Ratio Maps and Correspondence Analysis

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Summary. We compare two methods for visualising contingency tables and develop a method called the ratio map which combines the good properties of both. The first is a biplot based on the logratio approach to compositional data analysis. This approach is founded on the principle of subcompositional coherence, which assures that results are invariant to considering subsets of the composition. The second approach, correspondence analysis, is based on the chi-square approach to contingency table analysis. A cornerstone of correspondence analysis is the principle of distributional equivalence, which assures invariance in the results when rows or columns with identical conditional proportions are merged. Both methods may be described as singular value decompositions of appropriately transformed matrices. Correspondence analysis includes a weighting of the rows and columns proportional to the margins of the table, If this idea of row and column weights is introduced into the logratio biplot, we obtain a method which obeys both principles of subcompositional coherence and distributional equivalence.

Keywords: Biplot; compositional data; contingency tables; correspondence analysis; distributional equivalence; logratio transformation; singular value decomposition; spectral map.

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1 Introduction

This article considers different ways of visualizing contingency tables in the form of a map, where the rows and columns of the table are depicted as points in a low-dimensional Euclidean space, usually a two-dimensional plane. As a special case we shall look at tables of compositional data, that is positive data with row sums (or column sums) equal to a constant, usually 1 if the data are proportions or 100 if they are percentages. A method called the *ratio map* is introduced, which can be considered as a type of fusion of ideas emanating from compositional data analysis and correspondence analysis.

Correspondence analysis (Benzécri, 1973; Greenacre, 1984, 1993) has become a popular method for graphically displaying tables of nonnegative data, applicable primarily to contingency tables. The method, popular in the social and environmental sciences, has several equivalent definitions. One definition, in a nutshell, is the following (see, for example, Greenacre 1993). First, transform the rows of the table into *profiles*, that is the rows divided by their row totals. Second, assign weights to the row profiles proportional to the marginal row totals of the contingency table (these weights which sum to 1 are called "masses" in correspondence analysis). Third, perform a standardization of the profile elements by dividing them by values proportional to the square root of the marginal column totals of the contingency table. The third step implies a special distance function between the profiles, called the *chi-squared distance*. Finally, perform a weighted principal component analysis on the row profiles, identifying the plane, for example, which best fits the row profiles by minimizing the weighted sum of squared (chi-squared) distances from the points to the plane. Then project the profile points onto this plane and interpret their relative positions. An identical and completely symmetric analysis can be performed of the column profiles, and the two analyses are equivalent in that their solutions are based on the singular value decomposition (SVD) of the same matrix (Greenacre, 1984).

As emphasised often by Benzécri, who originally developed correspondence analysis as a method for exploring large frequency tables in linguistics, one of the founding principles of the method is the *principle of distributional equivalence*: "Our first principle is that of distributional equivalence" (Benzécri, 1973, vol.I, p. 23). This principle can be stated simply as follows: if two rows (or two columns) have the same relative values, that is they have the same profile, then merging them does not affect the results in any way. As an illustration of this principle, suppose that words have been counted in a sample of texts, including the two articles "the" and "a", and that the frequencies are collected in a texts$\times$words table. Suppose that it turns out that in each text, the relative occurrence of these two articles is identical, for example "a" always occurs 25% of the times "the" occurs. This means that the (column) profile of "a" is identical to that of "the". The principle of distributional equivalence states that it should make no difference to the analysis if we merge two such columns with identical profiles, adding together the frequencies to obtain one column, which could be labelled "articles" and where we make no distinction between its two components.

Geometrically, two identical profiles are points lying at identical positions and the result of the merger is a single point with mass equal to the sum of the masses. Trivially, it is clear that all distances between column profiles are unaffected by this merger, since the row margins are unaffected by the merger and thus all interpoint column distances stay the same. Less trivially, however, the chi-squared distances between all text (row) points is also unaffected, thus assuring distributional equivalence (for a proof, see Greenacre, 1984, section 4.1.17). The principle of distributional equivalence similarly guarantees invariance of all results if a row (or column) were split into parts in constant proportions. For example, if one column is split into three columns in fixed proportions 70:20:10, the interpoint row distances remain invariant, as well as the correspondence analysis solution.

Compositional data analysis (Aitchison, 1986) is concerned with data vectors of non-negative values summing to one. This methodology has become popular in the physical sciences, especially geology and chemistry, rather than the social sciences. For example, chemical samples are typically analyzed into constituent components by weight, or volume, expressed as proportions of the total sample. One of the founding principles of compositional data analysis is that of *subcompositional coherence*. Suppose that a chemical sample has inorganic and organic components, and that scientist A is investigating all of these components, whereas scientist B is investigating just the organic components of the same samples, that is B's data are the organic components expressed as proportions of total organic material. Subcompositional coherence means that statistical analysis by scientist B on the subcomposition of organic components should be the same as that of scientist A, unaffected by the fact that B is looking at a reduced data set. This principle has led to the study of ratios of the components, which are clearly unaffected by looking at subcompositions.

Aitchison (1986) defined a variant of principal component analysis for compositional data, based on logarithmically transforming the component ratios, called *logratios*. Later Aitchison (1990) introduced the biplot associated with this approach, calling it the "relative variation biplot", displaying both the samples (usually rows) and the components (columns) in a joint map. This biplot has several interesting properties, summarized by Aitchison and Greenacre (2001), who show that it is equivalent to analyze all the pairwise logratios or to analyze the logarithms of the components for each sample relative to their geometric mean. Computationally, the relative variation biplot is derived directly from the SVD of the components which have been first logarithmically transformed and then double-centred with respect to row and column means. This methodology can be applied in exactly the same way to crosstabulations and other tables of counts. But although the relative variation biplot has subcompositional coherence, it does not have distributional equivalence. This is unfortunate for compositional data analysis, because if two components were always occurring in the same proportion in every sample, then the analysis should be unaffected by considering these two components taken as one. Or, putting this in a different way in another context, suppose we were measuring the proportion of species in a biological sample, and later decided to distinguish between males and females of each species. Then if the male-to-female ratio were actually constant within each species across all samples, there should be no change at all to our analysis whether we distinguished between male and female or not, since no new information is introduced at all apart from the constant sex ratio.

So we have at our disposal two methods, justifiable in their own contexts, but which do not have the basic properties of the other: the logratio biplot in compositional data analysis has subcompositional coherence but not distributional equivalence, whereas correspondence analysis has the latter but not the former. We will show, however, that the simple introduction of the correspondence analysis concept of row and column weighting into the logratio biplot leads to a method of visualization that has both subcompositional coherence and distributional equivalence. This method, which we call the *ratio map*, can be used to analyze contingency tables as well as compositional data. As far as analyzing positive compositional data is concerned, the ratio map is a significant improvement over existing methods. As far as analyzing contingency tables is concerned, the ratio map forms an interesting alternative correspondence analysis, and could enjoy much wider use outside the natural sciences. But it does have a few disadvantages, for example zero frequencies are problematic since the data are log-transformed, and zero frequencies occur frequently in the social and environmental sciences.

In section 2 the ratio map is defined in the context of contingency table analysis. In section 3 the map's properties are listed and illustrated in the context of an application. Section 4 deals with the special case of compositional data and Section 5 closes with a comparison with correspondence analysis.

2 The ratio map

Suppose that $\mathbf{N} = \{n_{ij}\}$ denotes an $I \times J$ contingency table, with row totals, column totals and grand total denoted by n_{i+}, n_{+j} and n_{++} respectively. Let $r_i = n_{i+}/n_{++}$ and $c_j = n_{+j}/n_{++}$ be the respective row and column *masses*. Let $\mathbf{r}$ be the vector of row masses, $\mathbf{c}$ the vector of column masses and $\mathbf{D}_r$ and $\mathbf{D}_c$ the corresponding diagonal matrices. Denote by $\mathbf{L}$ the matrix of logarithms of the frequencies, $\ell_{ij} = \log(n_{ij})$. Aitchison's relative variation biplot consists of double-centring the matrix $\mathbf{L}$ with respect to simple arithmetic averages of the rows and columns, followed by a SVD to obtain least-squares matrix approximations. In the ratio map the row and column masses are introduced into the double-centring stage, so that centring is with respect to weighted averages, as well as into the matrix approximation stage, so that fitting is by weighted least squares. This simple modification of the algorithm, giving differential importances to the rows and columns in the centring and fitting, will be shown to bestow on the method the principle of distributional equivalence.

The computational steps to find the coordinates of the rows and columns in the ratio map are as follows:

Step 1. Double-centre the matrix $\mathbf{L}$ with respect to its weighted row and column averages, the order of centring being invariant. That is, calculate the weighted averages of the rows of $\mathbf{L}$, using the column masses to weight each column element: $\ell_{i\cdot} = \sum_j c_j \ell_{ij}$ ($i = 1, \ldots, I$), and then subtract these averages from all the elements in the corresponding row. Then centre the resultant matrix, with general element $\ell_{ij} - \ell_{i\cdot}$, with respect to weighted averages of the columns, using the row masses to weight each element: $\sum_i r_i(\ell_{ij} - \ell_{i\cdot})$ ($j = 1, \ldots, J$), and then subtract these averages from all the elements in the corresponding columns. The result of this operation is a double-centred matrix with elements $z_{ij} = \ell_{ij} - \ell_{i\cdot} - \ell_{\cdot j} + \ell_{\cdot\cdot}$, where the dot subscript indicates weighted averaging over the corresponding subscript. In matrix notation, this double-centring can be written as:

$$\mathbf{Z} = (\mathbf{I} - \mathbf{1}\mathbf{r}^{\mathsf{T}})\mathbf{L}(\mathbf{I} - \mathbf{c}\mathbf{1}^{\mathsf{T}})$$

Step 2. Multiply z_{ij} by $(r_i c_j)^{1/2}$, that is multiply the rows and columns by the square root of their respective masses:

$$\mathbf{S} = \mathbf{D}_r^{1/2}\mathbf{Z}\mathbf{D}_c^{1/2}$$

Step 3. Perform the SVD of this transformed matrix:

$$\mathbf{S} = \mathbf{U}\Gamma\mathbf{V}^{\mathsf{T}} \qquad \text{where } \mathbf{U}^{\mathsf{T}}\mathbf{U} = \mathbf{V}^{\mathsf{T}}\mathbf{V} = \mathbf{I}$$

and singular values in descending order: $\gamma_1 \geq \gamma_2 \geq \cdots > 0$.

Step 4. Divide the rows of the matrix of left singular vectors by $r_i^{1/2}$, and divide the rows of the matrix of right singular vectors by $c_j^{1/2}$:

$$\widetilde{\mathbf{U}} = \mathbf{D}_r^{-1/2}\mathbf{U} \qquad \widetilde{\mathbf{V}} = \mathbf{D}_c^{-1/2}\mathbf{V}$$

Step 5. The rows of the matrices $\widetilde{\mathbf{U}}$ and $\widetilde{\mathbf{V}}$ are the standard coordinates of the rows and columns respectively, while the same coordinates scaled by the corresponding singular values define the principal coordinates (Greenacre, 1984, 1993):

	Principal coordinates	*Standard coordinates*
Rows :	$\mathbf{F} = \widetilde{\mathbf{U}}\Gamma$	$\widetilde{\mathbf{U}}$
Columns :	$\mathbf{G} = \widetilde{\mathbf{V}}\Gamma$	$\widetilde{\mathbf{V}}$

As in all methods of this type, we can choose to represent either of two so-called *asymmetric maps*, using either $\mathbf{F}$ and $\widetilde{\mathbf{V}}$ for the asymmetric map which is "row-principal" or "row-metric-preserving", or $\widetilde{\mathbf{U}}$ and $\mathbf{G}$ for the asymmetric map which is "column-principal" or "column-metric-preserving"; or, alternatively, the *symmetric map* using $\mathbf{F}$ and $\mathbf{G}$ where both rows and columns are in principal coordinates. The asymmetric maps are biplots in the strict sense (Gabriel, 1971), but not the symmetric map (see, for example, Greenacre (1993)). Sometimes we use another symmetric solution which is a biplot, which we refer to as the *symmetric biplot*, with row coordinates $\tilde{\mathbf{U}}\Gamma^{1/2}$ and column coordinates $\tilde{\mathbf{V}}\Gamma^{1/2}$. The symmetric biplot, however, favours neither the rows nor the columns in the strict sense of preserving the metric between rows or between columns.

Steps 2 to 4 are what Greenacre (1984) has called the "generalized singular value decomposition", with row and column weights given by the row and column masses. These steps are equivalent to the following single step in which the singular vectors are constrained to have a weighted normalization.

Steps 2–4. Perform the generalized SVD of $\mathbf{Z}$:

$$\mathbf{Z} = \widetilde{\mathbf{U}}\Gamma\widetilde{\mathbf{V}}^{\mathsf{T}} \qquad \text{where } \widetilde{\mathbf{U}}^{\mathsf{T}}\mathbf{D}_r\widetilde{\mathbf{U}} = \widetilde{\mathbf{V}}^{\mathsf{T}}\mathbf{D}_c\widetilde{\mathbf{V}} = \mathbf{I}$$

In this weighted version of the SVD, low-rank approximations of the matrix $\mathbf{Z}$ are weighted least-squares approximations, where the rows and columns are weighted by their corresponding row masses. This is exactly what is done in correspondence analysis, where the matrix being approximated by weighted least squares has elements $(n_{ij}-n_{i+}n_{+j})/(n_{i+}n_{+j})$. This is the only algorithmic difference between correspondence analysis and the ratio map. In Section 5 we shall comment in more detail on this relationship with correspondence analysis.

3 Application and properties of the ratio map

Consider the contingency table in Table 1, the frequencies of eight occupational categories in each of the 41 Catalan counties (*comarcas*). The table appears in Vives and Villarroya (1996) and the original source of the data is the *Institut d'Estadística de Catalunya*. This is an interesting table for our present purpose since it has rows and columns of widely differing totals, so the effect of weighting will be of relevance.

Insert Table 1 about here

The asymmetric ratio map favouring the display of the rows is given in Figure 1. In the terminology of Aitchison and Greenacre (2001) this type of asymmetric map is also called the form biplot.

Insert Figure 1 about here

We now list all the properties of a ratio map, using this example as an illustration. Vectors drawn from the origin of the display to a point are called *rays*, and vectors joining two row points or two column points are called *links*.

Property 1. The row points and column points are both centred in terms of weighted averages at the origin. This is a direct consequence of the weighted double-centring transformation of the matrix. Thus the weighted average row point in the display is at the origin and the weighted average column point as well. For example, in Figure 1 the origin is clearly not at the ordinary average row point, but well to the right because of the large mass of the point Bn (Barcelona).

Property 2. The ratio map, based on the SVD of a double-centred matrix, optimally represents the all inter-row differences and inter-column differences. This result has been shown for the unweighted case by Aitchison and Greenacre (2001, Appendix 1), who point out that it is really these differences which are of interest, and that the computational algorithm using the centred logratios is just a short cut to the analysis of all differences. For example, in Figure 1 the rays (1) and (3) indicate the directions of the biplot axes for the columns "Professional/Technical" and "Services/Administration" respectively. If we are interested in the ratio between these two categories, then we simply look at the direction of the link connecting (1) and (3), which is practically vertical. From this we can deduce that PS (Pallars Sobirà) has one of the highest values of this ratio (from Table 1 it is 280/200=1.400) and BL (Baix Llobregat) one of the lowest (12371/31296=0.395). All the ratios between pairs of categories will be optimally displayed in this way.

Another way of thinking of this which is particularly useful in the case of contingency tables is to consider the matrix $\mathbf{Y}$ with $\frac{1}{2}n(n-1)$ rows and $\frac{1}{2}p(p-1)$ columns, having general element

$$y_{ii',jj'} = \log\left(\frac{n_{ij}n_{i'j'}}{n_{ij'}n_{i'j}}\right)$$

that is, the log-transformed odds ratio based on the four elements in rows i, i' and columns j, j'. If we assign weights $r_i r_{i'}$ to the rows and $c_j c_{j'}$ to the columns, and perform a weighted SVD as before, this is equivalent to performing a singular value decomposition of the smaller $I \times J$ matrix $\mathbf{Z}$ of double-centred log-frequencies, as in the ratio map. The total sum of squares is identical, the singular values are identical and the map coordinates of the rows or columns of $\mathbf{Y}$ may be obtained from the differences in the corresponding coordinates of the respective row or column pairs in the ratio map. Thus the ratio map is optimally displaying all the odds ratios that can be calculated on the contingency table and a particular odds ratio can be estimated by considering the two links connecting the pair of rows and pair of columns.

In other words, in the ratio map not only are the points themselves optimally displayed but also all the links are optimal representations of the true links in higher-dimensional space. The proof of this result is very similar to that given by Aitchison and Greenacre (2001), with the variation of including the weights in the process of fitting.

Property 3. Distances between row points and between column points in principal coordinates are approximations of *weighted Aitchison distances* between rows and between columns. These distances are defined in terms of the logarithms of ratios between data values. Consider, for example, the distances between row points i and i', corresponding to rows $[\, n_{i1}\, n_{i2}\, \ldots\, n_{ip} \,]$ and $[\, n_{i'1}\, n_{i'2}\, \ldots\, n_{i'p} \,]$ of the data matrix. Each row of J elements can be re-expressed as the set of $\frac{1}{2}J(J-1)$ ratios between all pairs of elements, for example n_{i1}/n_{i2}, n_{i1}/n_{i3}, n_{i2}/n_{i3}, ... and so on, that is the ratios $n_{ij}/n_{ij'}$ for $j < j'$. This vector of ratios describes the corresponding row, and since these ratios are considered to be on a multiplicative scale they are logarithmically transformed to logratios $\tau_{i,jj'} = \log(n_{ij}/n_{ij'})$. If the columns are not differentially weighted, the Aitchison distance between two rows is proportional to the Euclidean distance between the vectors of logratios $\boldsymbol{\tau_i} = [\tau_{i,jj'}]$ and $\boldsymbol{\tau_{i'}} = [\tau_{i',jj'}]$. It is convenient here to define the Aitchison distance as:

$$
\begin{aligned}
d_{ii'}^2 &= \frac{1}{p^2} \sum\sum\nolimits_{j<j'} \left(\log \frac{n_{ij}}{n_{ij'}} - \log \frac{n_{i'j}}{n_{i'j'}} \right)^2 \\
&= \frac{1}{p^2} \, (\boldsymbol{\tau_i} - \boldsymbol{\tau_{i'}})^{\mathsf{T}} (\boldsymbol{\tau_i} - \boldsymbol{\tau_{i'}})
\end{aligned}
$$

so that each ratio term is weighted by the product $\frac{1}{p} \times \frac{1}{p}$ of constant weights for each of the p columns (notice that the difference in the logratios is just the log-odds ratio $y_{ii',jj'}$ defined previously in Property 2). The introduction of the differential masses c_j for the columns leads to the weighted Aitchison distance between rows:

$$
\begin{aligned}
\tilde{d}_{ii'}^2 &= \sum\sum\nolimits_{j<j'} c_j c_{j'} \left(\log \frac{n_{ij}}{n_{ij'}} - \log \frac{n_{i'j}}{n_{i'j'}} \right)^2 \\
&= (\boldsymbol{\tau_i} - \boldsymbol{\tau_{i'}})^{\mathsf{T}} \mathbf{D}_{cc} (\boldsymbol{\tau_i} - \boldsymbol{\tau_{i'}}) \qquad (1)
\end{aligned}
$$

where $\mathbf{D}_{cc}$ is the diagonal weighting matrix of products c_1c_2, c_1c_3, c_2c_3, ..., $c_jc_{j'}$, ... $(j < j')$. Thus the (jj')-th logratio term is weighted by the product $c_jc_{j'}$ of the masses.

In the case of the unweighted Aitchison distance it is possible to show that the distance may be expressed more parsimoniously in terms of the so-called *centred logratios*, where centring and weighting of each term is by the constant mass $\frac{1}{p}$:

$$
d_{ii'}^2 = \frac{1}{p} \sum_j \left(\log \frac{n_{ij}}{g(\mathbf{n}_i)} - \log \frac{n_{i'j}}{g(\mathbf{n}_{i'})} \right)^2
$$

where $g(\mathbf{n}_i) = (n_{i1}n_{i2} \cdots n_{ip})^{1/p}$ is the geometric mean of the i-th row of data. In the same way, we can show that the weighted Aitchison distance can be expressed in terms of centred logratios with respect to a weighted mean:

$$\tilde{d}^2_{ii'} = \sum_j c_j \left(\log \frac{n_{ij}}{\tilde{g}(\mathbf{n}_i)} - \log \frac{n_{i'j}}{\tilde{g}(\mathbf{n}_{i'})} \right)^2$$

where $\tilde{g}(\mathbf{n}_i) = n_{i1}^{c_1} n_{i2}^{c_2} \cdots n_{ip}^{c_p}$ is the weighted geometric mean of the i-th row.

The above description applies in a completely symmetric way to distances between columns in terms of pairwise or centred logratios defined down columns. The matrix can be simply transposed and all the above results apply in an identical fashion.

Zero distance between a pair of rows (or between a pair of columns) means that all ratios are equal, that is the rows (or columns) have the same relative values, or *profile*: $n_{ij}/n_{i+} = n_{i'j}/n_{i'+}$. Thus if the link between rows i and i' is short in the display, and assuming that the display is an accurate representation of the data, this indicates that the logratios are approximately the same for all pairs (j, j'): $\tau_{i,jj'} = \log(n_{ij}/n_{ij'}) \approx \tau_{i',jj'} = \log(n_{i'j}/n_{i'j'})$. This is equivalent to saying $\log(n_{ij}/n_{i'j}) \approx \log(n_{ij'}/n_{i'j'})$, where the logratios are now calculated between row elements of the same column, and it can be shown that when the rows are displayed in principal coordinates, the distance from row i to row i' approximates the standard deviation of the logratios $\log(n_{ij}/n_{i'j})$ across the J columns. Similarly, the distance between columns j and j' in principal coordinates is an approximation of the standard deviation of the logratios $\log(n_{ij}/n_{ij'})$ across the I rows, where a small distance again indicates similar column profiles or compositions.

For example, in Figure 1 the row points No (Noguera) and TA (Terra Alta) are close together, which can be interpreted in two equivalent ways. First, thinking row-wise, all the 28 ratios between pairs of professional categories in Noguera are similar to their counterparts in Terra Alta. Second, thinking column-wise, the 8 ratios between these counties for the 8 professional categories are relatively constant, that is their standard deviation is low. Both interpretations indicate that these two counties have similar profiles, or compositions.

Property 4. The ratio map obeys the principle of distributional equivalence. Suppose two columns j and j' have the same profile, that is the ratios $n_{ij}/n_{ij'}$ are identical for all i. Without loss of generality we can assume that $j = 1$ and $j' = 2$, and that these ratios are equal to a constant K, say, so that $n_{i1} = K n_{i2}$. The ratio c_1/c_2 of column masses is also equal to K, so that $c_1 = K c_2$. Let us amalgamate these two columns into one column with values equal to $n_{i1} + n_{i2} = (1+K)n_{i2}$ $(i = 1,\ldots,n)$, and mass equal to $c_1 + c_2 = (1+K)c_2$.

Clearly, the weighted Aitchison distances between columns are unaffected by this amalgamation, since the row masses are unaffected by the merger. As far as the row distances are concerned, all terms with logratios not involving the first two columns are unaffected by the merger, so we just need to compare the terms involving columns 1 and 2. Before the merger the first term of the squared distance in (1) is equal to 0 since the ratios are equal and have zero difference. The other terms involving columns 1 and 2 can be written as

$$\begin{aligned}
& \sum_{j'=3}^{p} c_1 c_{j'} \left(\log \frac{n_{i1}}{n_{ij'}} - \log \frac{n_{i'1}}{n_{i'j'}} \right)^2 + \sum_{j'=3}^{p} c_2 c_{j'} \left(\log \frac{n_{i2}}{n_{ij'}} - \log \frac{n_{i'2}}{n_{i'j'}} \right)^2 \\
= & \sum_{j'=3}^{p} K c_2 c_{j'} \left(\log \frac{K n_{i2}}{n_{ij'}} - \log \frac{K n_{i'2}}{n_{i'j'}} \right)^2 + \sum_{j'=3}^{p} c_2 c_{j'} \left(\log \frac{n_{i2}}{n_{ij'}} - \log \frac{n_{i'2}}{n_{i'j'}} \right)^2 \\
= & \sum_{j'=3}^{p} (1+K) c_2 c_{j'} \left(\log \frac{n_{i2}}{n_{ij'}} - \log \frac{n_{i'2}}{n_{i'j'}} \right)^2
\end{aligned}$$

since the factor K disappears in the subtraction of the logratios. After the merger, there is no column 1, only a column 2 formed by the amalgamation of the previous first two columns, and the terms in the distance function corresponding to this new column are

$$\begin{aligned}
& \sum_{j'=3}^{p} (1+K) c_2 c_{j'} \left(\log \frac{(1+K) n_{i2}}{n_{ij'}} - \log \frac{(1+K) n_{i'2}}{n_{i'j'}} \right)^2 \\
= & \sum_{j'=3}^{p} (1+K) c_2 c_{j'} \left(\log \frac{n_{i2}}{n_{ij'}} - \log \frac{n_{i'2}}{n_{i'j'}} \right)^2
\end{aligned}$$

where the factor $(1+K)$ disappears from the logratio differences for the same reason, giving the same result obtained before the merger. Hence the distances between rows are unaffected by the amalgamation of these columns and the principle of distributional equivalence is satisfied.

Property 5. Just as in the unweighted logratio biplot, row or column points lying in a straight line reveal logratios of high correlation. Thus the collinearity of column rays (1) and (7), but pointing in opposite directions indicates a high negative correlation between professional categories "Professional/technical" and "Industry". So-called logcontrast models summarizing the interdependency between collinear points can be diagnosed from the relative lengths of the links between the points. In addition, four points which form a parallelogram also indicate a constant logcontrast model, since all the links can be transferred to the origin. Aitchison and Greenacre (2001) give more details about model diagnosis and an application.

Property 6. In an asymmetric map, which is a biplot, if a subset I of the individuals (rows) and a subset J of the components columns lie approximately on respective straight lines that are orthogonal, then the compositional submatrix formed by the rows I and columns J has approximately constant logratios amongst the components, that is the double-centred submatrix of log(compositions) has near-zero entries. This property of logratio constancy in submatrices of the data can be deduced directly from the concept of biplot calibration, also explained in detail and illustrated by Aitchison and Greenacre (2001). The rays or links in either biplot can be calibrated on a linear scale in logratio units or on a logarithmic scale in ratio units. Thus any points lying on a line perpendicular to a link will have constant estimated values of the corresponding ratios.

Property 7. The data matrix can be reconstructed approximately from either biplot, but we need to know the weighted geometric means of the rows to be able to "uncentre" the estimated centred logratios. This can be thought of as calibrating each one of the rays representing a column, for example, for which we need to know the average centred logratio to be able to anchor the scale at the origin. Then projecting each row i onto the ray for column j we obtain the estimate of the centred logratio $\log[n_{ij}/\tilde{g}(\mathbf{n}_i)]$, and with knowledge of $\tilde{g}(\mathbf{n}_i)$ we can eventually arrive at an estimate of n_{ij} itself.

4 Compositional data

Instead of analyzing the raw frequencies, we can convert the data to profiles and analyze them as compositional data. Table 2 shows the profiles in percentage form as well as the average percentages. If we apply the ratio map to these compositional data, the row masses are equal and the counties are not differentially weighted. The column masses, however, are different and this distinguishes the ratio map presented here from Aitchison's method of displaying compositional data.

Insert Table 2 about here

Figure 2 shows the ratio map of Table 2. The 41 rows now receive an equal weight of 1/41 in the analysis, whereas weights previously varied from 0.0006 (*Alta Ribagorça*) to 0.3803 (*Barcelona*).

Insert Figure 2 about here

Both Figures 1 and 2 represent the same logratios, since these are unaffected by expressing the data in profile form. The main difference between the analysis shown in Figure 1 and the one in Figure 2 is the change in the weights assigned to the rows. The effect can be seen in the position of the origin of the map, which is now at the arithmetic average of the row points. In Figure 2 the column weights are proportional to the average percentages, which are similar to the column weights based on the marginal frequencies which were used in Figure 1.

The column weighting is essential when one considers a column such as *Armed forces* (column 8), which has very low frequencies, but has ratios across the counties as high as 200% or more. Such ratios would dominate the display if their influence were not toned down by applying the small weight for that column.

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Table 1 Frequencies of 8 professional groups in Catalan counties

	COUNTY	Prof./ Tech.	Managmt	Admin. Services	Shops/ Sales	Hotel/ Other	Agric./ Fish.	Industry	Armed forces	*Total*
(AC)	Alt Camp	1231	243	1446	1420	875	1265	6286	25	*12791*
(AE)	Alt Empordà	2948	793	5040	5510	4823	3509	12083	317	*35023*
(AP)	Alt Penedés	2419	502	3667	3077	2000	1827	13118	36	*26646*
(AU)	Alt Urgell	778	135	835	1020	798	1068	2777	79	*7490*
(AR)	Alta Ribagorça	175	23	98	131	199	163	469	1	*1259*
(An)	Anoia	2764	614	3462	3556	2408	1124	17472	43	*31443*
(Ba)	Bages	6274	1022	6485	7095	4570	1755	28255	171	*55627*
(BC)	Baix Camp	5699	989	6165	7029	5221	3270	18436	110	*46919*
(Be)	Baix Ebre	2446	383	2311	2808	1994	3682	8846	65	*22535*
(BE)	Baix Empordà	2810	737	3716	4900	4635	2747	14519	127	*34191*
(BL)	Baix Llobregat	12371	4009	31296	26849	24955	2605	110826	274	*213185*
(BP)	Baix Penedés	1116	320	1705	1997	1762	785	6305	49	*14039*
(Bn)	Barcelona	146521	24845	182813	126740	95496	3462	274395	1258	*855530*
(Be)	Berguerà	1373	164	1207	1555	1131	1129	6910	78	*13547*
(Ce)	Cerdanya	492	116	462	679	786	670	1695	38	*4938*
(Co)	Conca de Barberà	563	124	636	631	488	1068	3018	7	*6535*
(Gf)	Garraf	3484	549	3419	3875	3559	836	11448	43	*27213*
(Ga)	Garrigues	539	79	524	619	424	2338	2286	13	*6822*
(Gx)	Garrotxa	1909	390	2064	2037	1420	1264	9712	32	*18828*
(Gi)	Gironès	7315	1187	8884	7173	5127	1727	19917	269	*51599*
(Ma)	Maresma	12837	3475	15056	15560	10867	4504	45818	189	*108306*
(Mo)	Montsià	1329	282	1600	2046	1394	4588	7716	77	*19032*
(No)	Noguera	1131	185	931	1226	824	3215	7911	35	*15458*
(Os)	Osona	4901	901	5277	5423	3238	3076	26436	50	*49302*
(PJ)	Pallars Jussà	567	79	479	465	410	955	1530	101	*4586*
(PS)	Pallars Sobirà	280	27	200	148	307	497	620	6	*2085*
(PU)	Pla d'Urgell	863	169	1019	1020	597	2570	4200	24	*10462*
(PE)	Pla de l'Estany	923	187	1036	881	587	804	4004	8	*8430*
(Pr)	Priorat	287	34	245	255	232	1063	1179	10	*3305*
(RE)	Ribera d'Ebre	936	75	684	657	592	1318	3263	27	*7552*
(Ri)	Ripollès	1012	193	905	1106	1006	801	5908	27	*10958*
(Sa)	Segarra	654	125	653	560	415	1152	3023	6	*6588*
(Se)	Segrià	7841	1279	8280	8294	6253	8678	18970	577	*60172*
(Sl)	Selva	2776	744	4106	4720	5758	2149	17562	66	*37881*
(So)	Solsonès	431	61	330	315	348	900	1854	6	*4245*
(Ta)	Tarragonès	8047	1201	9403	7294	7309	1640	21352	348	*56594*
(TA)	Terra Alta	217	41	220	324	209	1757	1710	16	*4494*
(Ur)	Urgell	1020	235	1099	1431	758	1991	4699	31	*11264*
(VA)	Val d'Aran	295	182	286	360	562	143	779	32	*2639*
(Vc)	Vallès Occidental	28614	5383	34772	31343	21310	1610	114191	231	*237454*
(Vr)	Vallès Oriental	9550	2250	13548	11619	8395	2499	54530	122	*102513*
	total	*287738*	*54332*	*366364*	*303748*	*234042*	*82204*	*916028*	*5024*	*2249480*

Professional groups are: Professional and technical, Management, Administrative services, Shopkeepers and salespersons, Hotel and other, Agriculture and fisheries, Industry, Armed forces.

Figure 1

Form biplot of logratios of Table 1

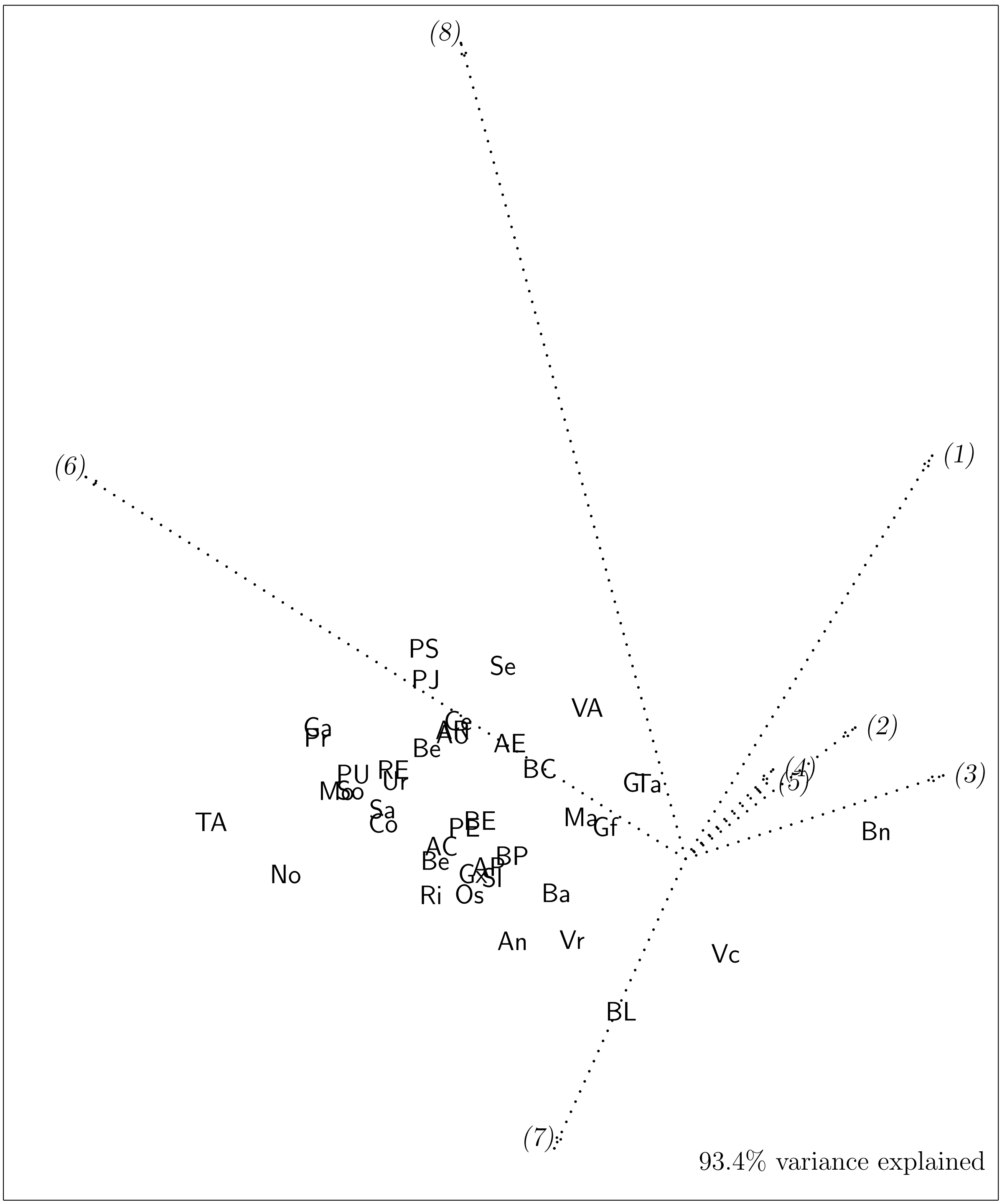


Table 2 Percentages of 8 professional groups in Catalan counties

COUNTY	Prof./ Tèc.	Pers. Dir.	Serveis admin.	Comerc. Vened.	Hotel. altres	Agric. Pesc.	Indust.	Forces arm.	*total*
(AC) Alt Camp	9.6	1.9	11.3	11.1	6.8	9.9	49.1	0.2	*100*
(AE) Alt Empordà	8.4	2.3	14.4	15.7	13.8	10.0	34.5	0.9	*100*
(AP) Alt Penedés	9.1	1.9	13.8	11.5	7.5	6.9	49.2	0.1	*100*
(AU) Alt Urgell	10.4	1.8	11.1	13.6	10.7	14.3	37.1	1.1	*100*
(AR) Alta Ribagorça	13.9	1.8	7.8	10.4	15.8	12.9	37.3	0.1	*100*
(An) Anoia	8.8	2.0	11.0	11.3	7.7	3.6	55.6	0.1	*100*
(Ba) Bages	11.3	1.8	11.7	12.8	8.2	3.2	50.8	0.3	*100*
(BC) Baix Camp	12.1	2.1	13.1	15.0	11.1	7.0	39.3	0.2	*100*
(Be) Baix Ebre	10.9	1.7	10.3	12.5	8.8	16.3	39.3	0.3	*100*
(BE) Baix Empordà	8.2	2.2	10.9	14.3	13.6	8.0	42.5	0.4	*100*
(BL) Baix Llobregat	5.8	1.9	14.7	12.6	11.7	1.2	52.0	0.1	*100*
(BP) Baix Penedés	7.9	2.3	12.1	14.2	12.6	5.6	44.9	0.3	*100*
(Bn) Barcelona	17.1	2.9	21.4	14.8	11.2	0.4	32.1	0.1	*100*
(Be) Berguerà	10.1	1.2	8.9	11.5	8.3	8.3	51.0	0.6	*100*
(Ce) Cerdanya	10.0	2.3	9.4	13.8	15.9	13.6	34.3	0.8	*100*
(Co) Conca de Barberà	8.6	1.9	9.7	9.7	7.5	16.3	46.2	0.1	*100*
(Gf) Garraf	12.8	2.0	12.6	14.2	13.1	3.1	42.1	0.2	*100*
(Ga) Garrigues	7.9	1.2	7.7	9.1	6.2	34.3	33.5	0.2	*100*
(Gx) Garrotxa	10.1	2.1	11.0	10.8	7.5	6.7	51.6	0.2	*100*
(Gi) Gironès	14.2	2.3	17.2	13.9	9.9	3.3	38.6	0.5	*100*
(Ma) Maresma	11.9	3.2	13.9	14.4	10.0	4.2	42.3	0.2	*100*
(Mo) Montsià	7.0	1.5	8.4	10.8	7.3	24.1	40.5	0.4	*100*
(No) Noguera	7.3	1.2	6.0	7.9	5.3	20.8	51.2	0.2	*100*
(Os) Osona	9.9	1.8	10.7	11.0	6.6	6.2	53.6	0.1	*100*
(PJ) Pallars Jussà	12.4	1.7	10.4	10.1	8.9	20.8	33.4	2.2	*100*
(PS) Pallars Sobirà	13.4	1.3	9.6	7.1	14.7	23.8	29.7	0.3	*100*
(PU) Pla d'Urgell	8.2	1.6	9.7	9.7	5.7	24.6	40.1	0.2	*100*
(PE) Pla de l'Estany	10.9	2.2	12.3	10.5	7.0	9.5	47.5	0.1	*100*
(Pr) Priorat	8.7	1.0	7.4	7.7	7.0	32.2	35.7	0.3	*100*
(RE) Ribera d'Ebre	12.4	1.0	9.1	8.7	7.8	17.5	43.2	0.4	*100*
(Ri) Ripollès	9.2	1.8	8.3	10.1	9.2	7.3	53.9	0.2	*100*
(Sa) Segarra	9.9	1.9	9.9	8.5	6.3	17.5	45.9	0.1	*100*
(Se) Segrià	13.0	2.1	13.8	13.8	10.4	14.4	31.5	1.0	*100*
(Sl) Selva	7.3	2.0	10.8	12.5	15.2	5.7	46.4	0.2	*100*
(So) Solsonès	10.2	1.4	7.8	7.4	8.2	21.2	43.7	0.1	*100*
(Ta) Tarragonès	14.2	2.1	16.6	12.9	12.9	2.9	37.7	0.6	*100*
(TA) Terra Alta	4.8	0.9	4.9	7.2	4.7	39.1	38.1	0.4	*100*
(Ur) Urgell	9.1	2.1	9.8	12.7	6.7	17.7	41.7	0.3	*100*
(VA) Val d'Aran	11.2	6.9	10.8	13.6	21.3	5.4	29.5	1.2	*100*
(Vc) Vallès Occidental	12.1	2.3	14.6	13.2	9.0	0.7	48.1	0.1	*100*
(Vr) Vallès Orinetal	9.3	2.2	13.2	11.3	8.2	2.4	53.2	0.1	*100*
average	*12.8*	*2.4*	*16.3*	*13.5*	*10.4*	*3.7*	*40.7*	*0.2*	100

Figure 2

Form biplot of logratios of Table 2

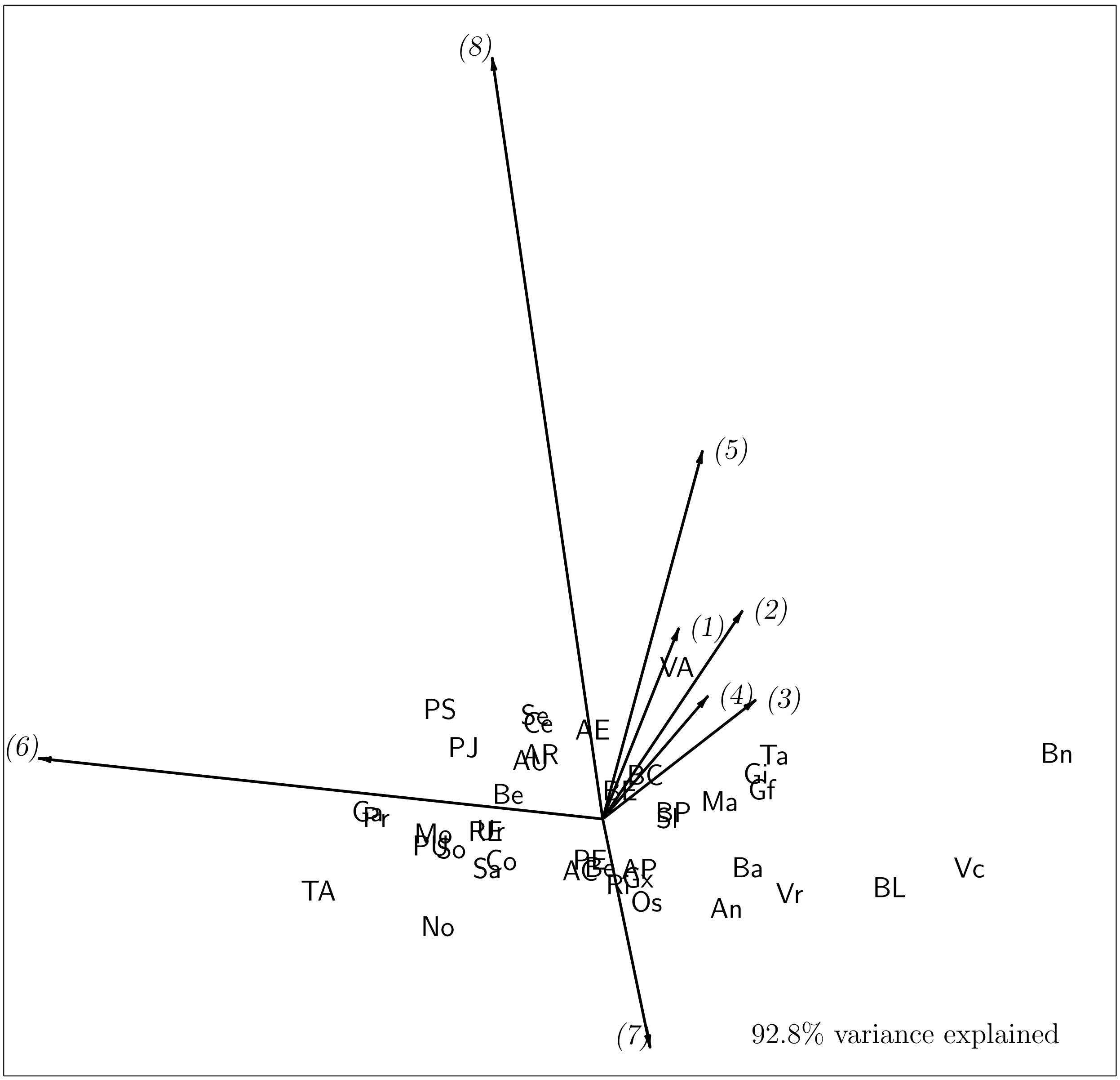
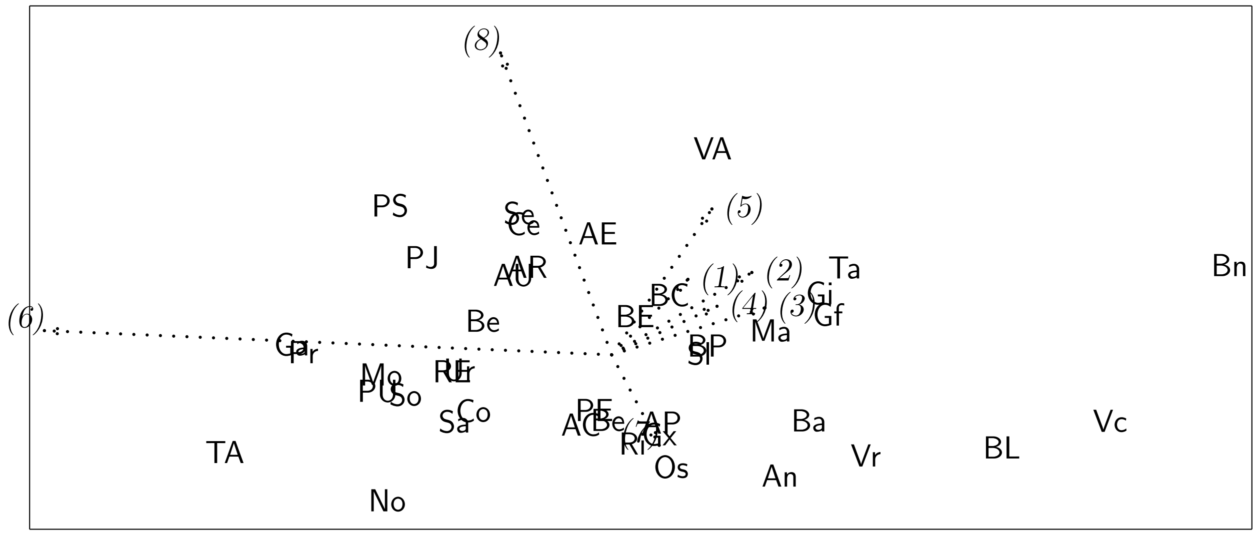


Figure 3

Symmetric map of logratios of Table 2



92.8% variance explained