous adjustment system (which is the most used in the literature) where each firm changes its production proportionally to the difference between profit maximizing production and actual production:

$$\dot{q}_i = k_i(q_i^* - q_i). \tag{1}$$

The $k_i(> 0)$ are considered the 'speeds' of the adjustment.

Under this adjustment system, the classical Cournot model is stable no matter the number of firms in the industry nor the values of the speeds of the adjustment.

At the same time, Fisher (1961) analyzed also Theocharis' adjustment system and reached the same conclusions as McManus and Quandt. Fisher commented that despite his result, "the tendency to instability does rise with the number of sellers", whatever that means.

Shortly after that, Hahn (1962) undertook the question of the stability and found a sufficient condition to establish it under the continuous adjustment system (1) of McManus and Fisher. Hahn's condition is general enough to be widely applicable. In short it says that if demand (d), cost (C_i), and q^*, q_i^* are, respectively, total production and firm i's production at equilibrium and

$$-d'(q^*) + C_i''(q_i^*) > 0, \tag{2}$$

Cournot equilibrium —when it exists and is unique— is stable.

Hahn's condition was generalized in Okuguchi (1964) who proved its validity even if the adjustment system were not linear but simply a sign–preserving function with respect to the difference between profit maximizing production and actual one.

The existence of Cournot equilibrium has been proved under very varied conditions. The most cited reference is perhaps Frank Jr. & Quandt (1963) (though existence had already been proved under more restrictive assumptions, see McManus (1962)). A more recent (and more general) proof can be found in Novshek (1985) and also in Szidarovszky & Yakowitz (1982).

Uniqueness is more difficult to prove and the number of references diminishes. We cite Ruffin (1971), Okuguchi & Suzumura (1971) Szidarovszky & Yakowitz (1982) and Schlee (1993). A more recent contribution is Gaudet & Salant (1991). It is important to remark that Okuguchi & Suzumura (1971) links uniqueness of the equilibrium with Hahn's condition and proves that this last condition implies not only stability but also the uniqueness of the Cournot solution.

The question of convergence to perfect competition was dealt with in Frank Jr. (1965) or McManus (1962). The former attempted to prove that quasi–competitiveness was sufficient for convergence and the latter questioned the relationship between these two issues. Later Ruffin (1971) detached the question of convergence to perfect competition from quasi–competitiveness and proved that the former issue is only related to the convexity of the cost function.

Quasi–competitiveness is at the heart of Cournot model. In fact, the mathematical model was expected to confirm the general opinion that *competition lowers prices*. This was not the case: Frank Jr. & Quandt (1963), besides proving the existence of equilibrium, present an example of passage from monopoly to duopoly in which quasi–competitiveness is lost. Their demand function, though, is not strictly decreasing. Frank Jr. (1965) gives conditions to ensure quasi–competitiveness. Later, these conditions were thoroughly investigated by Ruffin (1971), Okuguchi (1974) and Szidarovszky & Yakowitz (1982). It is worth reminding that Ruffin (1971), though mainly concerned with long-run competitive behaviour, addresses the other three issues mentioned at the beginning of this introduction: quasi–competitiveness, existence and uniqueness of equilibrium, and stability. Ruffin points out that Hahn's condition ensures not only stability but also quasi–competitiveness and provides an example in which quasi–competitiveness and stability break down with a large number of firms in the market.

In short, when an equilibrium exists, Hahn's condition implies uniqueness of the equilibrium, stability and quasi–competitiveness.

If Hahn's stability condition is violated, (Ruffin, 1971, p. 498) remarks:

> [...] it is probable that in this case the Cournot model would become dynamically unstable before the long-run equilibrium could be attained.

Later, Seade (1980) studies new entry in a Cournot market and assuming continuity in the number of firms, proves that "industry output unambiguously expands [...] as entry into *stable* equilibria takes place".

Fisher's comment and Ruffin's and Seade's papers seemed to indicate that strong evidence existed linking non-quasicompetitiveness and instability: If a Cournot market is stable, a new entry cannot rise the equilibrium price and, viceversa, in a stable Cournot market where new entries occur, there gets to a point where the new entry provokes instability.

For a discrete number of firms, specifically from monopoly to duopoly, de Meza (1985) noticed that a rise in price could happen without the loss of local stability. A previous note by some of the authors of this article, Villanova et al. (2001), offers a model very similar to de Meza's showing that equilibrium can be reached with *global* stability.

Nevertheless, the question of the rise in the industry price and stability under an indefinite number of entries, was still open.

An important contribution was made in 2000 by Amir & Lambson (2000). These authors retake the question of quasi-competitiveness for a Cournot oligopoly using lattice-theoretic methods. Their analysis is quite illuminating and reduces greatly the conditions needed to draw conclusions about the existence of Cournot equilibria and their relation to the issue of quasi-competitiveness. The global sign of the function

$$\Delta = -d'(q) + C''(q_i)$$

is the key element in their work. This is precisely Hahn's condition mentioned before. Essentially, Amir and Lambson's results are the following:

(1) If $\Delta > 0$, the n-poly is quasi-competitive.
(2) If $\Delta < 0$, the n-poly is non-quasi-competitive.

The case in which Δ changes signs is what they call the *hybrid* case. This case is not treated in depth.

It is worth mentioning that, in spite of the coincidence of Amir and Lambson's Δ with Hahn's condition, stability is not mentioned in their paper. A quick calculation shows that the case $\Delta < 0$ leads *both* to non-quasi-competitiveness and instability.

This confirms partly Fisher's and Ruffin's suspicions about the close relationship between both questions. But only in part because the hybrid case leaves the door open to the co-existence of non-quasi-competitiveness and a stable equilibrium. In the words of Amir and Lambson, "[...] these cases [the hybrid ones] would be characterized by a lack of monotonic relationship between the number of firms and the endogenous variables of interest (per-firm output, price level)". This is where our contribution enters the scene showing that this is not necessarily so.

The models dealt with by these authors vary slightly in their assumptions concerning demand and cost functions: some require differentiability, others only continuity or even semi–continuity. Others consider increasing marginal costs, others not. Some consider all the firms identical and others consider different costs for each firm, etc. A very good summary of these results can be found in Okuguchi (1976) and a good reference for the generalization to multi–product firms can be found in Okuguchi & Szidarovsky (1999).

A very good introduction to the subject is Friedman (1983). The state of the art can be found in Daughety (2008) which updates a previous important compilation, Daughety (1988). Other good summaries are Okuguchi & Szidarovsky (1999) and Vives (2000).

In our previous work, Villanova et al. (2001), we built a model in which the passage from monopoly to duopoly caused, at equilibrium, a loss of quasi–competitiveness keeping at the same time the stability. Obviously, the conditions of our model, though general enough, did not go against the known results in this area. The main feature of our model was an increasing two–piece linear cost function which was *concave* throughout. Concavity was in order as the convexity of the cost function causes directly the quasi–competitiveness of the model as had been shown in several occasions (see Szidarovszky & Yakowitz (1982) —who prove quasi–competitiveness assuming strictly convex cost functions— or the previously mentioned Amir & Lambson (2000)). This result of ours showed the possibility of losing quasi–competitiveness without losing at the same time the global stability of the equilibrium in a duopoly but could not be generalized to an r-poly with free entry maintaining the main characteristics of the model. This has been noticed by Hoernig (2003) who studies Cournot's comparative statics under differentiated goods markets.

In this paper, we change the model in order to extend our results to an r–poly, where r is any given number of firms. We prove that an oligopoly equilibrium may be non–quasi–competitive and, at the same time, be (locally) stable under the adjustment system given by equation (1) above. Besides, under a concave cost function, we prove that as r increases, marginal cost and market price tend to be equal which means perfect competition in the limit. Starting from any linear decreasing demand function we find an increasing piecewise linear cost function with an infinite number of pieces such that the model has the following unique features:

(i) A unique non-trivial symmetric Cournot equilibrium point exists for any number of firms.

Other trivial asymmetric equilibria may exist, in the sense that if a non-trivial symmetric equilibrium exists for an r-poly, any number of entries with zero per-firm output are also equilibria.

(ii) Industry price increases monotonically with the number of firms in the market; that is to say, if p_i denotes price at equilibrium when there are i

firms competing in the market,

$$p_1 < p_2 < \cdots < p_r < \cdots .$$

(iii) The successive equilibria are locally asymptotically stable.
(iv) The oligopoly is viable no matter the number of firms in the market, that is to say, at equilibrium, profit for an individual firm is always positive.
(v) The model converges to perfect competition as the number of firms tends to infinity; or, what amounts to the same, industry price p_r tends to marginal cost under perfect competition, which coincides with limiting marginal cost as output tends to zero.

In section 2, after building the basic functions (demand and cost) of our model in a very abstract way, we discuss the reaction curve and the necessary assumptions required to achieve our results. We find the different Cournot points of our model. In section 3, we use the parameters obtained in the previous section to determine completely the cost function of our model. In section 4 we prove the existence and uniqueness of a symmetric Cournot solution as well the asymmetric ones which allow for no-output firms in the industry. Section 5 covers the loss of quasi–competitiveness and the convergence to perfect competition as the number of oligopolists tends to infinity. Perhaps the more relevant feature of our model is presented in section 6 where we study the asymptotic stability of the different equilibria under (1). The Appendix collects most of the proofs.

2. The model

We will assume the following. There are r firms in a classical Cournot market with identical cost functions and linear demand. The demand and cost functions are:

i) Inverse linear demand function, $p = a - b\,q$, $(a,\, b > 0)$. Defined on the interval $[0, a/b]$.
ii) A concave continuous piecewise linear cost function with an infinite number of pieces:

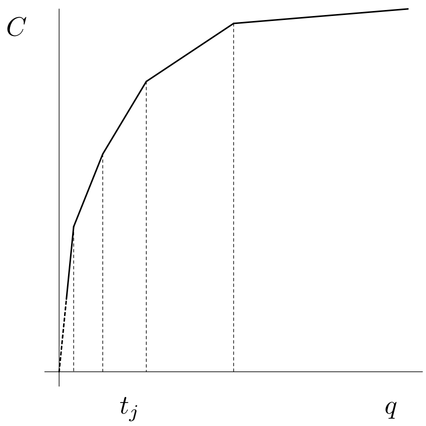
$$C(q) = \begin{cases} \vdots & \vdots \\ c_i + d_i\,q & \text{if} \quad t_i \le q < t_{i-1} \\ \vdots & \vdots \\ c_1 + d_1\,q & \text{if} \quad t_1 \le q \end{cases} \tag{3}$$

where
– the t_i satisfy $0 < \cdots < t_i < t_{i-1} < \cdots < t_1$;
– the c_i satisfy $c_1 > \cdots > c_i > c_{i+1} > \cdots > 0$;
– the d_i satisfy $0 < d_1 < \cdots < d_i < d_{i+1} < \cdots$.

As we require the continuity of the cost function at each $q = t_i$,

$$c_i - c_{i+1} = (d_{i+1} - d_i)\,t_i, \qquad i = 1, 2, \ldots, \tag{4}$$

We will also impose $c_i \to 0$. See Figure 1.



Cost function

Figure 1

As we have already mentioned in the introduction, the non-convexity of our cost function is necessary in order to achieve our goal since for linear demand and a *convex* cost function, the classical Cournot model is quasi–competitive.

Under these assumptions, the profit function of firm k $(k = 1, \ldots, r)$ is

$$\Pi_k(\mathbf{q}) = \Pi_k(q_1, \ldots, q_r) = [a - b(q_1 + \cdots + q_r)]\, q_k - C(q_k). \tag{5}$$

If $\hat{q}_k$ denotes the production of the whole industry except for firm k:

$$\hat{q}_k = q_1 + \cdots + q_{k-1} + q_{k+1} + \cdots + q_r, \quad (k = 1, 2, \ldots, r), \tag{6}$$

then function (5) can be written as

$$\Pi_k(\mathbf{q}) = \Pi_k(q_k, \hat{q}_k) = [a - b(q_k + \hat{q}_k)]\, q_k - C(q_k).$$

Displaying the different values of $C(q_k)$ and rearranging somewhat the result, we have

$$\Pi_k(\mathbf{q}) = \begin{cases} \qquad\vdots & \qquad\vdots \\ \Pi_{k,i} = -b\, q_k^2 + [(a - d_i) - b\, \hat{q}_k]\; q_k - c_i & \text{if } t_i \le q_k < t_{i-1} \\ \qquad\vdots & \qquad\vdots \\ \Pi_{k,1} = -b\, q_k^2 + [(a - d_1) - b\, \hat{q}_k]\; q_k - c_1 & \text{if } t_1 \le q_k \le a/b \end{cases} \tag{7}$$

For our purposes, it will be convenient to modify the way of presenting function (7). Let us define q_i^c as

$$q_i^c := \frac{a - d_i}{2b}. \tag{8}$$

We can now write expression (7) as

$$\Pi_k(\mathbf{q}) = \begin{cases} \vdots & \vdots \\ \Pi_{k,i} = -b\,q_k^2 + 2b\,(q_i^c - \hat{q}_k/2)\,q_k - c_i & \text{if } t_i \le q_k < t_{i-1} \\ \vdots & \vdots \\ \Pi_{k,1} = -b\,q_k^2 + 2b\,(q_1^c - \hat{q}_k/2)\,q_k - c_1 & \text{if } t_1 \le q_k \le a/b. \end{cases} \tag{9}$$

We remark that, as $d_i > d_{i-1}$, the q_i^c satisfy

$$0 < \cdots < q_{i+1}^c < \cdots < q_i^c < \cdots < q_1^c < a/(2b). \tag{10}$$

The q_i^c are the values of q where, under monopoly ($r = 1$), the different parabolas that constitute (9) have their vertexes (see Figure 2 below).

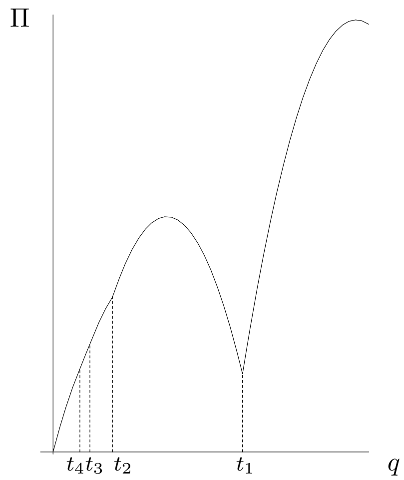


Figure 2: Monopoly profit

As usual in Cournot's model, each firm maximizes its own profit considering the production of the rest of the market, $\hat{q}_k$, constant. In our model, given $\hat{q}_k$ the profit function is not concave throughout $[0, a/b]$. It consists of an infinite number of parabolas, each one defined on the corresponding interval

$$\Pi_{k,i} = -b\,q_k^2 + 2b\,(q_i^c - \frac{1}{2}\hat{q}_k)\,q_k - c_i \qquad \text{if } t_i \le q_k < t_{i-1}.$$

The vertex of this parabola is found at

$$\left(q_i^c - \frac{1}{2}\hat{q}_k,\, b\left(q_i^c - \frac{1}{2}\hat{q}_k\right)^2 - c_i \right). \tag{11}$$

Let us denote the maximum of the profit function in $[t_i, t_{i-1}]$ as $\Pi_{k,i}^{max}$. Its value will depend on the situation of $q_i^c - \hat{q}_k/2$ with respect to the defining interval $[t_i, t_{i-1}]$:

(i) If $t_i \le q_i^c - \hat{q}_k/2 \le t_{i-1}$, then $\Pi_{k,i}^{max} = b\,(q_i^c - \hat{q}_k/2)^2 - c_i$ (the y–ordinate of the vertex of the parabola).

(ii) If $t_{i-1} \le q_i^c - \hat{q}_k/2$, then $\Pi_{k,i}^{max} = \Pi_{k,i}(t_{i-1}, \hat{q}_k)$.

(iii) If $q_i^c - \hat{q}_k/2 \le t_i$, then $\Pi_{k,i}^{max} = \Pi_{k,i}(t_i, \hat{q}_k)$.

The graph of $\Pi_{k,i}$ in each one of the previous situations is shown in Figure 3.

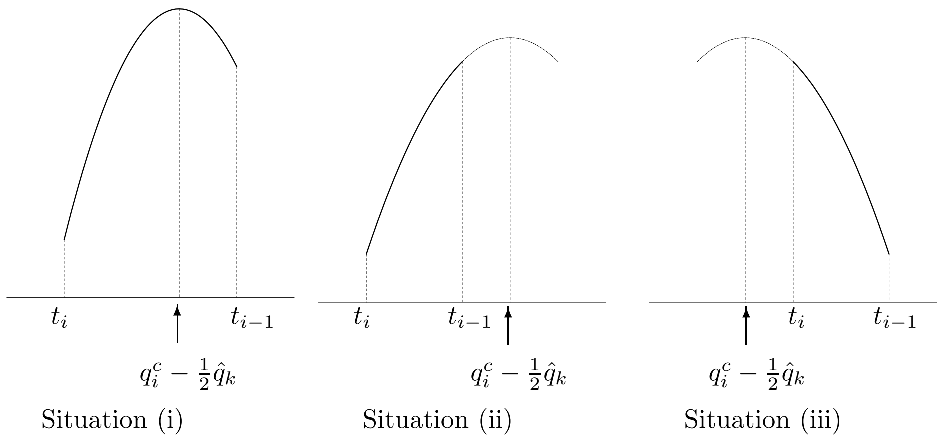


Figure 3

Thus the value (or values) of q_k that maximize $\Pi_k(\mathbf{q})$ depend on $\hat{q}_k$ through the corresponding reaction curve R_k:

$$(12) \qquad R_k(\hat{q}_k) = \begin{cases} \vdots \\ \begin{cases} t_i & \text{if } 0 \le q_i^c - \frac{1}{2}\hat{q}_k < t_i \\ q_i^c - \frac{1}{2}\hat{q}_k & \text{if } t_i \le q_i^c - \frac{1}{2}\hat{q}_k < t_{i-1} \\ t_{i-1} & \text{if } t_{i-1} \le q_i^c - \frac{1}{2}\hat{q}_k \le q_i^c \end{cases} \\ \vdots \\ \begin{cases} t_1 & \text{if } 0 \le q_1^c - \frac{1}{2}\hat{q}_k < t_1 \\ q_1^c - \frac{1}{2}\hat{q}_k & \text{if } t_1 \le q_1^c - \frac{1}{2}\hat{q}_k \le q_1^c. \end{cases} \end{cases}$$

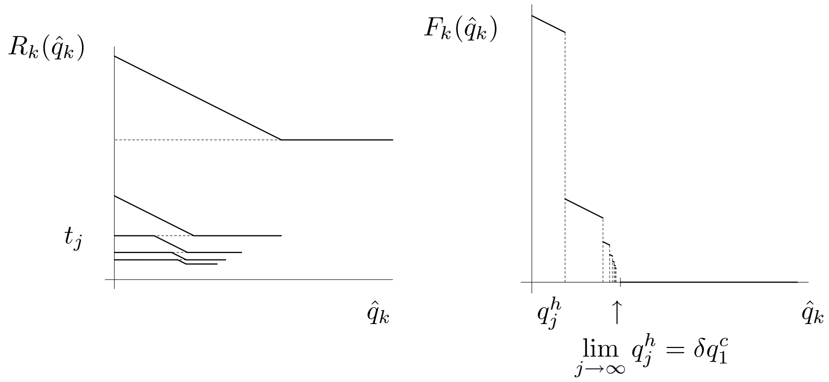
The inequalities above can be re-written in terms of $\hat{q}_k$:

- $0 \le q_i^c - \hat{q}_k/2 < t_i$ is equivalent to $2q_i^c - 2t_i < \hat{q}_k \le 2q_i^c$;
- $t_i \le q_i^c - \hat{q}_k/2 < t_{i-1}$ is equivalent to $2q_i^c - 2t_{i-1} < \hat{q}_k \le 2q_i^c - 2t_i$;
- $t_{i-1} \le q_i^c - \hat{q}_k/2 \le q_i^c$ is equivalent to $0 \le \hat{q}_k \le 2q_i^c - 2t_{i-1}$;
- $0 \le q_1^c - \hat{q}_k/2 < t_1$ is equivalent to $2q_1^c - 2t_1 < \hat{q}_k \le 2q_1^c$;
- $t_1 \le q_1^c - \hat{q}_k/2 \le q_1^c$ is equivalent to $0 \le \hat{q}_k \le 2q_1^c - 2t_1$.

Thus, a different way of writing (12) is

$$(13) \qquad R_k(\hat{q}_k) = \begin{cases} \vdots \\ \begin{cases} t_{i-1} & \text{if } 0 \le \hat{q}_k \le 2q_i^c - 2t_{i-1} \\ q_i^c - \frac{1}{2}\hat{q}_k & \text{if } 2q_i^c - 2t_{i-1} < \hat{q}_k \le 2q_i^c - 2t_i \\ t_i & \text{if } 2q_i^c - 2t_i < \hat{q}_k \le 2q_i^c \end{cases} \\ \vdots \\ \begin{cases} q_1^c - \frac{1}{2}\hat{q}_k & \text{if } 0 \le \hat{q}_k \le 2q_1^c - 2t_1 \\ t_1 & \text{if } 2q_1^c - 2t_1 < \hat{q}_k \le 2q_1^c. \end{cases} \end{cases}$$

This reaction curve (see Figure 4a), $R_k(\hat{q}_k)$, is in fact a multivalued function or correspondence on the variable $\hat{q}_k$. Among all the possible values of $R_k(\hat{q}_k)$ for a given $\hat{q}_k$, firm k will choose, as $\hat{q}_k$'s image, the value (or one of the values) that maximizes its profit.



a) Reaction curve b) Reaction function; $\lim_{j\to\infty} q_j^h$ is found in Lemma 4

Figure 4

The graphs in Figures 1 to 4 correspond to the numerical example in footnote 3 on page 14.

Consider now the vertexes of the infinity of parabolas that, given a fixed $\hat{q}_k$ constitute Π_k (see equation (11):

$$V_i(\hat{q}_k) = b\left(q_i^c - \frac{1}{2}\,\hat{q}_k\right)^2 - c_i, \qquad i = 1, 2, \ldots. \tag{14}$$

In order to determine the reaction function in our model we will impose the following condition on the V_i's:

For each $i = 1, 2, \ldots$, we demand that

$$V_i \geq V_{i+1}. \tag{15}$$

Let us determine the values of $\hat{q}_k$ that make this possible.

Inequality (15) leads to:

$$c_i - c_{i+1} \;\leq\; b\left((q_i^c - \frac{1}{2}\hat{q}_k)^2 - (q_{i+1}^c - \frac{1}{2}\hat{q}_k)^2\right), \tag{16}$$

and after some algebra and using relations (4) we get to

$$(d_{i+1} - d_i)\, t_i \;\leq\; b(q_i^c + q_{i+1}^c - \hat{q}_k)\,(q_i^c - q_{i+1}^c). \tag{17}$$

This last expression can be simplified using the relationship between d_i and q_i^c from equation (10):

$$2b(q_i^c - q_{i+1}^c)t_i \;\leq\; b(q_i^c + q_{i+1}^c - \hat{q}_k)\,(q_i^c - q_{i+1}^c), \tag{18}$$

which leads to

$$\hat{q}_k \;\leq\; q_i^c + q_{i+1}^c - 2t_i. \tag{19}$$

We denote the right hand side of equation (19) as q_i^h:

$$q_i^h := q_i^c + q_{i+1}^c - 2t_i, \quad (i = 1, 2, \ldots). \tag{20}$$

Consequently we can state the following

Lemma 1. *Given* $i = 1, 2, \ldots, V_i(\hat{q}_k) \geq V_{i+1}(\hat{q}_k)$ *if and only if* $\hat{q}_k \leq q_i^h$.

The next lemma is a direct consequence of the previous one:

Lemma 2. *If we can find a sequence of* q_i^h *satisfying definition (20) and satisfying*

$$0 < q_1^h < \cdots < q_i^h < q_{i+1}^h < \cdots,$$

then, for each $i = 1, 2, \ldots$ *we will have that*

$$\hat{q}_k \leq q_i^h \quad \Rightarrow V_i \geq V_{i+1} \geq V_{i+2} \geq \cdots,$$

and

$$\hat{q}_k \geq q_i^h \quad \Rightarrow V_1 \leq \cdots \leq V_{i-1} \leq V_i.$$

For the time being, let us suppose we have the increasing sequence of q_i^h needed in Lemma 2. We are now prepared for choosing firm k's reaction among the different set of values $R_k(\hat{q}_k)$.

Lemma 3. *Let* $i = 1, 2, \ldots$ *be given. If* $q_{i-1}^h \leq \hat{q}_k \leq q_i^h$ *we have*

$$V_1 \leq \cdots \leq V_{i-1} \leq V_i \geq V_{i+1} \geq V_{i+2} \geq \cdots,$$

and the situation of $\Pi_{k,i}$ *on interval* $[t_i, t_{i-1})$ *is exactly situation (i) as described in Figure 3. The situation in any interval on the left of* $[t_i, t_{i-1})$, *say* $[t_{i+m}, t_{i+m-1})$, *is either (i) or (ii) as described in Figure 3; lastly, the situation of any interval on the right of* $[t_i, t_{i-1})$, *say* $[t_{i-m}, t_{i-m-1})$ *is either (i) or (iii). Consequently,* $V_i(\hat{q}_k)$ *is the maximum value of* $\Pi_k(\cdot, \hat{q}_k)$.

The reaction function we finally get is:

$$F_k(\hat{q}_k) = \begin{cases} q_1^c - \frac{1}{2}\hat{q}_k & \text{if } \quad 0 \leq \hat{q}_k < q_1^h \\ \quad\vdots & \qquad\quad \vdots \\ q_i^c - \frac{1}{2}\hat{q}_k & \text{if } q_{i-1}^h \leq \hat{q}_k < q_i^h \\ \quad\vdots & \qquad\quad \vdots \\ 0 & \text{if } \ \delta q_1^c \leq \hat{q}_k \end{cases} \tag{21}$$

(see Figure 4b).

2.1. **Cournot equilibrium points.** Given the r reaction functions (21) we will call *potential* Cournot points the eventual intersections of the different lines $q_k = F_k(\hat{q}_k)$ that can be found without taking into consideration the constraints given by the inequalities

$$q_{i-1}^h \leq \hat{q}_k < q_i^h, (k = 1, \ldots, r).$$

There are an infinity of such points.

Among the potential intersections, those that satisfy the constraints will be called the *actual* Cournot points.

If we choose r indexes, $i_1, i_2, \ldots, i_r,$, among $\{1, 2, \ldots\}$, and restrict firm outputs to positive values, the general solution of the system formed by the r equations chosen is (see the Appendix):

$$I_{i_1,i_2,\ldots,i_r} :=$$

$$\left(2q_{i_1}^c - \frac{2\sum_{j=1}^r q_{i_j}^c}{r+1}, \ldots, 2q_{i_k}^c - \frac{2\sum_{j=1}^r q_{i_j}^c}{r+1}, \ldots, 2q_{i_r}^c - \frac{2\sum_{j=1}^r q_{i_j}^c}{r+1}\right). \tag{22}$$

If instead of r firms, we allow for s firms, $s-r$ of them with zero production output, the general solution of the system formed by the s equations is:

$$I_{i_1,i_2,\ldots,i_r,\underbrace{0,\ldots,0}_{s-r}} := \left(2q^c_{i_1} - \frac{2\sum_{j=1}^r q^c_{i_j}}{r+1}, \ldots, 2q^c_{i_r} - \frac{2\sum_{j=1}^r q^c_{i_j}}{r+1}, \overbrace{0,\ldots,0}^{s-r}\right). \tag{23}$$

We call these asymmetric solutions *trivial*, since they are obtained from solution (22) simply by adding firms with zero production.

3. Determination of the cost function

Let us keep in mind that our purpose is to build a non–quasi–competitive model. The demand function is given and known; we are now going to determine those values of the parameters used so far, q^c_i, c_i, d_i and t_i in such way that our goal is achieved. In the first place we will determine the q^c_i and the t_i. From them, we will find c_i and d_i using (8), (4) and the assumption that $\lim c_i = 0$.

The q^c_i are a decreasing sequence as seen in equation (10). We define them recursively from the first one, q^c_1.

Definition 1. *For $j = 1, 2, \ldots,$*

$$q^c_{j+1} = \frac{j+2}{j+1}\left(1 - \frac{1}{j+\delta}\right) q^c_j, \quad \textit{with } q^c_1 \leq a/2b \tag{24}$$

and $\delta \in (0, 2/3)$.

We will later see the reason for the parameter δ and its inclusion in the interval $(0, 2/3)$.

Replacing q^c_j by its corresponding expression in terms of q^c_{j-1} we eventually reach a second definition for the q^c_j:

Definition 2. *For $j = 1, 2, \ldots,$*

$$q^c_j = \frac{j+1}{2}\,\frac{\delta}{j-1+\delta}\, q^c_1, \tag{25}$$

where $\delta \in (0, 2/3)$.[1]

Definition 3. *For $j = 1, 2, \ldots,$*

$$t_j = \frac{1}{2}\, q^c_j \left(1 - \frac{j-2}{j+1}\,(1 - \frac{1}{j+\delta}) + \beta \frac{2\delta}{(j+1)(j+\delta)}\right) \tag{26}$$

where $\delta \in (0, 2/3)$ as before and[2]

$$0 < \beta < 1 - \frac{3}{2}\,\frac{3+\delta}{3+2\,\delta}\,\delta. \tag{27}$$

We are now ready to see the following Lemma which will be needed in the sequel:

Lemma 4. *The q^h_i defined in (20) satisfy*

$$0 < q^h_1 < \cdots < q^h_i < \cdots < a/(2b),$$

and $\lim_{i\to\infty} q^h_i = \delta\, q^c_1$.

---

[1]In the Appendix we check the monotony of the q^c_j sequence.

[2]In the Appendix we check the monotony of the t_j.

The lemma is proved in the Appendix.

Lastly, condition (4) says

$$c_i - c_{i+1} = (d_{i+1} - d_i)t_i \qquad i = 1, 2, \dots$$

Adding these last equations for $i = 1, 2, \dots, n$ we get

$$c_1 - c_n = \sum_{i=1}^{n} (d_{i+1} - d_i)t_i.$$

If we impose that $c_n \to 0$ we must have

$$c_1 = \sum_{i=1}^{\infty} (d_{i+1} - d_i)t_i. \tag{28}$$

We have that $d_{i+1} - d_i = 2b(q_i^c - q_{i+1}^c)$ and from (8),

$$d_i = a - \frac{i+1}{i-1+\delta} b\delta q_1^c. \tag{29}$$

Now, using for t_i expression (39) from the Appendix, we finally have

$$c_1 = \sum_{i=1}^{\infty} (2-\delta)\, b(\delta q_1^c)^2 \, \frac{i - 1/2 + (3\delta + 2\beta\delta)/4}{(i-1+\delta)^2 (i+\delta)^2}.$$

This series is obviously convergent and its sum can be obtained with the help of hypergeometric series,

$$c_1 = (2-\delta)\, b(\delta q_1^c)^2 \left(\frac{2\delta^2 - 4\beta\delta^2 + 2\beta\delta - \delta + 2}{4\delta^2} + \frac{\delta(2\beta - 1)}{2} \Psi(1, 1+\delta) \right) \tag{30}$$

where $\Psi(n, x)$ is the n–th polygamma function, i.e., the n–th derivative of the logarithmic derivative of $\Gamma(x)$, see Lebedev (1972).

4. Existence and uniqueness of equilibrium

We are now prepared to tackle the first of our aims: the existence and uniqueness of non-trivial Cournot's equilibrium no matter the number of oligopolists in the industry.

Theorem 1. *Under the previous conditions, given a fixed number of firms, $r \geq 1$ with identical cost function (3), there exists a unique non-trivial Cournot equilibrium point which is an actual solution of system (22). This solution is the one given by $i_1 = i_2 = \cdots = i_r = r$:*

$$I_{r,\dots,r} = \left(\frac{2}{r+1} q_r^c, \dots, \frac{2}{r+1} q_r^c \right). \tag{31}$$

If given a fixed number of firms, s, with identical cost function, there are $s - r$ with zero production, the unique trivial solution of system (23) is obtained for $i_1 = i_2 = \cdots = i_r = r$:

$$I_{r,\dots,r,\underbrace{0,\dots,0}_{s-r}} = \left(\frac{2}{r+1} q_r^c, \dots, \frac{2}{r+1} q_r^c, \overbrace{0, \dots, 0}^{s-r} \right). \tag{32}$$

The idea behind the proof (which is long and quite tedious and can be found in the Appendix) is to see that among the potential solutions to (22), only the symmetric one obtained for the system when the r equations are $i_1 = i_2 = \cdots = i_r = r$ is an actual solution. Let us recall that "actual" solutions are those that satisfy all the constraints given by the inequalities $q_{i-1}^h \leq \hat{q}_k < q_i^h, (k = 1, \dots, r)$.

5. The loss of quasi–competitiveness, perfect competition and viability

Theorem 2. *Under the same conditions as before, the oligopoly equilibrium reached is not quasi–competitive, that is to say, when a new firm enters the market and a new equilibrium is reached, the new market price is* ***greater*** *than the old one.*

Proof. Let us call Q_r the total production when there are r non-zero production firms in the market under equilibrium. We are going to prove that, for $r = 1, 2, \ldots$ we have $Q_r > Q_{r+1}$.

From (31) we have

$$Q_r = r\bar{q}_r = \frac{r\delta}{r-1+\delta} q_1^c = \delta q_1^c + \frac{\delta(1-\delta)}{r-1+\delta} q_1^c \tag{33}$$

which, as long as $0 < \delta < 2/3$, is strictly decreasing with r and tends to δq_1^c as $r \to \infty$.

Thus, industry price satisfies

$$\lim_{r\to\infty} p(Q_r) = a - b\delta q_1^c,$$

and marginal cost is the corresponding to $C'(0)$, that is $\lim d_r$. Now,

$$\lim_{r\to\infty} d_r = \lim_{r\to\infty} (a - 2bq_r^c) = a - b \lim_{r\to\infty} 2q_r^c = a - b\delta q_1^c,$$

as from definition 2, $\lim q_r^c = \delta q_1^c/2$. □

This result completes Ruffin's results on the subject as it proves that convexity of the cost function while a sufficient condition for perfect competition in the limit, is not a necessary one, Ruffin (1971).

Lastly, the oligopoly is always viable no matter the number of firms in the market. We will see that equilibrium profits constitute a strictly decreasing sequence ($r \to \infty$) with limit zero. This implies, consequently, that equilibrium profit is always positive for any number of firms in the market.

In order to check that, we consider an equilibrium point (31) (or (32)) and the profit there

$$\Pi_r^{\max} = \left(a - b\frac{2rq_r^c}{r+1}\right)\frac{2q_r^c}{r+1} - \left(c_r + d_r\frac{2q_r^c}{r+1}\right).$$

This, after using (2), (29), and some algebra becomes

$$\Pi_r^{\max} = b\left(\frac{\delta q_1^c}{r-1+\delta}\right)^2 - c_r. \tag{34}$$

As $c_r \to 0$ with r, it is obvious that

$$\lim_{r\to\infty} \Pi_r^{\max} = 0.$$

We must finally check that $\Pi_r^{\max}$ is a decreasing sequence:

$$\Pi_r^{\max} - \Pi_{r+1}^{\max} > 0.$$

Using (34) we have

$$\Pi_r^{\max} - \Pi_{r+1}^{\max} = b(\delta q_1^c)^2\left[\left(\frac{1}{r-1+\delta}\right)^2 - \left(\frac{1}{r+\delta}\right)^2\right] - (c_r - c_{r+1}).$$

From (4) we have

$$c_r - c_{r+1} = (d_{r+1} - d_r)t_r.$$

Replacing d_r and t_r by their values in (29) and (26), we finally obtain

$$\Pi_r^{\max} - \Pi_{r+1}^{\max} = \frac{b(\delta q_1^c)^2}{(r-1+\delta)^2(r+\delta)^2}\left[\delta(r-\beta) + \frac{\delta^2(3+2\beta)}{4}\right] > 0$$

for $r \geq 1$ (using (27) it is easy to see that $\beta < 1$).[3]

6. The stability of the successive equilibria

In order to prove the stability of our model we will use a commonly used adjustment system. We assume that each firm adjusts its output proportionally with the difference between its actual profit and its profit maximizing output:

$$\begin{cases} \dot{q}_1 = k_1 \left(F_1(\hat{q}_1) - q_1\right) \\ \quad\vdots \\ \dot{q}_i = k_i \left(F_i(\hat{q}_i) - q_i\right) \\ \quad\vdots \\ \dot{q}_s = k_s \left(F_s(\hat{q}_s) - q_s\right), \end{cases} \tag{36}$$

where the $k_i > 0$ $(i = 1, 2, \ldots, s)$ are the *speeds* of adjustment.

This is no standard system of differential equations. The piecewise character of $F_k(\hat{q_k})$ make (36) a very special dynamical system.

Nevertheless, we have already established that there are two kinds of steady states to our system:

- The symmetric ones for which the r firms in the industry have positive equal outputs.
- The asymmetric ones, where to the r firms above, another $s - r$ are in the industry with no output at all.

We will assume without loss of generality that the r firms with positive production are ordered according to their speeds of adjustment: $k_1 \leq k_2 \leq \cdots \leq k_r$.

Using our reaction function (21) the system of s differential equations (36) becomes

$$\dot{q}_i = \begin{cases} k_i(q^c_r - \frac{1}{2}\hat{q}_i - q_i) & i = 1, \ldots, r; \\ k_i(-q_i) & i = r + 1, \ldots, s. \end{cases} \tag{37}$$

The solution, our Cournot equilibrium, is, as we have already seen,

$$\left(\frac{2q^c_r}{r+1}, \ldots, \frac{2q^c_r}{r+1}, \overbrace{0, \ldots, 0}^{s-r} \right).$$

[3]We provide the data for a numeric example that satisfies all our assumptions showing thus the feasibility of our model. The figures provided in the paper are those that correspond to this numerical situation.

$$p = 100 - 2q \quad (0 \leq q \leq 50). \tag{35}$$

We choose $q^c_1 = 24$. If we fix $\delta = 1/3$ then $0 < \beta < 0.5454\ldots$; we choose $\beta = 1/2$.

For $r = 1, 2, \ldots, 5$ the successive productions and prices at equilibrium are:

r	Q_r/r	Q_r	p
1	24	24	52
2	6	12	76
3	3.43	10.29	79.43
4	2.4	9.6	80
5	1.85	9.25	81.5

When $r \to \infty$, $Q_r \to 8$ and this establishes the price $p = 84$. This is the perfect competition price that coincides with the marginal cost at $q = 0$: $C'(0) = \lim_{r\to\infty} d_r = 84$.

Let us recall that this equilibrium comes from solving system (43) which required that

$$
\begin{cases} \hat{q}_i \in [q^h_{r-1}, q^h_r) & \text{for } i = 1, \dots, r \\ \text{and} & \\ \hat{q}_j \geq \delta q^c_1 & \text{for } j = r+1, \dots, s. \end{cases} \tag{38}
$$

Denoting

$$\bar{q}_r := \frac{2q^c_r}{r+1}.$$

the equilibrium is written

$$(\bar{q}_r, \dots, \bar{q}_r, 0, 0, \dots, 0).$$

It cannot escape the reader that a difficult problem one faces when solving the dynamic system (36) is the control of the orbit of the solution. The piecewise character of $F_k(\hat{q}_k)$ partitions the phase space, $[0, a/b)^s$, into an infinity of regions in which a different system of differential equations rules the dynamics and, consequently, the orbit of a solution starting at a point $(q_1(0), \dots, q_s(0))$ is altered accordingly from one region to another.

We can establish without much difficulty that each one of these different systems of differential equations, when considered on its own (that is, without any constraint whatsoever), has a stationary solution which is globally asymptotically stable.

Theorem 3. *The stationary solution of the system of differential equations*

$$
\dot{q}_i = \begin{cases} k_i(q^c_r - \frac{1}{2}\hat{q}_i - q_i) & i = 1, \dots, r; \\ k_i(-q_i) & i = r+1, \dots, s. \end{cases}
$$

is globally asymptotically stable.

After that, we are ready to prove that all our equilibria, symmetric or not, are asymptotically stable for the dynamical system (36). This is done in the next Theorem whose proof (Appendix) is a bit elaborate. To give you a general idea of the line of reasoning, we prove that we can find a set, S, around the equilibrium for which any starting orbit tends back to the equilibrium **without abandoning** the region established by the constraints (38). This is essential as on the very moment an orbit leaves the region of the initial constraints it falls in the basin of attraction of a different attractor and we have no guarantee ot if going back to our original equilibrium.

Lemma 5. *Under the max norm,* $\|(x_1, x_2, \dots, x_s)\|_\infty = \max_i |x_i|$*, the ball*

$$B((\bar{q}_r, \dots, \bar{q}_r, 0, \dots, 0), \rho/(s-1))$$

of center $(\bar{q}_r, \dots, \bar{q}_r, 0, \dots, 0)$ *and radius* $\rho/(s-1)$*, where*

$$\rho = \min\left\{(r-1)\bar{q}_r - q^h_{r-1}, q^h_r - (r-1)\bar{q}_r, \frac{(s-1)(r\bar{q}_r - \delta q^c_1)}{r}\right\},$$

is entirely contained in the region established by the set of constraints (38).

Instead of the max norm, Any other equivalent norm could be used, but this one is more convenient for our purposes.

Our main Theorem in this section is:

Theorem 4. *There exists a radius* $\eta > 0$ *such that the equilibrium*

$$(\bar{q}_r, \dots, \bar{q}_r, 0, \dots, 0)$$

is asymptotically stable for any initial conditions lying in the ball

$$B((\bar{q}_r, \dots, \bar{q}_r, 0, \dots, 0), \eta).$$

APPENDIX: PROOFS.

Proof of Lemma 3

Proof. If $q^h_{i-1} \le \hat{q}_k \le q^h_i$, replacing q^h_i by its value, $q^c_i + q^c_{i+1} - 2t_i$, and q^h_{i-1} by its value $q^c_{i-1} + q^c_i - 2t_{i-1}$, we have immediately that

$$\frac{1}{2}(q^c_i - q^c_{i+1}) + t_i \le q^c_i - \frac{1}{2}\,\hat{q}_k \le t_{i-1} - \frac{1}{2}(q^c_{i-1} - q^c_i).$$

As $q^c_{i+1} < q^c_i < q^c_{i-1}$, we have

$$t_i < q^c_i - \frac{1}{2}\,\hat{q}_k < t_{i-1},$$

and the vertex V_i is exactly within (t_i, t_{i-1}) as shown in situation (i). Now, as we are assuming that the q^h_i are strictly increasing, for any interval on the left of $[t_i, t_{i-1})$, say $[t_{i+m}, t_{i+m-1})$, the situation of $q^c_{i+m} - \frac{1}{2}\,\hat{q}_k$ is determined by the relationship $\hat{q}_k < q^h_{i+m}$. Replacing, as before, q^h_{i+m} by its value $q^c_{i+m} + q^c_{i+m+1} - 2t_{i+m}$, we obtain

$$q^c_{i+m} - \frac{1}{2}\,\hat{q}_k > t_{i+m}.$$

Consequently, situation (iii) in Figure 3 is ruled out, and as $V_i \ge V_{i+m}$ and $V_{i+m} \ge \Pi_k(t_{i+m-1})$, V_i is the greatest value of Π_k on $[0, t_{i-1})$. Now, on the right of $[t_i, t_{i-1})$, say $[t_{i-m}, t_{i-m-1})$, the situation of $q^c_{i-m} - \frac{1}{2}\,\hat{q}_k$ is, as before, determined by the inequality $q^h_{i-m-1} < \hat{q}_k$. Replacing q^h_{i-m-1} by its value $q^c_{i-m-1} + q^c_{i-m} - 2t_{i-m-1}$ we obtain,

$$q^c_{i-m} - \frac{1}{2}\,\hat{q}_k < t_{i-m-1}.$$

As before, situation (ii) in Figure 3 is ruled out, and as $V_i \ge V_{i-m}$ and $V_{i-m} \ge \Pi_k(t_{i-m})$, V_i is the greatest value of Π_k on $[t_{i-1}, q^c_1])$. □

Monotony of q^c_j. From (24)

$$q^c_{j+1} = \left(1 + \frac{1}{j+1}\right)\left(1 - \frac{1}{j+\delta}\right) q^c_j,$$

As $0 < \delta < 2/3 < 1$ it is seen at once that

$$\left(1 + \frac{1}{j+1}\right)\left(1 - \frac{1}{j+\delta}\right) < \left(1 + \frac{1}{j+1}\right)\left(1 - \frac{1}{j+1}\right) = 1 - \frac{1}{(j+1)^2} < 1.$$

Monotony of t_j. From (26)

$$t_j = \frac{1}{2}\,q^c_j\,\left(1 - \frac{j-2}{j+1}\,(1 - \frac{1}{j+\delta}) + \beta\frac{2\delta}{(j+1)(j+\delta)}\right)$$

If we replace q^c_j by its expression (25) after some algebra we get to

$$t_j = \frac{1}{2}q^c_1\frac{\delta(4j + 3\delta + 2\beta\delta - 2)}{2(j-1+\delta)(j+\delta)} = q^c_1\delta\left(\frac{1}{j+\delta} + \frac{2 + 2\beta\delta - \delta}{4(j+\delta-1)(j+\delta))}\right). \quad (39)$$

The expression between parenthesis is clearly decreasing with j.

Proof of Lemma 4

Proof. The result is an immediate consequence of the following expressions of q_i^h obtained through (20), (24), (25) and (26):

$$q_{i-1}^h = 2\left(\frac{i-1}{i+1} - \frac{\beta\delta}{(i+1)(i-2+\delta)}\right) q_i^c \tag{40}$$

$$q_i^h = 2\left(\frac{i-1}{i+1} + \frac{(1-\beta)\delta}{(i+1)(i+\delta)}\right) q_i^c \tag{41}$$

Now

$$\lim_{i\to\infty} q_i^h = \lim_{i\to\infty} 2\left(\frac{i-1}{i+1} + \frac{(1-\beta)\delta}{(i+1)(i+\delta)}\right) q_i^c = 2\lim_{i\to\infty} q_i^c = \delta\, q_1^c$$

as by (25) $\lim q_i^c = \delta\, q_1^c/2$. □

Cournot points of the model. Given the r reaction functions (21):

$$F_k(\hat{q}_k) = \begin{cases} q_1^c - \frac{1}{2}\hat{q}_k & \text{if} \quad 0 \le \hat{q}_k < q_1^h \\ \quad\vdots & \qquad\vdots \\ q_i^c - \frac{1}{2}\hat{q}_k & \text{if } q_{i-1}^h \le \hat{q}_k < q_i^h \quad (k = 1, 2, \ldots, r), \\ \quad\vdots & \qquad\vdots \\ 0 & \text{if} \quad \delta q_1^c \le \hat{q}_k \end{cases}$$

we are interested in finding all the potential intersections.

Let $i_1, \ldots, i_r$ be any r-ple of indexes chosen among $1, 2, \ldots$. Let $q_k = F_k(\hat{q}_k) > 0$; the system of r equations that has to be solved is

$$\begin{cases} q_1 = q_{i_1}^c - \frac{1}{2}(q_2 + q_3 + \cdots + q_k + \cdots + q_r) \\ q_2 = q_{i_2}^c - \frac{1}{2}(q_1 + q_3 + \cdots + q_k + \cdots + q_r) \\ \quad\vdots \\ q_k = q_{i_k}^c - \frac{1}{2}(q_1 + q_2 + \cdots + q_{k-1} + q_{k+1} + \cdots + q_r) \\ \quad\vdots \\ q_r = q_{i_r}^c - \frac{1}{2}(q_1 + q_2 + \cdots + q_k + \cdots + q_{r-1}) \end{cases} \tag{42}$$

that can be written as

$$\begin{cases} 2q_1 + q_2 + \cdots + q_k + \cdots + q_r = 2q_{i_1}^c \\ q_1 + 2q_2 + \cdots + q_k + \cdots + q_r = 2q_{i_2}^c \\ \qquad\vdots \\ q_1 + q_2 + \cdots + 2q_k + \cdots + q_r = 2q_{i_k}^c \\ \qquad\vdots \\ q_1 + q_2 + \cdots + q_k + \cdots + 2q_r = 2q_{i_r}^c \end{cases}$$

Adding up both sides of the equations we have

$$(r+1)(q_1 + \cdots + q_r) = 2(q_{i_1}^c + \cdots + q_{i_r}^c),$$

and, as for any $k, (k = 1, 2, \ldots, r)$, $q_1 + \cdots + q_r = q_k + \hat{q}_k$,

$$\hat{q}_k = \frac{2}{r+1}(q_{i_1}^c + \cdots + q_{i_r}^c) - q_k,$$

which, replaced in (42) leads to

$$q_k = 2q_{i_k}^c - \frac{2}{r+1}(q_{i_1}^c + \cdots + q_{i_r}^c).$$

Thus the solution is

$$\left(2q_{i_1}^c - \frac{2}{r+1}\Sigma_{j=1}^r q_{i_j}^c, \ldots,\ 2q_{i_k}^c - \frac{2}{r+1}\Sigma_{j=1}^r q_{i_j}^c, \ldots,\ 2q_{i_r}^c - \frac{2}{r+1}\Sigma_{j=1}^r q_{i_j}^c\right).$$

Lets now suppose that we have a system of s equations with $q_k = F_k(\hat{q}_k) > 0$ if $k = 1, 2, \ldots, r$, and $q_m = 0$ when $m = r+1, \ldots, s$. The system is:

$$\left\{\begin{array}{lcl} q_1 & = & q^c_{i_1} - \frac{1}{2}(q_2 + q_3 + \cdots + q_k + \cdots + q_r) \\ & \vdots & \\ q_r & = & q^c_{i_r} - \frac{1}{2}(q_1 + q_2 + \cdots + q_k + \cdots + q_{r-1}) \\ q_{r+1} & = & 0 \\ & \vdots & \\ q_s & = & 0. \end{array}\right. \tag{43}$$

Proceeding along the same steps as before, we reach the solution

$$\left(2q^c_{i_1} - \frac{2}{r+1}\Sigma^r_{j=1} q^c_{i_j}, \ldots, 2q^c_{i_r} - \frac{2}{r+1}\Sigma^r_{j=1} q^c_{i_j}, \overbrace{0, \ldots, 0,}^{s-r}\right).$$

Proof of Theorem 1

Proof. To begin with, for reasons that will be clear in the sequel, we treat the monopoly case separately. Let us suppose then that $r = 1$.

From (22), the potential solutions are $I_{i_1} = q^c_{i_1}$, where $i_1 \in \{1, 2, \ldots\}$. As the $q^c_{i_1}$ constitute a decreasing sequence, and the maximum profit of the firm, (14), is proportional to the square of $q^c_{i_1}$, it is clear that our firm will choose the greatest possible value for $q^c_{i_1}$: q^c_1. Thus, when $r = 1$, the result is clear.

Let us now assume that there are $r \geq 2$ firms that have the same positive production and $s - r$ firms with zero production. Consequently the first r firms will all choose the same equation as their reaction and $i_1 = i_2 = \cdots = i_r = i$. We will prove that the only actual solution for system (23) is obtained when $i = r$. Let

$$I_{i,\ldots,i,\underbrace{0,\ldots,0}_{s-r}} = \left(\frac{2}{r+1} q^c_i, \ldots, \frac{2}{r+1} q^c_i, \overbrace{0, \ldots, 0}^{s-r}\right), \tag{44}$$

be the corresponding potential solution of system (23), which in this case is

$$\left\{\begin{array}{lll} q_1 & = q^c_i - \frac{1}{2}\hat{q}_1 & \text{if } q^h_{i-1} \leq \hat{q}_1 < q^h_i \\ \vdots & \quad\quad \vdots & \\ q_k & = q^c_i - \frac{1}{2}\hat{q}_k & \text{if } q^h_{i-1} \leq \hat{q}_k < q^h_i \\ \vdots & \quad\quad \vdots & \\ q_r & = q^c_i - \frac{1}{2}\hat{q}_r & \text{if } q^h_{i-1} \leq \hat{q}_r < q^h_i. \\ q_{r+1} & = 0 & \text{if } \delta q^c_1 \leq \hat{q}_{r+1} \\ \vdots & \quad\quad \vdots & \\ q_s & = 0 & \text{if } \delta q^c_1 \leq \hat{q}_s \end{array}\right.$$

From solution (44) we have that for firm $k, (k = 1, \ldots, r)$,

$$\hat{q}_k = 2\, q^c_i \frac{r-1}{r+1}. \tag{45}$$

As our solution must satisfy the necessary constraints, we must have $q^h_{i-1} \leq \hat{q}_k < q^h_i$, which using (40) and (41) can be written as

$$\begin{aligned} 2\left(\frac{i-1}{i+1} - \frac{\beta\delta}{(i+1)(i-2+\delta)}\right) q^c_i \quad &\leq \quad 2\, q^c_i \frac{r-1}{r+1} \\ &\leq \quad 2\left(\frac{i-1}{i+1} + \frac{(1-\beta)\delta}{(i+1)(i+\delta)}\right) q^c_i. \end{aligned}$$

Simplifying,

$$\frac{i-1}{i+1}-\frac{\beta\delta}{(i+1)(i-2+\delta)}\leq\frac{r-1}{r+1}\leq\frac{i-1}{i+1}+\frac{(1-\beta)\delta}{(i+1)(i+\delta)}. \tag{46}$$

The double inequality is obviously true for $i=r$, for any $r\geq 2$. We are now going to prove that in case $i>r$ the left inequality fails to be true and if $i<r$, the right one is not true.

Since the expression in the left hand side of the first inequality increases with i, it will suffice to prove that the inequality is not true for $i=r+1$ and our assertion will follow. In this case, replacing $i=r+1$ we have:

$$\frac{r-1+\delta}{r+1}<\frac{\beta\delta}{2}. \tag{47}$$

But $(r-1+\delta)/(r+1)$ increases with r which means that its value is always greater than the value obtained for $r=2$,

$$\frac{r-1+\delta}{r+1}\geq\frac{1+\delta}{3}>\frac{\delta}{2}>\frac{\beta\delta}{2} \tag{48}$$

The contradiction between (47) and (48) prove that $i\leq r$.

Let us now consider the second inequality in (46). The right hand expression increases again with i; we will thus prove our assertion if the inequality fails for $i=r-1$, the greatest possible value of i less than r. Replacing $i=r-1$, we have

$$\frac{r-1}{r+1}<\frac{r-2}{r}+\frac{(1-\beta)\delta}{r(r-1+\delta)}$$

from which we obtain

$$\frac{2}{r+1}<\frac{(1-\beta)\delta}{r-1+\delta}$$

and

$$\frac{r-1+\delta}{r+1}<\frac{(1-\beta)\delta}{2}. \tag{49}$$

But from (48) we have

$$\frac{r-1+\delta}{r+1}>\frac{\delta}{2}>\frac{(1-\beta)\delta}{2}. \tag{50}$$

The contradiction between (49) and (50) proves that $i\geq r$. We conclude that $i=r$ as we contented.

We now have to check that the $s-r$ zero-production firms also satisfy their constraints, i.e. $\hat{q}_m\geq\delta q_1^c$ for $m=r+1,\ldots,s$. This is trivial because we now know that, using (25),

$$\hat{q}_m=\frac{2r}{r+1}q_r^c=\frac{2r}{r+1}\frac{r+1}{2}\frac{\delta}{r-1+\delta}q_1^c\geq\delta q_1^c,$$

as long as $\delta\leq 1$, which is the case.

Obviously, allowing for $s=r$ leads to the solution (31).

Lastly, we are going to prove that if not all of the $i_1,i_2,\ldots,i_r$ are equal, there are no actual solutions of systems (22) and (23). The $s-r$ firms with zero production do not play any role in this part of the demonstration. Let us assume then that $i_1\leq i_2\leq\cdots\leq i_r$ and $i_1<i_r$. We recall that the constraints of solution (23) require

$$\begin{cases} q_{i_1-1}^h & \leq \quad \hat{q}_{i_1}<q_{i_1}^h \\ q_{i_r-1}^h & \leq \quad \hat{q}_{i_r}<q_{i_r}^h. \end{cases} \tag{51}$$

Now, from (23), we have

$$\begin{cases} \hat{q}_{i_1} = 2(q^c_{i_2} + q^c_{i_3} + \cdots + q^c_{i_r}) - (r-1)\dfrac{2}{r+1}\sum_{l=1}^r q^c_{i_l} \\ \quad\vdots \\ \hat{q}_{i_r} = 2(q^c_{i_1} + q^c_{i_2} + \cdots + q^c_{i_{r-1}}) - (r-1)\dfrac{2}{r+1}\sum_{l=1}^r q^c_{i_l}. \end{cases}$$

Consequently,

$$\hat{q}_{i_r} - \hat{q}_{i_1} = 2\,(q^c_{i_1} - q^c_{i_r}),$$

and (51) can be written as

$$\begin{cases} q^h_{i_1-1} & \leq \hat{q}_{i_r} - 2\,(q^c_{i_1} - q^c_{i_r}) < q^h_{i_1} \\ & \vdots \\ q^h_{i_r-1} & \leq \hat{q}_{i_r} < q^h_{i_r}. \end{cases} \tag{52}$$

It is now obvious that if we manage to establish that

$$q^h_{i_r} - q^h_{i_1-1} < 2\,(q^c_{i_1} - q^c_{i_r}), \tag{53}$$

it will be impossible to satisfy neither (52) nor (51).

To simplify notation and make the checking of (53) easier, we call $i_1 = i$, and $i_r = i + d$ $(d > 0)$. With this notation (53) becomes:

$$q^h_{i+d} - q^h_{i-1} < 2\,(q^c_i - q^c_{i+d}). \tag{54}$$

Changing sides,

$$q^h_{i+d} + 2q^c_{i+d} < q^h_{i-1} + 2q^c_i. \tag{55}$$

Using (25), (40) and (41) we have

$$\begin{aligned} q^h_{i+d} &+ 2q^c_{i+d} = \\ &= 2\frac{i+d-1}{i+d+1}q^c_{i+d} + \frac{2(1-\beta)\delta}{(i+d-1)(i+d+\delta)}q^c_{i+d} + 2q^c_{i+d} \\ &= \left(\frac{4(i+d)}{(i+d+1)} + \frac{2(1-\beta)\delta}{(i+d-1)(i+d+\delta)}\right) q^c_{i+d} \\ &= \left(\frac{2(i+d)\delta}{i+d-1+\delta} + \frac{(1-\beta)\delta^2}{(i+d-1+\delta)(i+d+\delta)}\right) q^c_1 \\ &= \left(2\delta + \frac{2\delta(1-\delta)}{i+d-1+\delta} + \frac{(1-\beta)\delta^2}{(i+d-1+\delta)(i+d+\delta)}\right) q^c_1. \end{aligned}$$

This last expression decreases with i and with d. Thus, in order to satisfy (55) for $d \geq 1$ it suffices to satisfy it for $d = 1$. For $d = 1$ the left hand side of (55) becomes

$$\left(2\delta + \frac{2\delta(1-\delta)}{i+\delta} + \frac{(1-\beta)\delta^2}{(i+\delta)(i+1+\delta)}\right) q^c_1, \tag{56}$$

and the right hand side,

$$\begin{aligned} q^h_{i-1} &+ 2q^c_i = \\ &= \left(2\frac{i-1}{i+1} - \frac{2\beta\delta}{(i+1)(i-2+\delta)} + 2\right) q^c_i \\ &= \left(\frac{4i}{(i+1)} - \frac{2\beta\delta}{(i+1)(i-2+\delta)}\right) q^c_i \\ &= \left(2\frac{\delta i}{i-1+\delta} - \frac{\beta\delta^2}{(i-2+\delta)(i-1+\delta)}\right) q^c_1 \\ &= \left(2\delta + \frac{2\delta(1-\delta)}{i+1+\delta} - \frac{\beta\delta^2}{(i-2+\delta)(i-1+\delta)}\right) q^c_1. \end{aligned}$$

Now, replacing the corresponding values in (55) and simplifying:

$$\frac{2(1-\delta)}{i+\delta} + \frac{(1-\beta)\delta}{(i+\delta)(i+1+\delta)} < \frac{2(1-\delta)}{(i-2+\delta)(i-1+\delta)}$$

or, equivalently,

$$\frac{(1-\beta)\delta}{(i+\delta)(i+1+\delta)} + \frac{\beta\delta}{(i-2+\delta)(i-1+\delta)} < \frac{2(1-\delta)}{i-1+\delta} - \frac{2(1-\delta)}{i+\delta} = \frac{2(1-\delta)}{(i+\delta)(i+\delta-1)},$$

which can be written

$$\frac{(1-\beta)\delta(i+\delta-1)}{i+\delta+1} + \frac{\beta\delta(i+\delta)}{i-2+\delta} < 2-2\delta,$$

or

$$(1-\beta)\delta\left(1-\frac{2}{i+\delta+1}\right) + \beta\delta\left(1+\frac{2}{i-2+\delta}\right) < 2-2\delta.$$

Rearranging,

$$\frac{2\beta\delta}{i-2+\delta} - \frac{2\delta(1-\beta)}{i+\delta+1} < 2-3\delta. \tag{57}$$

The condition $0 < \delta < 2/3$ ensures the positivity of the right hand side of last inequality. Now, as we want (57) to be true for $i \geq 2$, dividing through by 2δ we get

$$\frac{\beta}{i-2+\delta} - \frac{1-\beta}{i+\delta+1} < \frac{2-3\delta}{2\delta}$$

and if $i \geq 2$

$$\frac{\beta}{i-2+\delta} - \frac{1-\beta}{i+\delta+1} < \frac{2-3\delta}{2\delta} \leq \frac{\beta}{\delta} - \frac{1-\beta}{\delta+3} < \frac{2-3\delta}{2\delta}$$

this last inequality being equivalent to

$$\beta < 1 - \frac{3}{2}\delta\frac{3+\delta}{3+2\delta}$$

which is the condition we have demanded our β to satisfy all the time. □

Proof of Theorem 3

Proof. Let us recall that we are solving system (37), that is:

$$\dot{q}_i = \begin{cases} k_i(q_r^c - \frac{1}{2}\hat{q}_i - q_i) & i = 1,\dots,r; \\ k_i(-q_i) & i = r+1,\dots,s. \end{cases}$$

As the stationary solution is $(\bar{q}_r,\dots,\bar{q}_r,0,\dots,0)$ let us carry out a change of variables:

$$x_i = \begin{cases} q_i - \bar{q}_r & i = 1,\dots,r; \\ q_i & i = r+1,\dots,s. \end{cases} \tag{58}$$

Consequently, $\dot{x}_i = \dot{q}_i,\ \hat{x}_i = \sum_{j\neq i}(q_j - \bar{q}_r) = \hat{q}_i - (r-1)\bar{q}_r$ and using that

$$\bar{q}_r = \frac{2q_r^c}{r+1}$$

the system becomes

$$\dot{x}_i = \begin{cases} k_i(-\frac{1}{2}\hat{x}_i - x_i) & i = 1,\dots,r; \\ k_i(-x_i) & i = r+1,\dots,s. \end{cases} \tag{59}$$

which is a homogeneous system that can be written:

$$\dot{\mathbf{x}} = A\mathbf{x}$$

where A is

$$A = \begin{pmatrix} -k_1 & -k_1/2 & \cdots & -k_1/2 & 0 & \cdots & 0 \\ -k_2/2 & -k_2 & \cdots & -k_2/2 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -k_r/2 & -k_r/2 & \cdots & -k_r & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & -k_{r+1} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & -k_s \end{pmatrix}.$$

with $\mathbf{0}$ as stationary solution.

The stability of the system depends entirely on the eigenvalues of matrix A. These eigenvalues are exactly those of

$$B = \begin{pmatrix} 2k_1 & k_1 & \cdots & k_1 & 0 & \cdots & 0 \\ k_2 & 2k_2 & \cdots & k_2 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ k_r & k_r & \cdots & 2k_r & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 2k_{r+1} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 2k_s \end{pmatrix}$$

multiplied by $-1/2$. Now, the eigenvalues of matrix B are all real and positive. This can be seen just considering its characteristic polynomial,

$$P(x) = (2k_{r+1} - x) \cdots (2k_s - x) \begin{vmatrix} 2k_1 - x & k_1 & \cdots & k_1 \\ k_2 & 2k_2 - x & \cdots & k_2 \\ \cdots & \cdots & \cdots & \cdots \\ k_r & k_r & \cdots & 2k_r - x \end{vmatrix}. \tag{60}$$

Obviously, the $s - r$ eigenvalues $k_{r+1}, \dots, k_s$ are positive. The rest are the zeroes of the polynomial,

$$M(x) = \begin{vmatrix} 2k_1 - x & k_1 & \cdots & k_1 \\ k_2 & 2k_2 - x & \cdots & k_2 \\ \cdots & \cdots & \cdots & \cdots \\ k_r & k_r & \cdots & 2k_r - x \end{vmatrix} \tag{61}$$

which are also positive. This is proved as follows.

It is easy to see that

$$M(k_j) = k_j \prod_{\substack{i=1 \\ i \neq j}}^{r} (k_i - k_j).$$

As $k_1 < k_2 < \cdots < k_r$, we have that the signs of the sequence

$$M(k_1), M(k_2), \dots, M(k_r), M(\infty)$$

alternate. This fact guarantees that the r roots of $M(x)$ are one in each interval $(k_1, k_2), \dots, (k_{r-1}, k_r), (k_r, \infty)$. If ℓ consecutive k_i's are equal, k_i itself becomes a root of multiplicity $\ell - 1$, which can be seen differentiating ℓ times $M(x)$ from its determinant form, (61); the rest of the roots remain in the same intervals as before.

Consequently, A's eigenvalues are all strictly negative and therefore the stationary solution of our system is asymptotically stable. □

Proof of Lemma 5

Proof. Let $(q_1, \dots, q_r, q_{r+1}, \dots, q_s) \in B((\bar{q}_r, \dots, \bar{q}_r, 0, \dots, 0), \rho/(s-1))$. We can write

$$\begin{cases} q_i = \bar{q}_r + \xi_i & \text{for} \quad i = 1, \dots, r \\ q_j = \varepsilon_j & \text{for} \quad j = r+1, \dots, s \end{cases} \quad \text{with} \quad \left\| \overrightarrow{(\xi_i, \varepsilon_j)} \right\|_\infty < \frac{\rho}{s-1}.$$

Then, for $i = 1, \ldots, r$,

$$\hat{q}_i = (r-1)\bar{q}_r + \hat{\xi}_i + \sum_j \varepsilon_j.$$

But

$$|\hat{q}_i - (r-1)\bar{q}_r| \leq \left|\hat{\xi}_i\right| + \sum_j \varepsilon_j < (r-1)\frac{\rho}{s-1} + (s-r)\frac{\rho}{s-1} = \rho$$

and by $\rho' s$ definition, $\rho \leq \min\{(r-1)\bar{q}_r - q^h_{r-1}, q^h_r - (r-1)\bar{q}_r\}$ and so for $i = 1, \ldots, r,\ \hat{q}_i \in (q^h_{r-1}, q^h_r)$.

As for $j = r+1, \ldots, s$,

$$\hat{q}_j = r\bar{q}_r + \sum_i \xi_i + \hat{\varepsilon}_j = \delta q^c_1 + (r\bar{q}_r - \delta q^c_1) + \sum_i \xi_i + \hat{\varepsilon}_j.$$

But

$$(r\bar{q}_r - \delta q^c_1) + \sum_i \xi_i + \hat{\varepsilon}_j \geq 0$$

and using ρ's definition,

$$\rho \leq \frac{(s-1)(r\bar{q}_r - \delta q^c_1)}{r},$$

we have

$$\sum_i \xi_i + (r\bar{q}_r - \delta q^c_1) \geq \sum_i \xi_i + \frac{r}{s-1}\rho \geq 0$$

since $\sum_i |\xi_i| \leq r\dfrac{\rho}{s-1}$.

Consequently, for $j = r+1, \ldots, s-r,\ \hat{q}_j \geq \delta q^c_1$. □

Proof of Theorem 4

Proof. In the proof we consider both equilibria at the same time,

$$(\underbrace{\bar{q}_r, \ldots, \bar{q}_r}_{r}, \underbrace{0, \ldots, 0}_{s-r}).$$

For the symmetric case, we just assume that $s = r$ and there are no 0-producing firms.

Theorem 3 proved that the stationary solution of our system (37) is globally asymptotically stable. As we have already mentioned, the problem is that system (37) rules the dynamics of our model only if the orbit of the solution satisfy the constraints (38). We are going to find an open neighborhood such that for any initial conditions contained in such a neighborhood, the solution to (37) is entirely contained in the ball of Lemma 5, and consequently tends to the equilibrium.

Let us retake the change of variables (58) used in the proof of Theorem 3 that led to the homogeneous system written as

$$\dot{\mathbf{x}} = A\mathbf{x},$$

and $\mathbf{0}$ as its steady state.

The solution that satisfies the initial conditions

$$\overrightarrow{(\xi, \varepsilon)} = (\xi_1, \ldots, \xi_r, \varepsilon_{r+1}, \ldots, \varepsilon_s)$$

is

$$\mathbf{x}(t) = e^{At} \cdot \overrightarrow{(\xi, \varepsilon)}. \tag{62}$$

Now, as the system is asymptotically stable, and consequently any solution (62) must tend to the stationary state, $\mathbf{0}$, it is immediate to see that there exists a constant, $M > 0$ such that for all $t \geq 0$, we have

$$\left\| e^{At} \right\|_\infty \leq M.$$

In this way, for all $t \geq 0$ we have

$$\|\mathbf{x}(t)\|_\infty = \left\|e^{At} \cdot \overrightarrow{(\xi, \varepsilon)}\right\|_\infty \leq \|e^{At}\|_\infty \cdot \left\|\overrightarrow{(\xi, \varepsilon)}\right\|_\infty \leq M \cdot \left\|\overrightarrow{(\xi, \varepsilon)}\right\|_\infty .$$

Undoing the change of variables, we have the solution to (37)

$$\mathbf{q}(t) = \mathbf{x}(t) + (\bar{q}_r, \ldots, \bar{q}_r, 0, \ldots, 0))$$

with the initial conditions

$$(q_1(0), \ldots, q_s(0)) = (\bar{q}_r + \xi_1, \ldots, \bar{q}_r + \xi_r, \varepsilon_{r+1}, \ldots, \varepsilon_s). \tag{63}$$

(We take the $\varepsilon_j \geq 0$ as we do not consider negative outputs.)

This solution satisfies

$$\|\mathbf{q}(t) - (\bar{q}_r, \ldots, \bar{q}_r, 0, \ldots, 0)\|_\infty \leq M \cdot \left\|\overrightarrow{(\xi, \varepsilon)}\right\|_\infty$$

Now, let $B = B((\bar{q}_r, \ldots, \bar{q}_r, 0, \ldots, 0), \rho/(s-1))$ be the ball of Lemma 5 and let $\eta = \rho/[(s-1)M]$.

Taking $\overrightarrow{(\xi, \varepsilon)}$ such that $\left\|\overrightarrow{(\xi, \varepsilon)}\right\|_\infty \leq \eta$, our orbit $\{q(t)\}_{t \geq 0} \subset B$ as for any $t \geq 0$

$$\|\mathbf{q}(t) - (\bar{q}_r, \ldots, \bar{q}_r, 0, \ldots, 0)\|_\infty \leq M \cdot \rho/[(s-1)M] = \rho/(s-1).$$

□

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