Economics Working Paper 55

# Heterogeneous Beliefs, Wealth Accumulation, and Asset Price Dynamics*

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February, 1993
Revised: June, 1993

## Abstract

A model of asset markets with two types of investors is developed and its dynamic properties are analyzed. "Optimists" expect on average higher returns on the risky assets than "pessimists" do. The stochastic process for equilibrium asset return changes over time as the distribution of wealth between the two types of investors changes. In the long run, the share of wealth held by one type of investor may become negligible, but it is also possible for both types to co-exist, depending on the parameter values of the model. Relations between this model and some econometric models with time varying parameters, such as the ARCH (Autoregressive Conditional Heteroskedasticity) model and the STAR (Smooth Transition Autoregressive) model, are examined. The dynamic properties of another model, regarding investors who use strategies that are a bit more complex, are also analyzed. "Fundamentalists" believe that the asset returns follow a process that is solely determined by fundamentals and "contrarians" assume the market is wrong and choose a portfolio that is exactly opposite of the market portfolio. Again, depending on parameters, both types can co-exist even in the long run.

# 1. INTRODUCTION

Recent studies in finance, which include Shiller (1984), Black (1986), Frankel and Froot (1988, 1990a, 1990b), and De Long, Shleifer, Summers, and Waldmann (DSSW henceforth) (1990) among others, stressed the importance of heterogeneous beliefs in asset markets. These studies show that models with heterogeneous investors are useful in explaining some empirical puzzles that cannot be explained by a model with a representative investor.

This paper studies a similar model with heterogeneous beliefs to analyze the asset price dynamics. We consider an asset market which includes two types of investors, who follow two different rule of thumb strategies. Depending on the relative success of each strategy, the proportion of total wealth held by each type of investor fluctuates. If strategy A is more successful than the other strategy B, for example, the investors who use strategy A increase their wealth more than the other investors, invest more in the asset, and increase their influence on asset pricing. Thus the changes in the distribution of wealth between the two types of investors affect the asset prices, which in turn influence the relative success of each strategy and again change wealth distribution. Profits, which are random, are used for buying back the shares of the risky asset to create capital gains for the owners, or for paying out dividends to create income gains. Thus random profit shocks influence the dynamics of the asset price and wealth distribution. The asset price and wealth distribution themselves also become stochastic processes. This paper studies the limiting behavior of these stochastic processes. Under some parameter values, we show that the limiting behavior of those dynamics does not depend on the initial condition.

The most important empirical implication of our model is smooth shifts of the

price dynamics. Both the conditional mean and the conditional variance of the asset price change over time reflecting changes in the wealth distribution between two types of investors. For some periods, strategy A may do better than strategy B and enhance its influence. Following some random shocks to profits, strategy B may become more profitable, and the proportion of wealth held by investors who use strategy B will increase. Following some other shocks, strategy A may again become more profitable and regain its influence. In this way, wealth distribution fluctuates over time and asset price dynamics show smooth shifts between two extreme regimes: one where all the wealth is owned by investors who use strategy A and the other where all the wealth is owned by investors who use strategy B. Although these extremes never happen if the limiting distribution of the proportion of wealth is ergodic, the price dynamics still can show substantial variation over time without reaching an extreme.

Thus our model is useful in explaining an apparent empirical regularity in financial markets: namely time varying conditional variance of asset returns. Time varying conditional variance has been detected by many researchers who estimated ARCH (Autoregressive Conditional Heteroskedasticity) models to financial data. Bollerslev, Chou, Jayaraman, and Kroner (1990) surveyed numerous successful applications of ARCH models in finance. The model with the representative agent does not usually have strong implications for the dynamics of conditional variance of asset returns. Thus, in order to interpret empirical success of ARCH models in finance, the standard model has to assume ARCH at the fundamental level, for example in the process of cash flow. The model in this paper can explain time varying conditional variance without assuming ARCH at the fundamental level.

The model also implies that the mean growth rate of asset price changes over time, and

changes in the mean are related to changes in the conditional variance. Thus a type of ARCH models called the ARCH-M model, which is used to capture shifts in the conditional mean associated with changes in the conditional variance, becomes especially relevant for our model.

Our model is also useful in motivating another statistical model with time varying parameters called STAR (Smooth Transition Autoregressive) model. The STAR model was applied to some aggregate variables like GNP and industrial production by Anderson and Teräsvirta (1991). A STAR model assumes that a process is a weighted average of the two distinct AR processes and the weight changes over time. Our model also implies that the asset price process shifts between two extremes, although the variable that determines the weight in our model (proportion of wealth) enters the price process in a complex way.

There are several studies of the dynamics of asset prices in the asset market with heterogeneous agents. DSSW (1991) considers the wealth accumulation by two types of investors, noise traders and sophisticated investors. The noise traders have random and biased forecasts of the rate of return and its variance, whereas the sophisticated investors have the correct forecasts. They show, under some parameter values, that the noise traders' wealth may grow faster than that of the sophisticated investors and eventually the noise traders may dominate the market. One problem of DSSW (1991) is that they ignore the noise traders' influence on the asset price, which is the most important point made by models with noise traders, such as DSSW (1990). Our model explicitly studies the way the heterogeneous beliefs influence asset prices and examines the price dynamics.

Another important difference between our paper and DSSW's research is that we do not assume the presence of sophisticated investors who have rational expectations and maximize their

expected utility. Although the agents in our model use portfolio strategies that maximize an expected utility function under some assumptions about beliefs, they do not have rational expectations. Thus the agents in this paper deviate from the standard rational consumers in economics. In this sense, our approach is similar to that followed by Blume and Easley (1992). They consider the dynamic process of an asset market with heterogeneous investors, which is similar to ours. Each type of investor uses different portfolio rules, and the market eventually selects the most "fit" rules, in the sense that wealth held by such investors grows faster than that of the other investors. One difference between our model and Blume and Easley (1992) is the formulation of asset return. Blume and Easley (1992) assume an exogenous probability distribution over the possible pay-out of the asset. Thus the current price of the asset does not influence the rate of return on an asset that was bought last period. In other words, the assets in their model can have only income gains and not capital gains. Our model includes both income gains and capital gains of asset holding.

The paper is organized as follows. The next section presents a model of an asset market with two types of investors: optimists who expect a high rate of return for the risky asset and pessimists who expect a low rate of return for the risky asset. The dynamics of the asset price and wealth distribution are derived and characterized. Section 3 studies the asymptotic distribution of the proportions of wealth held by each agent. Section 4 discusses the implications on conditional variance and relates our model to ARCH models in econometrics. This section also discusses the model's relation to another statistical model with time varying parameters called STAR. Section 5 considers the investors who use slightly more complex strategies. "Fundamentalists" behave similarly to optimists and pessimists in Section 2. They maximize the

expected utility given their beliefs about the process of the asset returns. "Contrarians" start with the assumption that the market on average is wrong, and invest a proportion of their wealth in the risky asset which is opposite to the average proportion of wealth invested in that asset. The price dynamics in a market with fundamentalists and contrarians is examined. Finally, Section 6 presents our conclusions.

## 2. OPTIMISTS, PESSIMISTS AND THE ASSET PRICE

We consider an asset market where there are two types of investors. One type of investors called "optimists", hold higher expectation about the rate of return of the risky asset than the other type of investors called "pessimists." Besides this assumption of heterogeneous beliefs, our model is very much a standard model of asset pricing model used in macro/finance literature. Time is continuous and investors have infinite horizons. Both types of agents are assumed to have identical preferences with constant relative risk aversion, so their instantaneous

utility function is $u(C) = \dfrac{C^{1-\tau}}{1-\tau}$ , where $\tau > 0$, $\tau \neq 1$, and where C stands for

consumption. There are two types of assets, a riskless asset with an instantaneous rate of return r and a risky asset whose price is P(t). Let D(t) be the total dividends accumulated for a share of the risky asset from time 0 to t. Instantaneous dividends are thus dD(t). Let $W^o(t)$ be the wealth of the optimists, $W^p(t)$ the wealth of the pessimists and $W(t) = W^o(t) + W^p(t)$ is the total wealth in the economy. Let $\lambda^i(t)$ be the proportion of wealth that an agent of type i invests in the risky asset and $c^i(t)$ the proportion consumed out of total income. Given the assumptions about the return of the assets, wealth evolves according to the following equation:

$$dW^i(t) = \left(\lambda^i(t)\frac{dP(t)+dD(t)}{P(t)} + (1-\lambda^i(t))rdt - c^i(t)dt\right)W^i(t),$$  (1)

$$i = o, p$$

The agents believe that the process followed by the returns of the risky asset is determined by,

$$\frac{dP(t)+dD(t)}{P(t)} = \alpha_i dt + \beta dB(t)$$

where i is equal to o for the optimists and p for the pessimists and $\alpha_o > \alpha_p$. Thus, optimists expect higher returns than pessimists for a given level of risk. We assume that the agents never change their beliefs. The influence of each belief on the asset price, however, changes as the wealth distribution between the two types of agent changes. It may be more realistic to assume that some agents change their beliefs if their strategy yields much lower returns than the alternative one, but such an assumption will not change the qualitative results. Under this assumption, a successful strategy would increase the influence through two channels. The wealth of the agents that use the strategy grows faster, and, at the same time, the strategy gains some new converts. Thus, the dynamics of the model would qualitatively look the same.

The objective of the agents is to maximize $E\left\{\int_0^\infty e^{-\rho s}U[C(s)]ds\right\}$ subject to the

budget constraint (1), and $C^i(t) > 0$; $W^i(t) > 0$; $W^i(0) = W_0 > 0$. Under these conditions and

assuming $\rho > (1-\tau)\left[\frac{(\alpha_i-r)^2}{2\sigma^2\tau} + r\right]$, which always holds when the agent is more risk

averse than an agent with logarithmic utility, the demand for the risky asset by each type of agent will be a constant proportion of their total wealth, as Merton (1969) shows. Merton (1969) also shows that letting $\lambda^i$ be the proportion of the risky asset in the portfolio of a type i agent

$\lambda^i = \dfrac{\alpha_i - r}{\tau\beta^2}$ , and letting $c^i$ be the proportion of consumption on wealth,

$$c^i = \frac{1}{\tau}\left(\rho - (1-\tau)\left[\frac{(\alpha_i-r)^2}{2\sigma^2\tau} + r\right]\right).$$

If we let N(t) be the total number of shares outstanding at time t, market clearing implies,

$$\lambda^o W^o(t) + \lambda^p W^p(t) = P(t)N(t) , \quad (2)$$

or if we let $q(t) = \dfrac{W^o(t)}{W(t)}$ ,

$$\lambda^o q(t) + \lambda^p(1-q(t)) = \frac{N(t)P(t)}{W(t)} . \quad (2')$$

The firm that issued the shares is assumed to have instantaneous profits of

$$d\pi^*(t) = (\pi^* dt + \sigma dB(t))W(t) .$$

This will be the case if, for example, demand is a fixed proportion of total wealth, the price of the good is constant and the average cost, which fluctuates randomly over time, is the same for all levels of production at any given instant of time. The firm uses the profits to buy back its

shares and pay out dividends, so that

$$d\pi^*(t) = -dN(t)(P(t)+dP(t)) + N(t)dD(t) \quad (3)$$

Using the market equilibrium condition (2) and the relation between asset returns and the profits process given by (3), we can solve for the process of equilibrium asset returns. Then by using (1) and Itô's formula, we can derive the stochastic process q(t), the proportion of wealth held by the optimists[1]. This is done with the following proposition.

**Proposition A.**

a) $$\frac{dP(t)+dD(t)}{P(t)} = \frac{\pi^*-\lambda^o c^o q(t) - \lambda^P c^P (1-q(t))}{\lambda^o(1-\lambda^o)q(t) + \lambda^P(1-\lambda^P)(1-q(t))}dt + rdt$$

$$+ \frac{1}{\lambda^o(1-\lambda^o)q(t) + \lambda^P(1-\lambda^P)(1-q(t))}\sigma dB(t). \quad (5)$$

b) $$dq(t) = q(t)(1-q(t))(\lambda^o-\lambda^P)\left[\frac{\pi^*-\lambda^o c^o q(t) - \lambda^P c^P (1-q(t))}{\lambda^o(1-\lambda^o)q(t) + \lambda^P(1-\lambda^P)(1-q(t))}\right.$$

$$\left.-\frac{c^o-c^P}{\lambda^o-\lambda^P}\right]dt - \frac{q(t)(1-q(t))(\lambda^o-\lambda^P)(\lambda^o q(t)+\lambda^P(1-q(t)))\sigma^2}{(\lambda^o(1-\lambda^o)q(t) + \lambda^P(1-\lambda^P)(1-q(t)))^2}dt$$

$$+\frac{q(t)(1-q(t))(\lambda^o-\lambda^P)\sigma}{\lambda^o(1-\lambda^o)q(t) + \lambda^P(1-\lambda^P)(1-q(t))}dB(t),$$

**Proof:** See the Apppendix.

The proposition shows that the equilibrium asset returns depend on random shocks to the profits. More importantly, the proportion of optimists' wealth, q(t), affects the equilibrium asset

returns. The dynamics of q(t) is given by the latter half of the proposition.

The next section studies the asymptotic properties of the dynamics of q(t). Since the dynamics of asset returns critically depends on that of q(t), the knowledge of asymptotic properties of q(t) is necessary for us to understand the long run properties of asset return dynamics. To simplify the notation, we hereafter write the dynamics of q(t) as,

$$dq(t) = a(q(t))dt + b(q(t))dB(t).$$

It is obvious from the proposition that,

$$a(q(t)) = q(t)(1-q(t))(\lambda^o - \lambda^p)\left[\frac{\pi^* - \lambda^o c^o q(t) - \lambda^p c^p (1-q(t))}{\lambda^o(1-\lambda^o)q(t) + \lambda^p(1-\lambda^p)(1-q(t))} - \frac{c^o - c^p}{\lambda^o - \lambda^p}\right]$$

$$-\frac{q(t)(1-q(t))(\lambda^o - \lambda^p)(\lambda^o q(t) + \lambda^p(1-q(t))\sigma^2}{(\lambda^o(1-\lambda^o)q(t) + \lambda^p(1-\lambda^p)(1-q(t)))^2}$$

$$b(q(t)) = \frac{q(t)(1-q(t))(\lambda^o - \lambda^p)\sigma}{\lambda^o(1-\lambda^o)q(t) + \lambda^p(1-\lambda^p)(1-q(t))}.$$

## 3. ASYMPTOTIC BEHAVIOR OF THE MODEL

This section investigates some asymptotic properties of the process q(t). Let us first define the following,

$$I_1 = \int_0^{q_1(0)} \exp\left[-\int_s^x \frac{2a(q)}{b(q)^2}dq\right]dx, \quad I_2 = \int_{q_1(0)}^1 \exp\left[-\int_s^x \frac{2a(q)}{b(q)^2}\right]dx$$

and

$$M(x) = \frac{2}{b(x)^2} \exp\left[\int_z^x \frac{2a(q)}{b(q)^2} dq\right]$$

where z is an arbitrary number in (0, 1). As Gihman and Skorohod (1972) show $I_1$, $I_2$ and

$\int_0^1 M(x)\,dx$ are quite useful in studying the asymptotic properties of the process q(t). This

approach was used by Fudenberg and Harris (1992) to study the asymptotic properties of a class

of evolutionary game dynamics.

If $I_1$ is infinite and $I_2$ is finite the system converges to one almost surely. If $I_1$ is finite

and $I_2$ is infinite the system converges to zero almost surely. If $I_1$ and $I_2$ are infinite and

$\int_0^1 M(x)\,dx$ is finite the system has a unique ergodic distribution with density $\dfrac{M(x)}{\int_0^1 M(y)\,dy}$.

If $I_1$ and $I_2$ are finite the system converges to one with probability $\dfrac{I_1}{I_1+I_2}$.

The function $I_1$ measures the difficulty of converging to the state where q is zero. When

a(0) is positive the deterministic drift tends to push the dynamics away from zero, thus, it is

difficult for the dynamics to converge to that state. In $I_1$ this is reflected in the fact that the

exponent is positive (since x will be smaller than z the integral and the sign in front of it are

reversed) and very large, since $a(0)/b(0)^2$ is infinite. When a(0) is negative the deterministic drift

tends to push the dynamics to zero, but it might still be that there is no convergence if the

variance (diffusion) term is very large. This is so because a run of good luck for the optimists,

which eventually will happen because of the large variance, will make their wealth grow fast for

a while, thereby making it impossible that their wealth becomes negligible with respect to the

pessimists' wealth.

The function $I_2$ measures the difficulty of converging to the state where q is one. When a(1) is negative the deterministic drift tends to push the dynamics away from one, thus, it is difficult to converge to that state. In $I_2$ this is reflected in the fact that the exponent is positive and very large, since $-a(1)/b(1)^2$ is infinite. When a(1) is positive the deterministic drift tends to push the dynamics to one, but it might still be that there is no convergence if the variance (diffusion) term is very large. This is so because a run of good luck for the pessimists which eventually will happen because of the large variance, will make their wealth grow fast for a while, thereby making it impossible that their wealth becomes negligible with respect to the optimists' wealth.

For our purposes, defining a function $D(q) = \dfrac{2a(q)\,q(1-q)}{b(q)^2}$ makes the discussion easier. Note that $D(q)$ is a bounded, continuously differentiable function in the interval [0, 1]. Note also that both the a(.) and the b(.) functions have a factor of q(1-q) which in D(.) cancel out so that D(.) is well defined and need not be zero when q is zero or one.

Depending on the values of D(q) at q=0 and q=1, there are four possibilities or the asymptotic dynamics of q which are identified in the following proposition.

**Proposition 1.**  a) If $D(0) > 1$ and $D(1) < -1$ the process is ergodic[2].  b) If $D(0) > 1$ and $D(1) > -1$ the process converges to 1 almost surely. c) If $D(0) < 1$ and $D(1) < -1$ the process converges to 0 almost surely.  d) If $D(0) < 1$ and $D(1) > -1$ the process converges to 1 with probability $\dfrac{I_1}{I_1 + I_2}$ and to 0 with complementary probability.

**Proof:** See the Appendix.

The process under consideration has two absorbing states, zero and one, because there

is no deterministic drift at these states and no stochastic variation either. Once you are broke you have nothing to start over with. That is why the values of $D(0)$ and $D(1)$ are so important. The function $D(.)$ compares the strength of the deterministic drift to the variance of the noise term. When the variance is very large compared to the drift there can be no convergence to the absorbing states because sooner or later a run of good luck will happen for the less wealthy agents which will then become a little wealthier. When the drift is strong compared to the variance the stochastic effects can be overcome and convergence can be achieved.

Table 1 gives a visual exposition of Proposition 1. The table clearly shows that the values of $D(0)$ and $D(1)$ are important in determining the asymptotic properties of the stochastic process $q(t)$. The value of $D(0)$ is important in determining whether the process converges to zero with positive probability. If $D(0) < 1$, then the process $q(t)$ converges to zero with a probability greater than zero. The probability becomes one when $D(1) < -1$ also holds. If $D(0) > 1$, however, the process never converges to zero. Similarly, the value of $D(1)$ is important in determining whether the process $q(t)$ converges to one with a positive probability. If $D(1) > -1$, then the process converges to one with positive probability. The probability becomes one when $D(0) > 1$ also holds. If $D(1) < -1$, the process never converges to one.

We will now show that all of the cases announced in Proposition 1 are possible for some open nonempty set of the parameter values of this model. Suppose that $C^o = C^p = 0$. Since the inequalities in cases a) through d) are strict, showing that all the cases are possible for an open nonempty set of the subset of the parameter space where the above equalities hold guarantees that all cases are also possible for an open nonempty set of the whole parameter space. When the consumption rates are zero,

$$D(0) = \frac{2(\lambda^P(1-\lambda^P))^2}{(\lambda^O-\lambda^P)\sigma^2}\left[\frac{\pi^*}{\lambda^P(1-\lambda^P)} - \frac{\lambda^P\sigma^2}{(\lambda^P(1-\lambda^P))^2}\right]$$

$$= \frac{2\lambda^P}{\lambda^O-\lambda^P}\left[\frac{\pi^*(1-\lambda^P)}{\sigma^2} - 1\right]$$

Thus $D(0) > 1$ if and only if $2\dfrac{\pi^*}{\sigma^2} > \dfrac{\lambda^O+\lambda^P}{\lambda^P(1-\lambda^P)}$ .

$$D(1) = \frac{2(\lambda^O(1-\lambda^O))^2}{(\lambda^O-\lambda^P)\sigma^2}\left[\frac{\pi^*}{\lambda^O(1-\lambda^O)} - \frac{\lambda^O\sigma^2}{(\lambda^O(1-\lambda^O))^2}\right]$$

$$= \frac{2\lambda^O}{\lambda^O-\lambda^P}\left[\frac{\pi^*(1-\lambda^O)}{\sigma^2} - 1\right]$$

So $D(1) < -1$ if and only if $2\dfrac{\pi^*}{\sigma^2} < \dfrac{\lambda^O+\lambda^P}{\lambda^O(1-\lambda^O)}$ . Thus we can distinguish the

four cases a) through d) in the proposition by comparing $\dfrac{2\pi^*}{\sigma^2}$ to $\dfrac{\lambda^O+\lambda^P}{\lambda^P(1-\lambda^P)}$ and

$\dfrac{\lambda^O+\lambda^P}{\lambda^O(1-\lambda^O)}$ .

a) $\dfrac{\lambda^O+\lambda^P}{\lambda^O(1-\lambda^O)} > 2\dfrac{\pi^*}{\sigma^2} > \dfrac{\lambda^O+\lambda^P}{\lambda^P(1-\lambda^P)}$

b) $2\dfrac{\pi^*}{\sigma^2} > \dfrac{\lambda^O+\lambda^P}{\lambda^O(1-\lambda^O)}$ and $2\dfrac{\pi^*}{\sigma^2} > \dfrac{\lambda^O+\lambda^P}{\lambda^P(1-\lambda^P)}$

c) $2\dfrac{\pi^*}{\sigma^2} < \dfrac{\lambda^O+\lambda^P}{\lambda^O(1-\lambda^O)}$ and $2\dfrac{\pi^*}{\sigma^2} < \dfrac{\lambda^O+\lambda^P}{\lambda^P(1-\lambda^P)}$

d) $\dfrac{\lambda^O+\lambda^P}{\lambda^O(1-\lambda^O)} < 2\dfrac{\pi^*}{\sigma^2} < \dfrac{\lambda^O+\lambda^P}{\lambda^P(1-\lambda^P)}$

It is obvious that all cases are possible for a nontrivial set of parameters. As an example,
Figure 1 identifies the four cases on $\lambda^o - \lambda^p$ space. The case depicted in the figure assumes

$$\frac{2\pi^*}{\sigma^2} > 3 + 2\sqrt{2} \ .$$

Although the discussion above is useful in establishing that all the cases are indeed
possible, the conditions derived there are hard to interpret. By rewriting the inequalities in D(0)
and D(1) in Proposition 1, we can get a more intuitive interpretation of those conditions. For
example, noting that:

$$D(0) = \frac{2(\lambda^p(1-\lambda^p))^2}{(\lambda^o-\lambda^p)\sigma^2}\left[\frac{\pi^*-c^p\lambda^p}{\lambda^p(1-\lambda^p)} - \frac{\lambda^p\sigma^2}{(\lambda^p(1-\lambda^p))^2} - \frac{c^o-c^p}{\lambda^o-\lambda^p}\right]$$

$D(0) > 1$ is equivalent to:

$$\frac{1}{(\lambda^o-\lambda^p)^2}\left[(\lambda^o-\lambda^p)\left(\frac{2(\pi^*-c^p\lambda^p)}{\sigma^2}\lambda^p(1-\lambda^p)-2\lambda^p-(\lambda^o-\lambda^p)\right) - \frac{2(\lambda^p(1-\lambda^p))^2(c^o-c^p)}{\sigma^2}\right] > 0$$

or multiplying $(\lambda^o-\lambda^p)^2 > 0$ on both sides and rearranging,

$$(\lambda^p-\lambda^*(0))^2 - (\lambda^o-\lambda^*(0))^2 - 2(c^o-c^p)\frac{(\lambda^p(1-\lambda^p))^2}{\sigma^2} > 0 \quad (7)$$

where $\lambda^*(0) = \frac{\lambda^p(1-\lambda^p)(\pi^*-c^p\lambda^p)}{\sigma^2}$ .

Note that $\lambda^*(0)$ is the proportion of wealth that the agents would invest in the risky
asset if they had logarithmic utility and their beliefs were correct when q=0, that is, no optimists[3].
Inequality (7) is useful for interpreting the results in Proposition 1 in an intuitive way. Suppose,
for the moment, that $c^o$ is equal to $c^p$, that is the optimists and the pessimists consume at the
same rate. Then the last term on the left hand side of equation (7) vanishes, and whether the

asset if they had logarithmic utility and their beliefs were correct at the state where q = 1, that is, when there are no pessimists.

Again, let us start by assuming that $c^o = c^p$. Whether inequality (8) holds is determined by whether $\lambda^p$ is closer to $\lambda^*(1)$ than $\lambda^o$. If the pessimists' strategy is in fact closer to $\lambda^*(1)$, which is the optimal strategy of an agent with logarithmic utility and correct beliefs when there are no pessimists, then the pessimists start to do better whenever their share of wealth becomes close to zero. Thus the share of pessimists' wealth never converges to zero; that is, the share of the optimists' wealth never converges to one. If the optimists' strategy is closer to $\lambda^*(1)$, then the optimists dominate when the pessimists' share of wealth is small, and the optimists' share of the wealth converges to one with positive probability.

If the rate of consumption differs between two types of investors, and $c^o > c^p$, then the inequality (8) becomes easier to satisfy. If the optimists consume at a higher rate, it is less likely for their share of wealth to converge to one.

Our model may appear similar to the one developed by Blume and Easley (1992). Our results are, however somewhat different from those by Blume and Easley. In their model, if two types of investors invest constant, but different proportions of wealth in the risky asset, as they do in our model, one type of investors see their wealth eventually become very small relative to the other types' wealth. (Blume and Easley (1992), Proposition 3.1 and 3.2, p.16). In our model, one type of investors may eventually dominate the other (cases (b) and (c) in Proposition 1 ), but two types may co-exist even in the long run (case (a)), or both types have a chance of dominating the market and the probability of one type eliminating the other depends on the initial

start to dominate when q gets small, pushing q away from zero. When q gets large, the pessimists start to dominate and keep q away from one. Thus the process of q converges to an ergodic distribution.

If we allow investors in our model to have non-constant portfolio strategy, we can prove a proposition that is similar to Theorem 5.2 in Blume and Easley (1992). If one type of investors chooses a portfolio strategy that always differs from the (non-constant) optimal strategy that would be chosen by an investor with correct beliefs and logarithmic utility by more than the other strategy does, then such an investor' share of the wealth becomes asymptotically negligible.

**Proposition 2.** Suppose we have a riskless asset with rate of return r, and a risky asset with rate of return,

$$\frac{dP(t) + dD(t)}{dP(t)} = \alpha(t) dt + \beta(t) dB(t)$$

where $\alpha(t)$, $\beta(t)$ are bounded, measurable functions for all t and $|\beta(t)| > \underline{\beta} > 0$ for all t. Let $\lambda_1(t)$, $\lambda_2(t)$ be the investment shares of agents 1 and 2 in the risky asset, which are also assumed to be bounded and measurable, and suppose that both agents' consumption rates are $c^1(t)$ and $c^2(t)$. We will denote $k_i(t) = \lambda_i(t) - \frac{\alpha(t) - r}{\beta^2(t)}$ for i = 1, 2.

If $|k_1(t)| - |k_2(t)| > k^* > 0$ and $c_1(t) - c_2(t) > c^* > 0$ almost surely, $\frac{W^1(t)}{W^2(t)}$ converges to zero almost surely.

**Proof:** See the Appendix.

conditions (case (d)).

What is responsible for the difference is the presence of changes in the price level in the asset returns in our model. The Blume and Easley model considers assets that pay state contingent payoffs, which do not depend on changes in the prices. Although changes in the price affect the rate of return by changing the amount that the investors have to pay to acquire the assets, the payouts are not affected.

In our model the payouts of the risky asset include increases in the price. Since the price changes as the distribution of wealth between the two types of investors changes, the true process of the rate of return also changes in our model. As it is clear from Proposition A-(a), the stochastic process of equilibrium rate of return has parameters that are functions of q, the optimists' share of the wealth. When q changes, the true process of the rate of return changes, which in turn modifies the optimal strategy that would be used by an investor with correct beliefs.

These effects of price changes are not present in the Blume and Easley model, because the price changes do not affect payouts in their model. Hence the optimal strategy that would be used by an investor with correct beliefs in their model is a constant portfolio strategy. Thus they can show that if investors behave in a way that resembles more closely the constant portfolio strategy that would be chosen by an investor with correct beliefs and logarithmic utility, then they tend to dominate the market. (Blume and Easley (1992), Theorem 5.2, p. 22).

The optimal strategy for an investor with correct beliefs and logarithmic utility is not a constant portfolio strategy in our model. The optimal strategy may be closer to the optimists' strategy when q is small and closer to the pessimists' when q is large. In this case, the optimists

Note that $k_i(t)$ is a measure of distance between $\lambda_1(t)$ , and $\dfrac{\alpha(t) - r}{\beta(t)^2}$ , which is the portfolio strategy that would be followed by an investor with logarithmic utility and correct beliefs. The parameters in the process of asset returns, $\alpha(t)$ and $\beta(t)$ , are functions of t, and $\dfrac{\alpha(t) - r}{\beta^2(t)}$ also changes over time. The assumption of time-varying $\alpha$ and $\beta$ reflects the results in our model (Proposition A-(a)). Proposition 2 shows that if strategy 1 is consistently more distant from the (time-varying) optimal strategy than strategy 2 is for all t, and if type 1 does not consume less than type 2 investors, then the share of wealth of type 1 investors eventually becomes negligible. As is clear from the proof, the proposition just establishes a sufficient condition for $W^1(t)/W^2(t)$ to converge to zero. If the strategy followed by type 1 investors is sufficiently more distant from the optimal portfolio than strategy 2, then even if the rates of consumption are the same (or even if type 2 investors consume at a slightly faster rate), the share of type 1 investors' wealth can still become negligible in the long run. It is also not necessary for the portfolio of 1 to be always more distant from the optimal portfolio than that of 2 for $W_1(t)/W_2(t)$ to converge to zero. It would suffice that the portfolio of 1 were more distant a sufficiently large proportion of the time for that to happen[4].

Comparing Proposition 2 to Theorem 5.2 in Blume and Easley (1992) clarifies the important difference between our model and theirs. In their model, "fitness" of a strategy can be measured by the distance between that strategy and the strategy that would be used by an investor with logarithmic utility and correct beliefs, which turns out to be a constant portfolio strategy. In our model, fitness of a strategy can be measured also by its distance from the optimal strategy of a logarithmic utility maximizer with correct beliefs. But the fittest strategy is not a constant portfolio strategy. Thus it is possible for the optimists' strategy to be closer to

the fittest strategy when the pessimists' share of wealth is small and the pessimist strategy becomes fitter when the pessimists' share of wealth is small, allowing two strategies to coexist even in the long run.

## 4. RELATIONSHIP WITH ARCH AND ARCH-M

Our model implies that the process followed by asset returns has a time varying variance. As equation (5) suggests, the standard deviation of the returns depends on q(t), which fluctuates over time.

Time varying conditional heteroskedasticity of asset returns has been documented by much empirical research that applied ARCH models to financial data. Those empirical studies found that the conditional variance of asset return is well captured by a stochastic process that depends on its own past.

A class of ARCH models called ARCH-M (ARCH in mean) model considers the dependence of the mean rate of return on the conditional variance. Thus the mean as well as the variance of rate of return becomes time varying in ARCH-M model. Our model also predicts the time varying average returns.

This section examines the stochastic process of the conditional variance of the asset returns implied by our model and argues that our model can give a theoretical justification for an ARCH-like model in finance.

Given what we found in section 2 the returns process can be written as:

$$\frac{dP(t) + dD(t)}{P(t)} = \alpha(q(t)) dt + V(q(t)) dB(t)$$

where

$$\alpha(q) = \frac{\pi^* - \lambda^o c^o q - \lambda^p c^p (1-q)}{\lambda^o (1-\lambda^o) q + \lambda^p (1-\lambda^p) (1-q)} + r$$

$$V(q) = \frac{\sigma}{\lambda^o (1-\lambda^o) q + \lambda^p (1-\lambda^p) (1-q)}$$

Since V is a monotonic function of q we can write q as a function of V,

$$q = \frac{1}{\lambda^o (1-\lambda^o) - \lambda^p (1-\lambda^p)} \left( \frac{\sigma}{V} - \lambda^p (1-\lambda^p) \right) \equiv g(V) \quad . \text{ By Itô's Lemma,}$$

$$dV = V'(q(t)) dq(t) + \frac{1}{2} V''(q(t)) V^2(q(t)) dt$$

and noting that $dq(t) = a(q(t)) dt + b(q(t)) dB(t)$, and $q = g(V)$,

$$dV(t) = V'(g(V(t))) \alpha(g(V(t))) dt + \frac{1}{2} V''(g(V(t))) V^2(t) dt$$

$$+ V'(g(V(t))) V(t) dB(t)$$

which is a nonlinear, continuous time version of ARCH, since the conditional variance of the asset returns is time varying (thus heteroskedastic) and depends on its past (thus autoregressive).

The relationship goes further since the conditional mean of the returns, $a(q(t))$, also depends on $q(t)$, and therefore on the variance since $q = g(V)$. So we can write,

$$\frac{dP(t) + dD(t)}{P(t)} = a(g(V(t))) dt + V(t) dB(t)$$

This is a nonlinear version of ARCH-M.

Another type of statistical models that capture a stochastic process with time varying parameters are STAR models. A typical STAR model can be expressed as:

where A(L) and B(L) are polynomials of lag operators, u is a white noise disturbance term, and

$$y_t = (1-F(y_{t-d}))A(L)y_{t-1} + F(y_{t-d})B(L)y_{t-1} + u_t$$

F is a transition function, which is bounded by zero and one. Thus a STAR model assumes that the process of the variable in consideration is actually a weighted average of two distinct AR processes and the weight depends on a lagged value of the variable and hence changes over time. This type of model was applied to some aggregate variables by Teräsvirta and Anderson (1991) and Anderson (1992), but we do not know any application to financial data.

Our model also implies that the dynamics of the asset returns moves between two extreme cases (the case when $q = 0$ and that when $q = 1$). The relationship with the STAR model can be better seen by rewriting the asset price dynamics as

$$\frac{dD(t) + dP(t)}{dP(t)}$$

$$= \frac{\lambda^o(1-\lambda^o)q(t)}{\lambda^o(1-\lambda^o)q(t)+\lambda^P(1-\lambda^P)(1-q(t))}\left(\frac{\pi^*-\lambda^o c^o}{\lambda^o(1-\lambda^o)}dt+\frac{\sigma}{\lambda^o(1-\lambda^o)}dB(t)\right)$$

$$+ \frac{\lambda^P(1-\lambda^P)(1-q(t))}{\lambda^o(1-\lambda^o)q(t)+\lambda^P(1-\lambda^P)(1-q(t))}\left(\frac{\pi^*-\lambda^P c^P}{\lambda^P(1-\lambda^P)}dt+\frac{\sigma}{\lambda^P(1-\lambda^P)}dB(t)\right)$$

One difference between our model and STAR model is the conditional variance of the return process. As we saw above, our model implies time varying conditional variance, while a STAR model assumes a constant variance.

One may be able to slightly modify our model to find a more direct link to the STAR model. This is one of the agendas for future research.

## 5. FUNDAMENTALISTS AND CONTRARIANS

This section studies a model in which some agents use marketwide information when making their investment decisions. The first type of agents are called fundamentalists because they know the process which generates the profits of the firm and they act in a way that would be optimal if they were the only type of agents in the market. The second type of agents are called contrarians. They subscribe to the theory that the majority of agents are adopting wrong positions in the financial markets at all times, thus they adopt the opposite position to the average investment shares in the risky asset.

As with the previous model there are two types of assets in the economy, a riskless asset with instantaneous rate of return r and a risky asset whose price is P(t). Let D(t) be the total amount of dividends accumulated for a share of the risky asset from time 0 to t, and dD(t) instantaneous dividends. At time t there are N(t) shares outstanding of the risky asset. Let $\lambda^F$ be the proportion invested by the first type of agents in the risky asset out of their wealth $W^F(t)$, and $\lambda_c(t)$ the proportion invested by the contrarians in the risky asset out of their wealth $W^C(t)$. The total wealth in the economy, is $W(t) = W^F(t) + W^C(t)$. The assumption about the contrarians is that $\lambda^c(t) = 1 - \dfrac{P(t)N(t)}{W(t)}$. Contrarians assume that the market is wrong Since the market portfolio puts $\dfrac{P(t)N(t)}{W(t)}$ into the risky asset, contrarians put $1 - \dfrac{P(t)N(t)}{W(t)}$ into the risky asset and put $\dfrac{P(t)N(t)}{W(t)}$ into the safe asset.

We proceed very much in the same fashion as we did in Section 2, and derive the

process for the equilibrium rate of return in this economy and the process for the contrarians'

share of wealth, which is denoted by $q(t) \equiv \dfrac{W^c(t)}{W^c(t) + W^F(t)}$ .

Market clearing implies:

$$\left(1 - \frac{P(t)N(t)}{W(t)}\right)q(t) + \lambda^F(1 - q(t)) = \frac{P(t)N(t)}{W(t)}$$

or

$$\left(1 - \lambda^F + \frac{2\lambda^F - 1}{1 + q(t)}\right)W(t) = P(t)N(t) \quad (12)$$

Consumption for both types of agents is assumed to be a fixed proportion of their wealth, $c^F$ for the fundamentalists and $c^C$ for the contrarians. The evolution of wealth for each type of investors given the assumptions about the asset returns and consumption, is

$$dW^i(t) = \left(\lambda^i(t) \frac{dP(t) + dD(t)}{P(t)} + (1 - \lambda^i(t))rdt - c^idt\right)W^i(t), \quad (13)$$
$$i = C, F$$

The process for instantaneous profits is again assumed to be:

$$d\pi^*(t) = (\pi^*dt + \sigma dB(t))W(t)$$

which is used to buy back shares or pay dividends.

$$d\pi^*(t) = -dN(t)(P(t) + dP(t)) + N(t)dD(t) \quad (14)$$

Following similar steps to those taken in section 2 to derive Proposition A, we can prove the following proposition which establishes the process of equilibrium asset returns and the contrarians' share of wealth.

**Proposition B.** Let

$$\frac{dP(t) + dD(t)}{P(t)} = a(t)\,dt + b(t)\,dB(t) \quad (15)$$

and

$$dq(t) = c(t) + g(t)\,dB(t). \quad (16)$$

Then,

$$g(t) = -\frac{q(t)(1-q(t))\dfrac{2\lambda^F-1}{1+q(t)}}{(1-\lambda^F)\lambda^F+(2\lambda^F-1)^2\dfrac{2q(t)^2}{(1+q(t))^3}}\sigma$$

$$b(t) = \frac{\sigma - g(t)\dfrac{2\lambda^F-1}{(1+q(t))^2}}{(1-\lambda^F)\lambda^F+(2\lambda^F-1)^2\dfrac{q(t)}{(1+q(t))^2}}$$

$$c(t) = \frac{g(t)}{\sigma}\Bigg[\pi^* + \frac{2\lambda^F-1}{(1+q(t))^3}g(t)^2$$

$$+\left(1-\lambda^F+\frac{2\lambda^F-1}{1+q(t)}\right)(-q(t)c^C-(1-q(t))c^F)$$

$$-\frac{2\lambda^F-1}{(1+q(t))^2}g(t)b(t)\left(1-\lambda^F+\frac{2\lambda^F-1}{1+q(t)}\right)\Bigg]$$

$$-q(t)(1-q(t))(c^C-c^F)\frac{(1-\lambda^F)\lambda^F+\dfrac{(2\lambda^F-1)^2q(t)}{(1+q(t))^2}}{(1-\lambda^F)\lambda^F+\dfrac{(2\lambda^F-1)^22q(t)^2}{(1+q(t))^3}}$$

$$+ q(t) (1 - q(t)) \left( \frac{2\lambda^F - 1}{1 + q(t)} \right) \left( 1 - \lambda^F + \frac{2\lambda^F - 1}{1 + q(t)} \right) b(t)^2$$

$$a(t) = \frac{\pi^* + \dfrac{2\lambda^F - 1}{(1 + q(t))^3} g(t)^2}{(1 - \lambda^F)\lambda^F + \dfrac{(2\lambda^F - 1)q(t)}{(1 + q(t))^2}} - \frac{\dfrac{2\lambda^F - 1}{(1 + q(t))^2} g(t) b(t)}{\lambda^F - \dfrac{2\lambda^F - 1}{1 + q(t)}}$$

$$- \frac{\dfrac{2\lambda^F - 1}{(1 + q(t))^2}}{(1 - \lambda^F)\lambda^F + \dfrac{(2\lambda^F - 1)q(t)}{(1 + q(t))^2}} c(t) - \frac{c^C q(t) + c^F(1 - q(t))}{\lambda^F - \dfrac{2\lambda^F - 1}{1 + q(t)}} + r dt$$

**Proof:** See the appendix.

The expressions for c(t) and g(t) show that the drift and the diffusion terms for the stochastic process q(t) are functions of q(t) itself, so we can write

$$dq(t) = a_1(q(t)) dt + b_1(q(t)) dB(t)$$

The limiting properties of the process q(t) can be found again by analyzing the function $D(q) = \dfrac{a_1(q) q(1-q)}{b_1(q)^2}$ . Since the function D(q) is again bounded and continuously differentiable in [0, 1] Proposition 1 directly applies.

We can show that all four cases within the proposition are possible. We again show this for the case $c^C = c^F = 0$. Given that the inequalities in the proposition are strict, showing that all the cases are possible for an open nonempty set of the subset of the parameter space where

$c^C = c^F = 0$ guarantees that all cases are also possible for an open nonempty subset of the whole parameter space.

If $c^C = c^F = 0$, $D(0) > 1$ if and only $\dfrac{2\pi^*}{\sigma^2} > \dfrac{1}{\lambda^F(1-\lambda^F)}$ when $\lambda^F < \dfrac{1}{2}$ and $D(1)$ > -1 if and only if $\dfrac{2\pi^*}{\sigma^2} < 2+4\lambda^F$ when $\lambda^F < \dfrac{1}{2}$ . When $\lambda^F > \dfrac{1}{2}$ these inequalities are reversed.

$$a) \quad \lambda^F > \frac{1}{2} \;\wedge\; 2+4\lambda^F < \frac{2\pi^*}{\sigma^2} < \frac{1}{\lambda^F(1-\lambda^F)}$$

$$b) \quad \frac{2\pi^*}{\sigma^2} > 2+4\lambda^F \;\wedge\; \frac{2\pi^*}{\sigma^2} > \frac{1}{\lambda^F(1-\lambda^F)} \quad if \;\; \lambda^F < \frac{1}{2}$$

$$\frac{2\pi^*}{\sigma^2} < 2+4\lambda^F \;\wedge\; \frac{2\pi^*}{\sigma^2} < \frac{1}{\lambda^F(1-\lambda^F)} \quad if \;\; \lambda^F > \frac{1}{2}$$

$$c) \quad \frac{2\pi^*}{\sigma^2} < 2+4\lambda^F \;\wedge\; \frac{2\pi^*}{\sigma^2} < \frac{1}{\lambda^F(1-\lambda^F)} \quad if \;\; \lambda^F < \frac{1}{2}$$

$$\frac{2\pi^*}{\sigma^2} > 2+4\lambda^F \;\wedge\; \frac{2\pi^*}{\sigma^2} > \frac{1}{\lambda^F(1-\lambda^F)} \quad if \;\; \lambda^F > \frac{1}{2}$$

$$d) \quad 2+4\lambda^F < \frac{2\pi^*}{\sigma^2} < \frac{1}{\lambda^F(1-\lambda^F)} \quad if \;\; \lambda^F < \frac{1}{2}$$

$$2+4\lambda^F > \frac{2\pi^*}{\sigma^2} > \frac{1}{\lambda^F(1-\lambda^F)} \quad if \;\; \lambda^F > \frac{1}{2}$$

All of these four cases are possible for some parameter values of the model, as Figure 2 clearly demonstrates.

By rewriting the inequalities that incorporate D(0) and D(1), we can again intuitively understand the results. Noting that

$$D(0) = \frac{2}{(1-2\lambda^F)\sigma^2}\left[(1-\lambda^F)\lambda^F(\pi^*-\lambda^F c^F) - \frac{(c^C-c^F)}{1-2\lambda^F}((1-\lambda^F)\lambda^F)^2 - \lambda^F\sigma^2\right]$$

$$D(1) = \frac{4}{(1-2\lambda^F)\sigma^2}\left[\frac{1}{4}\left(\pi^*-\frac{1}{2}c^C\right) - \frac{(c^C-c^F)}{1-2\lambda^F}\frac{1}{8} - \frac{1}{2}\sigma^2\right]$$

in this case, D(0) > 1 if and only if:

$$\left(\lambda_F^*(0)-\lambda^F\right)^2 - \left(\lambda_F^*(0)-(1-\lambda^F)\right)^2 - \frac{(c^C-c^F)((1-\lambda^F)\lambda^F)^2}{\sigma^2} > 0 \quad (22)$$

where $\lambda_F^*(0) = \dfrac{\lambda^F(1-\lambda^F)(\pi^*-\lambda^F c^F)}{\sigma^2}$.

Note that $\lambda_F^*(0)$ is the proportion of wealth that the agents with logarithmic utility would invest in the risky asset if their beliefs were correct at the state where q=0, that is, when there are no contrarians. Suppose that both types of investors have the same propensity to consume. Then whether D(0) > 1 holds depends on whether the contrarians' investment strategy is closer to $\lambda_F^*(0)$ than the fundamentalists' investment strategy. If this is the case, the share of the contrarians' wealth never converges to zero, because the contrarians do better when their share of wealth is small. But if the fundamentalists' investment strategy is closer to $\lambda_F^*(0)$ than the contrarians' investment strategy, then the share of wealth of the fundamentalists will end up being one, with positive probability, because the fundamentalists do better when the contrarians' share of the wealth is small. If their rates of consumption are different and if $c^C$ is

larger than $c^F$ the inequality in (22) will be more difficult to satisfy if everything else remains constant, since the contrarians' higher propensity to consume slows down their wealth accumulation.

Similarly $D(1) < -1$ if and only if

$$\left(\lambda_F^{\bullet}(1) - \lambda^F\right)^2 - \left(\lambda_F^{\bullet}(1) - \frac{1}{2}\right)^2 - \frac{(c^C - c^F)\frac{1}{8}}{\sigma^2} < 0 \quad (23)$$

where $\quad \lambda_F^{\bullet}(1) = \dfrac{\pi^{\bullet} - \dfrac{1}{2}c^C}{4\sigma^2}$ .

Note that $\lambda_F^{\bullet}(1)$ is the proportion of wealth that an agent with logarithmic utility would invest in the risky asset if their beliefs were correct at the state where q=1, that is, no fundamentalists. Suppose both types of investors have the same consumption rate. Then the fundamentalists' strategy is closer to $\lambda_F^{\bullet}(1)$ than the contrarians' strategy, and the share of the wealth held by the fundamentalists will never converge to zero. But if the contrarians' strategy is closer to $\lambda_F^{\bullet}(1)$ than the fundamentalists' strategy, the share of the wealth held by the contrarians will end up being one with positive probability. A higher consumption rate means a lower rate of wealth accumulation and therefore if $C^F$ is larger than $C^C$ the inequality in (23) will be more difficult to satisfy, if everything else remains constant.

## 6. CONCLUDING REMARKS

Models with heterogeneous beliefs are potentially very important in explaining anomalies in the asset markets. With a few exceptions, however, the past literature on the asset markets

with heterogeneous investors did not pay much attention to the dynamic aspect of the models. This paper is an attempt to advance the important research on the dynamic properties of the models of asset markets with heterogeneous investors, originated by De Long et al. (1990) and Blume and Easley (1992) among others.

In this paper, we showed that the introduction of simple heterogeneity of beliefs into a simple model of asset pricing produces a rich dynamic of asset returns. Depending on the parameter values of the model, one type of investor may eventually have an infinitely larger wealth than the other investors, but it is also possible that two types of investors co-exist even in the long run while they keep influencing the asset returns.

One important empirical implication from our analysis is the presence of smooth regime shifts in the dynamics of asset returns, which are caused by the changes in the distribution of wealth among heterogeneous investors. As we have shown in Section 4, this implication is roughly consistent with a well known empirical regularity in financial markets: we often find ARCH in asset returns. We have not, however, put this implication through rigorous testing, which is obviously an important agenda for future work.

**Table 1.** Values of D(0), D(1) and the asymptotic properties of the q(t) process.

|  | D(1) < -1 | D(1) > -1 |
|---|---|---|
| D(0) > 1 | (a) ergodic distribution. | (b) a. s. convergence to 1. This is the limiting case of d) when $I_1$ is infinite and $I_2$ is finite. |
| D(0) < 1 | (c) a. s. convergence to 0. This is the limiting case of d) when $I_1$ is finite and $I_2$ is finite. | (d) convergence to 1 with probability $\dfrac{I_1}{I_1 + I_2}$, and to 0 with probability $\dfrac{I_2}{I_1 + I_2}$ |

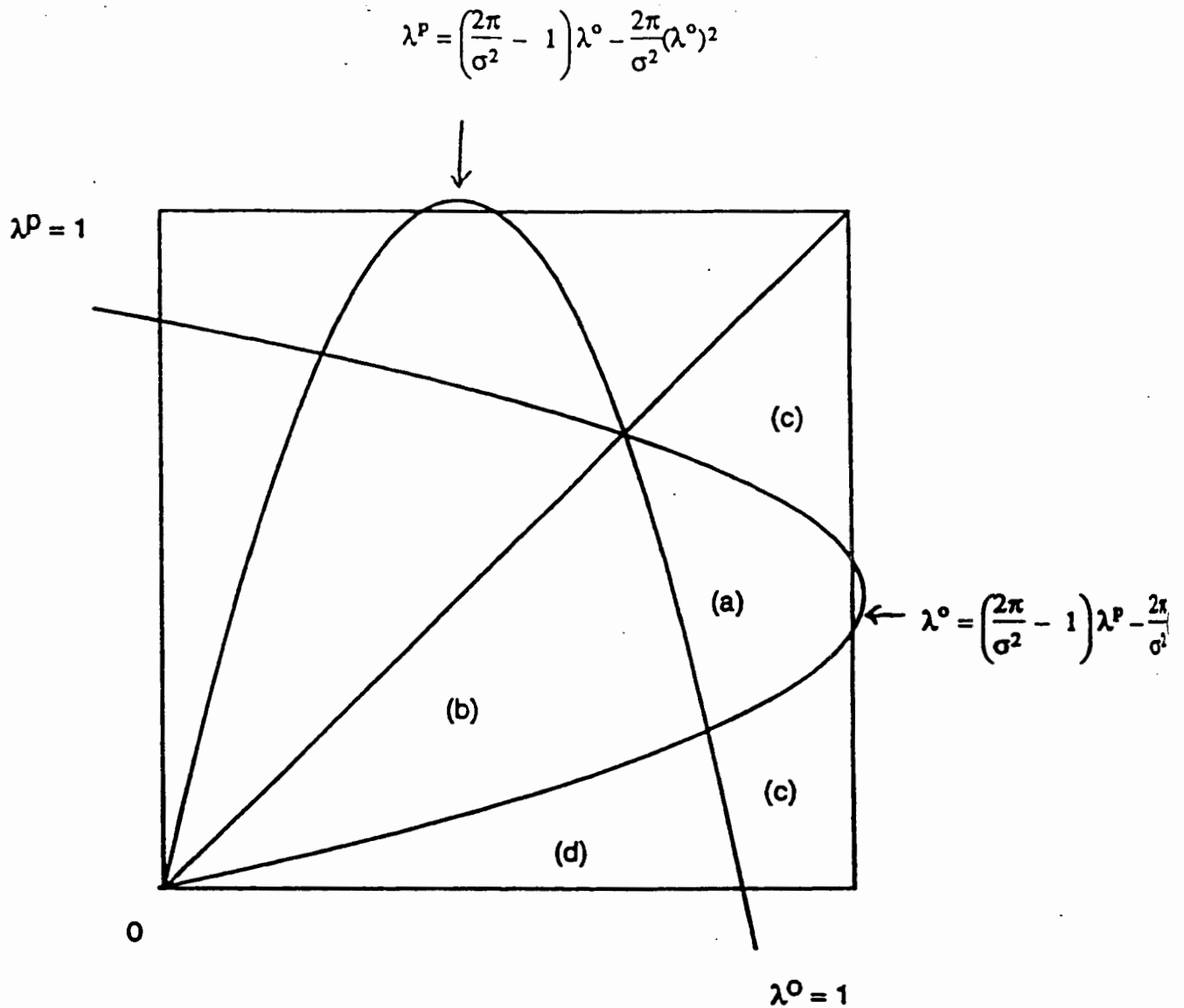**Figure 1.** The four cases from Proposition 1 for optimists and pessimists.



$$\lambda^P = \left(\frac{2\pi}{\sigma^2} - 1\right)\lambda^\circ - \frac{2\pi}{\sigma^2}(\lambda^\circ)^2$$

$\lambda^P = 1$

(c)

(a)

$\leftarrow \lambda^\circ = \left(\frac{2\pi}{\sigma^2} - 1\right)\lambda^P - \frac{2\pi}{\sigma^2}$

(b)

(c)

(d)

0

$\lambda^\circ = 1$

Figure 1

**Figure 2.** The four cases from Proposition 2 for fundamentalists and contrarians.

$$\frac{1}{\lambda^F(1-\lambda^F)}$$

$$\frac{2\pi^*}{\sigma^2}$$

(b)

(c)

(a)

$2+4\lambda^F$

$$\frac{2\pi}{\sigma^2}(\lambda$$

(d)

(d)

(c)

(b)

$\lambda^F = 0$

$\lambda^F = \frac{1}{2}$

$\lambda^F = \frac{1+\sqrt{7}}{6}$

$\lambda^F = 1$
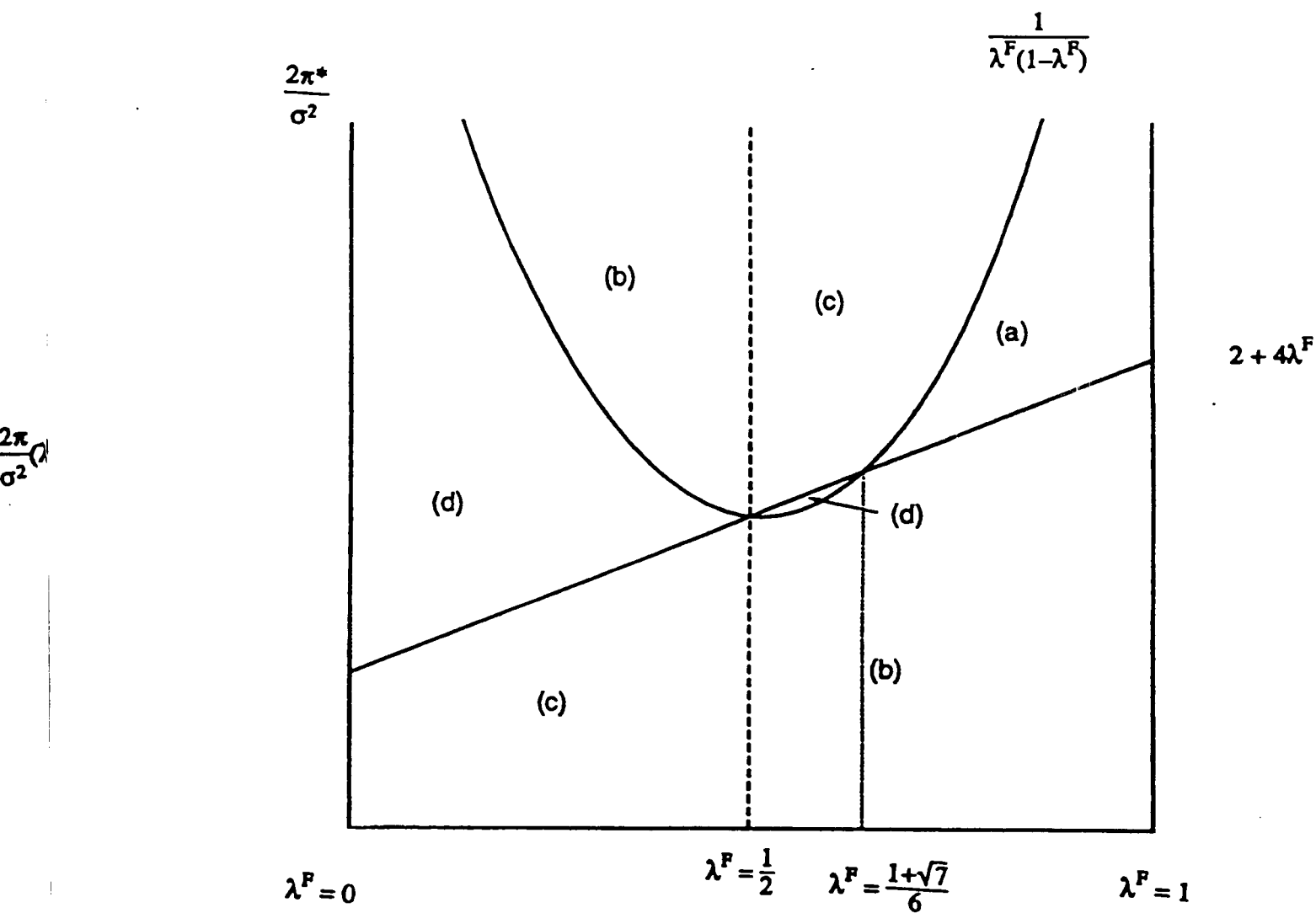
Figure 2

**APPENDIX**

**Proposition A.**

$$a) \quad \frac{dP(t)+dD(t)}{P(t)} = \frac{\pi^{\bullet}-\lambda^{o}c^{o}q(t)-\lambda^{P}c^{P}(1-q(t))}{\lambda^{o}(1-\lambda^{o})q(t)+\lambda^{P}(1-\lambda^{P})(1-q(t))}dt + rdt$$

$$+ \frac{1}{\lambda^{o}(1-\lambda^{o})q(t)+\lambda^{P}(1-\lambda^{P})(1-q(t))}\sigma dB(t) . \quad \textbf{(5)}$$

$$b) \quad dq(t)=q(t)(1-q(t))(\lambda^{o}-\lambda^{P})\left[\frac{\pi^{\bullet}-\lambda^{o}c^{o}q(t)-\lambda^{P}c^{P}(1-q(t))}{\lambda^{o}(1-\lambda^{o})q(t)+\lambda^{P}(1-\lambda^{P})(1-q(t))}\right.$$

$$\left. - \frac{c^{o}-c^{P}}{\lambda^{o}-\lambda^{P}}\right]dt - \frac{q(t)(1-q(t))(\lambda^{o}-\lambda^{P})(\lambda^{o}q(t)+\lambda^{P}(1-q(t))\sigma^{2}}{(\lambda^{o}(1-\lambda^{o})q(t)+\lambda^{P}(1-\lambda^{P})(1-q(t)))^{2}}dt$$

$$+ \frac{q(t)(1-q(t))(\lambda^{o}-\lambda^{P})\sigma}{\lambda^{o}(1-\lambda^{o})q(t)+\lambda^{P}(1-\lambda^{P})(1-q(t))}dB(t) ,$$

**Proof:**

Equation (2) implies:

$$\lambda^{o}dW^{o}(t)+\lambda^{P}dW^{P}(t) = dN(t)(P(t)+dP(t))+N(t)dP(t) . \quad \textbf{(4)}$$

Dividing equation (4) by W(t) and substituting (1), (2') and (3) into (4),

$$\lambda^{o}\left[\lambda^{o}\frac{dP(t)+dD(t)}{P(t)}+(1-\lambda^{o})rdt-c^{o}dt\right]q(t)$$

$$+ \lambda^P \left[ \lambda^P \frac{dP(t) + dD(t)}{P(t)} + (1-\lambda^P) rdt - c^P dt \right] (1-q(t))$$

$$= -(\pi^* dt + \sigma dB(t)) + \frac{dP(t) + dD(t)}{P(t)} (q(t) \lambda^o + (1-q(t)) \lambda^P)$$

Collecting the terms that are multiplied by $\dfrac{dP(t) + dD(t)}{P(t)}$ we get,

$$\frac{dP(t) + dD(t)}{P(t)} [\lambda^o (1-\lambda^o) q(t) + \lambda^P (1-\lambda^P) (1-q(t))] = \pi^* dt + \sigma dB(t)$$

$$+ [(\lambda^o (1-\lambda^o) q(t) + \lambda^P (1-\lambda^P) (1-q(t))) r - \lambda^o c^o q(t) - \lambda^P c^P (1-q(t))] dt,$$

hence

$$\frac{dP(t) + dD(t)}{P(t)} = \frac{\pi^* - \lambda^o c^o q(t) - \lambda^P c^P (1-q(t))}{\lambda^o (1-\lambda^o) q(t) + \lambda^P (1-\lambda^P) (1-q(t))} dt + rdt \quad (5)$$

$$+ \frac{1}{\lambda^o (1-\lambda^o) q(t) + \lambda^P (1-\lambda^P) (1-q(t))} \sigma dB(t).$$

Applying Itô's formula to the definition of q(t) and simplifying the resulting expression,

$$dq(t) = q(t) (1-q(t)) (\lambda^o - \lambda^P) \left[ \frac{dP(t) + dD(t)}{P(t)} - rdt - \frac{c^o - c^P}{\lambda^o - \lambda^P} \right] \quad (6)$$

$$-q(t)(1-q(t))(\lambda^o-\lambda^P)(\lambda^o q(t)+\lambda^P(1-q(t)))\left(\frac{dP(t)+dD(t)}{P(t)}\right)^2.$$

Substituting (5) into (6),

$$dq(t)=q(t)(1-q(t))(\lambda^o-\lambda^P)\left[\frac{\pi^*-\lambda^o c^o q(t)-\lambda^P c^P(1-q(t))}{\lambda^o(1-\lambda^o)q(t)+\lambda^P(1-\lambda^P)(1-q(t))}-\frac{c^o-c^P}{\lambda^o-\lambda^P}\right]dt$$

$$-\frac{q(t)(1-q(t))(\lambda^o-\lambda^P)(\lambda^o q(t)+\lambda^P(1-q(t)))\sigma^2}{(\lambda^o(1-\lambda^o)q(t)+\lambda^P(1-\lambda^P)(1-q(t)))^2}dt$$

$$+\frac{q(t)(1-q(t))(\lambda^o-\lambda^P)\sigma}{\lambda^o(1-\lambda^o)q(t)+\lambda^P(1-\lambda^P)(1-q(t))}dB(t),$$

or using more compact notation,

$$dq(t)=a(q(t))dt+b(q(t))dB(t).$$

**Proposition B.** Let

$$\frac{dP(t)+dD(t)}{P(t)}=a(t)dt+b(t)dB(t) \quad \text{(15)}$$

and

$$dq(t)=c(t)+g(t)dB(t). \quad \text{(16)}$$

Then,

$$g(t) = - \frac{q(t)(1 - q(t))\frac{2\lambda^F - 1}{1 + q(t)}}{(1 - \lambda^F)\lambda^F + (2\lambda^F - 1)^2\frac{2q(t)^2}{(1 + q(t))^3}}\sigma$$

$$b(t) = \frac{\sigma - g(t)\frac{2\lambda^F - 1}{(1 + q(t))^2}}{(1 - \lambda^F)\lambda^F + (2\lambda^F - 1)^2\frac{q(t)}{(1 + q(t))^2}}$$

$$c(t) = \frac{g(t)}{\sigma}\left[\pi^* + \frac{2\lambda^F - 1}{(1 + q(t))^3}g(t)^2\right.$$

$$+ \left(1 - \lambda^F + \frac{2\lambda^F - 1}{1 + q(t)}\right)(-q(t)c^C - (1-q(t))c^F)$$

$$- \frac{2\lambda^F - 1}{(1 + q(t))^2}g(t)b(t)\left(1 - \lambda^F + \frac{2\lambda^F - 1}{1 + q(t)}\right)\Bigg]$$

$$- q(t)(1 - q(t))(c^C - c^F)\frac{(1-\lambda^F)\lambda^F + \frac{(2\lambda^F - 1)^2 q(t)}{(1 + q(t))^2}}{(1-\lambda^F)\lambda^F + \frac{(2\lambda^F - 1)^2 2q(t)^2}{(1 + q(t))^3}}$$

$$+ q(t)(1 - q(t))\left(\frac{2\lambda^F - 1}{1 + q(t)}\right)\left(1 - \lambda^F + \frac{2\lambda^F - 1}{1 + q(t)}\right)b(t)^2$$

$$a(t) = \frac{\pi^* + \frac{2\lambda^F - 1}{(1 + q(t))^3}g(t)^2}{(1 - \lambda^F)\lambda^F + \frac{(2\lambda^F - 1)q(t)}{(1 + q(t))^2}} - \frac{\frac{2\lambda^F - 1}{(1 + q(t))^2}g(t)b(t)}{\lambda^F - \frac{2\lambda^F - 1}{1 + q(t)}}$$

**Proof:** Using Itô's rule on equation (12) we get,

$$- \frac{\frac{2\lambda^F - 1}{(1 + q(t))^2}}{(1 - \lambda^F)\lambda^F + \frac{(2\lambda^F - 1)q(t)}{(1 + q(t))^2}} c(t) - \frac{c^C q(t) + c^F(1-q(t))}{\lambda^F - \frac{2\lambda^F - 1}{1 + q(t)}} + rdt$$

$$\left( -\frac{2\lambda^F-1}{(1+q(t))^2}dq(t) + \frac{2\lambda^F-1}{(1+q(t))^3}(dq(t))^2 \right)W(t) + \left( 1-\lambda^F + \frac{2\lambda^F-1}{1+q(t)} \right)dW(t)$$

$$-\frac{2\lambda^F-1}{(1+q(t))^2}dq(t)\,dW(t) = (dP(t)+P(t))\,dN(t) + dP(t)N(t) \quad (17)$$

Now substituting equations (12) and (14) into (17) and dividing by W(t), we get

$$- \frac{2\lambda^F - 1}{(1+q(t))^2}dq(t) + \frac{2\lambda^F - 1}{(1+q(t))^3}(dq(t))^2 + \left( 1 - \lambda^F + \frac{2\lambda^F - 1}{1+q(t)} \right)\frac{dW(t)}{W(t)}$$

$$- \frac{2\lambda^F - 1}{(1+q(t))^2}dq(t)\frac{dW(t)}{W(t)} \quad (18)$$

$$= -\pi^* dt - \sigma dB(t) + \frac{dP(t)+dD(t)}{P(t)}\left( 1-\lambda^F + \frac{2\lambda^F-1}{1+q(t)} \right)$$

Since $dW(t) = dW^C(t) + dW^F(t)$

$$\frac{dW(t)}{W(t)} = q(t)\left( (1 - \lambda^C(t))rdt + \lambda^C(t)\frac{dP(t)+dD(t)}{P(t)} - c^C dt \right)$$

$$+ (1 - q(t))\left((1 - \lambda^F) r dt + \lambda^F \frac{dP(t) + dD(t)}{P(t)} - c^F dt\right),$$

using equation (13). Then, combined with equation (12), the above equation reduces to the following:

$$\frac{dW(t)}{W(t)} = \left(\frac{dP(t) + dDt)}{P(t)} - r dt\right)\left(1 - \lambda^F + \frac{2\lambda^F - 1}{1 + q(t)}\right) \quad (19)$$

$$+ r dt - q(t) c^c dt - (1 - q(t)) c^F dt$$

Substituting (15), (16) and (19) into (18) we obtain

$$- \frac{2\lambda^F - 1}{(1 + q(t))^2} dq(t) + \frac{2\lambda^F - 1}{(1 - q(t))^3} g(t)^2 dt +$$

$$+ \left(1 - \lambda^F + \frac{2\lambda^F - 1}{1 + q(t)}\right)\left[\left(\frac{dP(t) + dD(t)}{P(t)} - r dt\right)\left(1 - \lambda^F + \frac{2\lambda^F - 1}{1 + q(t)}\right)\right.$$

$$+ r dt - q(t) c^c dt - (1 - q(t)) c^F dt]$$

$$- \frac{2\lambda^F - 1}{(1 + q(t))^2} g(t) b(t)\left((1 - \lambda^F) + \frac{2\lambda^F - 1}{1 + q(t)}\right) dt$$

$$= - \pi^* dt - \sigma dB(t) + \frac{dP(t) + dD(t)}{P(t)}\left(1 - \lambda^F + \frac{2\lambda^F - 1}{1 - q(t)}\right)$$

Collecting the terms that multiply $\dfrac{dP(t)+dD(t)}{P(t)}$ we find that,

$$\frac{dP(t)+dD(t)}{P(t)}\left(1-\lambda^F+\frac{2\lambda^F-1}{1+q(t)}\right)\left(\lambda^F-\frac{2\lambda^F-1}{1+q(t)}\right)=\pi^{*}dt+\sigma dB(t)$$

$$-\frac{2\lambda^F-1}{(1+q(t))^2}dq(t)\ +\ \frac{2\lambda^F-1}{(1+q(t))^3}g(t)^2dt\ +\ (20)$$

$$+\left(1-\lambda^F+\frac{2\lambda^F-1}{1+q(t)}\right)\left[\left(\lambda^F-\frac{2\lambda^F-1}{1+q(t)}\right)rdt\ -\ q(t)c^Cdt\ -\ (1-q(t))c^Fdt\right]$$

$$-\frac{2\lambda^F-1}{(1+q(t))^2}g(t)b(t)\left((1-\lambda^F)\ +\ \frac{2\lambda^F-1}{1+q(t)}\right)dt$$

By Itô's rule,

$$dq(t)\ =\ q(t)(1-q(t))\left(\frac{-(2\lambda^F-1)}{1-q(t)}\right)\left(\frac{dP(t)+dD(t)}{P(t)}-rdt+(c^C-c^F)\frac{1-q(t)}{2\lambda^F-1}\right)$$

$$-\ q(t)(1-q(t))\left(\frac{-(2\lambda^F-1)}{1+q(t)}\right)\left(1-\lambda^F\ +\ \frac{2\lambda^F-1}{1+q(t)}\right)\left(\frac{dP(t)}{P(t)}\right)^2\ (21)$$

Substituting (20) into (21) and then identifying coefficients in the resulting expression with (15) and (16) the proposition follows.

**Proposition 1.** a) If $D(0)>1$ and $D(1)<-1$ the process is ergodic. b) If

$D(0) > 1$ and $D(1) > -1$ the process converges to 1 almost surely. c) If $D(0) < 1$ and $D(1) < -1$ the process converges to 0 almost surely. d) If $D(0) < 1$ and $D(1) > -1$ the process to 1 with probability $\dfrac{I_1}{I_1+I_2}$ and to 0 with complementary probability.

**Proof:** The proof requires establishing two lemmas.

**Lemma 1.** a) If $D(0) > 1$, $I_1$ is infinite. b) If $D(0) < 1$, $I_1$ is finite.

c) If $D(1) < -1$, $I_2$ is infinite. d) If $D(1) > -1$, $I_2$ is finite.

**Proof:** a) If $D(0) > 1$, there is $\delta > 1$ and $q_\delta > 0$ such that $D(q) > \delta \; \forall q < q_\delta$.

$$I_1 = \exp\left(-\int_x^{q_\delta} \frac{D(q)}{q(1-q)}\,dq\right)\left\{\int_0^{q_\delta} \exp\left[\int_x^{q_\delta} \frac{D(q)}{q(1-q)}\,dq\right]dx\right\}$$

$$+ \int_{q_\delta}^{q(0)} \exp\left[-\int_x^x \frac{D(q)}{q(1-q)}\,dq\right]dx,$$

but

$$\int_0^{q_\delta} \exp\left[\int_x^{q_\delta} \frac{D(q)}{q(1-q)}\,dq\right]dx \;\geq\; \int_0^{q_\delta} \exp\left[\int_x^{q_\delta} \frac{\delta}{q(1-q)}\,dq\right]dx$$

$$\geq \int_0^{q_\delta} \exp\left[\int_x^{q_\delta} \frac{\delta}{q}\,dq\right]dx \;=\; \int_0^{q_\delta} \exp\delta\,(\ln q_\delta - \ln x)\,dx \;=\; \int_0^{q_\delta} \left(\frac{q_\delta}{x}\right)^\delta dx,$$

which is infinite since $\delta > 1$. Since $D(x)$ is bounded $\displaystyle\int_{q_\delta}^{q(0)} \exp\left[-\int_z^x \frac{D(q)}{q(1-q)}\,dq\right]dx$ and

$\displaystyle\exp\left[-\int_z^{q_\delta} \frac{D(q)}{q(1-q)}\,dq\right]$ are strictly positive. Thus $I_1$ is infinite.

If $D(0) < 1$, there is $q_\delta > 0$ and $\delta < 1 - q_\delta$ such that $D(q) < \delta$ for all $q < q_\delta$.

$$I_1 = \exp\left(-\int_z^{q_\delta} \frac{D(q)}{q(1-q)}\,dq\right)\left\{\int_0^{q_\delta}\exp\left[\int_x^{q_\delta} \frac{D(q)}{q(1-q)}\,dq\right]dx\right\}$$

$$+ \int_{q_\delta}^{q(0)} \exp\left[-\int_z^x \frac{D(q)}{q(1-q)}\,dq\right]dx,$$

but

$$\int_0^{q_\delta} \exp\left[\int_x^{q_\delta} \frac{D(q)}{q(1-q)}\,dq\right]dx \;\leq\; \int_0^{q_\delta} \exp\left[\int_x^{q_\delta} \frac{\delta}{q(1-q)}\,dq\right]dx$$

$$\leq \int_0^{q_\delta} \exp\left[\int_x^{q_\delta} \frac{\delta}{(1-q_\delta)q} dq\right] dx = \int_0^{q_\delta} \exp\frac{\delta}{1-q_\delta}(\ln q_\delta - \ln x) dx = \int_0^{q_\delta}\left(\frac{q_\delta}{x}\right)^{\frac{\delta}{1-q_\delta}} dx$$

which is finite since $\dfrac{\delta}{1-q_\delta} < 1$ . Since D(x) is bounded, and $q_\delta$ and z are in the interval

(0, 1), $\displaystyle\int_{q_\delta}^{q(0)} \exp\left[-\int_z^x \frac{D(q)}{q(1-q)} dq\right] dx$ and $\exp\left[-\int_z^{q_\delta} \frac{D(q)}{q(1-q)} dq\right]$ are bounded. Thus $I_1$

is finite. One can show c) and d) in a similar fashion.

**Lemma 2:** If $D(0) > 1$ and $D(1) < -1$ , $\displaystyle\int_0^1 M(x)\,dx$ is finite.

**Proof:** If $D(0) > 1$ and $D(1) < -1$ , there are $1 > q_{\delta_2} > q_{\delta_1} > 0$ , $\delta_1 > 1$ and

$\delta_2 < -1$ such that $D(q) > \delta_1$ for all $q < q_{\delta_1}$ and $D(q) < \delta_2$ for all $q > q_{\delta_2}$ .

$$\int_0^1 M(x)\,dx = \exp\left[\int_z^{q_{\delta_1}} \frac{D(q)}{q(1-q)} dq\right]\left\{\int_0^{q_{\delta_1}} \frac{2}{b(x)^2}\exp\left[-\int_x^{q_{\delta_1}} \frac{D(q)}{q(1-q)} dq\right] dx\right\}$$

$$+ \int_{q_{\delta_1}}^{q_{\delta_2}} \frac{2}{b(x)^2}\exp\left[\int_z^x \frac{D(q)}{q(1-q)} dq\right] dx$$

$$+ \exp\left[\int_z^{q_{\delta_2}} \frac{D(q)}{q(1-q)} dq\right]\left\{\int_{q_{\delta_2}}^1 \frac{2}{b(x)^2}\exp\left[-\int_x^{q_{\delta_2}} \frac{D(q)}{q(1-q)} dq\right] dx\right\}$$

But

$$\int_0^{q_{\delta_1}} \frac{2}{b(x)^2} \exp\left[-\int_x^{q_{\delta_1}} \frac{D(q)}{q(1-q)} dq\right] \le \int_0^{q_{\delta_1}} \frac{2}{b(x)^2} \exp\left[-\delta_1 \ln\left(\frac{q_{\delta_1}}{x}\right)\right] dx$$

$$\le \int_0^{q_{\delta_1}} \frac{2}{b(x)^2} \left(\frac{x}{q_{\delta_1}}\right)^{\delta_1} dx < \int_0^{q_{\delta_1}} \frac{2(|\lambda^o(1-\lambda^o)|+|\lambda^p(1-\lambda^p)|)^2}{\sigma^2(\lambda^o-\lambda^p)^2 x^2(1-q_{\delta_1})^2} \left(\frac{x}{q_{\delta_1}}\right)^{\delta_1} dx$$

where the last inequality follows because $0 < q(t) < 1$ and in that integral $1-x > 1-q_{\delta_1}$

$$= \frac{2(|\lambda^o(1-\lambda^o)|+|\lambda^p(1-\lambda^p)|)^2}{\sigma^2(\lambda^o-\lambda^p)^2(1-q_{\delta_1})^2 q_{\delta_1}^{\delta_1}} \int_0^{q_{\delta_1}} x^{\delta_1-2} dx$$

which is finite since $2-\delta_1<1$ . We also have,

$$\int_{q_{\delta_2}}^1 \frac{2}{b(x)^2} \exp\left[\int_{q_{\delta_2}}^x \frac{D(q)}{q(1-q)} dq\right] \le \int_{q_{\delta_2}}^1 \frac{2}{b(x)^2} \exp\left[-\delta_2 \ln\left(\frac{1-x}{1-q_{\delta_2}}\right)\right] dx$$

$$\le \int_{q_{\delta_2}}^1 \frac{2}{b(x)^2} \left(\frac{1-x}{1-q_{\delta_2}}\right)^{-\delta_2} dx < \int_{q_{\delta_2}}^1 \frac{2(|\lambda^o(1-\lambda^o)|+|\lambda^p(1-\lambda^p)|)^2}{\sigma^2(\lambda^o-\lambda^p)^2(1-x)^2 q_{\delta_2}^2} \left(\frac{1-x}{1-q_{\delta_2}}\right)^{-\delta_2} dx$$

where the last inequality follows because $0 < q(t) < 1$ and in that integral $x > q_{\delta_2}$

$$= \frac{2(|\lambda^o(1-\lambda^o)| + |\lambda^p(1-\lambda^p)|)^2}{\sigma^2(\lambda^o-\lambda^p)^2 q_{\delta_2}^2(1-q_{\delta_2})} \int_{q_{\delta_2}}^1 (1-x)^{-\delta_2-2} dx$$

which is finite since $-\delta_2-2 > -1$ .

$$\exp\left[\int_z^{q_{\delta_1}} \frac{D(q)}{q(1-q)}dq\right] \cdot \int_{q_{\delta_1}}^{q_{\delta_2}} \frac{2}{b(x)^2}\exp\left[\int_z^x \frac{D(q)}{q(1-q)}dq\right] \qquad \text{and} \qquad \exp\left[\int_z^{q_{\delta_2}} \frac{D(q)}{q(1-q)}dq\right]$$

are finite because $D(x)$ is bounded and $z$, $q_{\delta_1}$ and $q_{\delta_2}$ are in $(0, 1)$. Then the Lemma follows.

Lemma 1 and Lemma 2 establish the proposition.

**Proposition 2.** Suppose we have a riskless asset with rate of return $r$, and a risky asset with rate of return,

$$\frac{dP(t)+dD(t)}{dP(t)} = \alpha(t)dt + \beta(t)dB(t)$$

where $\alpha(t)$, $\beta(t)$ are bounded, measurable functions for all $t$, and $|\beta(s)| > \underline{\beta}$ for all $s$. Let $\lambda_1(t)$, $\lambda_2(t)$ be the investment shares of agents 1 and 2 in the risky asset, which

are also assumed to be bounded and measurable, and suppose that both agents consumption rates are $c^1(t)$ and $c^2(t)$. We will denote $k_i(t) = \lambda_i(t) - \dfrac{\alpha(t)-r}{\beta^2(t)}$ for $i = 1, 2$.

If $|k_1(t)| - |k_2(t)| > k^* > 0$ and $c_1(t) - c_2(t) > c^* > 0$ almost surely, $\dfrac{W^1(t)}{W^2(t)}$ converges to zero almost surely.

**Proof:** By equation (1) the evolution of wealth is given by,

$$dW^i(t) = \left(\lambda_i(t)\frac{dP(t)+dD(t)}{P(t)} + (1-\lambda_i(t))rdt - c^i(t)dt\right)W^i(t), \quad (9)$$

$$i = 1, 2$$

Let $h(t) = \log\left(\frac{W^1(t)}{W^2(t)}\right)$. By Itô's lemma, and equation (9)

$$dh(t) = \left((\lambda_1(t)-\lambda_2(t))(\alpha(t)-r) - \frac{1}{2}\beta(t)^2(\lambda_1(t)^2-\lambda_2(t)^2)\right)dt$$

$$- (c_1(t)-c_2(t))dt + (\lambda_1(t)-\lambda_2(t))\beta(t)dB(t)$$

$$= \frac{\beta(t)^2}{2}\left(\left(\lambda_2(t) - \frac{\alpha(t)-r}{\beta(t)^2}\right)^2 - \left(\lambda_1(t) - \frac{\alpha(t)-r}{\beta(t)^2}\right)^2\right)dt$$

$$-(c^1(t)-c^2(t))dt + (\lambda_1(t)-\lambda_2(t))\beta(t)dB(t)$$

$$h(t) = h(0) + \int_0^t \left[\frac{\beta(s)^2}{2}((k_2(s)^2-k_1(s)^2) - (C^1(s)-C^2(s)))\right]ds$$

$$+ \int_0^t (\lambda_1(s)-\lambda_2(s))\beta(s)dB(s)$$

Let $\overline{\beta} > |\beta(s)| > \underline{\beta}$, $\forall s$, and $\overline{\lambda} > |\lambda_1(s)-\lambda_2(s)| > k^*$, $\forall s$ almost surely.

By Gihman and Skorohod's (1972) theorem 1.1 p.20

Then by Chebyshev's inequality, for all $\epsilon > 0$

$$E\left[\sup_{2^n \le t \le 2^{n+1}}\left(\frac{\int_0^t (\lambda_1(s) - \lambda^2(s))\beta(s)\,dB(s)}{t}\right)^2\right]$$

$$\le \frac{1}{(2^n)^2}E\left[\sup_{2^n \le t \le 2^{n+1}}\left(\int_0^t (\lambda_1(s) - \lambda_2(s))\beta(s)\,dB(s)\right)^2\right]$$

$$\le \frac{1}{(2^n)^2}4\int_0^{2^{n+1}} E\left[(\lambda_1(s) - \lambda_2(s))^2\beta(s)^2\right]ds \le \frac{1}{(2^n)^2}4(\overline{\lambda\beta})^2 2^{n+1} \le \frac{8(\overline{\lambda\beta})^2}{2^n}$$

$$P\left(\sup_{2^n \le t \le 2^{n+1}}\frac{\left|\int_0^t (\lambda_1(s) - \lambda_2(s))\beta(s)\,dB(s)\right|}{t} > \epsilon\right) \le \frac{8(\overline{\lambda\beta})^2}{\epsilon^2}2^{-n}$$

Since $\displaystyle\sum_{n=1}^{\infty} \frac{8(\overline{\lambda\beta})^2}{\epsilon^2}2^{-n} < \infty$ , by the Borel-Cantelli lemma

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t (\lambda_1(s) - \lambda_2(s))\beta(s)\,dB(s) \to 0, a.s. \quad (24)$$

By the assumption of the proposition,

$$\frac{1}{t}\int_0^t \left[\frac{\beta(s)^2}{2}((k_2(s)^2-k_1(s)^2)-(c^1(s)-c^2(s)))\right]ds \quad (25)$$

$$\leq -\frac{\beta^2}{2}(k^\bullet)^2-c^\bullet < 0$$

Equations (24) and (25) imply that almost surely

$$\limsup_{t\to\infty}\left\{\frac{1}{t}h(t)\right\} \leq -\frac{\beta^2}{2}k^\bullet-c^\bullet$$

Therefore $h(t) \to -\infty$ almost surely, which implies that $\dfrac{W^1(t)}{W^2(t)} \to 0$ almost surely, which establishes the proposition.

## ENDNOTES

1.      Itô's rule is the analog in stochastic differential calculus to the chain rule in ordinary calculus. Let $X(t)$ be a stochastic integral (an Itô stochastic integral with respect to a continuous time stochastic process is defined in a similar fashion as a Riemann integral, taking limits of summations of the discrete time processes with time intervals whose length decreases to zero):

$$dX(t) = u\,dt + v\,dB(t)$$

Let $(t,x) \sim g(t,x) \in \mathbf{R}; \ t \in [0, \infty), \ x \in \mathbf{R}$ be $C^2$ (i.e. twice continuously differentiable) on $[0, \infty) \times \mathbf{R}$. Then $Y(t) = g(X(t))$ is again a stochastic integral, and

$$dY(t) = \frac{\partial g}{\partial t}(t, X(t))\,dt + \frac{\partial g}{\partial x}(t, X(t))\,dX(t) + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t, X(t))(dX(t))^2,$$

where $dt.dt = dt.dB(t) = dB(t).dt = 0$, $dB(t)dB(t) = dt$. See, for example, Øksendal (1989).

2.      When $c^o = c^p = 0$ then

$$\exp\int_z^x \frac{2a(q)}{b(q)^2} = \exp\int_0^z 2\,\frac{\pi^*(\lambda^o(1-\lambda^o)\,q + \lambda^p(1-\lambda^p)\,(1-q)) - (\lambda^o q + \lambda^p(1-q))\,\sigma^2}{(\lambda^o - \lambda^p)\,q(1-q)\,\sigma^2}\,dq$$

$$= \exp\int_z^x \frac{2}{(\lambda^o - \lambda^p)\,\sigma^2}\left[\frac{(\pi^*(1-\lambda^o) - \sigma^2)\,\lambda^o}{1-q} + \frac{(\pi^*(1-\lambda^p) - \sigma^2)\,\lambda^p}{q}\right]dq$$

$$= \exp\int_z^x\left(\frac{D(1)}{1-q} + \frac{D(0)}{q}\right)dq = \left(\frac{1-x}{1-z}\right)^{-D(1)}\left(\frac{x}{z}\right)^{D(0)}$$

In the case when $D(1) < -1$ and $D(0) > 1$, then, the density of the ergodic distribution is proportional to

$$M(x) = \frac{2\left((\lambda^o(1-\lambda^o)\,x+\lambda^P(1-\lambda^P)\,(1-x))^2\right)}{(\lambda^o-\lambda^P)^2\sigma^2 x^2(1-x)^2}\left(\frac{1-x}{1-z}\right)^{-D(1)}\left(\frac{x}{z}\right)^{D(0)}$$

3.  If an investor has logarithmic utility and expects the risky asset to follow a stochastic process given by $\alpha(t)\,dt + \beta(t)\,dB(t)$, and the riskless asset return to be r, then the optimal proportion of wealth invested in the risky asset is $\dfrac{\alpha(t)-r}{\beta(t)^2}$. In our model, if the investor has correct beliefs, then

$$\alpha(t) = \frac{\pi^*-\lambda^o c^o q(t)-\lambda^P c^P(1-q(t))}{\lambda^o(1-\lambda^o)q(t)+\lambda^P(1-\lambda^P)(1-q(t))}+r$$

$$\beta(t) = \frac{\sigma}{\lambda^o(1-\lambda^o)q(t)+\lambda^P(1-\lambda^P)(1-q(t))}$$

substituting this into $\dfrac{\alpha(t)-r}{\beta(t)^2}$. , we get

$$\frac{1}{\sigma^2}\left[\pi^*-\lambda^o c^o q(t)-\lambda^P c^P(1-q(t))\right]\left[\lambda^o(1-\lambda^o)q(t)+\lambda^P(1-\lambda^P)(1-q(t))\right]$$

as the proportion of the risky asset in the portfolio of such an investor. When $q = 0$, the above expression simplifies to:

$$\frac{1}{\sigma^2} \lambda^P (1 - \lambda^P) (\pi^{\bullet} - \lambda^P c^P)$$

which is the definition of $\lambda^{\bullet}(0)$ .

4.      "Sufficiently large" depends on the variance term, as equation (25) in the appendix makes clear. Stating precisely the condition would lead to a cumbersome Proposition 2 and add little to the intuition, therefore it is omitted.

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