ECONOMICS WORKING PAPERS SERIES

Economics Working Paper 6

Simulation Analysis of Dynamic Stochastic Models: Applications to Theory and Estimation

Albert Marcet*

March 1993†

UNIVERSITAT POMPEU FABRA

Balmes, 132
Telephone (343) 484 97 00
Fax (343) 484 97 02
08008 Barcelona
e- mail econwp@upf.es

Economics Working Paper 6

Simulation Analysis of Dynamic Stochastic Models: Applications to Theory and Estimation

Albert Marcet*

March 1993†

Keywords: Survey, Simulations, Numerical analysis.

Journal of Economic Literature classification: E00, E3, G00, C63.

---

* Universitat Pompeu Fabra.

† Final draft (2d version), First version: August 1990.

Abstract

In this paper we survey the last developments of simulation techniques of dynamic stochastic models with rational expectations. We concentrate the discussion on applications to macroeconomics and financial economics, and we argue that the use of simulations techniques has permitted progress in many different topics, including asset pricing models with heterogeneous agents, asset pricing and endogenous production, distortionary taxation, game theory, mechanism design and monetary economics. We describe the classes of algorithms that have been applied in the last few years, including some new algorithms that permit the analysis of new types of models. We argue that theoretical work can be done with 'theory on the computer', in the sense of studying the properties of a model by simulation, when analytic solutions are not available; in this case, some of the problems that have to be adressed are the choice of the parameter values of the model, how to report the results, and how to make the results easy to reproduce by the reader. Finally, some estimations and testing methods used in empirical work are discussed, including maximum likelihood, the method of simulated moments, and calibration.

1. Introduction

The use of dynamic stochastic models in economics has grown very quickly during the last fifteen years. The importance of this type of models became evident in macroeconomics after the paper of Lucas [1972]; he argued that the basic relations that were taken as given in traditional macroeconomic models (whether Keynesian or Monetarist), such as the money demand function, the consumption function, the investment function, were not invariant to the very type of policy intervention that those models were designed to analyze. Furthermore, these relations were often mutually inconsistent. The way around this problem was to analyze models where objects like preferences of consumers, production technology, information dissemination etc. were fixed, and where a well specified concept of equilibrium determined the outcome of the model. The research program, then, was to analyze the equilibrium of the model under different environments, (for example, under different policy rules) in order to study the effect of changes in the economic environment or policy interventions, taking consumption function, money demand etc. as endogenous.

Nowadays, dynamic stochastic models of equilibrium are being used in virtually all field in economics.

One crucial element of how dynamic models behave is the assumption about how agents form their expectations. Nowadays, the standard assumption is that agents behave as if they had rational expectations. This avoids ad-hoc assumptions about expectations that would not be likely to stay constant under policy changes if agents were rational. Furthermore, many recent papers argue that, in many models (though not all) the rational expectations equilibrium can be justified as the limit of a learning process[1].

Economists have made a lot of progress on such basic issues as formulating equilibrium concepts for these models, finding conditions for existence and uniqueness of equilibria, determining the state variables, etc. But in order to use our models, that is, in order to characterize equilibria, evaluate different policy rules, learn about the importance of

certain assumptions and so on, it is necessary to obtain a solution for the law of motion of the equilibrium stochastic process. Unfortunately, progress in finding closed form solutions of dynamic stochastic models has been very slow; the only model that can be solved in any generality is the linear-quadratic model[2]. But in non-linear models, very special assumption shave to be made in order to obtain analytic solutions; in effect, these assumptions limit the interest of the exercise.

Closed form solutions are also needed in empirical applications, for example those using maximum likelihood estimation. The GMM procedure of Hansen and Singleton [1982] can be used to estimate and test certain rational expectations models from the Euler equations, even if a closed form solution is not available, but this technique can not be used if there are unobservables in the Euler equations. This is often the case in models with exogenous shocks to preferences, data aggregated over time or over agents, and models with inequality constraints.

In the last five years, given the difficulties for finding closed form solutions in models of interest, a growing number of researchers has turned to studying dynamic stochastic equilibrium models using computer simulations. With simulations, it is possible to implement empirical tests of the model and study the behavior of the model under different environments. In this paper we will discuss the usefulness of the simulation techniques, the progress that has been made in computer algorithms recently, and many applications of these techniques. Obviously, it is impossible to cite all the applications that have been done in economics, and we will discuss some applications to macroeconomics and financial economics.

With the new algorithms, and refinements on the old ones, we are now able to simulate very complicated models on desktop computers. Besides increasing computational speed, the new algorithms have enlarged the class of models that can be approached by simulation; more precisely, it is no longer necessary to cast an equilibrium model into a planner's problem in order to solve it.

Studies of theoretical interest can be performed using computer simulations not only for illustrative purposes, as a complement to analytic results, but as the main tool being for reaching conclusions of theoretical interest. We will argue that it is possible to do 'theory on the computer', much in the way that it is possible to do theory by proving theorems. In fact, this is probably the only way of making any progress in many models of interest, since analytic solutions are so difficult to obtain. Some issues that arise in doing theory on the computer are how to report results, how to choose the parameter values and issues about accuracy of the solution; we will discuss these below. On the issue of choosing parameter values, one alternative is to use those parameters that make the equilibrium of the model close to the observed series; in this respect, dynamic stochastic models are easy to handle, because they have very clear-cut implications for observed time series.

In section 2 we illustrate the difficulties for obtaining analytic closed form solutions by studying a simple asset pricing model; we discuss the limitations of the model and some extensions that have been performed using simulation; we also review the literature on other topics where simulation studies have proved useful. Section 3 reviews some of the recent progress in solution algorithms and the usefulness of the new approaches. Section 4 discusses the use of simulations in theoretical exercises; we argue that this kind of exercise is as valid as analytic techniques and review some of the applications; we also discuss how to report the results, choose parameter values and issues about accuracy of the solution. Section 5 discusses how to apply simulations in empirical studies; we discuss applications to maximum likelihood, the method of simulated moments and calibration. Section 6 is a response to Ken Judd's discussion of this paper.

2. Analyzing Dynamic Economies with Simulation. Some Applications.

First of all, we want to illustrate the need for simulation in equilibrium dynamic stochastic models. To this end, we show the limitations of analytic solutions to a well known asset pricing model and discuss how this and other models have been enriched by the use of simulations.

2.1 Analytic Solution to Lucas' Asset Pricing Model

Let us consider analytic solutions for asset prices in the model of Lucas [1978]. Since this is a well known model, the description will be brief.

There is an exogenous, stochastic stream of dividends $\{d_t\}$ that is produced exogenously by an infinitely lived productive unit. Agents have a right to this dividend if they hold shares of ownership of the productive unit for ever; shares can be bought and sold costlessly at any time period and all markets are perfectly competitive. There is only one type of agent, so it will simplify notation if we assume that there is only one representative agent who behaves competitively. Dividends are the only source of the only consumption good in this economy. At time t, the agent observes all current and past variables. The representative agent chooses streams of consumption and share holdings in order to solve[3]

$$\max \; E_0 \sum_{t=0}^{\infty} \delta^t \, u(c_t)$$

$$\text{s.t.} \qquad c_t + p_t \, s_t = (p_t + d_t) \, s_{t-1} \quad \text{for } t=0,1,\ldots; \quad s_{-1} = 1$$

taking the process for dividends and stock prices $\{d_t, p_t\}$ as given. Normalizing the number of shares to one, the equilibrium conditions in this model are

$$s_t = 1 \quad \text{and} \quad c_t = d_t \quad .$$

The first order conditions are

$$(2.1) \qquad p_t\, u'(c_t) = \delta\, E_t \left[(p_{t+1}+d_{t+1})\, u'(c_{t+1}) \right] \quad ,$$

which, using recursive substitution and the equilibrium condition for the consumption good can be rewritten as

$$(2.2) \qquad p_t = E_t \left[\sum_{l=1}^{\infty} \delta^l\, d_{t+l}\, u'(d_{t+l}) \,/\, u'(d_t) \right] \quad .$$

To describe the behavior of equilibrium asset prices in this economy we need to find a closed form solution for the asset prices in terms of the dividend process. The above formulas are not closed form solutions because they are written in terms of a conditional expectation that is difficult to solve unless special assumptions are introduced. So, we have to specialize this already very simple model in order to find a closed form solution. Here are a few possibilities:[4]

Example 2.1.

Assume that $\{d_t\}$ is identically independently distributed. Letting $C = E\left[d_t\, u'(d_t) \right]$, we obtain from (2.2)

$$(2.3) \qquad p_t = C\delta \,/\, [\, u'(d_t)\,(1-\delta)\,] \quad .$$

This is a closed form solution up to the constant C, which could be found explicitely if more assumptions were placed on the functional form of u(.) and on the distribution of d_t. Alternatively, this constant can be easily calculated by numerical integration.

Example 2.2

Assume that $u(c_t) = \log(c_t)$. Then $u'(d_t)=1/d_t$ and (2.2) becomes

$$(2.4) \qquad p_t = \delta\, d_t/(1-\delta) \quad .$$

This formula holds for any dividend process.

Example 2.3

Assume that $u(c_t) = (c_t)^{\gamma+1}/(\gamma+1)$ and that $d_t=d_{t-1}\varepsilon_t$, where ε_t is i.i.d. Let $\mu=E(\varepsilon_t^{\gamma+1})$; under the additional assumption that $\delta\mu<1$ (which is guaranteed, for example, if $E(\varepsilon_t)=1$ and $\gamma<-1$), then the formula for asset prices is

$$(2.5) \qquad p_t = \sum_{i=1}^{\infty} (\delta\mu)^i \, d_t^{\gamma+1} / d_t^{\gamma} = \delta\mu \, d_t / (1-\delta\mu) \quad .$$

The point of having analytic examples in a paper that deals with simulation techniques is that, even in this simple model, very strong assumptions have to be made in order to find closed form solutions. To the non-expert, the kind of exercise done in the above examples seems pure magic: we start with a complicated, hard to interpret formula like (2.2) and out come these neat, elegant formulas (2.3), (2.4) and (2.5). To the untrained eye, these formulas seemed to appear out of nowhere, but we all know that we put just the right cards in our sleeves (i.e., we made enough assumptions) to make the conditional expectation in (2.2) disappear.

Finding closed form solutions after making these very extreme assumptions has certainly improved our knowledge about equilibrium asset pricing, but it turns out to be an exercise with limited possibilities. We would like to explore versions of that model where more general utility functions are used, perhaps allowing for shocks to preferences, non time-separable utility functions, and for dividend processes that come closer to the dividend series observed in real data.

More importantly, there are many questions that can not be addressed even in the most general version of Lucas' model. For example, the effects of liquidity constraints, market incompleteness, private information, heterogeneity of consumers, effects of transaction costs, limited enforcement of contracts are some of the features that play a potentially important role in securities markets, but they can not be introduced in

Lucas' model without major changes.

Also, one of the reasons that securities markets receive so much attention is probably that they may be the first markets to reflect news in the productive sector. But any relation between productivity and asset prices can not be analyzed in Lucas' model because production is exogenous. Brock [1982] provided a theoretical framework for formulating asset pricing models with endogenous production, but the properties of these models have not been explored until very recently.

Nevertheless, there is no general way of obtaining closed form solutions in models with these type of generalizations and so the effect of market incompleteness, heterogeneity of agents, etc. has been largely unknown for a long time. There are some papers that have made just the right assumptions to solve models of this type, for example Scheinkman and Weiss [1986], Hansen [1987] and Hansen and Sargent [1990], but the generality of their results is almost impossible to explore.

2.2 Beyond the Representative Agent Asset Pricing Model.

The main reason that so many important issues can not even be addressed in asset pricing models with a representative agent is that there is no trading of securities in a representative agent model; for example, introducing or eliminating securities markets never has an effect on asset prices or on welfare; predictions about trading make no sense, etc.

Nevertheless, these are relevant problems in economics, nowadays. For example, there has been recently some political pressure to reduce the amount of securities trading in U.S. stock exchanges by introducing a tax on securities trading, but in a representative agent model it is clear that this measure would not have any effect because there is no trading of assets to begin with. Another example: recently an options market and a futures market started operating in Spain; presumably these markets are being established because somebody thinks they are useful, but the representative agent model would say that they serve no purpose and they will have no effect on the economy.

On the other hand, there are well documented and abundant empirical failures of the representative agent model: from the very popular equity premium puzzle of Mehra and Prescott [1985], to the excessive volatility of individual consumptions, to the excess volatility of asset prices of Grossman and Shiller [1981].

Only during last year, the study of equilibrium asset pricing models with heterogeneous agents, incomplete markets and liquidity constraints seems to have taken off. The papers by D. Lucas [1990], Marcet and Singleton [1990] and Ketterer and Marcet [1989] have two types of agents that can differ in their preferences or income processes. Only a few securities exist in this economy (stocks, bonds or call options), agents face liquidity constraints on the securities. Perfect competition and perfect information is assumed. The first two papers study empirical issues like volatility of individual consumption, how often the liquidity constraints are binding, and the risk premium puzzle. The third paper dicusses the effects of introducing derivative securities in a model with incomplete markets. Other papers by Hansen and Imrohoroglu [1992] and Diaz and Prescott [1990] study monetary models where money has value because of market incompleteness. Rios [1990] studies an overlapping generations model with uncertainty and a large number of generations alive at any point in time; he studies issues the life-cicle, insurance among generations and the effects of market incompleteness. Finally, Brock and le Baron [1989] introduce liquidity constraints on the side of the firms.

Although it is impossible to summarize all the results from these papers in a short space we will remark two features they seem to have in common. The first feature is methodological: all of the above papers rely heavily on simulations for their results. The second point is a more substantive one: in most of these models market incompleteness on its own is not capable of generating results that are very different from the complete contingent markets case. So, it turns out that there is barely any risk premium, individual consumptions are not very volatile, and the losses in utility from market incompleteness and liquidity constraints are not very large. It appears as if agents can do a great deal of consumption

smoothing just by buying and selling stocks of their securities; in the terminology of Deaton [1989], assets act as a buffer stock that can be adjusted to cope with unforeseen shocks.

This is a nice example of how simulation can enhance our understanding of dynamic modelling. It seems as if these generalizations of Lucas' model do not automatically produce much better results than the representative agent, at least in the cases that have been explored up to now. This is striking because so many papers had suggested that model like this would easily explain some empirical puzzles, for example Mehra and Prescott [1985], Hayashi [1987]. Also, from a theoretical point of view many negative results seemed to arise from papers with incomplete markets; for example, many papers with analytic examples argued that incomplete market models would generate strong suboptimalities and would have a very large number of equilibria. We will discuss this further in section 4.1.

2.3 Recent Applications of Simulation Techniques,

Besides asset pricing with heterogenous agents, there have been many other recent applications of simulation in economics. One example is asset prices and endogenous production, such as in the model of Brock [1982]. In section 3 we will use this model as our main example for how different algorithms work, and we will look at simulations of this model. Some papers in this area are: Rowenhorst [1990] studies the effects of leverage on asset prices, the relationship between productivity and asset returns and studies some empirical implications; den Haan [1990b] studies the shape of the term-structure of interest rates in an equilibrium monetary model with leisure; Marcet [1989] argues that a simple model with endogenous production, can produce a complicated covariance structure for asset prices and can generate humped term structures of the interest rate.

Besides the more radical departures that we have discussed in the previous subsection, there have been many extensions of the representative agent asset pricing model where equilibria have been studied by simulations. Some of these applications are the following: Novales [1990] studies the effects of introducing habit persistence of consumption for the

behavior of interest rates; Ingram [1986] studies a model with myopic agents in asset markets and Heaton [1990] studies the effects of habit persistence and time aggregation.

Other applications have been in Monetary Economics. A few years ago there was a large body of research on how to endogeneize the existence of money in an economy. Early papers by Wallace, Townsend and Bewley studied models with heterogeneous agents where money is held because it facilitates exchange across generations or across individuals in different locations. The cash in advance papers of Lucas and Stokey and the Sidrausky-type models were introduced a long time ago. But most applications of these models were in very specialized setups where uncertainty was often ruled out. Recently, uncertainty has been introduced in many monetary models thanks to simulation techniques. Cooley and Hansen [1989] introduce monetary shocks in a real business cycle model; den Haan [1990a] studies the optimal monetary policy in a model where money reduces the shopping-time; the Diaz and Prescott paper mentioned above is also a paper on monetary theory. Coleman [1989] and Baxter [1990] study a cash-in-advance model, and Marshall [1988] studies the empirical implications for inflation of a representative agent where money is valued because of exogenously imposed transaction costs.

There have been also several applications to models with distortionary taxes. Braun [1989], Chang [1989] and McGratten [1989] study the business cycle properties of these models from an empirical point of view, Bizer and Judd [1988] discuss the optimality of a stochastic taxation scheme and Otker [1990] studies the welfare loss due to the presence of distortionary taxation. Jones, Manuelli and Rossi [1990] and Chari, Christiano and Kehoe [1990] study optimal distortionary taxes.

The formulation of models with incentive compatibility constraints and mechanism design has received great attention since the seventies. The implications of these models for empirical data and characterization of equilibria in dynamic settings, however, have been difficult to study analytically. Using simulation, Phelan and Townsend [1989] characterize the sequentially optimal contracts in a non-growth economy with limited

information and they document the welfare loss from private information; Phelan [1991] argues that the consumption choices of individual agents are better explained by incentive compatibility models; Marcet and Marimon [1992] study the effects of limited information and limited enforcement on the growth path of an economy.

We also find applications to game theoretical models. Rotemberg and Woodford [1992] argue that some features of the business cycle can be better explained with a model of monopolistic competition; Marimon, McGratten and Sargent [1990] show that in a monetary model with a multiplicity of Nash equilibria, if agents learn how to trade and how to maximize utility using rules that make them take more frequently those actions that provide higher rewards (more precisely, by using some genetic algorithms that have been used in biology to study the survival of species) then the economy would converge to the optimal Nash equilibrium. Judd [1989] compares the equilibrium concepts of Bertrand and Cournot.

3. Algorithms for Solving Non-Linear Dynamic Stochastic Models.

3.1 New Developments in Algorithms

Recently a large amount of research effort has been devoted to the development of new approaches to solving rational expectations equilibrium models. The fact that there exist several alternatives for finding numerical solutions may be confusing to the non-expert but it has several advantages. First of all, when several algorithms can be applied one can check the solutions that different algorithms provide and see if they are similar or not. Perhaps more importantly, different approaches work best in different models; for example, there is usually a trade-off between algorithms that can handle complicated models (with strong non-linearities) and algorithms that can solve larger models with higher speed.

When choosing an algorithm for application on a given model, one has to take into account several factors. Speed of computation is one of these factors, but not the only one; it is very easy for a researcher who is introducing a new approach to make it appear as if this approach is very fast by illustrating its speed with the right model, where his own algorithm has some comparative advantages. For example, one algorithm might solve a model with one state variable very efficiently but, if this method relies on a discretization of the state variables, the computation time will increase exponentially as the number of state variables increases.

Another important feature of a method is its flexibility. If the same method can be applied to many different and interesting models the researcher will be able to use his programs and his expertise in other applications.

It is important to determine the advantages and disadvantages of different algorithms to solve particular models. In this section we will review some algorithms that have been used in a number of papers. The model we are going to use is Brock's model of asset pricing with endogenous production. There are two parts to solving this model: the first is to determine the consumption and investment allocations; this has to be done by solving the following growth model

$$\max E_0 \sum_{t=0}^{\infty} \delta^t \, u(c_t)$$

s.t.

$$c_t + k_t - (1-d)\, k_{t-1} = k_{t-1}^{\alpha}\, \theta_t \tag{3.1}$$

$$\log(\theta_t) = \rho \log(\theta_{t-1}) + \varepsilon_t \quad , \tag{3.2}$$

$$\varepsilon_t \sim N(0,\sigma^2), \text{ i.i.d.}$$

The first order condition for this model is:

$$(3.3) \qquad c_t^{\gamma} = \delta \, E_t \left[c_{t+1}^{\gamma} \left[\theta_{t+1} \, k_t^{\alpha-1} \, \alpha + (1-d) \right] \right] \quad ;$$

With some side conditions that we are going to ignore in this paper, a sufficient condition for an equilibrium consumption, investment and capital series is that the system (3.1)-(3.2)-(3.3) be satisfied. It is typical of equilibrium dynamic stochastic models to take the following general form:

$$(3.4) \qquad g \left(E_t[\phi(z_{t+1})], \, z_t, \, z_{t-1}, \, \varepsilon_t \right) = 0 \quad ,$$

of which the system (3.1)-(3.2)-(3.3) is a special case[5]; here z_t represents all the serially correlated variables in the model, so that in the above simple growth model $z_t=[k_t,c_t,\theta_t]$. A system like (3.4) is difficult to solve because it involves conditional expectations; we can not solve for z_t untii we know the conditional expectation, but we do not know this conditional expectation until we know the solution for z_t.

We will discuss briefly how to solve the simple growth model with five different methods. The first two methods are the linear-quadratic approximation and the value function iterations approach; these are fairly standard and they have been widely used in economics. The other three methods are much more recent: backsolving, iterations on Euler equations and parameterized expectations. Unfortunately, we will necessarily be unfair to each of these methods since they have been applied in much more sophisticated ways than the three line description we are forced to give in this paper.

Linear-Quadratic Approximation

Substituting the technology restrictions in the objective function we obtain

$$(3.5) \qquad \max E_0 \sum_{t=0}^{\infty} \delta^t \, u[\, k_{t-1}^{\alpha} \, \theta_t - k_t + (1-d) \, k_{t-1} \,] \quad .$$

This objective function can be replaced by a linear-quadratic approximation to the term multiplying δ^t:

$$\max E_0 \sum_{t=0}^{\infty} \delta^t [\, x_t A x_t' + x_t' B \,] \quad , \tag{3.6}$$

where $x_t = [k_t, k_{t-1}, \theta_t]$, and A,B are chosen so that the objective function in (3.6) is a good approximation of the objective function in (3.5).

As we said in the introduction, the linear-quadratic model is easy to solve (almost) analytically, so we can use these traditional techniques to solve the approximated problem (3.6).

Value Function Iterations

Using dynamic programming, the Bellman equation for the simple growth model is

$$V(k_{t-1}, \theta_t) = \max_{(c_t, k_t)} \left\{ u(c_t) + \delta E_t V(k_t, \theta_{t+1}) \right\} \tag{3.7}$$

$$\text{s.t.} \quad c_t + k_t - (1-d) k_{t-1} = k_{t-1}^{\alpha} \theta_t$$

If we replace V in the right side of (3.7) by an arbitrary V^0, solve for the max at every possible point of the state variables (k_{t-1}, θ_t), find the implied V^1 and we iterate on the Bellman equation, this series of value functions will converge to the value function that solves the Bellman equation. To find the max, one usually imposes a grid of values on the space of (k_{t-1}, θ_t) and searches on this grid for this max. The expectation in the right side of (3.7) is evaluated as an expectation on a random variable that can take only finitely many values, so it can be evaluated as a simple sum.

Backsolving

The idea here is to start by assuming a process for the endogenous variables. If this is done appropriately, then it may be quite easy to solve for all the variables in the system because we know the expectation in (3.3). For example, letting

$$\lambda_{t+1} = c_{t+1}^{\gamma}[\ \theta_{t+1}\ k_t^{\alpha-1}\ \alpha + (1-d)\ ], \tag{3.8}$$

we assume a process for λ_t, for example

$$\lambda_t = \xi\ \lambda_{t-1} + \eta_t \quad ,$$

and then solve for consumption

$$c_t^{\gamma} = \delta\ \xi\ \lambda_t \quad .$$

The solution for θ_t and k_t is found from the formula for λ_t and (3.1). The idea here is that the exogenous process that is finally backed out is the one that would be consistent with the assumption on λ_t.

Iterations on Euler Equations

Several authors use methods with the following steps:

-Impose a grid on (k_{t-1},θ_t).

-start at a law of motion for k_t, say $f^0(k_{t-1},\theta_t)$; this function is chosen from a family of functions that can approximate a continuous function and that depends on only finitely many parameters. Different types of polynomials, splines, linear interpolation or neural networks would qualify.

-At each point in the grid evaluate the conditional expectation in (3.3) by quadrature integration or other discrete expectations methods.

-Iterate on the (finitely many) parameters of the law of motion until the left side of the first order condition (3.3) equals the integral in the right side.

Parameterized Expectations

This method also works from the Euler equation. The idea is to parameterize the conditional expectations and then iterate until the series generated is such that the assumed conditional expectation is actually the best prediction of λ_t (where λ_t is given by (3.8)). The steps to follow are these:

-substitute the conditional expectation on right side of the first order condition (3.3) by a parameterized function $\psi(\beta;\ k_{t-1},\theta_t)$, to obtain

$$c_t^{\gamma} = \delta\ \psi(\beta;\ k_{t-1},\theta_t) \tag{3.9}$$

- Create a long series of c_t and k_t with (3.9) and (3.1)
- Run a non-linear regression of

$$\lambda_{t+1} \quad \text{on} \quad \psi(\beta;\ k_{t-1},\theta_t)$$

- Iterate until β coincides with the result of non-linear regression.

This ends our dicussion of the algorithms. For a more detailed description the reader is referred to the January 1990 issue of the Journal of Business and Economic Statistics, where different authors described how to solve this simple growth model with alternative algorithms; comparisons are made by Taylor and Uhlig [1990], and some companion articles discuss how to solve it with different approaches in more detail. The methods discussed in that issue include different types of linear-quadratic approximations (by Christiano and McGratten), iterations on the Bellman equation (by Christiano), iterations on the Euler equation (Coleman and Baxter, Crucini and Rowenhorst), backwards solution (by Sims and Ingram), parameterized expectations approach (by den Haan and Marcet), the extended path algorithm (by Fair and Taylor), the Euler equation method of Labadie, and a quadrature method (by Tauchen).

Some of these methods provide an approximation to the rational expectations equilibrium, but they do not have ways of obtaining arbitrary

accuracy in their approximations. The linear-quadratic approximations, the extended path method and the backwards solution procedure are such methods. They have the disadvantage that they can not be used to approximate the equilibrium arbitrarily well, and they may not produce solutions to models with complicated non-linearities like inequality constraints that are binding in some periods and non-binding in others. On the other hand, they tend to be much faster than the other methods discussed above; either they do not require any iterative procedure to find the equilibrium law of motion or the iterative procedure is very fast. This higher speed of computation may be the key to some applications that deal with very large models or that perform very computer intensive tasks like estimating parameters by the method of simulated moments.

Many of the other iterative methods can obtain arbitrary accuracy to non-linear models by refining the approximations. For example, the value function iterations and the methods of Coleman, Bizer, Judd and Baxter provide arbitrarily good approximations by refining the grid on the state variables and on the stochastic shocks that these authors impose; the method of Marcet [1989] and Judd [1989] would obtain arbitrary accuracy by increasing the degree of the polynomials and calculating integrals with arbitrary accuracy. It is fair to say, however, that these refinements can be done at a very high computing cost and it is not clear if arbitrary accuracy can be actually obtained in practice.

3.2 Solutions Without a Planner's Problem.

There are many models in economics that yield suboptimal equilibria. This is usually the case, for example, in models with externalities, public goods, distortionary taxation, imperfect competition, monetary models, models with incomplete markets, etc. Until recently, a large amount of effort was devoted to finding some planner's problem whose solution would coincide with the equilibrium of the suboptimal model at hand. Jones and Manuelli [1989] show how to cast a large number of models in a planner's problem. Part of the reason for these efforts was that the most widely used techniques were the linear-quadratic approximation and the value function iterations, two techniques borrowed from other sciences that had been

constructed with the purpose of solving dynamic, stochastic maximization problems, like the one involved in the planner's decision. To our knowledge, for some models (for example, models with liquidity constraints) nobody has yet found the corresponding planner's problem.

All the methods we have described above other than the linear-quadratic and value function iterations are designed to work independently of the specification of a planner's problem. These methods work directly from the Euler equations and the equilibrium conditions of the model, which are easy to find even if no equivalent planner's problem is at hand. Although casting suboptimal equilibria in some planner's problem is often useful, it is no longer a prerequisite for computing equilibria.

Actually, even LQ and VFI have been used in setups without a planner's problem. For example, Cooley and Hansen [1989] solve a cash-in-advance model, where one of the processes that are exogenous to the agent is the price level, but this process is in itself endogenous. Cooley and Hansen proceed by assuming a linear law of motion for the price level, and they iterate on this law of motion until it is consistent with the consumption allocations; in principle, the linearization of the law of motion for prices introduces more inaccuracies in the solution, since they end up approximating not only the objective function of the agents but also the law of motion of the prices (which, in principle, is non-linear) but it[6] permits the use of linear-quadratic techniques which, as we said before, are very fast.

Also, Rios [1990] uses the linear-quadratic approach to solve an overlapping generations model. The equilibrium is again suboptimal, but application of linear-quadratic is still possible. In Rios' model agents live one-hundred and fifty periods (trying to mimick the number of trimesters of active life), so that the wealth of all agents are state variables of the model. With such a high dimensional model it is inconceivable to use most iterative methods, but linear-quadratic techniques make this model tractable.

Finally, Diaz and Prescott [1989] solve a suboptimal equilibrium using value function iterations. They assume a process for the price level and back out the level of government spending that is consistent with such a process. Once they have assumed a process for the price level they can do the value function iterations in the usual way. This idea of imposing a process for variables that are endogenous to the model has been used previously in the backwards solution procedure of Sims-Novales and Ingram; these authors, however, used the assumption on the endogenous processes to solve the Euler equation instead of the value function.

3.3 Discretization and the 'Curse of Dimensionality'

Another unfounded belief is that any solution algorithm that tries to capture non-linearities will necessarily face the so-called 'curse of dimensionality'. This term refers to the following problem: assume that we have a state variable that takes on continuous values; for example, in the simple growth model of section 3.1 the state variables, namely k_{t-1} and θ_t are both continuous. Even though the model would still be interesting if θ_t was assumed to follow a discrete-valued Markov process, the capital stock is by its own nature a continuous variable. For a solution algorithm that must discretize the state variables or use some kind of grid in the space where the state variables live, it is very expensive to find the solution as soon as more variables are added to the problem; for example, if we discretize the capital stock to take on 100 possible values, and if we add a second type of capital to the model,[7] the state variables now can take 10000 possible values; in other words, the computational cost of the model increases, roughly, exponential rate of 100 with the number of (continuous) state variables.

Some of the methods we mentioned that suffer from this problem in different degrees are the following: value function iterations can only solve discrete problems, so they need very fine grids to get anywhere close to the true continuous solution; the method of Coleman, Bizer and Judd can produce continuous-valued series, but it uses a grid on the state variables where the Euler equation is evaluated; also, this method and that of Tauchen use quadrature integration, which means that a second grid has to

be imposed on the space of stochastic shocks. This last group of methods does not need as fine a grid as the value function iteration approach, because by their own nature they produce continuous simulations, so the computation time does not explode as quickly. For example, Judd uses 20 possible values of the capital stock. Nevertheless, the computation time still grows exponentially and a model with two or three continuous state varaibles would be very costly to solve.

The reader should now get an idea why this is called a 'curse'. Discretization means that even models with only two state variables are very costly to solve means that we have to wait the time necessary for computer builders to increase computing power by 100 until we can add one more state variable.

There are, however, several methods that completely avoid the discretizations and, therefore, they avoid the 'curse of dimensionality'. Obviously, models with more state variables are harder to solve with any method, but the key here is to use a procedure where the computational cost of the problem does not increase exponentially.

Clearly the 'fast' methods described before: linear-quadratic, backsolving and extended path, do not need any discretization. For models that can provide arbitrary accuracy, it is possible to avoid the grids in evaluating integrals if quadrature is replaced by Monte-Carlo integration[8]. In effect, the parameterized expectations approach of Marcet [1989] and the method of Smith [1989] substitute quadrature integration by Monte-Carlo integration, since they evaluate expectations as time averages. It is true that as stochastic shocks are added to a model Monte-Carlo integration is more costly, because more observations are needed in the averages that make up the calculation; but the number of observations increases at a speed much lower than exponential.[9] The parameterized expectations approach, also avoids the discretization of the state variables by evaluating integrals using long run simulations of the endogenous series; in this way discretization of the state variables or of the integrands is avoided.

3.4 Endogenous Oversampling

One of the reasons for the 'curse of dimensionality' is that, when grids are imposed exogenously by the researcher, the algorithm explores all points in the grid, the space of state variables, giving equal importance to all possible points in this space, even though most of these points will rarely happen. This can be avoided with techniques that do endogenous oversampling, where the algorithm is designed in a way that only the relevant points of the state variables are explored by the algorithm.

Consider the simple growth model of section 3.1, assume that we are interested in finding the solution at the steady state distribution. If we use a method that has to impose a grid on the state variables and, in order for the productivity shock to have close to a continuous distribution we could impose a grid of, say, 100 points for the productivity shock and a grid of 100 points for the capital stock. Then, there are 10000 possible values of the state variables (or pairs of θ_t and k_{t-1}) and we will spend equal computing time in all of them. In this case, all points are equally important for the algorithm, so that no eversampling is done. This is a waste of computing power, since many of these pairs happen very rarely; in fact, when the grid is imposed exogenously, many of these pairs will never happen. Let us see this problem in some more detail.

First of all, it is very hard to establish reasonable bounds for the endogenous state variables before knowing the solution. In the above growth model one can easily place bounds on the productivity shock: we can limit θ_t to stay within, say, four standard deviations of the mean; we will denote these bounds by $\underset{\sim}{\theta}$ and $\tilde{\theta}$. But the capital stock, we know that it has to stay in the interval (0,k'), where k' is the maximum capital that is physically possible; this value depends on the depretiation rate 'd' and the maximum productivity shock. But in the steady state distribution and for most parameter values, the capital stock will never get anywhere close to these bounds.

To see this point, we can have a familiar picture with the optimal decision functions, represented by $f(.,\theta)$:

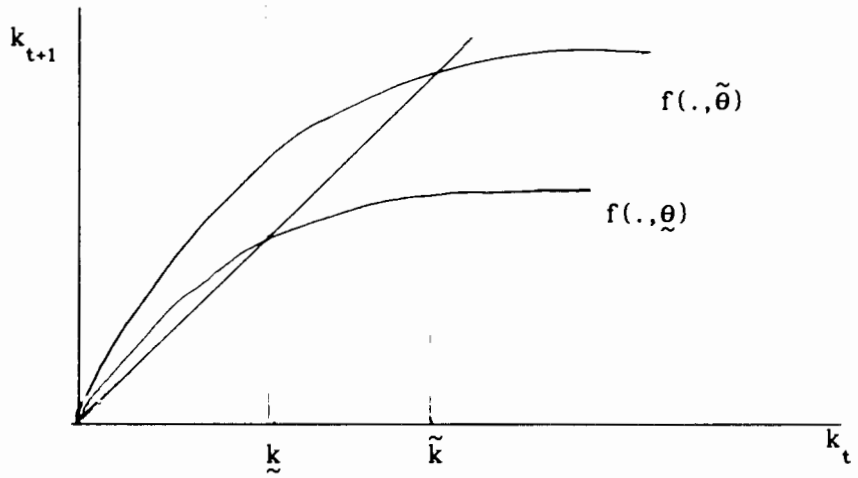

Figure 1

we know that the steady state distribution will never leave the interval $[\underset{\sim}{k},\tilde{k}]$. If the researcher <u>knew</u> the true law of motion, he could impose these bounds on the grid, but this law of motion <u>is</u> the solution we are seeking.

But even if we knew $\underset{\sim}{k}$ and $\tilde{k}$, we would still be wasting computer time by looking at all possible grid points within these bounds. The reason is that many combinations of the state variables never happen; for example, very high values of the shock θ_t never happen together with very low values of the capital stock, and viceversa. Figure 2 describes this situation: out of all the possibilities in the set $[\underset{\sim}{k},\tilde{k}]x[\underset{\sim}{\theta},\tilde{\theta}]$ only values in the parabola happen, but the discretization methods will tend to spend as much time computing the solution in points as B as in A, even though points near B never happen, and points near A happen very frequently.

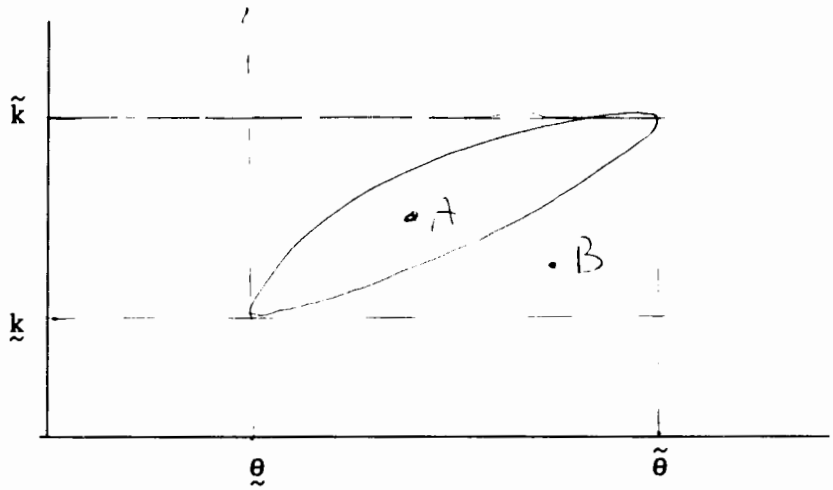

Figure 2

There has been a great deal of attention to this problem in other sciences. For example, the genetic algorithms used in biology and applied to economics by Marimon, McGratten and Sargent [1990] are designed to do endogenous oversampling. The parameterized expectations approach described above is also designed to do endogenous oversampling; in that method integrals are calculated by averaging out long run simulations of the endogenous variables; since in the long run the variables stabilize around the steady state distribution, the algorithm does not pay any attention to regions of the state space that do not happen and the polynomial approximations used are good approximations precisely at values that happen often in the steady state distribution.

Obviously, when one is not interested in the steady state distribution it is not correct to oversample values that happen often in the steady state distribution. For example, in the simple growth model one may be interested in calculating the growth path from a very low initial capital stock towards the steady state distribution. In this case, the initial capital stock may be so low that it never happens in the steady state distribution and a good approximation around the steady state is not valid. It is clear that some parts of the model behave very differently with low capital stocks; for example, the marginal productivity of capital and the marginal utility of consumption (which are the factors determining the investment decision) are much higher than at any point in the steady state

distribution. In this case we need to calculate the law of motion by oversampling at low capital stocks. With the parameterized expectations approach, this can be easily accomplished by using many independent draws of short simulations, each simulation with enough periods for the capital stock to go from its low initial level to the steady state distribution; letting the number of draws go to infinity one can obtain arbitrary accuracy in the integrals being evaluated. This scheme has been used in Marshall [1988] to deal with a non-stationary growth rate of the money supply, and in Marcet and Marimon [1992] to study a growth model where the capital grows from a very low initial capital stock to the steady state value.

Techniques like value function iteration and the method of Coleman, Bizer and Judd, as they are used by these authors, do not do any kind of oversampling, so that their grids are bound to become unmanageable as soon as they solve problems with a high number of state variables. These techniques would improve in speed by introducing some scheme that eliminated the grid-points that happen rarely, much in the way that genetic algorithms eliminate actions that give high payoffs with small probability.

On the other hand, these methods will perform better if a good approximation for any initial value of the state variables is needed. For example, in game theoretical models, if one looks for the perfect Nash equilibrium, then the equilibrium restrictions have to be satisfied at all nodes of the decision tree, even if they have zero probability in equilibrium; in this case endogenous oversampling would not look at equilibrium at those states[10].

4. Theoretical Research by Simulation.

4.1 Economic Theory on the Computer.

The use of computer simulations of structural models for empirical purposes, as will be discussed in section 5, is a common practice and its validity is generally accepted. More recent and perhaps more controversial is the use of simulations in theoretical work. There is a growing number of papers that study highly abstract models and reach conclusions about the behaviour of those models by studying computer simulations. These should be considered theoretical papers because they do not try to explain the real economy beyond, perhaps, some stylized facts.

For example, one result that seems to emerge from several papers we discussed in section 2.2, is that introducing market incompleteness by merely closing down some markets and introducing liquidity constraints does not produce many differences from the complete market or the representative agent case. This is a qualitative statement about how certain models behave, so it is a theoretical statement. It is also a statement that can be justified only because of the knowledge we have gained from computer simulations, and it stands in contrast with many negative theoretical results available in the literature about the behavior of incomplete market models. These results claimed that these models would have very different equilibria from the complete market case, and that strong non-optimalities would be present with incomplete markets. Perhaps other departures from the complete market assumption, like private information, the possibility of default, introducing different types of heterogeneity, or having costly exchange of securities will yield more striking results.

Another use of simulation for theorists may be as a tool for acquiring intuition about results that can be proved analytically. Theorists have regularly used ad-hoc models and graphical techniques for this purpose. For example, Mas Colell and Geanakopolous [ ] proved that removing only one security from a complete market setup, the model would have a very large number of equilibria. The reason why the above simulations pick out one equilibrium may be that the algorithms used impose

a time-invariant policy function and there is only one equilibrium in the Mas Colell- Geanakopoulos setup that has a time-invariant law of motion. As will be discussed in section 4.4, if one restricts the analysis to equilibria that can be the limit of a learning process, a restriction like time-invariance of the law of motion may be necessary.

There are many other striking theoretical 'results' we have learnt from simulation studies. For example, den Haan 1990] argues that in a stochastic monetary model with a shopping time technology the k-percent rule is nearly optimal; this contradicts some statements in Hahn [1971] that, because in a stochastic model money serves as insurance, money could have a lower return in equilibrium and it may be optimal to run an inflation. Bizer and Judd [1989] argue that stochastic taxation can improve welfare over deterministic taxation when the tax is distorting. In the incentive compatibility literature, Marcet and Marimon [1992] argue that limited enforcement of contracts has a very significant effect in reducing growth, while limited information has very little effect. Phelan and Townsend [1990] argue that the optimal incentive compatible contract causes a very small utility loss.

It is necessary for economists to start considering theory on the computer as a necessary tool, due to the evolution of the models that we are working with. In fact, this has been a routine practice in the natural sciences like physics, biology, chemistry etc. The discussion should not be about 'if', but 'how' this type of exercise should be performed. In other words, we must think of the standards that theory on the computer should meet. Some of the questions that come up in this respect are: how to report results, how to choose parameter values and accuracy of the solution. We will discuss these points in the remainder of this section. The aim of the following discussion is not to close the debate on these matters, but of highlighting their importance.

4.2 Reporting Simulation Results.

One of the reasons some people may be uncomfortable with theoretical results derived by simulation is that it seems easier to report results

that are incorrect with a simulation than with a theorem. While with analytic methods it is possible to fit all the derivations in a paper, it is impossible to report all the calculations and all the programs used in simulation in a journal article, so, the issue of how to report simulation results is an important one; besides preventing careless work, adequate reporting should make the results more convincing to the reader.

Similar problems in reporting results are faced by the experimental sciences and, closer to our experience, by empirical work in economics. Psychologists (and experimental economists) must detail carefully how their experiments have been conducted, how were the subjects of the experiment selected, what instructions were given to them etc. Economists doing empirical work must report what data sources the are using, what observations they ignored, what how was the data transformed ... Similarly, economic simulators should give as much detail as possible about how the solution was actually implemented, what was the level of the approximation, what parameter values were used, some measure of computation time etc.

Ideally, the actual law of motion should be reported in the paper; this would make it possible for some readers to easily write a computer program that calculates the simulations and study the simulation on his own, perform accuracy tests, and explore issues of the model that the authors may not have reported, without having to implement the whole iterative procedure that finds the corresponding fixed point.

4.3 Parameter Selection

In order to obtain a simulation one must assume certain values for the parameters of the model. This is, perhaps, more of a problem in economics than in natural sciences, since we tend to have more uncertainty about the parameters of our models.

The usual practice is to use parameter values generating equilibria that closely reproduce some features of the data. Economists have accumulated by now considerable experience on this type of exercise. Part of the research agenda should be to discover precisely what are interesting

parameter values to use in a given model.

Some parameters are easy to choose because they have direct implications on the technology and they can be measured in a fairly direct way; for example, the depretiation rate 'd' in the simple growth model can be easily measured by the depretiation of aggregate capital observed on average in real data; measures of this depretiation rate in the United States put its value to about 10% a year. Then we would have d=.9 in a yearly model, d=.975 in a quarterly model and so on.

Other parameters for the simple growth model can be set using the equilibrium of the model. These are usually harder to choose, because the implications on this dimension are less clear. For example, in the representative agent asset pricing model the inverse of δ (the discount factor of the utility function) is close to the average gross return of a real riskless bond. In U.S. data this return is around 1% so that, if we used this asset to pin down the value of δ, we should choose δ=.99 in a yearly model. But it turns out that in the simple growth model there is virtually no risk premium, so that δ^{-1} is also close to the average return of a risky stock that pays the return on capital. In U.S. data the average return of stocks is about 7% , so by this measure δ should be set equal to .93 . This is an example of how it may be hard topin down some parameters due to the fact that our models do not replicate closely some aspects of the data. Most authors choose a compromise and they set δ somewhere in between these two values, but ideally one would want to explore different values of parameters like this that are not easily pinned down.

Assuming that the instantaneous utility function is defined as

$$u(c_t) = \frac{c_t^{-\gamma+1}}{-\gamma+1} \quad ,$$

then γ is the coefficient of relative risk aversion. The implications of a parameter of this type are even harder to work out, so γ is even harder to choose convincingly. Mehra and Prescott [1985] report several microeconomic empirical work that sets γ around one; to be safe, they explore all values of γ between 1 and 10.

If a researcher does not feel comfortable with choosing fixed values for the parameters[11], then one should explore the parameter space in a systematic way. If this has to be done by imposing a grid on many parameters we will encounter similar problems as were discussed in section 3. More research on how to do this exploration would be useful.

4.4 Accuracy of Simulations.

So far there is no widely accepted measure of accuracy of numerical approximations. This may be an important issue since inaccurate solutions may distort considerably our conclusions.

One position is to be clear about what the approximation was, the solution reported can be taken as an approximation. For example, in methods that use polynomial approximations some authors just state that a second, or third degree polynomial was used.

If one wants to claim that the solution is accurate, though, some testing of the solution needs to be done. Unfortunately, testing the accuracy for these solutions beyond any doubt is an impossible task. Finding the solution amounts to finding a non-linear function that can be changed in uncountably many directions, and it is impossible to check all of them. The best we can do is to challenge our solution with different tests of accuracy that experience indicates are good a selecting inaccurate solutions. Another possibility is to find error bounds; ideally one would want to find variance bounds for the part of the model he is interested in, whether it is a covariance, a mean etc.

One such challenge is to compare solutions obtained with different methods that have been proved reliable in the past. The comparisons done by Taylor and Uhlig have been proved useful in this respect. Much has been made about the differences in the solutions obtained with different methods in that JBES issue; rather than just observe that there are some differences it may be more instructive to try to explain where those

differences arose. In particular, methods that simulated discrete series with relatively coarse grids tended to give different results; this is not surprising since the discreteness forces the solution for investment to be more variable.

Some methods have intrinsic ways of challenging the solution. For example, in the method of Coleman, Bizer and Judd one can change the grid points at which the Euler equation is evaluated and see if the solution changes considerably; the parameterized expectations approach can use different draws of stochastic shocks in order to find the non-linear regressions used in finding the law of motion in that method. These changes should not yield laws of motion that differed too much if the solution is accurate.

Another way of challenging a solution in order to test for accuracy is proposed in den Haan and Marcet [1990b]. They propose to test if the error in the Euler equation is orthogonal to functions of past variables; this amounts to testing the first order conditions of a maximization problem in certain directions. This test does not translate directly into bounds for the error in the simulated series, but in several examples it seems to select correctly the more accurate solution. Also, Taylor and Uhlig ran this test for the solutions of the different methods; they found that the methods that performed well in this test (namely, the solution of Coleman, backsolving, parameterized expectations and some of the linear-quadratic approximations) yielded similar solutions.

Finally, one can refine the aproximation and check whether the solution changes considerably. For example, the value function iterations would use a finer grid, the methods of Coleman, Bizer and Judd and parameterized expectations would use a more refined approximation scheme in the class of functions used to approximate the solution (which can be either polynomials or splines). Christiano (1990b) has made such comparisions between the linear-quadratic and the value-function iterations solution.

It is clear that, given a particular way of checking for accuracy, it

is possible to find examples where the test would not work, but reporting a number of these checks should convince the reader and the author himself that the solution is reasonably accurate.

4.5 Multiplicity of Equilibria and Learning Algorithms

It has been well documented in the literature of rational expectations that, in some models, there exists a multiplicity of equilibria. Often this multiplicity takes the form of a continuum of equilibria. This situation may present some problems in doing theory on the computer that are worth discussing.

If we study a model with a multiplicity of equilibria by simulation, depending on the model and on the algorithm, two things may happen: the solution algorithm can have as a limit point any one of the multiple equilibria, or the algorithm can have only one equilibrium as a limit point. In the first case, there will be numerical problems because the algorithm may start drifting from one equilibrium to the other; in the second case, if the algorithm does converge to one equilibrium, we will never know if that equilibrium is more interesting than the others.

One way to approach this problem is to use algorithms that replicate learning schemes. There is now a large literature discussing how learning schemes can select one rational expectations equilibrium as a limit point in certain models with a multiplicity of equilibria; that is, only one rational equilibrium is stable under learning in these models. Papers that discuss learning schemes that may converge to equilibria suggest computational algorithms that mimick the learning mechanism; these algorithms, then, will only converge to the equilibria that are stable under learning. The remaining equilibria will go unnoticed but, if one believes that we should concentrate our study on solutions that are stable under learning, ignoring the other equilibria is the correct alternative. One such algorithm has been used by Marcet [1989] among many others, where the iterative scheme mimicks the evolution of least squares learning about expectations and, therefore, that algorithm only converges to solutions

that are stable under learning.

5. Empirical Research by Simulation.

We will discuss two ways of estimating and testing dynamic models that are based on numerical simulations of structural models: maximum likelihood and the method of simulated moments.

5.1 Estimation: Maximum Likelihood and Method of Simulated Moments.

To see how maximum likelihood can be used and what its limitations are let us consider, again, the simple growth model. Assume we have data on only on the capital stock[12]. Given values for the parameter vector λ we have a law ofmotion of capital of the following form:

$$k_t = h_\lambda(k_{t-1}, \theta_t) \quad ; \tag{5.1}$$

here $\lambda = [\ \alpha,\ d,\ \rho,\ \sigma_\varepsilon^2,\ \delta,\ \gamma\ ]$, the fundamental parameters of the model. If we have a solution for the law of motion h_λ (which we have to find numerically), equation (5.1) allows us to back out θ_t from the observation on k_t and k_{t-1} ; more explicitely, we can find the function f_λ that satisfies

$$\theta_{t-1} = f_\lambda(k_{t-1}, k_{t-2}) \quad ; \tag{5.2}$$

that is consistent with (5.1). Then we can substitute this expression in (5.1) and write down the likelihood of k_t conditional on past observations (as is required in time series models). More applications of maximum likelihood estimation are surveyed in Rust [1988].

Unfortunately, in models with a larger number of unobservable than observable variables we can not back out solutions for the unobserved shocks as in (5.2); in this case there is a manifold of unobservables that

is consistent with the observations and a given parameter value and putting probabilities on this type of manifolds is very cumbersome. There are many cases of interest where this problem arises: models with private information will typically have 'too many' shocks; in applications with aggregation over agents have a similar problem, and the same happens with aggregation over time. Notice that even when we have too many unobservables the model can be identified if enough restrictions are imposed. This has been done, for example in the time aggregation literature of estimating continuous time models with discrete data.

In the cases where maximum likelihood can not be applied for the reasons just described, and if we are able to numerically solve our model, we can use the method of simulated moments. This procedure has been used in a time series framework by Ingram [1990] to test an asset pricing model with myopic agents, Garcia-Mila [1987] in a model with both private and public capital, Heaton [1990] in an asset pricing model with habit persistence in the instantaneous utility function, Bossaerts and Hillion 1989] in an option pricing model with early exercise, Smith [1989] in a capital accumulation model. Applications to microeconomic problems are reviewed in Pakes and Rust [1991].

The idea in the simulated method of moments is very simple: the estimator is determined by finding the value of the parameters that make certain moments of the simulated series as close as possible to the analogous moments of the observed series. More formally, assume we have T observations on m time series denoted $\{ z_t \}_{t=0}^{T}$, where each z_t is m-dimensional; assume that we can generate simulations of a the model to be estimated; let $\{ \bar{z}_t(\lambda) \}_{t=0}^{S}$ be a simulation of length S of the model we want to test at a given parameter vector λ. Let $h:R^m \longrightarrow R^q$ be a function such that $E(h(z_t))$ are the moments we want to match; typically, $q>n$, where n is the number of parameters to be estimated. Finally, let H and $\bar{H}$ be the actual and simulated moments:

$$H_T = (1/T) \sum_{t=0}^{T} h(z_t) \qquad \text{and} \qquad \bar{H}_S(\lambda) = (1/S) \sum_{t=0}^{S} h(\bar{z}_t(\lambda)) \quad ,$$

Then, the method of moments estimator λ_T is defined as

$$(5.4) \qquad \lambda_T = \underset{\lambda}{\text{argmin}} \; [H_T - \bar{H}_S(\lambda)]' \; W_{T,S} \; [H_T - \bar{H}_S(\lambda)] \quad ,$$

where $W_{T,S}$ is a positive definite qxq matrix that defines the distance between the simulated moments $\bar{H}_S(\lambda)$ and the actual moments H_T.

This is a complete method for doing statistical inference: one can obtain consistency and asymptotic distribution results as T and S go to infinity. These results have been proved in the case of serially independent observations by Pakes and Pollard [1989] and, in a slightly different setup, by Mcfadden [1989]; these authors allow for non-differentiable simulations[13].

The asymptotic results for the time series case has been analyzed by Ingram and Lee [1991], and Duffie and Singleton [1990]; the last paper allows for endogenous state variables, it handles the non-stationarity that arises in simulated series that start up at a fixed value for the state variable, and it provides conditions for checking ergodicity of the model. These authors have to assume that simulations are differentiable with respect to the parameter set λ; they need this assumption because they use the mean value theorem in obtaining the asymptotic distribution for the estimator λ_T. Asymptotic results for time series assuming only continuity (but not differentiability) of the simulations, has been partially analyzed by Bossaerts [1989]. Up to now, there exist no theorems allowing for time series dependence and discontinuous simulations.

Under standard assumptions of stationarity, ergodicity, differentiability of the simulations and assuming that $\underset{T \to \infty}{\text{Lim}} \; T/S = \tau$,

one can prove consistency and asymptotic normality of the method of simulated moments. Letting λ_0 be the true parameter value, and letting

$$S_w = \sum_{l=-\infty}^{\infty} E \left[[\, h(z_t) - Eh(z_t) \,] \circ [\, h(z_{t-l}) - Eh(z_{t-l}) \,]' \right] \quad \text{and}$$

$$B = \frac{\partial}{\partial \lambda'} E\, h(\bar{z}_t(\lambda_0)) \quad ,$$

where these expectations are calculated assuming that z_t is actually generated by the model at hand, and assume that $W_{T,S}$ converges in probability to a non-singular matrix W.

A necessary condition for identification is that B should have full column rank. This translates into the usual condition that the objective function in (5.4) have a unique local minimum at the true value of the parameters. With this condition it can be shown that the estimator is consistent and that $T^{1/2}\ [\lambda_T - \lambda_0]$ asymptotically has a normal distribution with mean zero and variance-covariance matrix equal to

$$[B'WB]^{-1}\ B'\ W\ S_W\ W'\ B\ [B'WB]^{-1}\ (1+\tau) \ .$$

The efficient matrix W in the sense of minimizing this variance-covariance matrix is $W=S_W^{-1}$. To test the overidentifying restrictions of the model one can use the fact that under the null hypothesis that the model is correct and using the optimal weighting matrix $W=S_W^{-1}$, the test-statistic

$$T\ [H_T - \bar{H}_S(\lambda_T)]'\ S_W^{-1}\ [H_T - \bar{H}_S(\lambda_T)] \tag{5.6}$$

converges in distribution to a χ^2_{q-n} . This is, then, a natural measure of goodness of fit; the interpretation is, as usual in these procedures, that we have n parameters so that we can fit q moments perfectly, but there are still q-n dimensions that do not perfectly fit but that, if the model is correct, they must be close to zero.

There are many aspects of this method that are yet to be explored. As we said, asymptotic results for the case of time series with discontinuous simulations have not yet been proved. Also, it is not known how to extend this procedure to non-stationary models that can not be normalized in an obvious way. Another area for research is the robustness of this estimator: MSM is a parametric method in the sense that the results hinge on

distributional assumptions about the underlying shocks. Finally, given that the numerical solutions are approximations and, therefore, they are not the correct solution, there is the issue of what effect these approximations have, if any, on the asymptotic results; as suggested by Pakes and Rust [1990], this problem can probably be handled by reinterpreting the parameter being estimated.

5.2 Calibration and the Method of Simulated Moments.

The paper by Kydland and Prescott [1982] has been one of the earliest and most influential applications of simulation techniques to dynamic stochastic models. They applied the linear-quadratic as an approximation to a non-linear stochastic model; they simulated a capital accumulation model with a complicated delay structure for converting investment into capital, and durability of the utility from leisure. That paper also introduced the so-called 'calibration' approach for validating dynamic stochastic models; using this validation approach Kydland and Prescott argued that their aggregate real business cycle model came close to matching several moments of the data and a large literature has developped since over the issue of whether or not the business cycle can be explained by real models or some monetary aspects are essential to the business cycle. Calibration has been used recently in many papers, among many others, Hansen [1986], Rogerson [ ], Prescott [1986], and there has been continued debate about wheter the calibration approach was called to replace traditional econometric procedures in testing dynamic models.

The principles of calibration can be summarized as follows: choose some moments of interest that the model at hand should explain, we will denote these as MTE (moments to be explained); choose the parameter values of the model from data or moments different from MTE; compare the MTE generated by the model at those parameters with the MTE from real data. To decide if the simulated moments are close enough, Kydland and Prescott obtain a large number of independent simulations and look at the dispersion of MTE's from these simulations; in this way they construct a 95% confidence interval for each MTE. If the corresponding MTE from the real data falls within this interval, they declare this particular moment as

being satisfactorily explained by the model.

This procedure can be interpreted as a simplified version of the method of simulated moments (MSM) and it may be instructive to look at it in this light. Using some of the moments for finding parameter estimates and the remaining moments for testing amounts to using a weighting matrix W in (5.4) with n entries in the main diagonal equal to one and all other entries equal to zero. This matrix will pick out the n moments used for estimation and match them perfectly; then, in the part that tests the goodness of fit, one could derive a chi-square statistic analogous to (5.6), that in effect tests if the remaning (q-n) moments are well explained by the simulated models.

Rather than testing all the overidentifying restrictions at once, calibration looks at each moment one by one. Also, instead of relying on asymptotic distribution as in the test statistic (5.6), it constructs short-sample confidence intervals by Monte-Carlo integration, fixing the parameter values as if they were known with certainty.

From the point of view of statistical inference, this procedure ignores several important issues: by testing the moments one by one and ignoring uncertainty on the parameters the confidence intervals are incorrect; by using an arbitrary weighting matrix W the estimates contain a larger amount of sampling error than is necessary and by not reporting the uncertainty on the parameter estimates it is difficult to know how reliable the results are. Recently, Burnside, Eichenbaum and Rebelo [1990] have argued that by ignoring the uncertainty in the parameter estimates of the productivity shock of a business cycle model, some inadequacies of the real business cycle models are overlooked.

The justification given by calibrators for using different moments for estimation and for the goodness of fit test is that if the same moments are used for estimation and testing somehow this favors acceptance of the model; in the language of statistical inference this translates into the familiar statement that the test for goodnes of fit is not powerful against reasonable alternatives. While this proposition may be true, it has not

been studied formally.

Nevertheles, calibration studies have helped us understand the working of dynamic models. The main advantage of calibration lies precisely in its simplicity. Estimating optimal weighting matrices, finding confidence intervals, reporting variance-covariance matrices etc. is very cumbersome. Perhaps it is more instructive for economists to spend time thinking about economic modelling instead of dwelling on statistical issues!.

Since we see econometricians too often forgetting about economic modelling, any voices that force econometricians to concentrate on the economics of their models are welcome influence. But calibration contains too many arbitrary choices to be considered as the final word in testing a model.[14]

6. Response to Judd's Comments

The discussion of this paper by Ken Judd (see his comments in this volume) is, to say the least, controversial. What follows is a detailed analysis of the substance of Judd's discussion.

6.1 Monte-Carlo Integration

Judd seems to disagree with my claim that 'Monte-Carlo integration is more efficient than quadrature in multidimensional integrals'; he writes that 'this claim is unsupported'. The reader will note, however, that his discussion argues that **quasi-Monte-Carlo** techniques are superior to straight **Monte-Carlo**. This is unrelated to my point, which was that 'Monte-Carlo integration is more efficient than quadrature in multidimensional integrals', a well known fact that is discussed in the

paper. Perhaps I should clarify that in most economic models of interest there are many random variables to integrate over: individual shocks, sectoral shocks, etc., so multidimensional integration is often needed in economics. I saw no need to discuss quasi-Monte-Carlo techniques in my paper due to the fact that they had not been applied in economics at the time of the Congress. In any event Ken Judd seems to further endorse Monte-Carlo methods in the secion on 'Theory by Computation'. This is a change from his original position expressed at the World Congress (among other places)where he argued that Monte-Carlo was not a good procedure. In the last two years, a growing number of economists are switching to Monte-Carlo based simulation methods, as they attack more complicated problems.[15]

6.2 Endogenous Oversampling

It is obviously true that endogenous oversampling will sometimes lead to bad approximations; this is also true from exogenous oversampling, and no oversampling at all. But endogenous oversampling reduces the amount of points in the state space that one looks at, and it permits solving models with a large number of continuous state variables, for the reasons explained in the main part of this paper.

Judd disagrees with my comments in favor of endogenous oversampling; let us analyze his reasoning. First, he gives an example where, if there is a change in tax policy then, using the law of motion calculated before the change in policy would yield wrong results. This is just saying that one needs a different law of motion after a policy change; anybody who is acquainted with the Lucas' critique knows that, and nothing in my paper would suggests that we should ignore the Lucas critique.

It is not true that 'Marcet acknowledges, (that) any method which uses endogenous oversampling ... is unsuitable to use in game-theoretic analysis'. What I say in the paper is that there may be problems in calculating *Nash perfect equilibria*, which is only one class of games. In fact, parameterized expectations has been used successfully in Marcet and Marimon [1992] in a mechanism design problem and in Rojas [1992] in a

differential game (of optimal taxation). Even for Nash perfect equilibria, the procedure can be modified in order to sample also regions that do not occur in equilibrium, so Judd's dismissal of techniques that use endogenous oversampling is unwarranted.

Judd claims that transitional paths to the steady state can not be calculated with endogenous oversampling. I discuss in enough detail how to use independent realizations in PEA to calculate this transition in subsection 3.4, and I will not repeat my arguments. I have also made this argument in the original version of this paper, in my presentation at the World Congress, and in many seminar presentations (the first one in January 89 at Stanford University). Furthermore, now there are two published papers (Marshall [1992] and Marcet and Marimon [1992]) that use this scheme. So, my arguments seem to have been well received in the profession.

Judd claims that endogenous oversampling does not guarantee that the oversampled region will be approximated more accurately and he cites counterexamples where the outer regions should be sampled more frequently. This can not be a generic case, since it is not hard to think of examples where oversampling the outer regions gives a bad approximation.

Judd cites an example that he does not work out, so it is very hard to respond precisely and I can only guess as to what is the reason for the bad approximations he has in mind. I suppose that those counterexamples are ones where the number of points in the fit is the same as the number of parametersin the polynomial. For example, if we fit five points on a function with a five degree polynomial, we may get the function fitting very poorly in some regions. But in algorithms that combine Monte-Carlo simulation and endogenous oversampling (such as PEA, Smith's or Heaton's), we usually face a situation where the order of the polynomial that we fit is much smaller than the number of points where the fit is performed (typically the number of points is of several thousand, while the number of parameters is ten or twenty). In this case, when we oversample in region A we get more accuracy than in an undersampled region B. More precisely, and using Judd's example in the discussion, if we have two intervals of equal length A and B, and we sample at n equally spaced points, it is easy to

show that for a given degree of the polynomial, if we choose the polynomial that minimizes mean square errors and we weight all points equally, as n goes to infinity, region A is better approximated than region B.

Also, our experience shows that this is the case. For example, in an earlier version of den Haan and Marcet [1990b] we compare the true solution of a Brock and Mirman model with the approximated solution and the levels near the steady state are better approximated.

6.3 The Curse of Dimensionality

I do not see any justification to the claim that my discussion of the curse of dimensionality is "at variance with conventional wisdom". If anything, Judd's discussion confirms that many methods suffer from this. I am sure that future adaptations of the current methods may partly solve the problem by performing some kind of endogenous oversampling.

6.4 Computational Experience

One must discuss issues of speed of computation with great care.

First, the comparisons reported by Taylor and Uhlig were made within the NBER Rational Expectations Modelling Group. For example, the solution time that was reported by parameterized expectations was, in fact, solving 10 different models, since it started with the solution for the capital growth model with 100% depretiation and it gradually moved to the 90% , 80% until it got to the zero depretiation case. The reason for this was that den Haan and Marcet [1990] wanted to demonstrate how the homotopy approach could work in the simplest case; but the solution had to travel a long way, from the very low capital stock in the 100% depretiation case to the very high capital stock of the no-depretiation case. By comparison, Judd performs the calculation by starting the law of motion at a constant level, equal to the deterministic steady state; if we use such excellent initial conditions, we cut the computation time by about 50.

Second, the Taylor and Uhlig comparisons were not meant as a race. Rather, the objective was to discern the applicability of the different methods to different models. Had we known we were in a race, we would have required much lower accuracy in the fixed point, we would have used better regression algorithms and used much smaller number of observations. Also, we would have thought very hard about how to get good initial conditions. Also, we would have probably asked the group to solve a model with more random variables and more state variables, where Monte-Carlo integration and endogenous oversampling perform better. Finally, anybody in the mailing list could have sent solutions for the comparison before the comparisons were made.

The fact is that the algorithms based on Monte-Carlo simulation and endogenous oversampling (the parameterized expectations approach among them) have produced many applications in a short period of time. Ultimately, the one and only test of an algorithm is if it produces interesting economic applications. There is not such a thing as a right or wrong algorithm; there are just algorithms that are more appropriate for certain models than others, and I tried to write the paper in a way so as to help the reader distinguish the advantages.

6.5 Misinterpretation of sentences

Judd clearly misinterprets some of my words.

I never give the impression that 'numerical simulations of rational expectations models is a new technique'. I cite many early papers. The only thing that is new in economics is the use of simulation in theoretical papers.

Judd criticises my claim that a linear-quadratic approximation does not provide arbitrary accuracy in non-linear models. His reasoning is that linear-quadratic is a special case of perturbation methods (although this is not the way in which it has been viewed in economics) and, therefore (?) my assertion is not correct. Surely, perturbation methods with higher order terms may provide arbitrary accuracy, but I never refer to these. One

detail that is missing from Judd's discussion is that perturbation methods of order **higher** than one is are considerably more complicated than the traditional linear-quadratic case; the reason is that in linear-quadratic we can apply certainty equivalence, and the problem is nearly a deterministic one in terms of computational costs, while the same is not true with higher order Taylor approximations.

In Den Haan and Marcet [1990], we never 'have difficulty solving a six parameter case'. As we explain in that paper and in Den Haan and Marcet [1990b], the fact that multicollinearity appears for high order terms just detects the fact that some of these high order terms are irrelevant and the solution is equally accurate if these are not introduced.

About liquidity constraints, it is clear that I refer to asset pricing models and the effect of liquidity constraints on issues such as risk premium, risk sharing, etc.

6.6 Review of the literature

The references that Judd finds missing from my paper are valuable, but they may give the impression that I was disrespectful with some authors. Any survey has to concentrate on a subset of the literature, and I say very clearly in the introduction to my paper and at the World Congress presentation, that I was going to review papers that have to do with simulation of dynamic, stochastic non-linear models, and I would concentrate on applications in the last few years to macroeconomics and financial economics.

The papers from Agricultural Economics are valuable references, but the fact is that it took a very long time for them to have an impact on the literature that I was reviewing. We have to thank Coleman, Baxter, Bizer and Judd for acquainting macroeconomists with these methods. The study of why this happened may be an interesting subject for a historian of economic thought; I can only advance that, perhaps, one of the reasons is that these papers use very simplified models: linear production functions, constant prices[16] etc. It may not have been obvious to macroeconomists in the 80's

interested in equilibrium models with non-linear utility and production function how to apply the algorithms in those papers.

Judd points out that many numerical procedures were not discussed in my paper. For obvious reasons, I limited myself to numerical procedures that had been applied to economics. Writing down a large number of numerical techniques before actually checking if they are useful in economics is not an exercise that I ever wanted to perform. I have strong doubts about the usefulness of such an exercise, especially since numerical analysis textbooks are readily available.[17]

Conclusion

Simulation techniques are available to solve many stochastic dynamic models that can not be handled with analytic methods. To the exent that these models can be used to ask interesting questions, and perhaps even to answer some of them, it seems necessary to use simulation techniques. Algorithms available for solving dynamic stochastic equilibrium models have progressed to a point that stumbling blocks of the past, like the 'curse of dimensionality' and the need for writing a planner's problem can be now side-stepped. Because of this we can now study very complicated models, with inequality constraints, suboptimal tax schemes, monetary models with uncertainty, discrete and continuous choices, incomplete markets, many types of agents, private information, limited enforcement of contracts etc.

There are several techniques available for empirical work that have been used extensively. The use of simulations for theoretical purposes, however, is still uncommon and is now starting to be accepted in the economics profession; there should be active discussion on how these exercises should be done, in particular, how to report the results, how to justify the choice of parameter values and what standards of accuracy are demanded.

References

Baxter, M. (1988); "Approximation Suboptimal Dynamic Equilibrium: An Euler Equation Approach" University of Rochester working paper.

Baxter, M., M. Crucini and K.G. Rowenhorst (1990); "Solving the Stochastic Growth Model by a Discrete-State-Space Euler Equation Approach" Journal of Business and Economics Statistics, January

Bizer, D. and K. Judd (1989); "Uncertainty and Taxation" Papers and proceedigns of the American Economic Review, May.

Bossaerts, P. (1989); "The Asymptotic Normality of Method of Simulated Moments Estimators of Option Pricing Models", working paper, Carnegie Mellon University.

Bossaerts, P. and P. Hillion, (1990); "Arbitrage Restrictions Across Financial Markets: Theory, Methodology and Tests" working paper, Carnegie Mellon University.

Braun, A. (1989); "The Dynamic Interaction of Distortionary Taxes and Aggregate Variables in Postwar U.S. Data", working paper, University of Virginia.

Brock, W.A. (1982) "Asset Prices in a Production Economy", in J.J McCall, ed. The Economics of Information and Uncertainty, University of Chicago Press.

Brock, W.A. and B. le Baron (1989); "Liquidity Constraints in Production Based Asset Pricing Models", NBER working paper.

Brock, W. A., L. J. Mirman (1972); "Optimal Economic Growth and Uncertainty: the Discounted Case", Journal of Economic Theory, vol. 4, 479-513.

Burnside, C., Eichenbaum, M. and Rebelo, S. (1990); "Labor Hoarding and

the Business Cycle" working paper, Northwestern University.

Chang, L.J. (1989); "Distortionary Taxes, Monetary Policies and Business Cycles" Ph. Dissertation, Carnegie Mellon University.

Chari, V.V., L. Christiano and P. Kehoe (1990); "Dynamic Ramsey Taxation in a Stochastic Growth Model", working paper, Federal Reserve Bank of Minneapolis.

Christiano, L.J. (1990); "Solving the Stochastic Growth Model by a Linear-Quadratic Approximation and Value-Function Iteration" Journal of Business and Economics Statistics, January

Christiano, L.J. (1990b); "Linear-Quadratic Approximation and Value-Function Iteration: A Comparison", Journal of Business and Economics Statistics, January

Christiano, L.J. and M. Eichenbaum (1990); "Current Real Business Cycle Theories and Aggregate Labor Market Fluctuations", working paper, Federal Reserve Bank of Minneapolis.

Coleman, W.J. (1989) "Equilibrium in a Production Economy with an Income Tax", working paper, Board of Governors of the Federal Reserve System.

Coleman. W.J. (1990): "Solving the Stochastic Growth Model by Policy Function Iteration" Journal of Business and Economics Statistics, January.

Cooley, T. and G. Hansen (1989); "The Inflation Tax in a Real Business Cycle Model" American Economic Review, vol. 79, pp. 733-748.

Deaton, A. (1989); "Saving and Liquidity Constraints" NBER, working paper series.

Den Haan, W. (1990a); "Optimal Inflation in a Sidrausky Model" forthcoming, Journal of Monetary Economics.

Den Haan, W. (1990b); "The Term Structure of Interest Rates in Real and Monetary Production Economies" working paper, Carnegie Mellon University.

Den Haan and Marcet (1990); "Solving a Simple Growth model by Parameterizing Expectations" Journal of Business Economics and Statistics, January 1990.

Den Haan, W. and Marcet A. (1990b); "Accuracy in Simulations" working paper, Carnegie Mellon University.

Diaz, J. and Prescott, E. (1989); "Asset Returns in Computable General Equilibrium Heterogeneous Agent Models" working paper, Federal Reserve Bank of Minneapolis.

Duffie, D. and Singleton, K. (1990); "Simulated Moments Estimation of Markov Models of Asset Prices" working paper, Stanford University.

Fair, R. C., and J. B. Taylor (1983); "Solution and Maximum Likelihood Estimation of Dynamic Nonlinear Rational Expectations Models", Econometrica, vol. 51, 1169-1185.

Gagnon, J. (1990); "Solving the Stochastic Growth Model by Deterministic Extended Path" Journal of Business and Economics Statistics, January

Garcia-Mila, T. (1987); "Government Purchases and Real Output: an Empirical Analysis and Equilibrium Model with Public Capital" working paper, Universitat Autònoma de Barcelona. Published in Spanish in Investigaciones Economicas.

Gregory, A.W. and Smith, G.W. (1991): "Calibration as Testing: Inference in Simulated Macroeconomic Models", Journal of Business and Economic Statistics, July.

Grossman, S. and R. Shiller; (1981); "The Determinants of the Variability of Stock Market Prices", American Economic Review, vol. 71, pp 222-227.

Hahn, F. (1971); "Professor Friedman's Views on Money", Economica, pp 61-80.

Hansen, G. (1985); "Indivisible Labor and the Business Cycle" Journal of Monetary Economics, vol. 16, pp 309-328.

Hansen, G. and Imrohoroglu, A. (1992); "The Role of Unemployment Insurance in an Economy with Liquidity Constraints and Moral Hazard", Journal of Political Economy, February.

Hansen, L.P. (1987); "Calculating Asset Prices in Three Example Economies" Advances in Econometrics, Fifth World Congress, Cambridge University Press, vol 1, chapter 6.

Hansen, L.P. and Sargent, T.J. (1990) Recursive Linear Models of Dynamic Economies manuscript, Hoover Institution.

Hansen, L.P. and Singleton, K. (1982) "Generalized Instrumental Variables Estimation of Non-linear Rational Expectations Models" Econometrica 50, 1269-1286.

Hayashi, F. (1987); "Tests for Liquidity Constraints: a Critical Survey and Some New Observations" Advances in Econometrics, Fifth World Congress, Cambridge University Press, vol 2, chapter 13.

Heaton, J. (1990); "The Interaction between Time Non-separable Preferences and Time Aggregation" working paper, Massachussetts Institute of Technology.

Ingram, B. (1990); "Equilibrium Modelling of Asset Prices: Rationality Versus Rules of Thumb", Journal of Business and Economic Statistics, January

Ingram, B. (1990); "Solving the Stochastic Growth Model by Backsolving with an Expanded State Space" Journal of Business and Economics Statistics, January

Jones, L.E., R.E. Manuelli and P.E. Rossi (1990); "Optimal Taxation in Convex Models of Equilibrium Growth", working paper, Northwestern University.

Jones, L.E. and R.E. Manuelli (1989); "Notes on a Model of Optimal Equilibrium Growth" working paper, Northwestern University.

Judd, K. (1989); "Minimum Weighted Residual Methods in Dynamic Economic Models" working paper, Hoover Institution.

Judd, K. (1989); "Cournot versus Bertrand: a Dynamic Resolution", working paper, Hoover Institution.

Ketterer and Marcet (1989); "Introducing Derivative Securities: A General Equilibrium Approach". Working Paper, Carnegie Mellon University.

Kydland, F. E. and E. C. Prescott (1982); "Time to Build and Aggregate Fluctuations", Econometrica, vol. 50, 1345-1370.

Labadie, P. (1990); "Solving the Stochastic Growth Model by a Recursive Mapping Based on Least Squares Projection" Journal of Business and Economics Statistics, January

Lee, B.S. and B.F. Ingram (1991); "Estimation by Simulation of Time Series Models" Journal of Econometrics, 47: 197-207.

Lucas, R.E. (1972); "Expectations and the Neutrality of Money" Journal of Economic Theory.

Lucas, R. E. (1978); "Asset Prices in an Exchange Economy", Econometrica, vol. 46, 1426-1446.

Lucas, D. (1990); "Asset Pricing with Undiversifiable Income Risk and Short Sales Constraints: Deepening the Equity Premium Puzzle" working paper, Northwestern University.

Marcet, A. and T. J. Sargent (1989a); "Convergence of Least Squares Learning Mechanisms in Self-Referential Linear Stochastic Models", Journal of Economic Theory, August.

Marcet, A. and T. J. Sargent (1989b); "Convergence of Least Squares Learning in Environments with Hidden State Variables and Private Information", Journal of Political Economy , December.

Marcet, A. (1989); "Solving Non-Linear Dynamic Models by Parameterizing Expectations" working paper, Carnegie Mellon University.

Marcet, A. and R. Marimon (1992); "Communication, Commitment and Growth", Journal of Economic Theory, December.

Marcet, A. and K. Singleton (1989); "Equilibrium Asset Prices and Savings of Heterogeneous Agents in the Presence of Portfolio Constraints", working paper, Carnegie Mellon University.

Marimon, R., McGratten, E. and Sargent, T.J. (1990); "Money as a Medium of Exchange in an Economy with Artificially Intelligent Agents", Journal of Economic Dynamics and Control, 14: 329-374.

Marshall, D. (1988); "Inflation and Asset Returns in a Monetary Economy with Transaction Costs", Working paper, Northwestern University. Forthcoming in the Journal of Finance.

McFadden, D. (1989); "A Method of Simulated Moments for Estimation of Discrete Response Models without Numerical Integration", Econometrica, 57: 995-1026

McGratten, E. (1989) "The Macroeconomic Effects of Tax Policy in an Equilibrium Model", working paper, Duke University.

McGratten, E. (1990); "Solving the Stochastic Growth Model by Linear-Quadratic Approximation" Journal of Business and Economics Statistics, January

Mehra, R. and E. C. Prescott (1985); "The Equity Premium: A Puzzle", Journal of Monetary Economics.

Novales, A. (1990):"A Stochastic Monetary Equilibrium Model of the Interest Rate" Econometrica

Otker, I. (1990); "The Welfare Implications of Fiscal Policy with Distortionary Taxation: A Numerical Analysis" working paper, Carnegie Mellon University.

Pakes, A. (1986); "Patents as Options: Some Estimates of the Value of Holding European Patent Stocks", Econometrica, vol. 54, 755-784.784.

Pakes, A. and Rust J. (1991); "Estimation of Dynamic Structural Models, Problems and Prospects" this volume.

Pakes, A. and D. Pollard (1989); "Simulation and the Asymptotics of Optimizations Estimators" Econometrica, 57-5: 1027-1057.

Phalen, C. (1990); "Incentives, Insurance and the Variability of Consumption and Leisure" working paper, University of Wisconsin.

Phalen, C. and Townsend, R. (1991); "Computing the Optimal Insurance-Incentive Tradeoff for Multi-Period Information Constrained Economies" Review of Economic Studies, October.

Prescott, E.C. (1986); "Theory Ahead of Business Cycle Measurement", Federal Reserve Bank of Minneapolis Quarterly Review, 10: 9-22.

Rios, J.V. (1990); "Life-Cycle Economies and Aggregate Fluctuations" working paper, Carnegie Mellon University.

Rogerson, R. (1988); "Indivisible Labor, Lotteries and Equilibrium", Journal of Monetary Economics, 21: 3-17.

Rotemberg, J. and Woodford, M. (1992); "Oligopolistic Pricing and the Effects of Military Purchases on Economic Activity" Journal of Political Economy, December.

Rowenhorst, K.G. (1989); "Asset Returns and Business Cycles: A General Equilibrium Approach". University of Rochester working paper.

Rust, J. (1988); "Maximum Likelihood Estimation of Discrete Control Processes", SIAM Journal of Control and Optimization, 26: 1006-1023.

Scheinkman, J. and L. Weiss (1986); "Borrowing Constraints and Aggregate Economic Activity" Econometrica, 54: 23-

Sims, C. A. (1985); "Solving Non-Linear Stochastic Equilibrium Models 'Backwards'", manuscript, University of Minnesota.

Sims, C. A. (1990); "Solving the Stochastic Growth Model by Backsolving with a Particular Non-linear Form for the Decision Rule" Journal of Business and Economics Statistics, January.

Smith, T. (1989); "Solving Non-linear Rational Expectations Models: A New Approach" working paper, Queen's University.

Tauchen, G. (1990); "Solving the Stochastic Growth Model by Using Quadrature Methods and Value-Function Iterations" Journal of Business and Economics Statistics, January.

Taylor, J. and H. Uhlig, (1990); "Solving Nonlinear Stochastic Growth Models: A Comparison of Alternative Solution Methods" Journal of Business Economics and Statistics, January.

---

[1] See, for example, Marcet and Sargent [1989a,1989b] and references therein.

[2] See Hansen and Sargent [1990] for a thorough review of applications of the linear quadratic framework.

[3] Strictly speaking, we have to add some constraints to the maximization problem in order to avoid Ponzi schemes, where the agent sells arbitrarily large amounts of shares. Lucas introduces a constraint of the type $s_t \geq K$ for all t, where K is a sufficiently low number for this constraint to never be binding.

[4] Some of these examples can be found, for example, in Sargent [1989].

[5] Even models with inequality constraints take the form of (3.4), when the Kuhn & Tucker conditions are one of the equations in (3.4). See Marcet and Singleton [1990].

[6] For example, their law of motion for prices may not hold exactly at the non-stochastic steady-state, unlike in most LQ applications.

[7] For example, by adding a second productive sector or by introducing public capital as in Garcia-Mila [1989].

[8]Obviously, the Monte-Carlo approach may give slightly different solutions for different draws of the random number generator. This is just a numerical error that has to be studied by the researcher in order to ensure that it is not marring the conclusions from his simulations. This is not more of a problem in Monte-Carlo methods than it is in any numerical solution. For example, iterations on the Euler equations will depend on the grid imposed, and sensitivity to the choice of the grid should be explored.

[9]See Geweke [1989] for some results on the increase of observations needed.

[10]This point can be found in Judd [1990].

[11]In which case he should probably feel uncomfortable with analytic solutions like that of Example 2.1, where the parameter $cov(d_t, d_{t-1})$ is arbitrarily set to zero.

[12]Strictly speaking, the above growth model could not be estimated by maximum likelihood if we had data on consumption and investment because, having only one unobserved shock in the model there exists a stochastic singularity.

[13]Discontinuous simulations arise whenever the model contains some discrete decision: for example, in the case of Pakes [1986], firms have to decide whether or not to renew a patent. Then, by changing slightly the parameters λ it may happen that, in a given period, the firm changes its decision on whether to renew or not, causing a discrete jump in the simulated series.

[14]Gregory and Smith [1991] have applied similar ideas to the ones presented here to evaluate the risk premium puzzle.

[15]The missing reference is Geweke, J. [1989]: "Modeling with Normal Polynomial Expansions" {\em Economic Complexity; Chaos, Sunspots, Bubbles and Nonlinearity}, ed. by W.A. Barnett, J. Geweke and K. Shell, International Symposia in Economic Theory and Econometrics, Cambridge University Press.

[16]This is true, at least, of the papers that I could access. The Gustafson paper (reference is missing from Judd's list) is a working paper from the Department of Agriculture that I could not obtain

[17]I did miss one application of perturbation methods with higher order terms that. By the time of the World Congress, there had only one application, in the unpublished manuscript Judd [1985a].

RECENT WORKING PAPERS

1. Albert Marcet and Ramon Marimon
Communication, Commitment and Growth. (June 1991)

2. Antoni Bosch
Economies of Scale, Location, Age and Sex Discrimination in Household Demand.(June 1991)

3. Albert Satorra
Asymptotic Robust Inferences in the Analysis of Mean and Covariance Structures. (June 1991)

4. Javier Andrés and Jaume Garcia
Wage Determination in the Spanish Industry. (June 1991)

5. Albert Marcet
Solving Non-Linear Stochastic Models by Parameterizing Expectations: An Application to Asset Pricing with Production. (July 1991)

6. Albert Marcet
Simulation Analysis of Dynamic Stochastic Models: Applications to Theory and Estimation. (November 1991)

7. Xavier Calsamiglia and Alan Kirman
A Unique Informationally Efficient and Decentralized Mechanism with Fair Outcomes. (November 1991)

8. Albert Satorra
The Variance Matrix of Sample Second-order Moments in Multivariate Linear Relations. (January 1992)

9. Teresa Garcia-Milà and Therese J. McGuire
Industrial Mix as a Factor in the Growth and Variability of States'Economies. (Jannuary 1992)

10. Walter Garcia-Fontes and Hugo Hopenhayn
Entry Restrictions and the Determination of Quality. (February 1992)

11. Guillem López and Adam Robert Wagstaff
Indicadores de Eficiencia en el Sector Hopitalario. (March 1992)

12. Daniel Serra and Charles ReVelle
The PQ-Median Problem: Location and Districting of Hierarchical Facilities. Part I (April 1992)

13. Daniel Serra and Charles ReVelle
The PQ-Median Problem: Location and Districting of Hierarchical Facilities. Part II: Heuristic Solution Methods. (April 1992)

27. Xavier Freixas
Short Term Credit Versus Account Receivable Financing. (March 1993)

28. Massimo Motta and George Norman
Does Economic Integration cause Foreign Direct Investment?
(March 1993)

29. Jeffrey Prisbrey
An Experimental Analysis of Two-Person Reciprocity Games.
(February 1993)

30. Hugo A. Hopenhayn and Maria E. Muniagurria
Policy Variability and Economic Growth. (February 1993)

31. Eva Ventura Colera
A Note on Measurement Error and Euler Equations: an Alternative to Log-Linear Approximations. (March 1993)

32. Rafael Crespí i Cladera
Protecciones Anti-Opa y Concentración de la Propiedad: el Poder de Voto.
(March 1993)

33. Hugo A. Hopenhayn
The Shakeout. (April 1993)