# On the Concept of Optimality Interval

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# 1 Introduction

The connection between continued fractions and best approximations is well known. It seems that Huygens (1629–1695) was the first to realise that the approximants of a continued fraction show the property that can be described as 'best approximation' to a number, see [1, p. 86 and ff.]. The following two definitions of best approximation are from Kintchine's *Continuous Fractions* ([5, pp. 27-35]):

Definition (Kintchine). ***a***) *A rational fraction $P/Q$ is a* best approximation of the first kind *to the real number $\omega$, if any other rational fraction having the same or smaller denominator differs from this number more than $P/Q$. In symbols,*

$$\left|\omega - \frac{p}{q}\right| > \left|\omega - \frac{P}{Q}\right|$$

*whenever $p/q \neq P/Q$, $0 \leq q \leq Q$.*
***b***) *A rational fraction $P/Q$ is a* best approximation of the second kind *to the real number $\omega$, if from $p/q \neq P/Q$, $0 \leq q \leq Q$ there follows*

$$|q\,\omega - p| > |Q\,\omega - P|\,.$$

*A best approximation of the second kind is also known as an* optimal *approximation.*

*Remark.* Best approximations of the second kind sometimes behave in a rather non intuitive way. For instance, it may happen that $p/q < P/Q < \omega$ while $p/q$ approximates $\omega$ *better* than $P/Q$.

The motivation for this paper comes from an interesting problem by Bill Gosper cited in [6, page 363, Ex. 39].

> If a baseball player's batting average is .334, what is the fewest possible number of times he has been at bat?

The problem, in a slightly more general setting is (see [3]):

> Given an interval, find in it the rational number with the smallest numerator and denominator.

Gosper's solution is the following:
"Express the endpoints as continued fractions. Find the first term where they differ and add 1 to the lesser term, unless it's last. Discard the terms to the right. What's left is the continued fraction for the smallest rational

in the interval. (If one fraction terminates but matches the other as far as it goes, append an infinity and proceed as above.)”

This problem gave us the ideas: what if we reverse the question? Given a rational number, $P/Q$, what is the set of real numbers for which $P/Q$ is a ‘best approximation’, either of the first kind or the second? Is it an interval or a more complicated set? In the case of best appoximations of the first kind, it seems quite natural that this set is an interval. But in the case of the best approximations of the second kind the remark that follows their definition makes it not so obvious.

We call these sets *Optimality Intervals* and the purpose of this paper is to prove that they are intervals indeed. More formally:

**Definition 1.** ***a***) *Given a positive proper rational fraction $P/Q$ we write $\mathcal{O}_1(P/Q)$ to refer to the set of real numbers to which $P/Q$ is a best approximation of the first kind and*
***b***) *$\mathcal{O}_2(P/Q)$ to refer to the set of real numbers to which $P/Q$ is a best approximation of the second kind.*

Before going any further, we need some results from the arithmetic theory of continued fractions:

## 2 Some Results on Continued Fractions

As usual we write

$$[a_0; a_1, a_2, \dots, a_n] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots + \cfrac{1}{a_n}}}$$

and since mostly we are going to stay in $[0,1]$, we drop the integer part $a_0$ so that $[a_1, a_2, \dots, a_n]$ is to be understood as $[0; a_1, a_2, \dots, a_n]$.

If continued fractions must represent numbers uniquely, the identity

$$[a_1, a_2, \dots, a_n, 1] = [a_1, a_2, \dots, a_n + 1]$$

must be dispensed with, usually by imposing that, apart from the continued fraction $[1]$ representing the number 1, the last partial quotient of all other terminating continued fractions has to be an integer greater than 1. That is the option adopted in [5].

A different strategy may be chosen which consists in accepting both representations as valid and use one or the other depending on the current setting (see [4, p. 133 140]). That is the convention we shall adopt in this paper.

For non terminating continued fractions, which represent irrational numbers, no ambiguity arises.

**Definition 2.** ***a***) *The set of real numbers in* $[0,1]$ *whose first* $n$ *partial quotients are given positive integers* $a_1, a_2, \ldots, a_n$ *constitute an interval with endpoints:* $[a_1, a_2, \ldots, a_n]$ *and* $[a_1, a_2, \ldots, a_n + 1]$ *which we shall refer to as a* fundamental interval of rank $n$*, or a* cylinder. *We write* $J_n(a_1, a_2, \ldots, a_n)$ *to denote such an interval and* $|J_n|$ *to refer to the its length.*

*With the usual notations for the approximants:*

$$\frac{p_{n-1}}{q_{n-1}} = [a_1, a_2, \ldots, a_{n-1}], \qquad \frac{p_n}{q_n} = [a_1, a_2, \ldots, a_n]$$

*it is seen at once that*

$$|J_n(a_1, a_2, \ldots, a_n)| = \frac{1}{q_{n-1}(q_{n-1} + q_n)}.$$

Any real number $\omega \in J_n(a_1, a_2, \ldots, a_n)$ admits a representation of the form

$$\omega = [a_1, a_2, \ldots, a_n + \theta], \quad \theta \in (0,1].$$

The identity

$$[m + \theta] = \frac{1}{m + \theta} = \frac{1}{m} - \frac{\theta}{m(m+\theta)}$$

admits the following generalization

$$[a_1, a_2, \ldots, a_n + \theta] = \frac{p_n + p_{n-1}\theta}{q_n + q_{n-1}\theta} = \frac{p_n}{q_n} + (-1)^{n-1}\frac{(1+\xi_n)\theta}{1+\xi_n\theta}\,|J_n|$$

with

$$\xi_n := \frac{q_{n-1}}{q_n} = [a_n, \ldots, a_1].$$

The ratios $\xi_n,\ n = 1, 2, \ldots$ were introduced by Paul Lévy in his 1929 paper on the Gauss-Kuzmin Theorem [8]. For the details see [9, p. 10 and pp. 155-158].

We shall also adhere to the following convention which we state in the form of a definition.

**Definition 3.** ***a***) *Given a positive proper rational fraction* $P/Q$ *we write*

$$\frac{P}{Q} = [a_1, a_2, \ldots, a_n, m]$$

*where $n \geq 0$ is an integer and $a_1, a_2, \dots, a_n, m$ are positive integers. It is to be understood that the case $n = 0$ implies that $P/Q$ is the unit fraction $1/m$. ($m$ is allowed to take the value 1.)*

**Proposition 1.** *Given two positive proper fractions $p/q$ and $P/Q$ the relation*

$$\left| \frac{P}{Q} - \frac{p}{q} \right| = \frac{1}{Qq}, \tag{1}$$

*holds if, and only if, for some integer $n \geq 0$ there are $n+1$ positive integers, $a_1, a_2, \dots, a_n, m$, which satisfy*

$$\frac{p}{q} = [a_1, a_2, \dots, a_n], \ \frac{P}{Q} = [a_1, a_2, \dots, a_n, m].$$

*which amounts to saying that $p/q$ and $P/Q$ are both consecutive approximants to the numbers of the fundamental interval $J_{n+1}(a_1, a_2, \dots, a_n, m)$.*

*Remark.* If $m = 1$ and only in that case $\frac{p}{q}$ and $\frac{P}{Q}$ are the endpoints of an interval of rank $n+1$. In symbols,

$$\frac{p}{q} = [a_1, a_2, \dots, a_n], \ \frac{P}{Q} = [a_1, a_2, \dots, a_n, 1].$$

It is perhaps worth to point out that in the simplest case, that of order zero, that is to say when

$$\frac{p}{q} = \frac{0}{1} = [0;]$$

then, necessarily,

$$\frac{P}{Q} = \frac{1}{m} = [m] = [m-1, 1]$$

for $m = 2, , 3 \dots$

When the order is equal to one, which means $\frac{p}{q} = \frac{1}{1} = [1]$ then, necessarily,

$$\frac{P}{Q} = \frac{m-1}{m} = \cfrac{1}{1 + \cfrac{1}{m-1}} = [1, m-1]$$

for $m = 2, 3, \dots$. The form $[a_1, a_2, \dots, a_n, m]$ has been chosen to stress the fact that the same pattern propagates to any order of depth.

For the proof, see [4, Theorem 172, p. 140]).

**Lemma 1.** *Given the positive proper fraction $P/Q = [a_1, a_2, \ldots, a_n, m]$, $n \geq 0$, $m \geq 2$, if we set*

$$\frac{p_n}{q_n} = [a_1, a_2, \ldots, a_n] \qquad \frac{P_{n+1}}{Q_{n+1}} = [a_1, a_2, \ldots, a_n, m-1]\,,$$

*then, obviously, $q_n < Q$ and $Q_{n+1} < Q$ while, except for $P/Q$ itself the rational numbers in the open interval with endpoints $p_n/q_n$ and $P_{n+1}/Q_{n+1}$ have denominators strictly larger than $Q$.*

*In other words $p_n/q_n$ and $P_{n+1}/Q_{n+1}$ are the closest fractions to $P/Q$ with denominators less than or equal to $Q$. In addition we have*

$$\begin{cases} \dfrac{p_n}{q_n} & < & \dfrac{P}{Q} & < & \dfrac{P_{n+1}}{Q_{n+1}} & n = 0, 2, 4, \ldots \\ \dfrac{P_{n+1}}{Q_{n+1}} & < & \dfrac{P}{Q} & < & \dfrac{p_n}{q_n} & n = 1, 3, 5, \ldots\,. \end{cases}$$

**Proof** Any number $p/q$ laying between $p_n/q_n$ and $P/Q$ may be written as

$$[a_1, a_2, \ldots, a_n, m + k + \theta]\,, \quad k \geq 1 \quad \theta \in (0, 1)$$

so that $[a_1, a_2, \ldots, a_n, m + k]$ is an approximant to $p/q$ and since the sequence of denominators is strictly increasing. $q > (m+k)q_n + q_{n-1} \geq mq_n + q_{n-1} = Q$.

On the other hand, the numbers laying between $P/Q$ and $P_{n+1}/Q_{n+1}$ constitute the fundamental interval $J_{n+1}(a_1, a_2, \ldots, a_n, m-1)$. $\square$

# 3 Calculation of $\mathcal{O}_1$.

**Proposition 2.** *The set of real numbers to which the positive proper fraction $P/Q = [a_1, a_2, \ldots, a_n, m]$ is a best approximation of the first kind is an interval with endpoints*

$$\begin{cases} r & := & [a_1, a_2, \ldots, a_n, 2m, a_n, \ldots, a_1]\,, \\ s & := & [a_1, a_2, \ldots, a_n, m-1, 2, m-1, a_n, \ldots, a_1]\,. \end{cases}$$

**Proof** It is obvious that given two fractions $p/q$ and $P/Q$ which we may suppose ordered from left to right, and any real number $\omega$ satisfying

$$\frac{1}{2}\left(\frac{p}{q} + \frac{P}{Q}\right) < \omega \leq \frac{P}{Q}$$

is better approximated by $P/Q$.

In view of Lema 1 we only have to show that

$$\frac{1}{2}\left(\frac{p_n}{q_n}+\frac{P}{Q}\right)=[a_1,a_2,\ldots,a_n,2m,a_n,\ldots,a_1]$$

Since $r$ belongs to the fundamental interval $J_n := J_n(a_1,a_2,\ldots,a_n)$ it can be written as

$$r=[a_1,a_2,\ldots,a_n+x]=\frac{p_n}{q_n}\pm\frac{(1+\xi_n)x}{1+\xi_n x}\,|J_n|$$

for some rational $x$, $0<x\leq 1$ and $\xi_n=[a_n,\ldots,a_1]$. Since $P/Q$ may be written as

$$\frac{P}{Q}=[a_1,a_2,\ldots,a_n,m]=\frac{p_n}{q_n}\pm\frac{(1+\xi_n)\frac{1}{m}}{1+\xi_n\frac{1}{m}}\,|J_n|$$

the arithmetic mean takes the form

$$\frac{1}{2}\left(\frac{p_n}{q_n}+\frac{P}{Q}\right)=\frac{p_n}{q_n}\pm\frac{1}{2}\,\frac{(1+\xi_n)}{m+\xi_n}\,|J_n|$$

and equating the two expressions,

$$\frac{x}{1+\xi_n x}=\frac{1}{2(m+\xi_n)}.$$

A simple calculation shows that the last equation is equivalent to

$$x=\frac{1}{2m+\xi_n}$$

and since $\xi_n=[a_n,\ldots,a_1]$ we finally get

$$x=[2m,a_n,\ldots,a_1]\iff r=[a_1,a_2,\ldots,a_n,2m,a_n,\ldots,a_1]\,.$$

As to the other endpoint $s$, again in view of Lema 1 setting

$$J_{n+1}:=J_{n+1}(a_1,a_2,\ldots,a_n,m-1)$$

the same argument applies and now we have

$$s:=\frac{1}{2}\left(\frac{P_{n+1}}{Q_{n+1}}+\frac{P}{Q}\right)=\frac{P_{n+1}}{Q_{n+1}}\pm\frac{1}{2}\,|J_{n+1}|$$

which, as the midpoint of $J_{n+1}$, can also be written as

$$[a_1,a_2,\ldots,a_n,m-1,y]=\frac{P_{n+1}}{Q_{n+1}}\pm\frac{(1+\xi_{n+1})y}{1+\xi_{n+1}y}\,|J_{n+1}|$$

for some rational $y$, $0 < y \leq 1$ and $\xi_{n+1} = [m-1, a_n, \dots, a_1]$. Equating, we get

$$\frac{1}{2} = \frac{(1+\xi_{n+1})y}{1+\xi_{n+1}y}$$

which is easily seen to imply

$$y = \frac{1}{2+\xi_{n+1}} = [2, m-1, a_n, \dots, a_1],$$

and, finally

$$s = [a_1, a_2, \dots, a_n, m-1, 2, m-1, a_n, \dots, a_1]$$

as was to be proved. $\square$

# 4 Calculation of $\mathcal{O}_2$.

For $P/Q$ falling short of being an optimal approximation to the real number $\omega$ there must exist a fraction $p/q$ with $q \leq Q$ and

$$|q\,\omega - p| \leq |Q\,\omega - P|, \tag{2}$$

which might be phrased saying that $p/q$ *prevents* $P/Q$ from being an optimal approximation to $\omega$.

The following remarks show that when studying approximations of the second kind, mediant fractions play the role formerly played by arithmetic mean. If we suppose for instance, that $p/q < \omega < P/Q$, then condition (2) amounts to

$$(q+Q)\,\omega \leq p+P \iff \omega \leq \frac{p+P}{q+Q},$$

while in the case that $p/q < P/Q < \omega$ then condition (2) is equivalent to

$$(P-p)\,\omega \geq Q-q \iff \omega \geq \frac{P-p}{Q-q}.$$

To sum up, if $p/q < P/Q$ the condition for $p/q$ not to prevent $P/Q$ from being an optimal approximation to $\omega$ is that $\omega$ belong to the open interval

$$\mathcal{I}_l := \Big(\frac{p+P}{q+Q}, \frac{P-p}{Q-q}\Big).$$

while if $P/Q < p/q$ the condition is

$$\omega \in \Big(\frac{P-p}{Q-q}, \frac{p+P}{q+Q}\Big) =: \mathcal{I}_r.$$

Since no ambiguity may arise, we may agree to write $\mathcal{I}(p/q \to P/Q)$ to refer to $\mathcal{I}_l$ or $\mathcal{I}_r$ depending on which side $p/q$ lies. With this notation it becomes obvious that

$$\mathcal{O}_2\left(\tfrac{P}{Q}\right) = \bigcap_{p \le q \le Q} \mathcal{I}\left(\tfrac{p}{q} \to \tfrac{P}{Q}\right)$$

which, in addition, proves that $\mathcal{O}_2(P/Q)$ is an interval. The following lema, where we switch to determinant notation, does most of the job.

**Lemma 2.** *Let $a/b, p/q, P/Q$ be three positive proper fractions.*

***a**) If $a/b < p/q < P/Q$, the condition*

$$\frac{P}{Q} - \frac{p}{q} = \frac{1}{Qq} \quad \Longleftrightarrow \quad \begin{vmatrix} P & p \\ Q & q \end{vmatrix} = 1 \tag{3}$$

*implies*

$$\textbf{\textit{i}}) \quad \frac{a+P}{b+Q} \le \frac{p+P}{q+Q} \quad \textit{and} \quad \textbf{\textit{ii}}) \quad \frac{P-p}{Q-q} \le \frac{P-a}{Q-b}. \tag{4}$$

***b**) In a similar way, if $P/Q < p/q < a/b$ then the condition*

$$\frac{p}{q} - \frac{P}{Q} = \frac{1}{Qq} \quad \Longleftrightarrow \quad \begin{vmatrix} p & P \\ q & Q \end{vmatrix} = 1 \tag{5}$$

*implies*

$$\textbf{\textit{i}}) \quad \frac{p+P}{q+Q} \ge \frac{a+P}{b+Q} \quad \textit{and} \quad \textbf{\textit{ii}}) \quad \frac{P-a}{Q-b} \le \frac{P-p}{Q-q}. \tag{6}$$

*In other words, in any case*

$$\mathcal{I}\left(\frac{p}{q} \to \frac{P}{Q}\right) \subseteq \mathcal{I}\left(\frac{a}{b} \to \frac{P}{Q}\right)$$

**Proof** We only give the details of case **a**). It is sufficient to show that

$$\textbf{i}) \quad \begin{vmatrix} p+P & a+P \\ q+Q & b+Q \end{vmatrix} \ge 0 \quad \text{and} \quad \textbf{ii}) \quad \begin{vmatrix} P-a & P-p \\ Q-b & Q-q \end{vmatrix} \ge 0. \tag{7}$$

Since (3) implies

$$\begin{vmatrix} p+P & P \\ q+Q & Q \end{vmatrix} = \begin{vmatrix} P-p & P \\ Q-q & Q \end{vmatrix} = -1$$

from

$$\begin{vmatrix} p+P & a+P \\ q+Q & b+Q \end{vmatrix} = \begin{vmatrix} p+P & a \\ q+Q & b \end{vmatrix} + \begin{vmatrix} p+P & P \\ q+Q & Q \end{vmatrix} = \begin{vmatrix} p+P & a \\ q+Q & b \end{vmatrix} - 1$$

and the fact that $\begin{vmatrix} p+P & a \\ q+Q & b \end{vmatrix}$ is a positive integer we get

$$\begin{vmatrix} p+P & a+P \\ q+Q & b+Q \end{vmatrix} \geq 1 - 1 = 0.$$

which proves case (7) **i**).

In an analogous way, from

$$\begin{vmatrix} P-a & P-p \\ Q-b & Q-q \end{vmatrix} = \begin{vmatrix} P & P-p \\ Q & Q-q \end{vmatrix} + \begin{vmatrix} -a & P-p \\ -b & Q-q \end{vmatrix} = -1 + \begin{vmatrix} P-p & a \\ Q-q & b \end{vmatrix}.$$

Since, obviously $a/b < P/Q < (P-p)/(Q-q)$, again $\begin{vmatrix} P-p & a \\ Q-q & b \end{vmatrix}$ is a positive integer and (7) **ii**) follows. $\square$

**Proposition 3.** *The set of real numbers to which the positive proper fraction $P/Q = [a_1, a_2, \ldots, a_n, m]$ is a best approximation of the second kind is an interval with endpoints*

$$\begin{cases} r & := & [a_1, a_2, \ldots, a_n, m+1] = [a_1, a_2, \ldots, a_n, m, 1] \\ s & := & [a_1, a_2, \ldots, a_n, m-1, 2]\,. \end{cases} \tag{8}$$

**Proof** By Lema 1, $p_n/q_n$ and $P_{n+1}/Q_{n+1}$ are the closest fractions to $P/Q$ with denominators less than or equal to $Q$ and, obviously verify (3), Lema 2 implies that no other fraction may prevent $P/Q$ from being an optimal approximation to the numbers of the set

$$\mathcal{I}\left(\frac{p_n}{q_n} \to \frac{P}{Q}\right) \bigcap \mathcal{I}\left(\frac{P_{n+1}}{Q_{n+1}} \to \frac{P}{Q}\right)$$

which is easily seen to be the interval with endpoints the mediant fractions

$$\frac{p_n + P}{q_n + Q} \quad \text{and} \quad \frac{P_{n+1} + P}{Q_{n+1} + Q}.$$

Two further calculations show that

$$\frac{p_n + P}{q_n + Q} = \frac{(m+1)p_n + p_{n-1}}{(m+1)q_n + q_{n-1}}$$

and that

$$\frac{P_{n+1}+P}{Q_{n+1}+Q}=\frac{(2m-1)p_n+2p_{n-1}}{(2m-1)q_n+2q_{n-1}}=\frac{(m-1+\frac{1}{2})p_n+2p_{n-1}}{(m-1+\frac{1}{2})q_n+2q_{n-1}},$$

which prove (8).

$\square$

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