Additive Utility with Intransitive Indifference and without Independence: A Homogeneous Case

Marc Le Menestrel* and Bertrand Lemaire†

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Abstract

In the homogeneous case of one type of goods or objects, we prove the existence of an additive utility function without assuming transitivity of indifference and independence. The representation reveals a positive factor $\alpha \leq 1$ that influences rational choice beyond the utility function and explains departures from these standard axioms of utility theory ($\alpha = 1$).

Keywords: Rationality, Utility, Maximization.

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*Universitat Pompeu Fabra, Departament d'Economia i Empresa, Barcelona, España.

†CNRS, Université Paris-Sud, Département de Mathématiques, Orsay, France.

1 Introduction

Standard theories of utility can be formulated as a collection of axioms about a nonempty ordering $\succ$ on a set A and a binary (commutative, associative) operation $\circ$ on A that permit the construction of a real-valued function u on A verifying

$$x \succ y \Longleftrightarrow u(x) > u(y), \tag{i}$$

$$u(x \circ y) = u(x) + u(y). \tag{ii}$$

Two groups of axioms are crucial to these theories. Firstly, the ordering is assumed to be *asymmetric:* $x \succ y \Rightarrow y \nsucc x$, and *negatively transitive:* $(x \nsucc y$ and $y \nsucc z) \Rightarrow x \nsucc z$. Note that these two properties imply that the ordering is also *transitive*: $(x \succ y$ and $y \succ z) \Rightarrow x \succ z$. Secondly, the combination of the ordering and the operation is assumed to verify a form of *independence* or cancellation law, also called monotonicity: $x \succ y \Leftrightarrow (x \circ z \succ y \circ z$ for all $z \in A)$. Note that this property of independence, joint to the asymmetry of the ordering, imply that the operation is $\succ$*-regular:* $(x \succ y$ or $y \succ x) \Rightarrow (x \circ z \neq y \circ z$ for all $z \in A)$. If there exists a real-valued function u on A verifying (i) and (ii), then all these axioms necessarily hold (because they hold for the triple $\langle \mathbb{R}, >, + \rangle$). In this sense, if a theory replaces negative transitivity with the weaker axiom of transitivity, allowing intransitive indifference, then (i) must be modified in

$$x \succ y \Longrightarrow u(x) > u(y). \tag{i'}$$

On the other hand, if a theory relaxes independence maintaining a two-way representation like (i), then (ii) cannot be satisfied. Those theories lose the additivity of the utility function. In both examples, the theory is significantly weakened.[1]

[1] For a presentation of the standard theory, see e.g. Fishburn (1970a); Krantz & Al. (1971); Barbera & Al. (1998). On the independence condition in preference theory, see Fishburn & Wakker (1995). A seminal reference on intransitive indifference is Luce (1956). For a review of intransitive indifference in preference theory: Fishburn (1970b) and also Krantz & Al. (1971). For the treatment of discrimination through interval orders, see e.g. Fishburn (1985). About additivity, see for instance Wakker (1988a); Luce & Al. (1990, Chap. 19). About empirical deviations from standard utility theory, see for instance Hogarth & Reder (1987); Kahneman & Tversky (2000).

Assuming transitivity (i.e. without assuming negative transitivity) and replacing independence by a weaker property (*replicated independence*, see Definition 1), we would show there exists a utility function u that verifies (ii) and a two-way representation (i'') more general than (i). More precisely, we expect there exists a function $\alpha : A \times A \to \mathbb{R}_{>0}$ (satisfying certain technical conditions ensuring the uniqueness of the pair (u, α) up to scalar) such that

$$x \succ y \Longleftrightarrow \alpha(x, y)u(x) > u(y). \qquad (i'')$$

In a discrete and *homogeneous* case (see Definition 1, section 2), we prove here that α is a constant ≤ 1 (in this case, no "technical condition" is needed). Further, we slightly generalize this result to a continuous setting (section 3).

With this model, we can, for instance, reflect a rational individual being indifferent between €100 and €101, and between €101 and €102, while strictly preferring €102 to €100. Moreover, an individual who is indifferent between €101 and €102 may not be indifferent between €1 and €2. Therefore, such a model allows one to reflect a lack of discrimination (intransitive indifference) and a diminishing marginal utility (violation of independence). For the factor α, we have had in mind a model of rational behavior that combines processes and consequences. In this interpretation, α would reflect intrinsic procedural concerns outside the utility function. Without doubt, other interpretations are possible.[2]

2 Utility Representation (Discrete Setting)

We start with three primitives: a nonempty set A, a nonempty binary relation $\succ$ on A, and a closed binary relation $\circ$ on A. We write $x \sim y$ if and only if ($x \not\succ y$ and $y \not\succ x$), and $x \succsim y$ if and only if ($x \succ y$ or $x \sim y$). We note $\mathbb{N}_{>0}$ the set of positive integers, $\mathbb{Q}_{>0}$ the set of positive rational numbers and $\mathbb{R}_{>0}$ the set of positive real numbers.

Definition 1 Let A be a nonempty set, $\succ$ a nonempty binary relation on A, and $\circ$ a closed binary operation on A. The triple $\langle A, \succ, \circ \rangle$ is a *partially ordered positive structure* if and only if the following five axioms are satisfied for all $x, y, z \in A$:

[2] For the relevance of a procedural dimension in rationality, see e.g. Simon (1978) and Sen (1997). A model of rational behavior combining processes and consequences is tentatively explored in Le Menestrel (1999, 2001a, 2001b). See also Le Menestrel & Van Wassenhove (2001). For a resembling (proportional) lack of discrimination in psychology (see e.g. Suppes & Al. 1989).

1. Strict Partial Order: $x \succ y \Rightarrow y \not\succ x$; $(x \succ y$ and $y \succ z) \Rightarrow x \succ z$.

2. Commutativity; Associativity: $x \circ y = y \circ x; (x \circ y) \circ z = x \circ (y \circ z)$.

3. Positivity: $x \succ y \Longrightarrow x \circ z \succ y$.

4. Replicated Independence: $x \succ y \Leftrightarrow (nx \succ ny$ for all $n \in \mathbb{N}_{>0})$, where nx is defined inductively by $1x = x$ and $(n+1)x = nx \circ x$.

5. Archimedean: If $x \succ y$, then there exists $n \in \mathbb{N}_{>0}$ such that $nx \succ (n+1)y$.

A partially ordered positive structure $\langle A, \succ, \circ \rangle$ is said to be *homogeneous* if it satisfies the following condition, for all $x, y \in A$:

6. Homogeneity: $mx = ny$ for some $(m, n) \in \mathbb{N}_{>0} \times \mathbb{N}_{>0}$.

A nonempty set A endowed with a closed associative and commutative binary operation $\circ$, is called a *commutative semigroup.*[3] A commutative semigroup A is said to be *regular* (respectively *replicated-regular*) if for all $x \in A$, the map $A \rightarrow A, y \longmapsto x \circ y$ (respectively the map $\mathbb{N}_{>0} \rightarrow A, n \longmapsto nx$) is injective. Let $\langle A, \succ, \circ \rangle$ be a partially ordered positive structure. Then (by replicated independence and asymmetry) the commutative semigroup A is *replicated-$\succ$-regular:* $(x \succ y$ or $y \succ x) \Rightarrow (nx \neq ny$ for all $n \in \mathbb{N})$. Clearly, the four notions of regularity we have introduced in this paper satisfy the following implications:

$$\text{regularity} \Rightarrow \succ\text{-regularity} \Rightarrow \text{replicated-}\succ\text{-regularity,}$$

and

$$\text{regularity} \Rightarrow \text{replicated-regularity} \Rightarrow \text{replicated-}\succ\text{-regularity.}$$

It is not difficult to verify (see the proof of Theorem 1 below) that if A is homogeneous, then it is also replicated-regular. In particular (always assuming A is homogeneous), this implies that for all $x, y \in A$, the set $\{\frac{m}{n} : m, n \in \mathbb{N}_{>0}, mx = ny\}$ is reduced to one element.

[3] See Fuchs (1963) for a seminal algrebraic treatment. There, axiom 5 is said to exclude "anomalous" pairs. It has been introduced by Alimov in 1950, see reference above (p. 162s) and also footnote 4 below. The name for axiom 4 has been suggested to us by Peter Fishburn.

Theorem 1 *Let $\langle A, \succ, \circ \rangle$ be a partially ordered positive homogeneous structure. Then there exist a function $u : A \to \mathbb{R}_{>0}$ and a real number $0 < \alpha \leqslant 1$ such that for all $x, y \in A$*

$$x \succ y \Longleftrightarrow \alpha u(x) > u(y), \tag{i''}$$

$$u(x \circ y) = u(x) + u(y). \tag{ii}$$

If (v, β) is another pair satisfying (i'') and (ii), then $\beta = \alpha$ and there exists a real number $\lambda > 0$ such that $v = \lambda u$. Moreover, u is injective if and only if A is regular, u can be chosen with values in $\mathbb{Q}_{>0}$, and $\alpha \in \mathbb{Q}$ if and only if there exist $x, y \in A$ such that $\alpha u(x) = u(y)$.

Proof. Since $\succ$ is not empty, there exist $x, y \in A$ such that $x \succ y$. Let $z, z' \in A$, and choose $(m, n), (m', n') \in \mathbb{N}_{>0} \times \mathbb{N}_{>0}$ such that $mx = nz$ and $m'y = n'z'$ (homogeneity). By replicated independence, we have $m'mx \succ mm'y$, i.e. $pz \succ qz'$ with $p = m'n$ and $q = mn'$. Take $z = z'$, and suppose there exists $(a, b) \in \mathbb{N}_{>0} \times \mathbb{N}_{>0}$ such that $a > b$ and $az = bz$. Then we have $(b + k(a - b))z = bz$ for all $k \in \mathbb{N}_{>0}$, hence $m''(b + k(a - b))z = m''bz$ for all $(m'', k) \in \mathbb{N}_{>0} \times \mathbb{N}_{>0}$. Taking $m'' = q$, we can choose k big enough so that $q(b + k(a - b)) > pb$. Since $pbz \succ qbz$ (replicated independence), by positivity we obtain $q(b + k(a - b))z \succ qbz$, which is impossible. This implies the replicated-regularity of A.

For $x \in A$, we define the subsets of $\mathbb{Q}_{>0}$

$$\begin{aligned} \mathcal{Q}_x &= \{\frac{m}{n} : mx \succsim nx, \exists (m, n) \in \mathbb{N}_{>0} \times \mathbb{N}_{>0}\}, \\ \mathcal{P}_x &= \{\frac{m}{n} : mx \succ nx, \exists (m, n) \in \mathbb{N}_{>0} \times \mathbb{N}_{>0}\}. \end{aligned}$$

By homogeneity and replicated independence, for all $x, y \in A$, we have $\mathcal{Q}_x = \mathcal{Q}_y$ and $\mathcal{P}_x = \mathcal{P}_y$. So we can drop the index x in the notation $\mathcal{Q}_x$ and $\mathcal{P}_x$. From the previous paragraph, $\mathcal{P}$ is not empty, and $1 \in \mathcal{Q}$. We also have $\mathbb{Q}_{>0} = \mathcal{Q} \cup \mathcal{P}^{-1} = \mathcal{Q}^{-1} \cup \mathcal{P}$ and $\mathcal{Q} \cap \mathcal{P}^{-1} = \mathcal{Q}^{-1} \cap \mathcal{P} = \varnothing$.

By positivity and replicated independence, we have $q \in \mathcal{Q} \Rightarrow \mathbb{Q}_{\geq q} \subset \mathcal{Q}$ and $q \in \mathcal{P} \Rightarrow \mathbb{Q}_{\geq q} \subset \mathcal{P}$.

We define $r = \inf_{\mathbb{R}} \mathcal{Q}$ and $s = \inf_{\mathbb{R}} \mathcal{P}$.

Because $1 \in \mathcal{Q}$, we have $0 \leq s \leq 1$. Because of positivity, we have $r \geq 1$.

If $s = 0$, then for all $(m, n) \in \mathbb{N}_{>0} \times \mathbb{N}_{>0}$, there exists $(m', n') \in \mathbb{N}_{>0} \times \mathbb{N}_{>0}$ such that $(m', n') \in \mathcal{Q}$ and $\frac{m'}{n'} < \frac{m}{n}$. Hence $\frac{m}{n} \in \mathcal{Q}$. Therefore $\mathcal{P} = \varnothing$, contradiction. Hence $0 < s \leq 1$. The same argument implies that $\mathbb{Q}_{>s} \subset \mathcal{Q}$.

Suppose $s \in \mathbb{Q} \setminus \mathcal{Q}$. Take $(m,n) \in \mathbb{N}_{>0} \times \mathbb{N}_{>0}$ such that $s = \frac{m}{n}$. Since $s \notin \mathcal{Q}$, we have $nx \succ mx$ and thus $pnx \succ (p+1)mx$ for some $p \in \mathbb{N}_{>0}$ (Archimedean). Therefore $\frac{(p+1)}{p} s \notin \mathcal{Q}$ which contradicts $\mathbb{Q}_{>s} \subset \mathcal{Q}$. Therefore, $s \in \mathbb{Q}$ implies $s \in \mathcal{Q}$.

Finally, we have $\mathcal{Q} = \mathbb{Q}_{\geq s}$, and also $\mathcal{P} = \mathbb{Q}_{>\frac{1}{s}}$. Hence, $r = \frac{1}{s}$.

By replicated-regularity, for all $x, y \in A$, there exists a unique $q_{x,y} \in \mathbb{Q}_{>0}$ such that $\{\frac{m}{n} : m, n \in \mathbb{N}_{>0}, mx = ny\} = \{q_{x,y}\}$. Let $x \in A$. We define a function $f_x : A \longrightarrow \mathbb{Q}_{>0}$ by $f_x = q_{x,y}$. Let $y, y' \in A$. We write $mx = ny$ and $m'x = n'y'$ for some $(m,n), (m',n') \in \mathbb{N}_{>0} \times \mathbb{N}_{>0}$. Since $(n'm + nm')x = nn'(y \circ y')$, we have $q_{x,y\circ y'} = \frac{n'm+nm'}{nn'} = \frac{m}{n} + \frac{m'}{n'}$, i.e. $f_x(y \circ y') = f_x(y) + f_x(y')$. Moreover,

$$y \succ y' \Leftrightarrow n'ny \succ nn'y \Leftrightarrow n'mx \succ nm'x \Leftrightarrow \frac{n'm}{nm'} \in \mathcal{P} \Leftrightarrow \frac{n'm}{nm'} > r$$

and

$$\frac{n'm}{nm'} > r \Leftrightarrow \frac{m}{n} > r\frac{m'}{n'} \Leftrightarrow sf_x(y) > f_x(y').$$

So we have proved that the pair $(u, \alpha) = (f_x, s)$ verifies the conditions (i'') and (ii) of Theorem 1. By construction u is $\mathbb{Q}_{>0}$-valued.

Let $f' : A \to \mathbb{R}_{>0}$ be a function such that $f'(y \circ z) = f'(y) + f'(z)$ for all $y, z \in A$. Let $y \in A$, and write $mx = ny$ for some $(m,n) \in \mathbb{N}_{>0} \times \mathbb{N}_{>0}$. Then we have $mf'(x) = f'(mx) = f'(ny) = nf'(y)$, i.e. $f'(y) = \lambda f_x(y)$ with $\lambda = f'(x)$. Then u is unique up to scaling transformation, which implies the uniqueness of α. Condition (ii) of the Theorem implies that u is injective if and only if A is regular. The last assertion of the Theorem is clear. ∎

Reciprocally, if $(A, \circ)$ is a commutative semigroup (not necessarily homogeneous) endowed with a nonempty binary relation $\succ$ such that there exist a function $u : A \to \mathbb{R}$ and a real number $0 < \alpha \leq 1$ satisfying the conditions (i'') and (ii) of Theorem 1, then the triple $\langle A, \succ, \circ \rangle$ is a partially ordered positive structure. The verification of this assertion is easy and left to the reader.

Theorem 1 implies that $\alpha = 1$ if and only if negative transitivity and independence hold. We recover the standard theory where (i) and (ii) are satisfied.[4] In general, the factor α may not equal to one, "twisting" the representation and preventing the interpretation that a rational individual acts as if he maximizes the utility function u.

[4]When $\alpha = 1$, the triple $\langle A, \succ, + \rangle$ is a *closed positive extensive structure* as defined by Krantz and Al. (1970, p. 73). With respect to the Archimedean axiom used there and axiom 5 here, see the discussion referred to in footnote 3.

3 A Continuous Setting Generalization

Formulated using a discrete algebraic approach, Theorem 1 can be generalized to a continuous set of goods or objects. Retaining the algebraic approach, we now introduce such a generalization.[5]

Let $R \subset \mathbb{R}_{>0}$ be a subset containing 1 such that for all $\lambda, \mu \in R$, we have $\lambda + \mu \in R$, $\lambda\mu \in R$, and $\lambda > \mu \Rightarrow \lambda - \mu \in R$. Since $1 \in R$, we have $\mathbb{N}_{>0} \subset R$. We call $R-$ *semimodule* a commutative semigroup $(A, \circ)$ endowed with a closed operation $R \times A \to A, (\lambda, \mu) \mapsto \lambda \cdot \mu$ such that for all $x, y \in A$ and $\lambda, \mu \in R$, we have:

$$\lambda \cdot (x \circ y) = (\lambda \cdot x) \circ (\lambda \cdot y),$$

$$(\lambda + \mu) \cdot x = (\lambda \cdot x) \circ (\mu \cdot x),$$

$$\lambda \cdot (\mu \cdot x) = (\lambda\mu) \cdot x,$$

$$1 \cdot x = x.$$

Because of the last condition, for $n \in \mathbb{N}_{>0}$, we have $n \cdot x = nx$. Therefore, the notions of commutative semigroup and $\mathbb{N}_{>0}$-semimodule coincide. An R-semimodule $(A, \circ, \cdot)$ is said to be $R-$ *regular* if for all $x \in A$ the map $R \to A, \lambda \mapsto \lambda \cdot x$ is injective.

Definition 2 Let A be a nonempty set, $\succ$ a nonempty binary relation on A, $\circ$ a closed binary operation on A, and $\cdot$ a closed operation of R on A. The quadruple $\langle A, \succ, \circ, \cdot \rangle$ is a *partially ordered positive* $R-$*structure* if and only if the following five axioms are satisfied for all $x, y \in A$:

1. Strict Partial Order (Definition 1, axiom 1).
2. $(A, \circ, \cdot)$ is a R-semimodule.
3. Positivity (Definition 1, axiom 3).
4. $R-$independence: $x \succ y \Leftrightarrow (\lambda x \succ \lambda y$ for all $\lambda \in R)$.
5. $R-$archimedean: If $x \succ y$, then there exist $\lambda, \mu \in R$ with $\lambda < \mu \in R$, such that $\lambda \cdot x \succ \mu \cdot y$.

A partially ordered positive R-structure $\langle A, \succ, \circ, \cdot \rangle$ is said to be *homogeneous* if it satisfies the following condition, for all $x, y \in A$:

6. R-homogeneity: $\lambda \cdot x = \mu \cdot y$ for some $(\lambda, \mu) \in R \times R$.

[5] The algebraic approach is slightly more general than the topological one (see Wakker 1988b).

Let $F(R) \subset \mathbb{R}_{>0}$ be the subset defined by $F(R) = \{\frac{\lambda}{\mu} : \lambda, \mu \in R\}$. Since $1 \in R$, we have the inclusions $\mathbb{N}_{>0} \subset R \subset F(R)$. And for all $\lambda, \mu \in F(R)$, we have $\lambda + \mu \in F(R)$, $\lambda\mu \in F(R)$, and $\lambda > \mu \Rightarrow \lambda - \mu \in F(R)$. In particular, we have:

- if $R \subset \mathbb{Q}$, then $F(R) = \mathbb{Q}_{>0}$,
- if $R = \mathbb{R}_{>0}$, then $F(R) = \mathbb{R}_{>0}$.

Theorem 2 *Let $\langle A, \succ, \circ, \cdot \rangle$ be a partially ordered positive homogeneous $R-$structure. Then there exist a function $u : A \to \mathbb{R}_{>0}$ and a real number $0 < \alpha \leqslant 1$ such that for all $x, y \in A$ and $\lambda \in R$, we have*

$$x \succ y \Longleftrightarrow \alpha u(x) > u(y), \tag{i''}$$

$$u(x \circ y) = u(x) + u(y), \tag{ii}$$

$$u(\lambda \cdot x) = \lambda u(x). \tag{iii}$$

If (v, β) is another pair satisfying (i''), (ii) and (iii), then $\beta = \alpha$ and there exists a real number $\gamma > 0$ such that $v = \gamma u$. Moreover, u is injective if and only if the semigroup $(A, \circ)$ is regular, u can be chosen with values in $F(R)$, and $\alpha \in F(R)$ if and only if there exist $x, y \in A$ such that $\alpha u(x) = u(y)$.

Proof. Roughly speaking, it suffices to replace $\mathbb{N}_{>0}$ by R and $\mathbb{Q}_{>0}$ by $F(R)$ in the proof of Theorem 1. We sketch this briefly. Let $z, z' \in A$. Since $\succ$ is nonempty, by R-homogeneity and R-independence, there exist $\lambda, \mu \in R$ such that $\lambda \cdot z \succ \lambda \cdot z'$. Take $z = z'$, and suppose there exists $(a, b) \in R \times R$ such that $a > b$ and $a \cdot z = b \cdot z$. Since $a - b \in R$, for all $k \in \mathbb{N}_{>0}$, we have $(b + k(a - b)) \cdot z = b \cdot z$. Choosing k big enough so that $\mu(b + k(a - b)) > \lambda b$, by R-independence and positivity, we obtain $\mu(b + k(a - b)) \cdot z \succ \mu b \cdot z$, which is impossible. This implies the R-regularity of the R-semimodule $(A, \circ, \cdot)$.

For $x \in A$, we define the (nonempty) subsets of $F(R)$

$$\begin{aligned} \mathcal{Q}_x &= \{\frac{\lambda}{\mu} : \lambda \cdot x \succsim \mu \cdot x, \exists (\lambda, \mu) \in R \times R\}, \\ \mathcal{P}_x &= \{\frac{\lambda}{\mu} : \lambda \cdot x \succ \mu \cdot x, \exists (\lambda, \mu) \in R \times R\}. \end{aligned}$$

By R-homogeneity and R-independence, we can drop the index x in the notation $\mathcal{Q}_x$ and $\mathcal{P}_x$. We have $F(R) = \mathcal{Q} \cup \mathcal{P}^{-1} = \mathcal{Q}^{-1} \cup \mathcal{P}$ and $\mathcal{Q} \cap \mathcal{P}^{-1} = \mathcal{Q}^{-1} \cap \mathcal{P} = \varnothing$. By positivity and R-independence, we have $q \in \mathcal{Q} \Rightarrow F(R)_{\geq q} \subset \mathcal{Q}$ and $q \in \mathcal{P} \Rightarrow F(R)_{\geq q} \subset \mathcal{P}$. We define $r = \inf_{\mathbb{R}} \mathcal{Q}$ and $s = \inf_{\mathbb{R}} \mathcal{P}$. Because $1 \in \mathcal{Q}$, we have $0 \leq s \leq 1$, and because $\succ$ is nonempty, we have $s > 0$ and $F(R)_{>s} \subset \mathcal{Q}$. This last inclusion, joint to the R-archimedean axiom, implies that if $s \in F(R)$, then $s \in \mathcal{Q}$. So we have $\mathcal{Q} = F(R)_{\geq s}, \mathcal{P} = F(R)_{\geq s^{-1}}$ and $r = s^{-1}$.

By R-regularity, for all $x, y \in A$, there exists a unique $q_{x,y} \in F(R)$ such that $\{\frac{\lambda}{\mu} : \lambda, \mu \in R, \lambda \cdot x = \mu \cdot y\} = \{q_{x,y}\}$. Let $x \in A$. We define a function $f_x : A \longrightarrow F(R)$ by $f_x(y) = q_{x,y}$. As in the proof of Theorem 1, we verify that the pair $(u, \alpha) = (f_x, s)$ verifies the conditions (i'') and (ii). By construction u is $F(R)$-valued and $u(\lambda \cdot x) = \lambda f_x(y)$ $(\lambda \in R, y \in A)$. The uniqueness of u up to scaling transformation is obtained as in the proof of Theorem 1, using R-homogeneity and condition (iii) . All the remaining assertions of Theorem 2 are clear. ■

Finally, if $(A, \circ, \cdot)$ is a R-semimodule (not necessarily R-homogeneous) endowed with a nonempty binary relation $\succ$ such that there exist a function $u : A \to \mathbb{R}$ and a real number $0 < \alpha \leq 1$ satisfying the conditions (i''), (ii) , and (iii) of Theorem 2, then the quadruple $\langle A, \succ, \circ, \cdot \rangle$ is a partially ordered positive R-structure.

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