The Daily Market for Funds in Europe: Mathematical Appendix*

Gabriel Pérez Quirós† European Central Bank

Hugo Rodríguez Mendizábal‡ Universitat Pompeu Fabra

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Abstract

This paper includes the derivations of the main expressions in the paper "The Daily Market for Funds in Europe: Has Something Changed With the EMU?" by G. Pérez Quirós and H. Rodríguez Mendizábal.

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* The views expressed in the paper are those of the authors and do not necessarily reflect the views of the European Central Bank or the European System of Central Banks.

† D.G. Research, European Central Bank. Kaiserstrasse, 29. 60311 Frankfurt am Main. Germany. E-mail: gabriel.perezquiros@ecb.int.

‡ Department of Economics and Business. Universitat Pompeu Fabra. Ramon Trias Fargas 25-27. 08005 Barcelona. Spain. E-mail: hugo.rodriguez@econ.upf.es.

1 Introduction

This paper includes the derivations of the main expressions in Pérez and Rodríguez [1]. The following results regarding derivatives of integrals will be used:

$$g_1(x) = \int_{-\infty}^{x} f(\epsilon)\, d\epsilon \rightarrow \frac{dg_1}{dx} = f(x),$$

$$g_2(x) = \int_{-\infty}^{x} \epsilon f(\epsilon)\, d\epsilon \rightarrow \frac{dg_2}{dx} = x f(x),$$

$$g_3[h(x)] = \int_{-\infty}^{h(x)} f(\epsilon)\, d\epsilon \rightarrow \frac{dg_3}{dx} = \frac{dh}{dx}\frac{dg_3}{dh},$$

$$g_4[h(x)] = \int_{-\infty}^{h(x)} \epsilon f(\epsilon)\, d\epsilon \rightarrow \frac{dg_4}{dx} = \frac{dh}{dx}\frac{dg_4}{dh},$$

$$g_5(x) = \int_{x}^{\infty} f(\epsilon)\, d\epsilon \rightarrow \frac{dg_5}{dx} = -f(x),$$

$$g_6(x) = \int_{x}^{\infty} \epsilon f(\epsilon)\, d\epsilon \rightarrow \frac{dg_6}{dx} = -x f(x),$$

$$g_7[h(x)] = \int_{h(x)}^{\infty} f(\epsilon)\, d\epsilon \rightarrow \frac{dg_7}{dx} = \frac{dh}{dx}\frac{dg_7}{dh},$$

$$g_8[h(x)] = \int_{h(x)}^{\infty} \epsilon f(\epsilon)\, d\epsilon \rightarrow \frac{dg_8}{dx} = \frac{dh}{dx}\frac{dg_8}{dh},$$

$$g_9(x) = \int_{-\infty}^{x} f(x,\epsilon)\, d\epsilon \rightarrow \frac{dg_9}{dx} = f(x,x) + \int_{-\infty}^{x} \frac{\partial f(x,\epsilon)}{\partial x} d\epsilon.$$

2 The Problem

This section develops a model of the overnight rate. It builds on the reserve management problem of a price-taking, representative bank. Implicitly, it is assumed that there exists a continuum of identical banks with measure one, each solving the same problem described here. The only perturbations hitting the system are aggregate shocks. Thus, there are no idiosyncratic risks and all aggregate variables coincide with their individual counterparts.

Assume the central bank requires financial institutions to maintain a total of reserves of R monetary units over a reserve maintenance period of T days. Denote by A_t the accumulated reserves at the beginning of day t by the representative bank. The initial wealth of this bank is divided into reserves voluntarily deposited at the central bank (M_t) and reserves loaned to other banks in the money market (B_t), that is,

$$A_t = M_t + B_t \tag{1}$$

with

$$M_t \geq 0.$$

Reserves are exchanged in the market at the interest rate i_t. Assume that after banks have gone to the market they receive an aggregate liquidity shock ϵ_t. This shock is i.i.d. with zero mean and probability distribution function $F(\epsilon)$. It takes the same value for all banks.

The representative bank ends up the day with a balance in the central bank of $M_t + \epsilon_t$. It is assumed that any financial institution has unrestricted access to the central bank's marginal lending facility at the interest rate i^l. This means that if the end of day's balance is negative, the bank has to borrow from the central bank the funds needed to set it back to zero. If the bank ends up with a positive balance, those reserves work towards satisfying the reserve requirement. Denote by R_t the increase in reserves accounted for the requirement,

$$R_t = \max\{0, M_t + \epsilon_t\}$$

and by L_t, the accumulated reserves accounted for the requirement up to time t,

$$L_t = \sum_{\tau=1}^{t} R_\tau.$$

The reserve requirement is fulfilled if

$$L_T \geq R. \tag{2}$$

Define by e_t the reserves needed in t to fulfill the reserve requirement for the whole maintenance period, that is,

$$e_t \equiv \max\{0, R - L_{t-1}\}.$$

Another way of writing (2) is

$$e_{T+1} = 0.$$

Once the reserve requirement is fulfilled, financial institutions can deposit excess reserves at the central bank. These deposits are remunerated at the interest rate i^d. It is assumed that $i^l > i^d$. In this model, required reserves do not earn any interest.

The objective of this bank is to decide on a sequence for $\{M_t\}_{t=1}^T$ to maximize the expected profits derived from managing its reserves within the maintenance period, that is,

$$\max_{\{M_t\}_{t=1}^T} E_1 (A_{T+1})$$

with

$$A_{t+1} = (1 + i_t) A_t + \epsilon_t - c_t$$

for $t = 1, 2, \ldots, T$, where c_t represents the net costs the bank incurs in managing its reserves. This term includes the opportunity cost of holding reserves ($i_t M_t$) but also comprises the interest paid on borrowing from the central bank net of the interest received from maintaining reserves there. The only information the bank needs to make its decision on M_t every day t, apart from interest rates, is its level of reserves A_t and its reserve deficiency e_t. The way to solve this problem is by backward induction. With this method, we first solve the problem at date T, and then work backwards towards the beginning of the maintenance period.

3 Problem at T

The solution of this problem is summarized by the function

$$V (A_T, e_T, i_T) = \max_{M_T} E_T (A_{T+1})$$

subject to

$$A_{T+1} = (1 + i_T) A_T + \epsilon_T - c_T.$$

The important point is to compute the variable c_T. We differentiate two cases depending on whether the reserve requirement has been satisfied already or not.

3.1 Case 1: $e_T = 0$

In this case, the bank has already satisfied the reserve requirement with the reserves accumulated up to time $T - 1$. We denote this situation by saying that the bank is "locked-in". The variable c_T is equal to:

$$c_T = i_T M_T - i^l (M_T + \epsilon_T) I \{\epsilon_T \leq -M_T\} - i^d (M_T + \epsilon_T) I \{\epsilon_T \geq -M_T\},$$

where $I\{X\}$ is an indicator function taking value 1 if event X occurs. Then, the problem is to maximize

$$\begin{aligned} E_T(A_{T+1}) &= (1+i_T)A_T - i_T M_T + i^l \int_{-\infty}^{-M_T} (M_T+\epsilon_T) f(\epsilon_T) d\epsilon_T \\ &\quad + i^d \int_{-M_T}^{\infty} (M_T+\epsilon_T) f(\epsilon_T) d\epsilon_T, \end{aligned}$$

or

$$\begin{aligned} E_T(A_{T+1}) &= (1+i_T)A_T - i_T M_T + i^l M_T F(-M_T) + i^l \int_{-\infty}^{-M_T} \epsilon_T f(\epsilon_T) d\epsilon_T \\ &\quad + i^d M_T [1-F(-M_T)] + i^d \int_{-M_T}^{\infty} \epsilon_T f(\epsilon_T) d\epsilon_T. \end{aligned}$$

The first order condition with respect to M_T is,

$$i_T = i^l F(-M_T) + i^d [1-F(-M_T)].$$

In equilibrium it has to be the case that $B_T = 0$, and $M_T = A_T$. This means that the equilibrium interest rate if the economy is locked-in is

$$i_T(e_T = 0) = i^l F(-A_T) + i^d [1-F(-A_T)].$$

Then, the value function will take the form

$$\begin{aligned} V(A_T, e_T, i_T | e_T = 0) &= A_T + i^l A_T F(-A_T) + i^l \int_{-\infty}^{-A_T} \epsilon_T f(\epsilon_T) d\epsilon_T \\ &\quad + i^d A_T [1-F(-A_T)] + i^d \int_{-A_T}^{\infty} \epsilon_T f(\epsilon_T) d\epsilon_T. \end{aligned}$$

3.2 Case 2: $e_T > 0$

In this case, the net cost of managing reserves is equal to

$$\begin{aligned} c_{hT} &= i_T M_T + i^l (e_T - M_T - \epsilon_T) I\{\epsilon_T \leq e_T - M_T\} \\ &\quad - i^d (M_T + \epsilon_T - e_T) I\{\epsilon_T \geq e_T - M_T\}. \end{aligned}$$

Then, the problem is to maximize

$$\begin{aligned} E_T(A_{T+1}) &= (1+i_T)A_T - i_T M_T - i^l \int_{-\infty}^{e_T - M_T} (e_T - M_T - \epsilon_T) f(\epsilon_T) d\epsilon_T \\ &+ i^d \int_{e_T - M_T}^{\infty} (M_T + \epsilon_T - e_T) f(\epsilon_T) d\epsilon_T, \end{aligned}$$

or

$$\begin{aligned} E_T(A_{T+1}) &= (1+i_T)A_T - i_T M_T + i^l \int_{-\infty}^{e_T - M_T} \epsilon_T f(\epsilon_T) d\epsilon_T \\ &- i^l (e_T - M_T) F(e_T - M_T) + i^d \int_{e_T - M_T}^{\infty} \epsilon_T f(\epsilon_T) d\epsilon_T \\ &- i^d (e_T - M_T) [1 - F(e_T - M_T)]. \end{aligned}$$

The first order condition for a maximum is

$$i_T = i^l F(e_T - M_T) + i^d [1 - F(e_T - M_T)].$$

And, in equilibrium, the interest rate satisfies

$$i_T(e_T > 0) = i^l F(e_T - A_T) + i^d [1 - F(e_T - A_T)].$$

Finally, the value function is

$$\begin{aligned} V(A_T, e_T, i_T | e_T > 0) &= A_T - i^l (e_T - A_T) F(e_T - A_T) \\ &- i^d (e_T - A_T) [1 - F(e_T - A_T)] \\ &+ i^l \int_{-\infty}^{e_T - A_T} \epsilon_T f(\epsilon_T) d\epsilon_T + i^d \int_{e_T - A_T}^{\infty} \epsilon_T f(\epsilon_T) d\epsilon_T. \end{aligned}$$

4 Problem at $T-1$

The problem is to maximize with respect to M_{T-1}

$$E_{T-1}(A_{T+1})$$

when the expectation is evaluated at the equilibrium level computed before. This problem can be expressed as

$$V(A_{T-1}, e_{T-1}, i_{T-1}) = \max_{M_{T-1}} E_{T-1}[E_T(A_{T+1})] = \max_{M_{T-1}} E_{T-1}[V(A_T, e_T, i_T)]$$

with

$$A_T = (1 + i_{T-1}) A_{T-1} + \epsilon_{T-1} - c_{T-1} \tag{3}$$

and

$$\begin{aligned} e_T &= \max\{0, R - L_{T-1}\} = \max\{0, R - L_{T-2} - R_{T-1}\} \\ &= \max\{0, e_{T-1} - \max[0, M_{T-1} + \epsilon_{T-1}]\}. \end{aligned}$$

4.1 Case 1: $e_{T-1} = 0$

In this case, the bank has already satisfied the reserve requirement with the reserves accumulated up to time $T - 2$. The variable c_T is equal to:

$$\begin{aligned} c_{T-1} &= i_{T-1} M_{T-1} - i^l (M_{T-1} + \epsilon_{T-1}) I\{\epsilon_{T-1} \leq -M_{T-1}\} \\ &\quad - i^d (M_{T-1} + \epsilon_{T-1}) I\{\epsilon_{T-1} \geq -M_{T-1}\}. \end{aligned}$$

Since the bank is locked-in at $T-1$ it will be locked-in at T. Then, the problem is to maximize

$$\begin{aligned} E_{T-1}[V(A_T, e_T, i_T | e_T = 0)] &= E_{T-1}\left[A_T + i^l \int_{-\infty}^{-A_T} (A_T + \epsilon_T) f(\epsilon_T) d\epsilon_T \right. \\ &\quad \left. + i^d \int_{-A_T}^{\infty} (A_T + \epsilon_T) f(\epsilon_T) d\epsilon_T \right]. \end{aligned}$$

Changing M_{T-1} will affect the amount of reserves available to the bank on the following period, A_T. The agent values this change for two reasons. First, the bank wants to maximize the return on managing its reseves. This means having more reserves on average next period, A_T. Second, the new level of reserves will affect the probabilities of going to the deposit or lending facility. To solve this problem we break this function down into these components. Define

$$\begin{aligned} V &\equiv E_{T-1}[V(A_T, e_T, i_T | e_T = 0)] = E_{T-1}(A_T) \\ &\quad + i^l E_{T-1}[A_T F(-A_T)] + i^l E_{T-1}\left[\int_{-\infty}^{-A_T} \epsilon_T f(\epsilon_T) d\epsilon_T\right] \\ &\quad + i^d E_{T-1}[A_T(1 - F(-A_T))] + i^d E_{T-1}\left[\int_{-A_T}^{\infty} \epsilon_T f(\epsilon_T) d\epsilon_T\right] \\ &= V_1 + i^l (V_2 + V_3) + i^d (V_4 + V_5) \end{aligned}$$

The first order condition for a maximum is

$$\frac{dV}{dM_{T-1}} = 0.$$

As functions of M_{T-1}, the components of V will take different forms depending on whether ϵ_{T-1} is larger or smaller than $-M_{T-1}$. This is because A_T will take different values. So,

- if $\epsilon_{T-1} < -M_{T-1}$,

$$A_T = (1 + i_{T-1}) A_{T-1} + \epsilon_{T-1} - i_{T-1} M_{T-1} + i^l (\epsilon_{T-1} + M_{T-1}) \tag{4}$$

- if $\epsilon_{T-1} > -M_{T-1}$,

$$A_T = (1 + i_{T-1}) A_{T-1} + \epsilon_{T-1} - i_{T-1} M_{T-1} + i^d (\epsilon_{T-1} + M_{T-1}) . \tag{5}$$

When computing the first order condition, we will take into account these results.

- V_1

  This term is defined as

$$V_1 = E_{T-1} (A_T) .$$

By substituting for (3), we get

$$\begin{aligned} V_1 \quad = \quad & (1 + i_{T-1}) A_{T-1} - i_{T-1} M_{T-1} \\ & + i^l M_{T-1} F(-M_{T-1}) + i^l \int_{-\infty}^{-M_{T-1}} \epsilon_{T-1} f(\epsilon_{T-1}) d\epsilon_{T-1} \\ & + i^d M_{T-1} [1 - F(-M_{T-1})] + i^d \int_{-M_{T-1}}^{\infty} \epsilon_{T-1} f(\epsilon_{T-1}) d\epsilon_{T-1} . \end{aligned}$$

We have that

$$\frac{dV_1}{dM_{T-1}} = -i_{T-1} + i^l F(-M_{T-1}) + i^d [1 - F(-M_{T-1})] .$$

When evaluated at the optimum ($M_{T-1} = A_{T-1}$), we get

$$\frac{dV_1^*}{dM_{T-1}} = -i_{T-1} + i^l F(-A_{T-1}) + i^d [1 - F(-A_{T-1})] .$$

- V_2

  This term is defined as

$$V = E_{T-1} [A_T F(-A_T)] = \int_{-\infty}^{\infty} A_T F(-A_T) f(\epsilon_{T-1}) d\epsilon_{T-1} .$$

First, split this integral in two:

$$V_{21} = \int_{-\infty}^{-M_{T-1}} A_T F\left(-A_T\right) f\left(\epsilon_{T-1}\right) d\epsilon_{T-1}$$

and

$$V_{22} = \int_{-M_{T-1}}^{\infty} A_T F\left(-A_T\right) f\left(\epsilon_{T-1}\right) d\epsilon_{T-1}.$$

Using the value for A_T in (4), V_{21} becomes

$$V_{21} = \int_{-\infty}^{-M_{T-1}} A_T F\left(-A_T\right) f\left(\epsilon_{T-1}\right) d\epsilon_{T-1} = V_{211} + V_{212} + V_{213}$$

with

$$V_{211} = \left(1 + i_{T-1}\right) A_{T-1} \int_{-\infty}^{-M_{T-1}} F\left(-A_T\right) f\left(\epsilon_{T-1}\right) d\epsilon_{T-1}$$

$$V_{212} = \left(1 + i^l\right) \int_{-\infty}^{-M_{T-1}} \epsilon_{T-1} F\left(-A_T\right) f\left(\epsilon_{T-1}\right) d\epsilon_{T-1}$$

$$V_{213} = -\left(i_{T-1} - i^l\right) M_{T-1} \int_{-\infty}^{-M_{T-1}} F\left(-A_T\right) f\left(\epsilon_{T-1}\right) d\epsilon_{T-1}.$$

Then,

$$\begin{aligned} \frac{dV_{211}}{dM_{T-1}} &= -\left(1 + i_{T-1}\right) A_{T-1} F\left[\left(1 + i_{T-1}\right)\left(M_{T-1} - A_{T-1}\right)\right] f\left(-M_{T-1}\right) \\ &\quad + \left(1 + i_{T-1}\right)\left(i_{T-1} - i^l\right) A_{T-1} \int_{-\infty}^{-M_{T-1}} f\left(-A_T\right) f\left(\epsilon_{T-1}\right) d\epsilon_{T-1}, \end{aligned}$$

$$\begin{aligned} \frac{dV_{212}}{dM_{T-1}} &= \left(1 + i^l\right) M_{T-1} F\left[\left(1 + i_{T-1}\right)\left(M_{T-1} - A_{T-1}\right)\right] f\left(-M_{T-1}\right) \\ &\quad + \left(1 + i^l\right)\left(i_{T-1} - i^l\right) \int_{-\infty}^{-M_{T-1}} \epsilon_{T-1} f\left(-A_T\right) f\left(\epsilon_{T-1}\right) d\epsilon_{T-1}, \end{aligned}$$

$$\frac{dV_{213}}{dM_{T-1}} = -\left(i_{T-1}-i^{l}\right)\int_{-\infty}^{-M_{T-1}} F\left[-A_T\right] f\left(\epsilon_{T-1}\right) d\epsilon_{T-1}$$
$$+\left(i_{T-1}-i^{l}\right) M_{T-1} F\left[\left(1+i_{T-1}\right)\left(M_{T-1}-A_{T-1}\right)\right] f\left(-M_{T-1}\right)$$
$$-\left(i_{T-1}-i^{l}\right)^{2} M_{T-1}\int_{-\infty}^{-M_{T-1}} f\left(-A_T\right) f\left(\epsilon_{T-1}\right) d\epsilon_{T-1}.$$

When evaluated at the optimum ($M_{T-1}=A_{T-1}$), we get

$$\frac{dV_{21}^{*}}{dM_{T-1}} = \left(1+i^{l}\right)\left(i_{T-1}-i^{l}\right) A_{T-1}\int_{-\infty}^{-A_{T-1}} f\left(-A_T\right) f\left(\epsilon_{T-1}\right) d\epsilon_{T-1}$$
$$+\left(1+i^{l}\right)\left(i_{T-1}-i^{l}\right)\int_{-\infty}^{-A_{T-1}} \epsilon_{T-1} f\left(-A_T\right) f\left(\epsilon_{T-1}\right) d\epsilon_{T-1}$$
$$-\left(i_{T-1}-i^{l}\right)\int_{-\infty}^{-A_{T-1}} F\left(-A_T\right) f\left(\epsilon_{T-1}\right) d\epsilon_{T-1}. \tag{6}$$

On the other hand, using the value for A_T in (5), V_{22} becomes

$$V_{22} = \int_{-M_{T-1}}^{\infty} A_T F\left(-A_T\right) f\left(\epsilon_{T-1}\right) d\epsilon_{T-1} = V_{221}+V_{222}+V_{223}$$

with

$$V_{221} = \left(1+i_{T-1}\right) A_{T-1}\int_{-M_{T-1}}^{\infty} F\left(-A_T\right) f\left(\epsilon_{T-1}\right) d\epsilon_{T-1}$$

$$V_{222} = \left(1+i^{d}\right)\int_{-M_{T-1}}^{\infty} \epsilon_{T-1} F\left(-A_T\right) f\left(\epsilon_{T-1}\right) d\epsilon_{T-1}$$

$$V_{232} = -\left(i_{T-1}-i^{d}\right) M_{T-1}\int_{-M_{T-1}}^{\infty} F\left(-A_T\right) f\left(\epsilon_{T-1}\right) d\epsilon_{T-1}$$

Then,

$$\frac{dV_{221}}{dM_{T-1}} = \left(1+i_{T-1}\right) A_{T-1} F\left[\left(1+i_{T-1}\right)\left(M_{T-1}-A_{T-1}\right)\right] f\left(-M_{T-1}\right)$$
$$+\left(1+i_{T-1}\right)\left(i_{T-1}-i^{d}\right) A_{T-1}\int_{-M_{T-1}}^{\infty} f\left(-A_T\right) f\left(\epsilon_{T-1}\right) d\epsilon_{T-1},$$

$$\frac{dV_{222}}{dM_{T-1}} = -\left(1+i^d\right) M_{T-1} F\left[\left(1+i_{T-1}\right)\left(M_{T-1}-A_{T-1}\right)\right] f\left(-M_{T-1}\right) + \left(1+i^d\right)\left(i_{T-1}-i^d\right) \int_{-M_{T-1}}^{\infty} \epsilon_{T-1} f\left(-A_T\right) f\left(\epsilon_{T-1}\right) d\epsilon_{T-1},$$

$$\frac{dV_{223}}{dM_{T-1}} = -\left(i_{T-1}-i^d\right) \int_{-M_{hT-1}}^{\infty} F\left[-A_T\right] f\left(\epsilon_{T-1}\right) d\epsilon_{T-1} - \left(i_{T-1}-i^d\right) M_{T-1} F\left[\left(1+i_{T-1}\right)\left(M_{T-1}-A_{T-1}\right)\right] f\left(-M_{T-1}\right) - \left(i_{T-1}-i^d\right)^2 M_{T-1} \int_{-M_{T-1}}^{\infty} f\left(-A_T\right) f\left(\epsilon_{T-1}\right) d\epsilon_{T-1}.$$

When evaluated at the optimum ($M_{T-1} = A_{T-1}$), we get

$$\frac{dV_{22}^*}{dM_{T-1}} = \left(1+i^d\right)\left(i_{T-1}-i^d\right) A_{T-1} \int_{-A_{T-1}}^{\infty} f\left(-A_T\right) f\left(\epsilon_{T-1}\right) d\epsilon_{T-1} + \left(1+i^d\right)\left(i_{T-1}-i^d\right) \int_{-A_{T-1}}^{\infty} \epsilon_{T-1} f\left(-A_T\right) f\left(\epsilon_{T-1}\right) d\epsilon_{T-1} - \left(i_{T-1}-i^d\right) \int_{-A_{T-1}}^{\infty} F\left(-A_T\right) f\left(\epsilon_{T-1}\right) d\epsilon_{T-1}. \tag{7}$$

- V_3

  This term is defined as

$$V_3 = E_{T-1}\left[\int_{-\infty}^{-A_T} \epsilon_T f\left(\epsilon_T\right) d\epsilon_T\right] = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{-A_T} \epsilon_T f\left(\epsilon_T\right) d\epsilon_T\right] f\left(\epsilon_{T-1}\right) d\epsilon_{T-1}$$

As before, split this integral in two:

$$V_{31} = \int_{-\infty}^{-M_{T-1}}\left[\int_{-\infty}^{-A_T} \epsilon_T f\left(\epsilon_T\right) d\epsilon_T\right] f\left(\epsilon_{T-1}\right) d\epsilon_{T-1}$$

and

$$V_{32} = \int_{-M_{T-1}}^{\infty}\left[\int_{-\infty}^{-A_T} \epsilon_T f\left(\epsilon_T\right) d\epsilon_T\right] f\left(\epsilon_{T-1}\right) d\epsilon_{T-1}.$$

Using the value for A_T in (4), we have

$$\frac{dV_{31}}{dM_{T-1}} = -f(-M_{T-1}) \int_{-\infty}^{(1+i_{T-1})(M_{T-1}-A_{T-1})} \epsilon_T f(\epsilon_T)\, d\epsilon_T + \left(i^l - i_{T-1}\right) \int_{-\infty}^{-M_{T-1}} A_T f(-A_T) f(\epsilon_{T-1})\, d\epsilon_{T-1}.$$

On the other hand,

$$\frac{dV_{32}}{dM_{T-1}} = f(-M_{T-1}) \int_{-\infty}^{(1+i_{T-1})(M_{T-1}-A_{T-1})} \epsilon_T f(\epsilon_T)\, d\epsilon_T + \left(i^d - i_{T-1}\right) \int_{-M_{T-1}}^{\infty} A_T f(-A_T) f(\epsilon_{T-1})\, d\epsilon_{T-1}.$$

These expressions evaluated at the optimum, yield

$$\begin{aligned} \frac{dV_3^*}{dM_{T-1}} &= \left(1+i^l\right)\left(i^l - i_{T-1}\right) A_{T-1} \int_{-\infty}^{-A_{T-1}} f(-A_T) f(\epsilon_{T-1})\, d\epsilon_{T-1} \\ &+ \left(1+i^l\right)\left(i^l - i_{T-1}\right) \int_{-\infty}^{-A_{T-1}} \epsilon_{T-1} f(-A_T) f(\epsilon_{T-1})\, d\epsilon_{T-1} \\ &+ \left(1+i^d\right)\left(i^d - i_{T-1}\right) A_{T-1} \int_{-A_{T-1}}^{\infty} f(-A_T) f(\epsilon_{T-1})\, d\epsilon_{T-1} \\ &+ \left(1+i^d\right)\left(i^d - i_{T-1}\right) \int_{-A_{T-1}}^{\infty} \epsilon_{T-1} f(-A_T) f(\epsilon_{T-1})\, d\epsilon_{T-1}. \end{aligned} \quad (8)$$

Summarizing, using (6), (7) and (8) we have

$$\frac{dV_2^*}{dM_{T-1}} + \frac{dV_3^*}{dM_{T-1}} = -\left(i_{T-1} - i^l\right) \int_{-\infty}^{-A_{T-1}} F(-A_T) f(\epsilon_{T-1})\, d\epsilon_{T-1} - \left(i_{T-1} - i^d\right) \int_{-A_{T-1}}^{\infty} F(-A_T) f(\epsilon_{T-1})\, d\epsilon_{T-1}$$

- V_4

Define

$$V_4 = \int_{-\infty}^{\infty} A_T \left[1 - F\left(-A_T\right)\right] f\left(\epsilon_{T-1}\right) d\epsilon_{T-1} = V_{41} + V_{42}$$

with

$$V_{41} = \int_{-\infty}^{\infty} A_T f\left(\epsilon_{T-1}\right) d\epsilon_{T-1}$$

and

$$V_{42} = -\int_{-\infty}^{\infty} A_T F\left(-A_T\right) f\left(\epsilon_{T-1}\right) d\epsilon_{T-1}.$$

With respect to V_{41}, we have

$$\frac{dV_{41}}{dM_{T-1}} = -i_{T-1} M_{T-1} + i^l F\left(-M_{T-1}\right) + i^d \left[1 - F\left(-M_{T-1}\right)\right].$$

Also,

$$\frac{dV_{42}}{dM_{T-1}} = -\frac{dV_2}{dM_{T-1}}.$$

Then,

$$\begin{aligned}
\frac{dV_4^*}{dM_{T-1}} &= -i_{T-1} + i^l F\left(-A_{T-1}\right) + i^d \left[1 - F\left(-A_{T-1}\right)\right] \\
&\quad - \left(1 + i^l\right)\left(i_{T-1} - i^l\right) A_{T-1} \int_{-\infty}^{-A_{T-1}} f\left(-A_T\right) f\left(\epsilon_{T-1}\right) d\epsilon_{T-1} \\
&\quad - \left(1 + i^l\right)\left(i_{T-1} - i^l\right) \int_{-\infty}^{-A_{T-1}} \epsilon_{T-1} f\left(-A_T\right) f\left(\epsilon_{T-1}\right) d\epsilon_{T-1} \\
&\quad + \left(i_{T-1} - i^l\right) \int_{-\infty}^{-A_{T-1}} F\left(-A_T\right) f\left(\epsilon_{T-1}\right) d\epsilon_{T-1} \\
&\quad - \left(1 + i^d\right)\left(i_{T-1} - i^d\right) A_{T-1} \int_{-A_{T-1}}^{\infty} f\left(-A_T\right) f\left(\epsilon_{T-1}\right) d\epsilon_{T-1} \\
&\quad - \left(1 + i^d\right)\left(i_{T-1} - i^d\right) \int_{-A_{T-1}}^{\infty} \epsilon_{T-1} f\left(-A_T\right) f\left(\epsilon_{T-1}\right) d\epsilon_{T-1} \\
&\quad + \left(i_{T-1} - i^d\right) \int_{-A_{T-1}}^{\infty} F\left(-A_T\right) f\left(\epsilon_{T-1}\right) d\epsilon_{T-1}. \qquad (9)
\end{aligned}$$

- V_5

Define

$$V_5 = \int_{-\infty}^{\infty} \left[\int_{-A_T}^{\infty} \epsilon_T f(\epsilon_T)\, d\epsilon_T \right] f(\epsilon_{T-1})\, d\epsilon_{T-1} = V_{51} + V_{52}$$

with

$$V_{51} = \int_{-\infty}^{-M_{T-1}} \left[\int_{-A_T}^{\infty} \epsilon_T f(\epsilon_T)\, d\epsilon_T \right] f(\epsilon_{T-1})\, d\epsilon_{T-1}$$

and

$$V_{52} = \int_{-M_{T-1}}^{\infty} \left[\int_{-A_T}^{\infty} \epsilon_T f(\epsilon_T)\, d\epsilon_T \right] f(\epsilon_{T-1})\, d\epsilon_{T-1}.$$

Using the value for A_T in (4), we have

$$\begin{aligned} \frac{dV_{51}}{dM_{T-1}} &= -f(-M_{T-1}) \int_{(1+i_{T-1})(M_{T-1}-A_{T-1})}^{\infty} \epsilon_T f(\epsilon_T)\, d\epsilon_T \\ &\quad - \left(i^l - i_{T-1}\right) \int_{-\infty}^{-M_{T-1}} A_T f(-A_T)\, f(\epsilon_{T-1})\, d\epsilon_{T-1}. \end{aligned}$$

On the other hand,

$$\begin{aligned} \frac{dV_{52}}{dM_{T-1}} &= f(-M_{T-1}) \int_{(1+i_{T-1})(M_{T-1}-A_{T-1})}^{\infty} \epsilon_T f(\epsilon_T)\, d\epsilon_T \\ &\quad - \left(i^d - i_{T-1}\right) \int_{-M_{T-1}}^{\infty} A_T f(-A_T)\, f(\epsilon_{T-1})\, d\epsilon_{T-1}. \end{aligned}$$

These expressions evaluated at the optimum, yield

$$\begin{aligned}
\frac{dV_5^*}{dM_{T-1}} &= -\left(1+i^l\right)\left(i^l - i_{T-1}\right) A_{T-1} \int_{-\infty}^{-A_{T-1}} f\left(-A_T\right) f\left(\epsilon_{T-1}\right) d\epsilon_{T-1} \\
&\quad -\left(1+i^l\right)\left(i^l - i_{T-1}\right) \int_{-\infty}^{-A_{T-1}} \epsilon_{T-1} f\left(-A_T\right) f\left(\epsilon_{T-1}\right) d\epsilon_{T-1} \\
&\quad -\left(1+i^d\right)\left(i^d - i_{T-1}\right) A_{T-1} \int_{-A_{T-1}}^{\infty} f\left(-A_T\right) f\left(\epsilon_{T-1}\right) d\epsilon_{T-1} \\
&\quad -\left(1+i^d\right)\left(i^d - i_{T-1}\right) \int_{-A_{T-1}}^{\infty} \epsilon_{T-1} f\left(-A_T\right) f\left(\epsilon_{T-1}\right) d\epsilon_{T-1}. \quad (10)
\end{aligned}$$

Summarizing, using (9), and (10) we have

$$\begin{aligned}
\frac{dV_4^*}{dM_{T-1}} + \frac{dV_5^*}{dM_{T-1}} &= -i_{T-1} + i^l F\left(-A_{T-1}\right) + i^d \left[1 - F\left(-A_{T-1}\right)\right] \\
&\quad + \left(i_{T-1} - i^l\right) \int_{-\infty}^{-A_{T-1}} F\left(-A_T\right) f\left(\epsilon_{T-1}\right) d\epsilon_{T-1} \\
&\quad + \left(i_{T-1} - i^d\right) \int_{-A_{T-1}}^{\infty} F\left(-A_T\right) f\left(\epsilon_{T-1}\right) d\epsilon_{T-1}.
\end{aligned}$$

The first order condition for this problem evaluated at the optimum is, therefore,

$$\frac{dV_1^*}{dM_{T-1}} + i^l \left(\frac{dV_2^*}{dM_{T-1}} + \frac{dV_3^*}{dM_{hT-1}}\right) + i^d \left(\frac{dV_4^*}{dM_{T-1}} + \frac{dV_5^*}{dM_{T-1}}\right) = 0.$$

This means

$$
\begin{aligned}
0 &= -i_{T-1} + i^l F(-A_{T-1}) + i^d [1 - F(-A_{T-1})] \\
&\quad - i^l \left(i_{T-1} - i^l\right) \int_{-\infty}^{-A_{T-1}} F(-A_T) f(\epsilon_{T-1}) d\epsilon_{T-1} \\
&\quad - i^l \left(i_{T-1} - i^d\right) \int_{-A_{T-1}}^{\infty} F(-A_T) f(\epsilon_{T-1}) d\epsilon_{T-1} \\
&\quad - i^d i_{T-1} + i^d i^l F(-A_{T-1}) + \left(i^d\right)^2 [1 - F(-A_{T-1})] \\
&\quad + i^d \left(i_{T-1} - i^l\right) \int_{-\infty}^{-A_{T-1}} F(-A_T) f(\epsilon_{T-1}) d\epsilon_{T-1} \\
&\quad + i^d \left(i_{T-1} - i^d\right) \int_{-A_{T-1}}^{\infty} F(-A_T) f(\epsilon_{T-1}) d\epsilon_{T-1}
\end{aligned}
$$

or,

$$
\begin{aligned}
0 &= -i_{T-1} + i^l F(-A_{T-1}) + i^d [1 - F(-A_{T-1})] \\
&\quad - i_{T-1} \int_{-\infty}^{\infty} \left[i^l F(-A_T) + i^d (1 - F(-A_T))\right] f(\epsilon_{T-1}) d\epsilon_{T-1} \\
&\quad + i^l \int_{-\infty}^{-A_{T-1}} \left[i^l F(-A_T) + i^d (1 - F(-A_T))\right] f(\epsilon_{T-1}) d\epsilon_{T-1} \\
&\quad + i^d \int_{-A_{T-1}}^{\infty} \left[i^l F(-A_T) + i^d (1 - F(-A_T))\right] f(\epsilon_{T-1}) d\epsilon_{T-1} \\
&= -i_{T-1} + i^l F(-A_{T-1}) + i^d [1 - F(-A_{T-1})] + E.
\end{aligned}
$$

with

$$
\begin{aligned}
E &= -i_{T-1} \int_{-\infty}^{\infty} i_T f(\epsilon_{T-1}) d\epsilon_{T-1} + i^l \int_{-\infty}^{-A_{T-1}} i_T f(\epsilon_{T-1}) d\epsilon_{T-1} \\
&\quad + i^d \int_{-A_{T-1}}^{\infty} i_T f(\epsilon_{T-1}) d\epsilon_{T-1}.
\end{aligned}
$$

It is easy to show that this term is bounded above and below by

$$
-\left(i^l\right)^2 < E < \left(i^l\right)^2,
$$

so, a good approximation of the equilibrium interest rate is given by the expression

$$i_{T-1} = i^l F(-A_{T-1}) + i^d [1 - F(-A_{T-1})].$$

In other words, the only important term in V is $E_{T-1}(A_T)$. The other terms are very close to being a constant. This means that the value function at $T-1$ when $e_{T-1} = 0$ can be approximated by

$$\begin{aligned}
& V(A_{T-1}, e_{T-1}, i_{T-1} | e_{T-1} = 0) \\
\simeq\ & A_{T-1} + i^l A_{T-1} F(-A_{T-1}) + i^l \int_{-\infty}^{-A_{T-1}} \epsilon_{T-1} f(\epsilon_{T-1}) d\epsilon_{T-1} \\
& + i^d A_{T-1} [1 - F(-A_{T-1})] + i^d \int_{-A_{T-1}}^{\infty} \epsilon_{T-1} f(\epsilon_{T-1}) d\epsilon_{T-1}.
\end{aligned}$$

4.2 Case 2: $e_{T-1} > 0$

In this case, the variable c_{T-1} is equal to:

$$\begin{aligned}
c_{T-1} =\ & i_{T-1} M_{T-1} - i^l (M_{T-1} + \epsilon_{T-1}) I\{\epsilon_{T-1} \leq -M_{T-1}\} \\
& - i^d (M_{T-1} + \epsilon_{T-1} - e_{T-1}) I\{\epsilon_{T-1} \geq e_{T-1} - M_{T-1}\}.
\end{aligned}$$

Then, the problem is to maximize

$$E_{T-1}[V(A_T, e_T, i_T)] = E_{T-1}\left[A_T - i^l \int_{-\infty}^{e_T - A_T} (e_T - A_T - \epsilon_T) f(\epsilon_T) d\epsilon_T \right.$$
$$\left. - i^d \int_{e_T - A_T}^{\infty} (e_T - A_T - \epsilon_T) f(\epsilon_T) d\epsilon_T \right].$$

As before, to solve this problem we break this function down into these components. Define

$$\begin{aligned}
V \equiv\ & E_{T-1}[V(A_T, e_T, i_T)] = E_{T-1}(A_T) \\
& - i^l E_{T-1}[(e_T - A_T) F(e_T - A_T)] + i^l E_{T-1}\left[\int_{-\infty}^{e_T - A_T} \epsilon_T f(\epsilon_T) d\epsilon_T\right] \\
& - i^d E_{T-1}[(e_T - A_T)(1 - F(e_T - A_T))] + i^d E_{T-1}\left[\int_{e_T - A_T}^{\infty} \epsilon_T f(\epsilon_T) d\epsilon_T\right] \\
=\ & V_1 + i^l(-V_2 + V_3) + i^d(-V_4 + V_5).
\end{aligned}$$

The first order condition for a maximum is

$$\frac{dV}{dM_{T-1}} = 0.$$

When evaluating the maximization problem, the value of the shock in $T-1$ will be important not only because the magnitude of A_T will be affected by it, as before, but also because e_T depends on ϵ_{T-1}, too. So,

- if $\epsilon_{T-1} < -M_{T-1}$,

$$A_T = (1+i_{T-1})\,A_{T-1} + \epsilon_{T-1} - i_{T-1}M_{T-1} + i^l\,(\epsilon_{T-1} + M_{T-1}) \tag{11}$$

$$e_T = e_{T-1}, \tag{12}$$

- if $-M_{T-1} < \epsilon_{T-1} < e_{T-1} - M_{T-1}$,

$$A_T = (1+i_{T-1})\,A_{T-1} + \epsilon_{T-1} - i_{T-1}M_{T-1} \tag{13}$$

$$e_T = e_{T-1} - \epsilon_{T-1} - M_{T-1}, \tag{14}$$

- if $\epsilon_{T-1} > e_{T-1} - M_{T-1}$,

$$\begin{aligned} A_T &= (1+i_{T-1})\,A_{T-1} + \epsilon_{T-1} - i_{T-1}M_{T-1} \\ &\quad + i^d\,(\epsilon_{T-1} + M_{T-1} - e_{T-1}) \end{aligned} \tag{15}$$

$$e_T = 0. \tag{16}$$

- V_1

  By substituting for (3), we get

$$\begin{aligned} V_1 &= (1+i_{T-1})\,A_{T-1} - i_{T-1}M_{T-1} + i^l M_{T-1}F\,(-M_{T-1}) \\ &\quad + i^d\,(M_{T-1} - e_{T-1})\,[1 - F\,(e_{T-1} - M_{T-1})] \\ &\quad + i^l \int_{-\infty}^{-M_{T-1}} \epsilon_{T-1} f\,(\epsilon_{T-1})\,d\epsilon_{T-1} + i^d \int_{e_{T-1}-M_{T-1}}^{\infty} \epsilon_{T-1} f\,(\epsilon_{T-1})\,d\epsilon_{T-1}. \end{aligned}$$

We have that

$$\frac{dV_1}{dM_{T-1}} = -i_{T-1} + i^l F\,(-M_{T-1}) + i^d\,[1 - F\,(e_{T-1} - M_{T-1})]\,.$$

When evaluated at the optimum ($M_{T-1} = A_{T-1}$), we get

$$\frac{dV_1^*}{dM_{T-1}} = -i_{T-1} + i^l F\,(-A_{T-1}) + i^d\,[1 - F\,(e_{T-1} - A_{T-1})]\,.$$

- V_2

Define

$$V_2 = \int_{-\infty}^{\infty} (e_T - A_T) F(e_T - A_T) f(\epsilon_{T-1}) d\epsilon_{T-1} = V_{21} + V_{22} + V_{23}$$

with

$$V_{21} = \int_{-\infty}^{-M_{T-1}} (e_T - A_T) F(e_T - A_T) f(\epsilon_{T-1}) d\epsilon_{T-1}$$

$$V_{22} = \int_{-M_{T-1}}^{e_{T-1}-M_{T-1}} (e_T - A_T) F(e_T - A_T) f(\epsilon_{T-1}) d\epsilon_{T-1}$$

and

$$V_{23} = \int_{e_{T-1}-M_{T-1}}^{\infty} (e_T - A_T) F(e_T - A_T) f(\epsilon_{T-1}) d\epsilon_{T-1}.$$

Using the value for A_T and e_T in (11) and (12), V_{21} becomes

$$V_{21} = \int_{-\infty}^{-M_{T-1}} (e_T - A_T) F(e_T - A_T) f(\epsilon_{T-1}) d\epsilon_{T-1} = V_{211} + V_{212} + V_{213}$$

$$V_{211} = [e_{T-1} - (1 + i_{T-1}) A_{T-1}] \int_{-\infty}^{-M_{T-1}} F(e_T - A_T) f(\epsilon_{T-1}) d\epsilon_{T-1}$$

$$V_{212} = -\left(1 + i^l\right) \int_{-\infty}^{-M_{T-1}} \epsilon_{T-1} F(e_T - A_T) f(\epsilon_{T-1}) d\epsilon_{T-1}$$

$$V_{213} = \left(i_{T-1} - i^l\right) M_{T-1} \int_{-\infty}^{-M_{T-1}} F(e_T - A_T) f(\epsilon_{T-1}) d\epsilon_{T-1}.$$

Then,

$$\begin{aligned}\frac{dV_{211}}{dM_{T-1}} &= [e_{T-1} - (1+i_{T-1}) A_{T-1}] \times \\ &\times \{-F[e_{T-1} - (1+i_{T-1})(M_{T-1} - A_{T-1})] f(-M_{T-1}) \\ &+ \left(i_{T-1} - i^l\right) A_{T-1} \int_{-\infty}^{-M_{T-1}} f(e_T - A_T) f(\epsilon_{T-1}) d\epsilon_{T-1}\},\end{aligned}$$

$$\begin{aligned}\frac{dV_{212}}{dM_{T-1}} &= -\left(1+i^l\right) M_{T-1} F[e_{T-1} - (1+i_{T-1})(M_{T-1} - A_{T-1})] f(-M_{T-1}) \\ &- \left(1+i^l\right)\left(i_{T-1} - i^l\right) \int_{-\infty}^{-M_{T-1}} \epsilon_{T-1} f(e_T - A_T) f(\epsilon_{T-1}) d\epsilon_{T-1},\end{aligned}$$

$$\begin{aligned}\frac{dV_{213}}{dM_{T-1}} &= \left(i_{T-1} - i^l\right) \times \left\{ \int_{-\infty}^{-M_{T-1}} F[e_T - A_T] f(\epsilon_{T-1}) d\epsilon_{T-1} \right. \\ &- M_{T-1} F[e_{T-1} - (1+i_{T-1})(M_{T-1} - A_{T-1})] f(-M_{T-1}) \\ &\left. + M_{T-1}\left(i_{T-1} - i^l\right) \int_{-\infty}^{-M_{T-1}} f(e_T - A_T) f(\epsilon_{T-1}) d\epsilon_{T-1} \right\}.\end{aligned}$$

When evaluated at the optimum ($M_{T-1} = A_{T-1}$), we get

$$\begin{aligned}\frac{dV_{21}^*}{dM_{T-1}} &= -e_{T-1} F(e_{T-1}) f(-A_{T-1}) + \\ &+ \left(i_{T-1} - i^l\right) e_{T-1} \int_{-\infty}^{-A_{T-1}} f(e_T - A_T) f(\epsilon_{T-1}) d\epsilon_{T-1} \\ &- \left(1+i^l\right)\left(i_{T-1} - i^l\right) A_{T-1} \int_{-\infty}^{-A_{T-1}} f(e_T - A_T) f(\epsilon_{T-1}) d\epsilon_{T-1} \\ &- \left(1+i^l\right)\left(i_{T-1} - i^l\right) \int_{-\infty}^{-A_{T-1}} \epsilon_{T-1} f(e_T - A_T) f(\epsilon_{T-1}) d\epsilon_{T-1} \\ &+ \left(i_{T-1} - i^l\right) \int_{-\infty}^{-A_{T-1}} F(e_T - A_T) f(\epsilon_{T-1}) d\epsilon_{T-1}. \qquad (17)\end{aligned}$$

On the other hand, using the value for A_T and e_T in (13) and (14), we have

$$V_{22} = \int_{-M_{T-1}}^{e_{T-1}-M_{T-1}} (e_T - A_T) F(e_T - A_T) f(\epsilon_{T-1}) d\epsilon_{T-1} = V_{221} + V_{222} + V_{223}$$

$$V_{221} = [e_{T-1} - (1 + i_{T-1}) A_{T-1}] \int_{-M_{T-1}}^{e_{T-1}-M_{T-1}} F(e_T - A_T) f(\epsilon_{T-1}) d\epsilon_{T-1}$$

$$V_{222} = -2 \int_{-M_{T-1}}^{e_{T-1}-M_{T-1}} \epsilon_{T-1} F(e_T - A_T) f(\epsilon_{T-1}) d\epsilon_{T-1}$$

$$V_{223} = -(1 - i_{T-1}) M_{T-1} \int_{-M_{T-1}}^{e_{T-1}-M_{T-1}} F(e_T - A_T) f(\epsilon_{T-1}) d\epsilon_{T-1}.$$

Then,

$$\begin{aligned} \frac{dV_{221}}{dM_{T-1}} &= [e_{T-1} - (1 + i_{T-1}) A_{T-1}] \times \\ &\times \{-F[(1 + i_{T-1})(M_{T-1} - A_{T-1}) - e_{T-1}] f(e_{T-1} - M_{T-1}) \\ &+ F[e_{T-1} - (1 + i_{T-1})(M_{T-1} - A_{T-1})] f(-M_{T-1}) \\ &- (1 - i_{T-1}) \int_{-M_{T-1}}^{e_{T-1}-M_{T-1}} f(e_T - A_T) f(\epsilon_{T-1}) d\epsilon_{T-1}\}, \end{aligned}$$

$$\begin{aligned} \frac{dV_{222}}{dM_{T-1}} &= 2(e_{T-1} - M_{T-1}) f(e_{T-1} - M_{T-1}) \times \\ &\times F[(1 + i_{T-1})(M_{T-1} - A_{T-1}) - e_{T-1}] \\ &+ 2M_{T-1} F[(1 + i_{T-1})(M_{T-1} - A_{T-1}) - e_{T-1}] f(-M_{T-1}) \\ &+ 2(1 - i_{T-1}) \int_{-M_{T-1}}^{e_{T-1}-M_{T-1}} \epsilon_{T-1} f(e_T - A_T) f(\epsilon_{T-1}) d\epsilon_{T-1}, \end{aligned}$$

$$\frac{dV_{223}}{dM_{T-1}} = -\left(1-i_{T-1}\right)\times\left\{\int_{-M_{T-1}}^{e_{T-1}-M_{T-1}} F\left(e_T - A_T\right) f\left(\epsilon_{T-1}\right) d\epsilon_{T-1}\right.$$
$$-M_{T-1}F\left[\left(1+i_{T-1}\right)\left(M_{T-1}-A_{T-1}\right)-e_{T-1}\right] f\left(e_{T-1}-M_{T-1}\right)$$
$$+M_{T-1}F\left[\left(1+i_{T-1}\right)\left(M_{T-1}-A_{T-1}\right)-e_{T-1}\right] f\left(-M_{T-1}\right)$$
$$\left.-\left(1-i_{T-1}\right)M_{T-1}\int_{-M_{T-1}}^{e_{T-1}-M_{T-1}} f\left(e_T - A_T\right) f\left(\epsilon_{T-1}\right) d\epsilon_{T-1}\right\}.$$

When evaluated at the optimum ($M_{T-1} = A_{T-1}$), we get

$$\begin{aligned}\frac{dV_{22}^*}{dM_{T-1}} = {} & e_{T-1}F\left(-e_{T-1}\right) f\left(e_{T-1}-A_{T-1}\right) + e_{T-1}F\left(e_{T-1}\right) f\left(-A_{T-1}\right) \\ & -\left(1-i_{T-1}\right)e_{T-1}\int_{-A_{T-1}}^{e_{T-1}-A_{T-1}} f\left(e_T - A_T\right) f\left(\epsilon_{T-1}\right) d\epsilon_{T-1} \\ & +2\left(1-i_{T-1}\right)A_{T-1}\int_{-A_{T-1}}^{e_{T-1}-A_{T-1}} f\left(e_T - A_T\right) f\left(\epsilon_{T-1}\right) d\epsilon_{T-1} \\ & +2\left(1-i_{T-1}\right)\int_{-A_{T-1}}^{e_{T-1}-A_{T-1}} \epsilon_{T-1} f\left(e_T - A_T\right) f\left(\epsilon_{T-1}\right) d\epsilon_{T-1} \\ & -\left(1-i_{T-1}\right)\int_{-A_{T-1}}^{e_{T-1}-A_{T-1}} F\left(e_T - A_T\right) f\left(\epsilon_{T-1}\right) d\epsilon_{T-1}. \qquad (18)\end{aligned}$$

Finally, using the value for A_T and e_T in (15) and (16), we have

$$V_{23} = -\int_{e_{T-1}-M_{T-1}}^{\infty} A_T F\left(e_T - A_T\right) f\left(\epsilon_{T-1}\right) d\epsilon_{T-1} = -\left(V_{231}+V_{232}+V_{233}\right)$$

with

$$V_{231} = \left[\left(1+i_{T-1}\right)A_{T-1} - i^d e_{T-1}\right]\int_{e_{T-1}-M_{T-1}}^{\infty} F\left(-A_T\right) f\left(\epsilon_{T-1}\right) d\epsilon_{T-1}$$

$$V_{232} = \left(1+i^d\right)\int_{e_{T-1}-M_{T-1}}^{\infty} \epsilon_{T-1} F\left(-A_T\right) f\left(\epsilon_{T-1}\right) d\epsilon_{T-1}$$

$$V_{233} = -\left(i_{T-1} - i^d\right) M_{T-1} \int\limits_{e_{T-1}-M_{T-1}}^{\infty} F\left(-A_T\right) f\left(\epsilon_{T-1}\right) d\epsilon_{T-1}.$$

Then,

$$\begin{aligned}
\frac{dV_{231}}{dM_{T-1}} &= \left[\left(1+i_{T-1}\right) A_{T-1} - i^d e_{T-1}\right] \times \\
&\times \{F\left[\left(1+i_{T-1}\right)\left(M_{T-1} - A_{T-1}\right) - e_{T-1}\right] f\left(e_{T-1} - M_{T-1}\right) \\
&+ \left(i_{T-1} - i^d\right) \int\limits_{e_{T-1}-M_{T-1}}^{\infty} f\left(-A_T\right) f\left(\epsilon_{T-1}\right) d\epsilon_{T-1}\},
\end{aligned}$$

$$\begin{aligned}
\frac{dV_{232}}{dM_{T-1}} &= \left(1+i^d\right)\left(e_{T-1} - M_{T-1}\right) f\left(e_{T-1} - M_{T-1}\right) \\
&\times F\left[\left(1+i_{T-1}\right)\left(M_{T-1} - A_{T-1}\right) - e_{T-1}\right] \\
&+ \left(1+i^d\right)\left(i_{T-1} - i^d\right) \int\limits_{e_{T-1}-M_{T-1}}^{\infty} \epsilon_{T-1} f\left(-A_T\right) f\left(\epsilon_{T-1}\right) d\epsilon_{T-1},
\end{aligned}$$

$$\begin{aligned}
\frac{dV_{233}}{dM_{T-1}} &= -\left(i_{T-1} - 1\right) \times \left\{ \int\limits_{e_{T-1}-M_{T-1}}^{\infty} F\left(-A_T\right) f\left(\epsilon_{T-1}\right) d\epsilon_{T-1} \right. \\
&+ M_{T-1} F\left[\left(1+i_{T-1}\right)\left(M_{T-1} - A_{T-1}\right) - e_{T-1}\right] f\left(e_{T-1} - M_{T-1}\right) \\
&\left. + \left(i_{T-1} - i^d\right) M_{T-1} \int\limits_{e_{T-1}-M_{T-1}}^{\infty} f\left(-A_T\right) f\left(\epsilon_{T-1}\right) d\epsilon_{T-1} \right\}.
\end{aligned}$$

When evaluated at the optimum ($M_{T-1} = A_{T-1}$), we get

$$\begin{aligned}
\frac{dV_{23}^*}{dM_{T-1}} &= -e_{T-1} F\left(-e_{T-1}\right) f\left(e_{T-1} - A_{T-1}\right) \\
&+ i^d \left(i_{T-1} - i^d\right) e_{T-1} \int\limits_{e_{T-1}-A_{T-1}}^{\infty} f\left(-A_T\right) f\left(\epsilon_{T-1}\right) d\epsilon_{T-1} \\
&- \left(1+i^d\right)\left(i_{T-1} - i^d\right) A_{T-1} \int\limits_{e_{T-1}-A_{T-1}}^{\infty} f\left(-A_T\right) f\left(\epsilon_{T-1}\right) d\epsilon_{T-1} \\
&- \left(1+i^d\right)\left(i_{T-1} - i^d\right) \int\limits_{e_{T-1}-A_{T-1}}^{\infty} \epsilon_{T-1} f\left(-A_T\right) f\left(\epsilon_{T-1}\right) d\epsilon_{T-1} \\
&+ \left(i_{T-1} - i^d\right) \int\limits_{e_{T-1}-A_{T-1}}^{\infty} F\left(-A_T\right) f\left(\epsilon_{T-1}\right) d\epsilon_{T-1}. \qquad (19)
\end{aligned}$$

- V_3

Define

$$V_3 = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{e_T - A_T} \epsilon_T f(\epsilon_T)\, d\epsilon_T \right] f(\epsilon_{T-1})\, d\epsilon_{T-1} = V_{31} + V_{32} + V_{33}$$

with

$$V_{31} = \int_{-\infty}^{-M_{T-1}} \left[\int_{-\infty}^{e_T - A_T} \epsilon_T f(\epsilon_T)\, d\epsilon_T \right] f(\epsilon_{T-1})\, d\epsilon_{T-1}$$

and

$$V_{32} = \int_{-M_{T-1}}^{e_{T-1} - M_{T-1}} \left[\int_{-\infty}^{e_T - A_T} \epsilon_T f(\epsilon_T)\, d\epsilon_T \right] f(\epsilon_{T-1})\, d\epsilon_{T-1}.$$

$$V_{32} = \int_{e_{T-1} - M_{T-1}}^{\infty} \left[\int_{-\infty}^{e_T - A_T} \epsilon_T f(\epsilon_T)\, d\epsilon_T \right] f(\epsilon_{T-1})\, d\epsilon_{T-1}.$$

Using the values for A_T and e_T in (13) and (14), we have

$$\begin{aligned} \frac{dV_{31}}{dM_{T-1}} &= -f(-M_{T-1}) \int_{-\infty}^{e_{T-1} - (1+i_{T-1})(A_{T-1} - M_{T-1})} \epsilon_T f(\epsilon_T)\, d\epsilon_T \\ &\quad - \left(i^l - i_{T-1}\right) \int_{-\infty}^{-M_{T-1}} (e_T - A_T) f(e_T - A_T) f(\epsilon_{T-1})\, d\epsilon_{T-1}. \end{aligned}$$

On the other hand,

$$\begin{aligned} \frac{dV_{32}}{dM_{T-1}} &= -f(e_{T-1} - M_{T-1}) \int_{-\infty}^{(1+i_{T-1})(M_{T-1} - A_{T-1}) - e_{T-1}} \epsilon_T f(\epsilon_T)\, d\epsilon_T \\ &\quad + f(-M_{T-1}) \int_{-\infty}^{e_{T-1} - (1+i_{T-1})(A_{T-1} - M_{T-1})} \epsilon_T f(\epsilon_T)\, d\epsilon_T \\ &\quad + (i_{T-1} - 1) \int_{-M_{T-1}}^{e_{T-1} - M_{T-1}} (e_T - A_T) f(e_T - A_T) f(\epsilon_{T-1})\, d\epsilon_{T-1}. \end{aligned}$$

Finally

$$\frac{dV_{33}}{dM_{T-1}} = f\left(e_{T-1} - M_{T-1}\right) \int_{-\infty}^{(1+i_{T-1})(M_{T-1}-A_{T-1})-e_{T-1}} \epsilon_T f\left(\epsilon_T\right) d\epsilon_T$$
$$- \left(i_{T-1} - i^d\right) \int_{e_{T-1}-M_{T-1}}^{\infty} A_T f\left(-A_T\right) f\left(\epsilon_{T-1}\right) d\epsilon_{T-1}.$$

These expressions evaluated at the optimum, yield

$$\begin{aligned}
\frac{dV_3^*}{dM_{T-1}} = & \left(i_{T-1} - i^l\right) e_{T-1} \int_{-\infty}^{-A_{T-1}} f\left(e_T - A_T\right) f\left(\epsilon_{T-1}\right) d\epsilon_{T-1} \\
& - \left(1 + i^l\right)\left(i_{T-1} - i^l\right) A_{T-1} \int_{-\infty}^{-A_{T-1}} f\left(e_T - A_T\right) f\left(\epsilon_{T-1}\right) d\epsilon_{T-1} \\
& - \left(1 + i^l\right)\left(i_{T-1} - i^l\right) \int_{-\infty}^{-A_{T-1}} \epsilon_{T-1} f\left(e_T - A_T\right) f\left(\epsilon_{T-1}\right) d\epsilon_{T-1} \\
& - \left(1 - i_{T-1}\right) e_{T-1} \int_{-M_{hT-1}}^{e_{T-1}-A_{T-1}} f\left(e_T - A_T\right) f\left(\epsilon_{T-1}\right) d\epsilon_{T-1} \\
& + \left(1 - i_{T-1}\right) 2A_{T-1} \int_{-M_{hT-1}}^{e_{T-1}-A_{T-1}} f\left(e_T - A_T\right) f\left(\epsilon_{T-1}\right) d\epsilon_{T-1} \\
& + \left(1 - i_{T-1}\right) 2 \int_{-M_{T-1}}^{e_{T-1}-A_{T-1}} \epsilon_{T-1} f\left(e_T - A_T\right) f\left(\epsilon_{T-1}\right) d\epsilon_{T-1} \\
& + i^d \left(i_{T-1} - i^d\right) e_{T-1} \int_{e_{T-1}-M_{T-1}}^{\infty} f\left(-A_T\right) f\left(\epsilon_{T-1}\right) d\epsilon_{T-1} \\
& - \left(1 + i^d\right)\left(i_{T-1} - i^d\right) A_{T-1} \int_{e_{T-1}-M_{T-1}}^{\infty} f\left(-A_T\right) f\left(\epsilon_{T-1}\right) d\epsilon_{T-1} \\
& - \left(1 + i^d\right)\left(i_{T-1} - i^d\right) \int_{e_{T-1}-M_{T-1}}^{\infty} \epsilon_{T-1} f\left(-A_T\right) f\left(\epsilon_{T-1}\right) d\epsilon_{T-1}.
\end{aligned} \tag{20}$$

Summarizing, using (17), (18), (19) and (20) we have

$$
\begin{aligned}
-\frac{dV_2^*}{dM_{T-1}} + \frac{dV_3^*}{dM_{T-1}} &= -\left(i_{T-1} - i^l\right) \int_{-\infty}^{-A_{T-1}} F\left(e_T - A_T\right) f\left(\epsilon_{T-1}\right) d\epsilon_{T-1} \\
&\quad - \left(i_{T-1} - 1\right) \int_{-A_{T-1}}^{e_{T-1}-A_{T-1}} F\left(e_T - A_T\right) f\left(\epsilon_{T-1}\right) d\epsilon_{T-1} \\
&\quad - \left(i_{T-1} - i^d\right) \int_{e_{T-1}-A_{T-1}}^{\infty} F\left(-A_T\right) f\left(\epsilon_{T-1}\right) d\epsilon_{T-1}
\end{aligned}
$$

- V_4

Define

$$V_4 = \int_{-\infty}^{\infty} \left(e_T - A_T\right)\left[1 - F\left(e_T - A_T\right)\right] f\left(\epsilon_{T-1}\right) d\epsilon_{T-1} = V_{41} + V_{42}$$

with

$$V_{41} = \int_{-\infty}^{\infty} \left(e_T - A_T\right) f\left(\epsilon_{T-1}\right) d\epsilon_{T-1}$$

and

$$V_{42} = -\int_{-\infty}^{\infty} \left(e_T - A_T\right) F\left(e_T - A_T\right) f\left(\epsilon_{T-1}\right) d\epsilon_{T-1}.$$

The expression V_{41} becomes

$$V_{41} = V_{411} + V_{412} + V_{413}$$

with

$$V_{411} = \int_{-\infty}^{-M_{T-1}} \left(e_T - A_T\right) f\left(\epsilon_{T-1}\right) d\epsilon_{T-1}$$

$$V_{412} = \int_{-M_{T-1}}^{e_{T-1}-M_{T-1}} \left(e_T - A_T\right) f\left(\epsilon_{T-1}\right) d\epsilon_{T-1}$$

$$V_{413} = \int_{e_{T-1}-M_{T-1}}^{\infty} (e_T - A_T) f(\epsilon_{T-1}) d\epsilon_{T-1}.$$

Then we have

$$\begin{aligned}
\frac{dV_{411}}{dM_{T-1}} &= -[e_{T-1} - (1+i_{T-1}) A_{T-1}] f(-M_{T-1}) + \left(i_{T-1} - i^l\right) F(-M_{T-1}) \\
&\quad - \left(i_{T-1} - i^l\right) M_{T-1} f(-M_{T-1}) - \left(1 + i^l\right) M_{T-1} f(-M_{T-1}).
\end{aligned}$$

Also,

$$\begin{aligned}
\frac{dV_{412}}{dM_{T-1}} &= [e_{T-1} - (1+i_{T-1}) A_{T-1}] [-f(e_{T-1} - M_{T-1}) + f(-M_{T-1})] \\
&\quad + (i_{T-1} - 1) [F(e_{T-1} - M_{T-1}) - F(-M_{T-1})] \\
&\quad + (i_{T-1} - 1) M_{T-1} [-f(e_{T-1} - M_{T-1}) + f(-M_{T-1})] \\
&\quad + 2 (e_{T-1} - M_{T-1}) f(e_{T-1} - M_{T-1}) + 2 M_{T-1} f(-M_{T-1}).
\end{aligned}$$

Finally,

$$\begin{aligned}
\frac{dV_{413}}{dM_{T-1}} &= -\left[(1+i_{T-1}) A_{T-1} - i^d e_{T-1}\right] f(e_{T-1} - M_{T-1}) \\
&\quad - \left(i^d - i_{T-1}\right) [1 - F(e_{T-1} - M_{T-1})] \\
&\quad - \left(i^d - i_{T-1}\right) M_{T-1} f(e_{T-1} - M_{T-1}) \\
&\quad - \left(1 + i^d\right) (e_{T-1} - M_{T-1}) f(e_{T-1} - M_{T-1}).
\end{aligned}$$

Then,

$$\begin{aligned}
\frac{dV_{41}^*}{dM_{T-1}} &= \left(1 - i^l\right) F(-A_{T-1}) + (i_{T-1} - 1) F(e_{T-1} - A_{T-1}) \\
&\quad - \left(i^d - i_{T-1}\right) [1 - F(e_{T-1} - A_{T-1})]. \qquad (21)
\end{aligned}$$

On the other hand,

$$\frac{dV_{42}^*}{dM_{T-1}} = -\frac{dV_2^*}{dM_{T-1}}. \qquad (22)$$

- V_5

  Define

$$V_5 = \int_{-\infty}^{\infty} \left[\int_{e_T - A_T}^{\infty} \epsilon_T f(\epsilon_T) d\epsilon_T \right] f(\epsilon_{T-1}) d\epsilon_{T-1} = V_{51} + V_{52} + V_{53}$$

with

$$V_{51} = \int_{-\infty}^{-M_{T-1}} \left[\int_{e_T - A_T}^{\infty} \epsilon_T f(\epsilon_T) d\epsilon_T \right] f(\epsilon_{T-1}) d\epsilon_{T-1}$$

$$V_{52} = \int_{-M_{T-1}}^{e_{T-1}-M_{T-1}} \left[\int_{e_T - A_T}^{\infty} \epsilon_T f(\epsilon_T)\, d\epsilon_T \right] f(\epsilon_{T-1})\, d\epsilon_{T-1}$$

and

$$V_{53} = \int_{e_{T-1}-M_{T-1}}^{\infty} \left[\int_{e_T - A_T}^{\infty} \epsilon_T f(\epsilon_T)\, d\epsilon_T \right] f(\epsilon_{T-1})\, d\epsilon_{T-1}.$$

Using the value for A_T and e_T in (11) and (12), we have

$$\begin{aligned} \frac{dV_{51}}{dM_{T-1}} &= -f(-M_{T-1}) \int_{e_{T-1}-(1+i_{T-1})(M_{T-1}-A_{T-1})}^{\infty} \epsilon_T f(\epsilon_T)\, d\epsilon_T \\ &\quad - \left(i^l - i_{T-1}\right) \int_{-\infty}^{-M_{T-1}} (e_T - A_T) f(e_T - A_T) f(\epsilon_{T-1})\, d\epsilon_{T-1}. \end{aligned}$$

On the other hand, using the value for A_T and e_T in (13) and (14), we have

$$\begin{aligned} \frac{dV_{52}}{dM_{T-1}} &= -f(e_{T-1} - M_{T-1}) \int_{(1+i_{T-1})(M_{T-1}-A_{T-1})-e_{T-1}}^{\infty} \epsilon_T f(\epsilon_T)\, d\epsilon_T \\ &\quad + f(-M_{T-1}) \int_{e_{T-1}-(1+i_{T-1})(M_{T-1}-A_{T-1})}^{\infty} \epsilon_T f(\epsilon_T)\, d\epsilon_T \\ &\quad - (1 - i_{T-1}) \int_{-M_{T-1}}^{e_{T-1}-M_{T-1}} (e_T - A_T) f(e_T - A_T) f(\epsilon_{T-1})\, d\epsilon_{T-1}. \end{aligned}$$

Finally, using the value for A_T and e_T in (15) and (16), we have

$$\begin{aligned} \frac{dV_{53}}{dM_{T-1}} &= f(e_{T-1} - M_{T-1}) \int_{(1+i_{T-1})(M_{T-1}-A_{T-1})-e_{T-1}}^{\infty} \epsilon_T f(\epsilon_T)\, d\epsilon_T \\ &\quad - \left(i^d - i_{T-1}\right) \int_{e_{T-1}-M_{T-1}}^{\infty} A_T f(-A_T) f(\epsilon_{T-1})\, d\epsilon_{T-1}. \end{aligned}$$

These expressions evaluated at the optimum, yield

$$\begin{aligned}
\frac{dV_5^*}{dM_{T-1}} &= -\left(i^l - i_{T-1}\right) \int\limits_{-\infty}^{-A_{T-1}} \left(e_T - A_T\right) f\left(e_T - A_T\right) f\left(\epsilon_{T-1}\right) d\epsilon_{T-1} \\
&\quad - \left(1 - i_{T-1}\right) \int\limits_{-A_{T-1}}^{e_{T-1}-A_{T-1}} \left(e_T - A_T\right) f\left(e_T - A_T\right) f\left(\epsilon_{T-1}\right) d\epsilon_{T-1} \\
&\quad - \left(i^d - i_{T-1}\right) \int\limits_{e_{T-1}-A_{T-1}}^{\infty} A_T f\left(-A_T\right) f\left(\epsilon_{T-1}\right) d\epsilon_{T-1}. \qquad (23)
\end{aligned}$$

Summarizing, using (21), (22), and (23) we have

$$\begin{aligned}
-\frac{dV_4^*}{dM_{T-1}} + \frac{dV_5^*}{dM_{T-1}} &= \left(i_{T-1} - i^l\right) \int\limits_{-\infty}^{-A_{T-1}} F\left(e_T - A_T\right) f\left(\epsilon_{T-1}\right) d\epsilon_{T-1} \\
&\quad - \left(i_{T-1} - 1\right) F\left(e_{T-1} - A_{T-1}\right) + \left(i^l - 1\right) F\left(-A_{T-1}\right) \\
&\quad + \left(i_{T-1} - 1\right) \int\limits_{-A_{T-1}}^{e_{T-1}-A_{T-1}} F\left(e_T - A_T\right) f\left(\epsilon_{T-1}\right) d\epsilon_{T-1} \\
&\quad + \left(i^d - i_{T-1}\right) \left[1 - F\left(e_{T-1} - A_{T-1}\right)\right] \\
&\quad + \left(i_{T-1} - i^d\right) \int\limits_{e_{T-1}-A_{T-1}}^{\infty} F\left(-A_T\right) f\left(\epsilon_{T-1}\right) d\epsilon_{T-1}
\end{aligned}$$

The first order condition for this problem evaluated at the optimum is, therefore

$$\frac{dV_1^*}{dM_{T-1}} + i^l \left(-\frac{dV_2^*}{dM_{T-1}} + \frac{dV_3^*}{dM_{T-1}}\right) + i^d \left(-\frac{dV_4^*}{dM_{T-1}} + \frac{dV_5^*}{dM_{T-1}}\right) = 0.$$

This means

$$
\begin{aligned}
0 \quad = \quad & -i_{T-1} + i^l F\left(-A_{T-1}\right) + i^d\left[1 - F\left(e_{T-1} - A_{T-1}\right)\right] \\
& -i^l\left(i_{T-1} - i^l\right) \int_{-\infty}^{-A_{T-1}} F\left(e_T - A_T\right) f\left(\epsilon_{T-1}\right) d\epsilon_{T-1} \\
& -i^l\left(i_{T-1} - 1\right) \int_{-A_{T-1}}^{e_{T-1}-A_{T-1}} F\left(e_T - A_T\right) f\left(\epsilon_{T-1}\right) d\epsilon_{T-1} \\
& -i^l\left(i_{T-1} - i^d\right) \int_{e_{T-1}-A_{T-1}}^{\infty} F\left(-A_T\right) f\left(\epsilon_{T-1}\right) d\epsilon_{T-1} \\
& +i^d\left(i_{T-1} - i^l\right) \int_{-\infty}^{-A_{T-1}} F\left(e_T - A_T\right) f\left(\epsilon_{T-1}\right) d\epsilon_{T-1} \\
& +i^d\left(i_{T-1} - 1\right) \int_{-A_{T-1}}^{e_{T-1}-A_{T-1}} F\left(e_T - A_T\right) f\left(\epsilon_{T-1}\right) d\epsilon_{T-1} \\
& +i^d\left(i_{T-1} - i^d\right) \int_{e_{T-1}-A_{T-1}}^{\infty} F\left(-A_T\right) f\left(\epsilon_{T-1}\right) d\epsilon_{T-1} \\
& -i^d\left(i_{T-1} - 1\right) F\left(e_{T-1} - A_{T-1}\right) + i^d\left(i^l - 1\right) F\left(-A_{T-1}\right) \\
& +i^d\left(i^d - i_{T-1}\right)\left[1 - F\left(e_{T-1} - A_{T-1}\right)\right]
\end{aligned}
$$

or,

$$\begin{aligned}
0 &= -i_{T-1} + i^l F(-A_{T-1}) + i^d [1 - F(e_{T-1} - A_{T-1})] \\
&\quad -i_{T-1} \int_{-\infty}^{\infty} \left[i^l F(e_T - A_T) + i^d (1 - F(e_T - A_T)) \right] f(\epsilon_{T-1}) \, d\epsilon_{T-1} \\
&\quad +i^l \int_{-\infty}^{-A_{T-1}} \left[i^l F(e_T - A_T) + i^d (1 - F(e_T - A_{hT})) \right] f(\epsilon_{T-1}) \, d\epsilon_{T-1} \\
&\quad +i^d \int_{-A_{T-1}}^{\infty} \left[i^l F(e_T - A_T) + i^d (1 - F(e_T - A_T)) \right] f(\epsilon_{T-1}) \, d\epsilon_{T-1} \\
&\quad +i^l \int_{-A_{T-1}}^{e_{T-1}-A_{T-1}} F(e_T - A_T) f(\epsilon_{T-1}) \, d\epsilon_{T-1} + i^d F(e_{T-1} - A_{T-1}) \\
&\quad -i^d \int_{-A_{T-1}}^{e_{T-1}-A_{T-1}} F(e_T - A_T) f(\epsilon_{T-1}) \, d\epsilon_{T-1} - i^d F(-A_{T-1}) \\
&= -i_{T-1} + i^l F(-A_{T-1}) + i^d [1 - F(-A_{T-1})] \\
&\quad + \int_{-A_{T-1}}^{e_{T-1}-A_{T-1}} i_T f(\epsilon_{T-1}) \, d\epsilon_{T-1} + E.
\end{aligned}$$

with

$$\begin{aligned}
E &= -i_{T-1} \int_{-\infty}^{\infty} i_T f(\epsilon_{T-1}) \, d\epsilon_{T-1} + i^l \int_{-\infty}^{-A_{T-1}} i_T f(\epsilon_{T-1}) \, d\epsilon_{T-1} \\
&\quad +i^d \int_{e_{T-1}-A_{T-1}}^{\infty} i_T f(\epsilon_{T-1}) \, d\epsilon_{T-1}.
\end{aligned}$$

It is easy to show that this term is bounded above and below by

$$-\left(i^l\right)^2 < E < \left(i^l\right)^2,$$

so, a good approximation of the equilibrium interest rate is given by the expression

$$i_{T-1} = i^l F(-A_{T-1}) + i^d [1 - F(-A_{T-1})] + \int_{-A_{T-1}}^{e_{T-1}-A_{T-1}} i_T f(\epsilon_{T-1}) \, d\epsilon_{T-1}.$$

As before, the only important term in V is $E_{T-1}(A_T)$. The other terms are very close to being a constant. This means that the value function at $T-1$ when $e_{T-1} > 0$ can be approximated by

$$\begin{aligned} & V\left(A_{T-1}, e_{T-1}, i_{T-1} \left| e_{T-1} > 0\right.\right) \\ \simeq\ & A_{T-1} + i^l A_{T-1} F\left(-A_{T-1}\right) + i^l \int_{-\infty}^{-A_{T-1}} \epsilon_{T-1} f\left(\epsilon_{T-1}\right) d\epsilon_{T-1} \\ & + i^d A_{T-1}\left[1 - F\left(e_{T-1} - A_{T-1}\right)\right] + i^d \int_{e_{T-1}-A_{T-1}}^{\infty} \epsilon_{T-1} f\left(\epsilon_{T-1}\right) d\epsilon_{T-1}. \end{aligned}$$

5 Problem at t

For any day t in the reserve maintenance period, the problem is to maximize with respect to M_t

$$E_t\left(A_{T+1}\right)$$

when the expectation is evaluated at the equilibrium level computed before. Using recursively the approximations derived above, this problem can be expressed as

$$V\left(A_t, e_t, i_t\right) = \max_{M_t} E_t\left[V\left(A_{t+1}, e_{t+1}, i_{t+1}\right)\right] \simeq \max_{M_t} E_t\left[E_{t+1}\left(A_{t+2}\right)\right].$$

Again, these expectations will be different depending on whether the bank is locked-in or not. If the bank is locked-in, the expectation will be equal to

$$\begin{aligned} E_t\left[V\left(A_{t+1}, e_{t+1}, i_{t+1}\right)\right] \quad = \quad & E_t\left[A_{t+1} + i^l \int_{-\infty}^{-A_{t+1}} \left(A_{t+1} + \epsilon_{t+1}\right) f\left(\epsilon_{t+1}\right) d\epsilon_{t+1}\right. \\ & \left. + i^d \int_{-A_{t+1}}^{\infty} \left(A_{t+1} + \epsilon_{t+1}\right) f\left(\epsilon_{t+1}\right) d\epsilon_{t+1}\right]. \end{aligned}$$

In this case, the equilibrium interest rate is, approximately,

$$i_t = i^l F\left(-A_t\right) + i^d\left[1 - F\left(-A_t\right)\right].$$

For the case where $e_t > 0$, the function to maximize is

$$\begin{aligned} E_t\left[V\left(A_{t+1}, e_{t+1}, i_{t+1}\right)\right] = E_{T-1}\left[A_T - i^l \int_{-\infty}^{e_T - A_T} \left(e_T - A_T - \epsilon_T\right) f\left(\epsilon_T\right) d\epsilon_T\right. \\ \left. - i^d \int_{e_T - A_T}^{\infty} \left(e_T - A_T - \epsilon_T\right) f\left(\epsilon_T\right) d\epsilon_T\right]. \end{aligned}$$

and the resulting equilibrium interest rate will be close to

$$i_t = i^l F\left(-A_t\right) + i^d \left[1 - F\left(-A_t\right)\right] + \int_{-A_t}^{e_t - A_t} i_{t+1} f\left(\epsilon_t\right) d\epsilon_t.$$

References

[1] Pérez Quirós, G. and H. Rodríguez Mendizábal (2000): "The Daily Market for Funds in Europe: Has Something Changed With the EMU?" Universitat Pompeu Fabra Working Paper 474.