

Forecasting and Turning Point Predictions in a Bayesian Panel VAR Model

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Abstract

We provide methods for forecasting variables and predicting turning points in panel Bayesian VARs. We specify a flexible model which accounts for both interdependencies in the cross section and time variations in the parameters. Posterior distributions for the parameters are obtained for a particular type of diffuse, for Minnesota-type and for hierarchical priors. Formulas for multistep, multiunit point and average forecasts are provided. An application to the problem of forecasting the growth rate of output and of predicting turning points in the G-7 illustrates the approach. A comparison with alternative forecasting methods is also provided.

Key Words: Forecasting Turning Points, Bayesian Methods, Panel VAR, Markov Chains Monte Carlo Methods

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1 Introduction

Panel VAR models have become increasingly popular in macroeconomics to study the transmission of shocks across countries (Ballabriga, Sebastian and Valles (1995)), the propagation effects of monetary policy in the European Union (Gertler and Smets (1996)) and the average differential response of developed and underdeveloped countries to domestic and external disturbances (Hörmann and Rudebusch (1997), Rudebusch (1998)). At the same time, recent developments in computer technology have permitted the estimation of increasingly complex multicountry VAR models in reasonable time, making them potentially usable for a variety of forecasting and policy purposes.

Despite this interest, the theory for panel VAR is somewhat underdeveloped. After the works of Chamberlain (1982, 1984) and Holtz-Eakin et al. (1988), who specify panel VAR models for micro data, to the best of our knowledge only Pesaran and Smith (1996), Canova and Marcellino (1997) and Hsiao et al. (1998) have considered problems connected with the specification and the estimation of (univariate) dynamic macro panels. García Ferrer et al. (1987), Zellner and Hong (1989), Zellner, Hong and Min (1991), on the other hand, have provided Bayesian shrinkage estimators and predictors for similar models. In general, a researcher focuses on the specification

$$y_{it} = A(L)y_{it-1} + \varepsilon_{it}$$

where y_{it} is a G -dimensional vector, $i = 1, \dots, N$; $A(L)$ is a matrix in the lag operator, $\varepsilon_{it} = \alpha_i + \beta_t + u_{it}$, where β_t is a time effect, α_i is a unit specific effect and u_{it} a disturbance term. In some cases (see e.g. Holtz-Eakin et al. (1988)) a specification with time varying slope coefficients and a fixed effect is used. Two main restrictions characterize this specification. First, it assumes common slope coefficients. Second, it does not allow for interdependencies across units. With these restrictions, the interest is typically in estimating the average dynamics of the system in response to shocks (the matrix $A(L)$).

García Ferrer et al., Canova and Marcellino and Pesaran and Smith, instead, use a univariate dynamic model of the form

$$y_{it} = \alpha_i + \beta_i y_{it-1} + \gamma_i' x_t + \delta_t' v_t + \varepsilon_{it}$$

where y_{it} is a scalar, x_t is a set of k exogenous unit specific regressors, v_t is a set of h exogenous regressors common to all units while β_i , γ_i and δ_t are unit specific vectors of coefficients. In some specifications these vectors of coefficients are assumed to have an exchangeable prior. Two restrictions are implicit also in this specification. First, no time variation is allowed in the parameters. Second, there are no interdependencies either among different variables within units or among the same variable across units.

The task of this paper is to relax these restrictions and study the issues of specification, estimation and forecasting in a macropanel VAR model with interdependencies. Our point of view is Bayesian. Such an approach has been widely used in the VAR literature since the works of Doan, Litterman and Sims (1984), Litterman (1986), and Sims and Zha (1998) and provides a convenient framework where one can allow for both interdependencies and meaningful time variations in the coefficients. The specification we consider has the general form

$$y_{it} = A_{it}(L)Y_{t-1} + \varepsilon_{it}$$

where Y_{it} ($s < t$) is a vector of G elements (there G variables for each unit $i = 1, \dots, N$). Because coefficients vary across units and along time, estimation of the parameters is impossible without imposing restrictions. However, instead of constraining the coefficients to be the same across units, we assume that they are random and a prior distribution on $A_{it}(\lambda)$ is introduced. We decompose the parameter vector into two components, one which is unit specific and the other which is time specific. We specify a flexible prior on these two components which parsimoniously takes into account possible interdependencies in the cross section and allows for time variations in the evolution of the parameters over time. The prior shares features with those of Lindley and Smith (1972), Doan, Litterman and Sims (1984) and Hsiao et al. (1998) and it is specified to have a hierarchical structure, which allows for various degrees of ignorance in the researcher's information about the parameters.

Besides important considerations concerning the specification of the model, Bayesian VARs are known produce better forecasts than unrestricted VAR and, in many situations, ARIMA or structural models (Canova (1995) for references). By allowing interdependencies and some degree of information pooling across units we introduce an additional level of flexibility which may improve the forecasting ability of these models.

We analyze several special cases of our specification and compute Bayesian estimators for the individual coefficients and for their mean values over the cross section. In some cases analytical formulas for the posterior mean are available using standard formulas. Whenever the parameters of the prior are unknown, we employ the predictive density of the model to estimate them and plug in our estimates in the relevant formulas in an empirical Bayesian fashion.

In the case of fully hierarchical priors, a Markov Chain Monte Carlo method (the Gibbs sampler) is employed to calculate posterior distributions. Such an approach is particularly useful in our setup since it exploits the recursive features of the posterior distribution. We provide recursive formulas for multistep, multiunit forecasts, consistent with the information available at each point in time using the posterior of the parameters or the predictive density of future observations. The predictive density of future observation is also used to compute turning point probabilities.

To illustrate the forecasting ability of the proposed approach, we apply the methodology to the problem of predicting output growth, of forecasting turning points in output growth and computing the probability of a recession in the G-7 using three variables (output growth, real stock returns and real money growth) for each country in the panel. To evaluate the performance of the model we also provide a forecasting comparison with other specifications suggested in the literature. We show that our panel VAR approach improves over existing univariate and simple BVAR models when we measure the forecasting performance using the Theil-U and the MAD criteria, both at the one step and at the four steps horizons. The improvements are of the order of 5-10% with the Theil-U and about 2-4% with the MAD. The forecasting performance of our specification is also slightly better than the one of a BVAR model which mechanically extends the Litterman prior to the panel case. In terms of turning point predictions, the two versions of our panel approach are able to recognize about 80% of turning points in the sample and they turn out to be the best for this task, along with Zellner's g-prior shrinkage approach. The simple extension of the Litterman's prior to the panel case does poorly along this dimension and, among all the procedures employed, is the second worst. Finally, we show that the proposed method is competitive with

the best specifications in predicting the peak in US economic activity occurred in 1990:3 when using the information available in 1988:4, a peak which was missed by many of the commercial and government forecasting procedures. Depending on the specification, our approach finds 20-55% probability of a downward turn at that date.

The rest of the paper is organized as follows. The next section gives the general model specification and the assumptions we make. Section 3 provides the generalities of Bayesian estimation of the model. Section 4 specifies the prior and discusses the computational issues involved. Section 5 describes formulas for multi-step, multi-units forecasting. Section 6 contains the forecasting application to a panel VAR model for the G-7. Section 7 concludes.

2 The general specification

The statistical reduced form model we use is of the form:

$$y_{it} = \sum_{j=1}^G \sum_{l=1}^G b_{it,l}^j y_{jt-l} + d_t v_t + u_{it} \quad (1)$$

where $i = 1, \dots, N$; $t = 1, \dots, T$; y_{it} is a G -dimensional vector for each i , $b_{it,l}^j$ are $G \times G$ matrices, d_t is $G \times q$, v_t is a $q \times 1$ vector of exogenous variables common to all units and u_{it} is a G -dimensional vector of random disturbances. Here p is the number of lags, G the number of endogenous variables and q the number of exogenous variables.

The generality of (1) comes from at least two features. First, the coefficients are allowed to vary both across units and across time. Second, there are interdependencies among units, since $b_{it,l}^j \neq 0$ for $j \neq i$ and for any l . Both features constitute the main difference with the literature (Holtz-Eakin et al. (1988), Roubini (1998)) that considers panel VAR models. It is easy to verify that if we set $d_t v_t = a_t$; $b_t = b \otimes I$; $u_{it} = \tilde{A}_t F_i + \varepsilon_{it}$ $b_{it,l}^j = 0$; $j \neq i$; $\forall l$; our specification collapses to the one used by Holtz-Eakin et al. (1988).

We rewrite (1) in a stacked regression manner

$$Y_t = W_t \theta_t + U_t \quad (2)$$

where $W_t = I_{NG} - X_t'$; $X_t = (y_{t-1}', y_{t-2}', \dots, y_{t-p}')'$; $v_t' = (v_{1t}, \dots, v_{qt})'$ and $\theta_t = (c_{1t}, \dots, c_{Gt})'$. Here y_s ($s < t$) is a $N \times G$ -dimensional vector, c_{it}^g are k -dimensional vectors, with $k = N \times p + q$ containing stacked, the g rows of the coefficient matrices b_t and d_t , while Y_t and U_t are $N \times G \times 1$ matrices containing the endogenous variables and the random disturbances of the model.

If the θ_{it} are different for each cross-sectional unit in different time periods, there is no way to obtain meaningful estimates of them. One possibility is to view each coefficient vector as random with a given probability distribution. We make the following assumptions:

1. For each i , the $G \times k \times 1$ vector θ_{it} has a time invariant and a time varying component, that is

$$\theta_{it} = \theta_i + \varepsilon_{it} \quad (3)$$

2. For each i , the $G \times 1$ vector of time invariant components α_i follows a normal distribution

$$\alpha_i \gg N(R_i \alpha; \Sigma_i) \tag{4}$$

where $R_i = I_G - E_i$, $\Sigma_i = V - E_i^{-1} E_i$, and the $G \times G$ matrix V and the $k \times k$ matrix E_i are symmetric and positive definite. Here E_i is a $k \times k$ matrix that commutes the coefficients of unit i for each of the G equations with those of unit one. We also assume that $\text{cov}(\alpha_i; \alpha_j) = 0$ for $i \neq j$.

3. The mean vector α is common to all units and is assumed to have a normal distribution

$$\alpha \gg N(1; a) \tag{5}$$

4. For each i we write the vector of the time varying components as $y_{it} = R_{i,t} y_t$, where y_t is independent of α_i for any i . The $G \times 1$ vector y_t evolves according to

$$y_t = B y_{t-1} + \epsilon_t \tag{6}$$

where $B = \frac{1}{2} \text{diag}(G_k)$ and, conditional on U_t and W_t , $\epsilon_t \gg N(0; S_\epsilon)$, with $S_\epsilon = V - \Sigma$, and Σ is a positive definite, symmetric matrix. The initial condition is such that $y_0 \gg N(y_0; \Sigma_0)$.

5. Conditional on W_t , the vector of random disturbances U_t has a normal distribution

$$U_t \gg N(0; S_u) \tag{7}$$

We assume that $S_u = S - H$, where S is a $N \times N$ matrix and H is a $G \times G$ matrix, both positive definite and symmetric.

Given the previous assumptions, the structure of the model (1) can be summarized with the following a priori hierarchical scheme

$$\begin{aligned} Y_t & \mid j \ F_t; \alpha_i; y_t \gg N(W_t \alpha + Z_{t,t}; S_u) \\ \alpha & \mid j \ F_t \gg N(S_N \alpha; \Sigma) \\ \alpha & \mid j \ F_t \gg N(1; a) \\ y_t & \mid j \ F_t \gg N(\hat{y}_{t|t-1}; \hat{\Sigma}_{t|t-1}) \end{aligned} \tag{8}$$

where F_t is the information set at t (which includes Y_0 , the presample information, and W_t); $S_N = \epsilon_N - R_i$; $Z_t = W_t S_N$; $\Sigma = \text{diag}(\Sigma_1; \dots; \Sigma_n)$, $\hat{y}_{t|t-1} = B \hat{y}_{t-1|t-1}$; $\hat{\Sigma}_{t|t-1} = B \hat{\Sigma}_{t-1|t-1} B' + S_\epsilon$, ϵ_N is a vector of ones of dimension N and the notation $t|j; 1$ indicates values at t predicted with information at $t; 1$.

Assumptions 1-4 decompose the parameters vector in 2 components: one is unit specific and constant over time, the other is common across units but varies with time. The prior possibility for time variation increases the flexibility of the specification and provides a general mechanism to account for structural shifts without explicitly modelling the source of the shift. The fact that the

time-varying parameter vector is common across units does not prevent unit-specific structural shifts, since α_{it} can be re-written as

$$\alpha_{it} = (\mathbf{1} - \frac{1}{2})\alpha_i + \frac{1}{2}\alpha_{it-1} + \epsilon_{it} \tag{9}$$

where unit-specific variations of time occur through the common coefficient $\frac{1}{2}$:

Assumptions 2 and 3 can be used to recover the vector α or the mean coefficient vector α . In this sense, we can distinguish between "fixed" and "random" effects, following the terminology of Lindley and Smith (1972). By fixed effects we mean the estimation of the vector α_{it} , while the term random effects refers to the estimation of $\alpha_t = \alpha + \nu_t$. For example, in the context of a VAR without interdependencies, (i.e. $\beta_{it,l}^j = 0; j \notin i$), we may be more interested in the relationships among the variables of the system for a "typical" unit, in which case interest centers in the estimation of the random effect α_t . If, instead, we are interested in the relationships across units, for example, wishing to find the effect of a shock in the g variable of unit j on the variables of unit i ; we better estimate α_{it} for each unit i . In the context of forecasting we may be concerned with point prediction using the average coefficient vector α_t or in predicting future values of the variables of interest using information available for each unit.

The assumed Kronecker structure for the variance-covariance matrices is convenient to nest interesting hypothesis. For instance, when $\beta_1 = 0$, there is no heterogeneity in the cross-sectional dimension of the panel. If $B = I_{Gk}$, coefficients evolve over time as a random walk, while when $B = I_{Gk}$ and $\beta_2 = 0$, the model reduces to a standard dynamic panel model with no time variation in the coefficient vector. Finally, when $V = 0$ neither heterogeneity nor time variation are present in the model.

The prior specification is fully symmetric in the sense that it is the same regardless of the variables and of the units we are considering. In some applications where it is interesting to consider some prior asymmetries, this restriction may not be needed. In that case we set $E_i = I_N$ so that $R_i = I_G - I_N$ and (3) becomes $\alpha_{it} = \alpha_i + \nu_{it}$ where $\alpha_i \gg N$ ($\alpha; \nu$) and the prior distributions for α and ν_{it} are the same as before.

As compared to standard BVAR models, we allow for some degree of a-priori pooling of cross-sectional information via the exchangeable prior on α . This may be important if there are some similarities in the time series characteristics of the vector of variables considered across units since coefficients of other units may contain useful information for estimating the coefficients of the unit under consideration. A single country VAR with fixed coefficients is nested in our specification and can be obtained by setting $\beta_{it,l}^j = 0; j \notin i; \beta$ and letting $\alpha; \nu; S_\epsilon$ go to zero.

3 Posterior Estimates

3.1 Fixed effects model

Given prior information on α_t , and assuming that β_0, β_1 and the covariance matrices are known, we can obtain the posterior distribution of the parameter vector by combining the likelihood function conditional on F_t with the prior distribution for α_t in the usual way. From (8) the likelihood is

$$L(Y_t; F_t) = N(W_t \alpha + L_{t,t} \nu; S_u)$$

and the prior, given information at $t-1$, is

$$p(\theta_t | F_t) = N(\hat{\theta}_{t-1}; \hat{H}_{t-1}) \quad (10)$$

where $\hat{\theta}_{t-1} = S_N^{-1} + \hat{\Sigma}_{t|t-1}$ and $\hat{H}_{t-1} = (S_N^{-1} S_N' + \zeta) + S_N^{-1} \hat{\Sigma}_{t|t-1} S_N'$:

Standard calculations give us that the posterior $\mathcal{N}(\theta_t | F_t; Y_t)$ is normal with mean θ_t^* and variance H_t^* where

$$\begin{aligned} \theta_t^* &= H_t^* W_t' S_u^{-1} Y_t + \hat{H}_{t-1}^{-1} \hat{\theta}_{t-1} \\ H_t^* &= \hat{H}_{t-1}^{-1} + W_t' S_u^{-1} W_t \end{aligned} \quad (11)$$

Hence θ_t^* is a standard weighted average of prior and sample information. With a known S_u and starting from initial conditions $\hat{\theta}_0$ and \hat{H}_0 we can also obtain posterior moments for θ_t using the following recursive formulas:

$$\begin{aligned} \theta_t^* &= \hat{\theta}_{t-1} + \hat{H}_{t-1}^{-1} W_t' W_t \hat{H}_{t-1}^{-1} W_t' + S_u^{-1} (Y_t - W_t \hat{\theta}_{t-1}) \\ H_t^* &= \hat{H}_{t-1}^{-1} - \hat{H}_{t-1}^{-1} W_t' W_t \hat{H}_{t-1}^{-1} W_t' + S_u^{-1} W_t \hat{H}_{t-1} \end{aligned} \quad (12)$$

Here information about θ_t^* and H_t^* is updated in a Kalman filter fashion.

In some cases attention may be centered in obtaining posterior distributions of θ^* and Σ_t separately. It is straightforward to show that

$$\begin{aligned} \begin{matrix} \tilde{A} \\ Y_t | F_t \end{matrix} &\gg N \left(\begin{matrix} S_N^{-1} \\ \tilde{A} \end{matrix}; \begin{matrix} \tilde{A} \\ \begin{matrix} \hat{A}_{11} & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_{22} \end{matrix} \end{matrix} \right) \end{aligned}$$

where $\hat{A}_{11} = (S_N^{-1} S_N' + \zeta)$; $\hat{A}_{12} = \hat{A}_{11} W_t'$; $\hat{A}_{21} = W_t \hat{A}_{11}$; $\hat{A}_{22} = W_t \hat{A}_{11} W_t' + Z_t \hat{\Sigma}_{t|t-1} Z_t' + S_u$.

Using the properties of multivariate normal distributions, the conditional marginal $\mathcal{N}_1(\theta^* | F_t; Y_t)$ is normal with mean $\theta^* = S_N^{-1} + \hat{A}_{12} \hat{A}_{22}^{-1} Y_t | Z_t^{-1} + \hat{\Sigma}_{t|t-1}$ and variance $V_\alpha^* = \hat{A}_{11} - \hat{A}_{12} \hat{A}_{22}^{-1} \hat{A}_{21}$.

Repeating the same argument we obtain that the conditional marginal $\mathcal{N}_2(\Sigma_t | F_t; Y_t)$ is normal with mean $\Sigma_t^* = \hat{\Sigma}_{t|t-1} + \hat{\Sigma}_{t|t-1} Z_t' \hat{A}_{22}^{-1} Z_t \hat{\Sigma}_{t|t-1}$ and variance $\Sigma_t^* = \hat{\Sigma}_{t|t-1} - \hat{\Sigma}_{t|t-1} Z_t' \hat{A}_{22}^{-1} Z_t \hat{\Sigma}_{t|t-1}$. As usual, the mean of the posterior distribution is used as a point estimate for the parameter vector while the variance provides a measure of dispersion.

For the formulas to be operational we need at time $t = 1$ a specification for S_u and for the prior distributions of θ^* and Σ_t , which in turn requires the specification of the matrices B , S_ε , ζ , α , γ_0 and of the vectors β^1 and β^2 . We will return on this issue in the next section.

3.2 Random effects model

When interest centers on the estimation of the mean vector $\theta^* = \theta^* + \Sigma_t$, we rewrite the model as

$$Y_t = Z_t \theta_t^* + \varepsilon_t \quad (13)$$

prior for all the elements of β leads to posteriors which contain the sample information summarized in a least square fashion. Also, as shown by Kadiyala and Karlsson (1997), such prior produces posterior dependence among the coefficients of different equations, i.e. the joint posterior for the $N \times K \times T$ vector of coefficients does not factor into the product of the posterior for the K coefficients of each of the N equations. Here we concentrate attention on two special cases of interest, one where there is no information on the location of the mean of the unit speed effect $\beta^{a-1} = 0$ and one where there is no information on the time varying component of the coefficients either at time zero $\beta_0^{-1} = 0$ or at a particular point in time $\beta_{t|t-1}^{-1} = 0$. All other components of the vector of parameters are assumed to be known.

4.1.1 Case 1: Ignorance about β

When the prior distribution of the second stage of the hierarchy is proportional to a constant, the posterior distribution changes according to the following proposition:

Proposition 4.1 Given the prior (8), if $\beta^{a-1} = 0$; conditional on Y_t and F_t ,

(i) The posterior distribution $\mathcal{N}_3(\beta | Y_t; F_t)$ is normal with mean β^{**} and variance α^{**} where

$$\beta^{**} = \alpha^{**} Z_t' [Z_t \hat{\Lambda}_{t|t-1} Z_t' + S_u + W_t \Phi W_t']^{-1} Y_t ; Z_t \hat{\Lambda}_{t|t-1}$$

$$\alpha^{**^{-1}} = Z_t' [Z_t \hat{\Lambda}_{t|t-1} Z_t' + S_u + W_t \Phi W_t']^{-1} Z_t$$

(ii) The posterior distribution $\mathcal{N}_1(\beta | Y_t; F_t)$ is normal with mean β^{**} and variance V_α^{**} where

$$\beta^{**} = V_\alpha^{**} W_t' [S_u + Z_t \hat{\Lambda}_{t|t-1} Z_t']^{-1} Y_t ; Z_t \hat{\Lambda}_{t|t-1} ;$$

$$V_\alpha^{**^{-1}} = W_t' [S_u + Z_t \hat{\Lambda}_{t|t-1} Z_t']^{-1} W_t + F \tag{18}$$

with $F = \Phi^{-1} ; \Phi^{-1} S_N^{-1} S_N' \Phi^{-1} S_N^{-1} S_N' \Phi^{-1}$.

(iii) The posterior distribution of $\beta_{s,t}$ is equal to the prior, i.e.,

$$\mathcal{N}_2(\beta_{s,t} | Y_t; X_t) = p(\beta_{s,t} | X_t):$$

(The proof of all propositions is in the appendix).

Notice that the diffuse prior on β does not allow to update the prior information we have on $\beta_{s,t}$. In fact, in this case, the posterior distribution of $\beta_{s,t}$ does not depend on the prior for $\beta_{s,t}$. To see this note that, with $\beta^{a-1} = 0$; we have that $H_{t-1} S_N^{-a} + \hat{\Lambda}_{t|t-1} S_N' + \Phi = F^{-1}$ and using the fact that $H_{t-1}^{-1} \hat{\Lambda}_{t-1}^{-1} = F S_N^{-1} + \beta_{s,t-1}^{-1} = 0$ we have $\beta_{s,t}^* = F + W_t' S_u^{-1} W_t^{-1} W_t' S_u^{-1} Y_t$ and $H_t^* = F + W_t' S_u^{-1} W_t^{-1}$ where no prior information on $\beta_{s,t}$ is involved.

4.1.2 Case 2: Ignorance about α_t

There are two simple ways of attaching a diffuse prior to the time varying component of the coefficient vector. One possibility is to consider lack of information at time zero ($\alpha_0^{-1} = 0$). When the prior distribution for α_0 is proportional to a constant, given the autoregressive structure for α_t , and provided $\lambda < 1$, the process tends to "forget" the initial condition. In other words, subsequent realizations of α_t make less and less uncertain our information on the time varying component of the coefficients so that $\alpha_0^{-1} = 0$ does not imply $\alpha_{t|t-1}^{-1} = 0$ at all points in time and, for large enough T , the posterior for α and α_t is the one presented in section 3.

Another possibility is to set $S_\varepsilon^{-1} = 0$. To implement this diffuse prior, we assume $\alpha_0^{-1} = 0$. Notice that $\alpha_{t|t-1}^{-1} = B' \alpha_{t-1|t-1}^{-1} B + S_\varepsilon$. Therefore if $S_\varepsilon^{-1} = 0$, $\alpha_{t|t-1}^{-1} = 0$. In this case it is possible to prove the following result

Proposition 4.2 Given the prior (8), if $S_\varepsilon^{-1} = 0$; then

- (i) The posterior distribution of α is equal to the prior, i.e.,

$$p_3(\alpha | Y_t; X_t) = p(\alpha | X_t)$$

- (ii) The posterior distribution $p_1(\alpha | Y_t; F_t)$ is normal with mean α^{**} and variance V_α^{**} where

$$\alpha^{**} = V_\alpha^{**} \left[\sum_{t=1}^T W_t' S_t^{-1} Y_t + \sum_{t=1}^T i' S_{N^a} S_N' + \zeta' \Phi^{-1} S_N^{-1} i \right]$$

$$V_\alpha^{**} = W_t' T W_t + \sum_{t=1}^T i' S_{N^a} S_N' + \zeta' \Phi^{-1}$$

$$\text{and } T = S_u^{-1} \left[\sum_{t=1}^T L_t' S_u^{-1} L_t + \zeta' \Phi^{-1} \zeta \right] S_u^{-1}$$

- (iii) The posterior distribution $p_2(\alpha_t | Y_t; F_t)$ is normal with mean α_t^{**} and variance $V_{\alpha_t}^{**}$ where

$$\alpha_t^{**} = V_{\alpha_t}^{**} \left[\sum_{t=1}^T L_t' W_t i' S_{N^a} S_N' + \zeta' W_t' + S_u^{\alpha-1} (Y_t | L_t) \right]$$

$$V_{\alpha_t}^{**} = L_t' W_t i' S_{N^a} S_N' + \zeta' W_t' + S_u^{\alpha-1} L_t$$

The assumption $S_\varepsilon^{-1} = 0$ implies that $\alpha_{t|t-1}^{-1} = 0$, at all points in time. This implication is unreasonable or, at least, excessively myopic, because it prevents researchers to learn from past realizations of α_t and to be less uncertain on its mean as times goes by. The assumption $\alpha_{t|t-1}^{-1} = 0$ can be more realistic if we attach this infinite uncertainty to the coefficients only at a particular point in time (let's say, $t = t_0$), perhaps to take care of a structural break, after which the process restarts and behaves as it did before the break.

It is worth noting that in both cases 1 and 2, the posterior mean and variance for α_t are the same as those obtained when only prior information on α is used. This is not surprising if we write (8) as a three stage hierarchy

$$Y_t | F_t; \alpha_t \sim N(W_t \alpha_t; S_u)$$

$$\alpha_t | F_t; \alpha; \alpha_t \sim N[E_N(\alpha + \alpha_t); \zeta]$$

$$(\alpha + \alpha_t) | F_t; \alpha; \alpha_{t|t-1} \sim N[\alpha + \alpha_{t|t-1}; i' S_{N^a} S_N' + \zeta' \Phi^{-1} \zeta]$$

Assuming $\alpha^{-1} = 0$ or $S_e^{-1} = 0$ is equivalent to assume a diffuse prior on the third stage of the hierarchy.

4.2 Litterman-type prior

Next, we modify the so-called Minnesota prior to account for the presence of multiple units in the VAR. The Minnesota prior, described in Litterman (1986), Doan, Litterman and Sims (1984), Ingram and Whiteman (1995), Ballabriga et al. (1998) among others is a way to account for the near nonstationarity of many macroeconomic time series and, at the same time, to weakly reduce the dimensionality of a VAR model. Given that the intertemporal dependence of the variables is believed to be strong the prior mean of the VAR coefficients on the first own lag is set equal to one and the mean of remaining coefficients is equal to zero. The covariance matrix of the coefficients is diagonal (so we have prior and posterior independence between equations) and the elements are specified in a way that coefficients of higher order lags are likely to be close to zero (the prior variance decreases when the lag length increases). Moreover, since most of the variations in the VAR variables is accounted for by own lags, coefficients of variables other than the dependent one are assigned a smaller relative variance. The prior on the constant term, other deterministic and exogenous variables is diffuse. Finally, the variance-covariance matrix of the error term is assumed to be fixed and known.

For a panel VAR setup we introduce the following modifications. The covariance matrices $\Sigma_{\alpha}; \Sigma_{\beta}; \Sigma_{\gamma}$, are assumed to have the same a-priori structure. Take, for example, $\Sigma_{\beta} = \text{diag}(\Sigma_{\beta_1}; \dots; \Sigma_{\beta_n})$, where $\Sigma_{\beta_i} = V - E_{i-1}E_i$.

The matrix Σ_{β} is assumed to be diagonal and its elements have the following structure

$$\Sigma_{\beta_{g;j_s}}^2 = \frac{\bar{A} \mu_{1\alpha} \mu_3^{\delta(g_i, j_s)} 1}{|\theta_2| \Sigma_{\beta_{j_s}}} \quad g, j = 1; \dots; G \quad i, s = 1; \dots; N \quad l = 1; \dots; p$$

where $\delta(g; j_s) = 0$ if $i = s$ and 1 otherwise and

$$\Sigma_{\beta_{gm}}^2 = (\mu_{1\alpha} \mu_4)^2 \quad m = 1; \dots; q$$

Here, g represents equation g of unit i , j_s the endogenous variable j of units, l the lag m exogenous or deterministic variables.

The hyperparameter $\mu_{1\alpha}$ controls the tightness of beliefs for the vector α ; μ_2 the rate at which the prior variance decays with the lag; μ_3 the degree of uncertainty for the coefficients of the variables of unit s in the equations of unit i ; μ_4 the degree of uncertainty of the coefficients of the exogenous variables and $\Sigma_{\beta_{j_s}}$ are the diagonal elements of the matrix S_u used as scale factors to account for differences in units of measurement. Also assume that $V = H$ (see equation (7)). Notice that we don't have prior independence between equations. Hence our prior information specifies that, for example, the coefficient on lag 1 of the GDP equation for the US may have some relationship with the same coefficient in the PRICE equation for US. Moreover, we have not specified a hyperparameter which controls the overall tightness of beliefs because the randomness of the coefficients depends on α_i and $\beta_{s,t}$ and we parametrize the uncertainty in each of them separately.

Finally, there is no distinction between own versus other countries variables. Because of this V and σ_{-1} are common to all units and the prior has a symmetric structure (see Sims and Zha (1998)).

The structures for α and σ_o are similar with $\mu_{1\alpha}$ being replaced by $\mu_{1\bar{\alpha}}$ and $\mu_{1\lambda}$, respectively.

To complete the specification we need to have a measure the elements of the matrix H and of the β 's. Following Litterman, these parameters are estimated from the data to tune up the prior to the specific application.

The prior time-varying features of the model are determined by specifying the matrices B , S_ε . We assume that B is diagonal and that each of the $k \times k$ diagonal blocks B_g satisfies: $B_g = \text{diag}(\mu_5)$. Furthermore, we assume $S_\varepsilon = \mu_6 \sigma_o$. Here μ_5 controls the evolution of the law of motion of s_t and μ_6 the heteroskedasticity in the coefficients. Note that a time-invariant model is obtained by setting $\mu_5 = 1$ and $\mu_6 = 0$. Homoscedastic time variations are obtained by setting $\mu_6 = 0$.

Finally, we assume that the $k \times 1$ vectors 1_g and $\tilde{\sigma}_{og}$ have the following structures:

$$1_g = \begin{matrix} 2 & 3 \\ 0 & \\ \vdots & \vdots \\ \mu_7 & \\ 0 & \\ \vdots & \vdots \\ 4 & 5 \\ 0 & \end{matrix}; \quad \tilde{\sigma}_{og} = \begin{matrix} 2 & 3 \\ 0 & \\ \vdots & \vdots \\ 1 & \mu_7 \\ 0 & \\ \vdots & \vdots \\ 4 & 5 \\ 0 & \end{matrix}$$

where 1_g and $\tilde{\sigma}_{og}$ are the g th elements of the mean vectors 1 and $\tilde{\sigma}_o$ and μ_7 controls the prior mean on the first own lag coefficient of the dependent variable in equation g for unit i .

Summing up, our prior information is a function of a 9-dimensional vector of hyperparameters $E = (\mu_{1\alpha}; \mu_{1\lambda}; \mu_{1\bar{\alpha}}; \mu_2; \mu_3; \mu_4; \mu_5; \mu_6; \mu_7)$. Estimates of E can be obtained by maximizing the predictive density of the model as in Doan, Litterman and Sims (1984). Posterior distributions for the parameters are then obtained by plugging in the resulting estimates for $1; \tilde{\sigma}_o; \sigma_o; S_u; S_\varepsilon; B; \alpha; \beta$ in the formulas we have derived in section 3 in an empirical Bayes fashion (see e.g. Berger (1985)).

Compared with Ballabriga et al. (1998), who used a Minnesota prior on a panel VAR model for the Spanish, German and French economies, our specification allows for unit specific time variations in the variance of the process ($\mu_6 \neq 0$); it separates the prior information for the time and the individual component (they have one parameter in place of $\mu_{1\alpha}; \mu_{1\lambda}; \mu_{1\bar{\alpha}}$) and introduces a further level of uncertainty by specifying a prior for μ_7 . Furthermore, our prior specification is symmetric and it allows for a-priori pooling of the information present in the cross sectional dimension of the panel. None of these features is present in their specification.

4.3 Informative priors

When the prior for the vector of parameters is informative, the posterior distribution for the parameter vector does not have an analytical closed form. Nevertheless, we can implement a hierarchical Bayes analysis using a sampling based approach, such as the Gibbs sampler, (see e.g. Gelman and Gelman (1984), Gelman and Smith (1990), Gelman and al. (1990) among others).

The basic idea of the approach is to construct a (computable) Markov chain on a general state space such that the limiting distribution of the chain is the joint posterior of interest. Suppose we

have a parameter vector $\#$ with k components $(\#_1; \#_2; \dots; \#_k)$ and that the posterior distributions $\% (\#_j | \#_s; s \in j)$ are available. Then the algorithm works as follows. We start from arbitrary values for $\#_1^{(0)}; \#_2^{(0)}; \dots; \#_k^{(0)}$. Setting $i = 1$; we cycle through the conditional distributions sampling $\#_1^{(1)}$ from $\% (\#_1 | \#_2^{(0)}; \dots; \#_k^{(0)})$, $\#_2^{(1)}$ from $\% (\#_2 | \#_1^{(1)}; \dots; \#_k^{(0)})$ up to $\#_k^{(i)}$ from $\% (\#_k | \#_1^{(1)}; \dots; \#_{k-1}^{(1)})$; Next, we set $i = 2$ and repeat the cycle. After iterating on this cycle, say, M times, the sample value $\#^{(M)} = (\#_1^{(M)}; \#_2^{(M)}; \dots; \#_k^{(M)})$ can be regarded as a drawing from the true joint posterior density. Once this simulated sample has been obtained, any posterior moment of interest or any marginal density can be estimated, using the ergodic theorem. Convergence to the desired distribution can be checked as suggested in Gelfand and Smith (1990).

In order to apply the Gibbs sampler to our panel VAR model we need to specify prior information so that the conditional posterior distribution for components of the parameter vector can be obtained analytically. Recall that our hierarchical model is given by:

$$\begin{aligned} Y_t &= W_t \otimes + Z_{t,s,t} + u_t; \\ \otimes_i &= S_N \otimes + \epsilon_i \\ \otimes &= 1 + v \\ s,t &= B_{s,t-1} + e \end{aligned}$$

where $u_t \gg N(0; S - H)$; $\epsilon_i \gg N(0; V - E_{i-1} E_i)$; $v \gg N(0; a)$; $s_o \gg N(0; V - -_2)$ $e \gg N(0; V - -_2)$ and ϵ is the tightness on time variation: if $\epsilon = 0$ and $B = I$ then s is time invariant. We assume that the covariance matrices are independent, that $V; a, \epsilon$, and 1 are known and that $S \gg iW_N(\%_o; M_o)$, $H \gg iW_G(h_b; P_o)$; $-_1 \gg iW_k(w_1; W_1)$, and $-_2 \gg iW_k(w_2; W_2)$, where the notation $\otimes \gg iW_p(v; L)$ means that the symmetric positive definite matrix \otimes follows a p dimensional inverted Wishart distribution with v degrees of freedom and scale matrix L : We also assume that for each of these distributions the degrees of freedom and the scale matrix are known. These assumptions are inconsequential and the analysis goes through, even when consistent estimates are substituted for the true ones.

Given this prior information, the posterior density of the parameter vector $\# = (\otimes; S; H; \otimes; -_1, f_{s,t} g_{t=0}^T; -_2)$ is given by

$$\% (\# | Y_T; F_T) / f(Y_T | \#; F_T) p(\# | F_T) \tag{19}$$

where $Y_T = (Y_1; \dots; Y_T)$ is the sample data and $p(\# | F_T)$ is the prior information available at T .

Given the difficulty to obtain marginal posteriors directly from the integration of (19), we iterate on the conditional distributions of the parameters, which can easily be obtained from the conditional posterior (19). To deal with the presence of time varying parameters we adapt the results of Carter and Khon (1994) and Chib and Gierensberg (1996). In fact, conditional on $f_{s,t} g_{t=0}^T$, the distribution of the remaining parameters can be derived without difficulty. Let \tilde{A}_{-x} be the vector $\#$ containing all the parameters but x . Then the conditional distributions for parameters other than $f_{s,t} g$ are:

$$-_1 | \tilde{A}_{-1}; Y_T; F_T \gg iW_k(w_1 + N G; \hat{W}_1)$$

$$\begin{aligned}
 & \text{G} \quad j \quad \tilde{A}_{-2}; Y_T; F_T \gg \text{iW}_k \quad \text{W}_2 + \text{TG}; \hat{W}_2 \\
 & \text{S} \quad j \quad \tilde{A}_{-\Sigma}; Y_T; F_T \gg \text{iW}_N \quad \text{W}_0 + \text{GT}; \hat{M}_0 \\
 & \text{H} \quad j \quad \tilde{A}_{-H}; Y_T; F_T \gg \text{iW}_G \quad \text{h}_0 + \text{NT}; \hat{P}_0 \\
 & \text{O} \quad j \quad \tilde{A}_{-\alpha}; Y_T; F_T \gg \text{N} \quad \text{O}; \hat{V}_\alpha \\
 & \text{O} \quad j \quad \tilde{A}_{-\bar{\alpha}}; Y_T; F_T \gg \text{N} \quad \text{O}^*; \hat{V}^*
 \end{aligned} \tag{20}$$

where the expressions for \hat{W}_1 ; \hat{W}_2 ; \hat{M}_0 ; \hat{P}_0 ; O ; \hat{V}_α ; O^* ; \hat{V}^* are given in the appendix

Following Chib (1996) the parameter vector \tilde{s}_t can be included in the Gibbs sampler via the distribution $\mathcal{N}(\tilde{s}_0; \dots; \tilde{s}_T | Y_T; F_T; \tilde{A}_T)$ where $\tilde{A}_t \in \mathbb{R}^{\#\{\lambda_t\}_t}$. We can rewrite such a distribution as

$$\mathcal{N}(\tilde{s}_T | Y_T; F_T; \tilde{A}_T) \propto \mathcal{N}(\tilde{s}_{T-1} | Y_T; F_T; \tilde{A}_{T-1}; \tilde{s}_T) \propto \dots \propto \mathcal{N}(\tilde{s}_0 | Y_T; F_T; \tilde{A}_0; \tilde{s}_1; \dots; \tilde{s}_T) \tag{21}$$

A draw from the joint distribution can be obtained by drawing \tilde{s}_T from $\mathcal{N}(\tilde{s}_T | Y_T; F_T; \tilde{A}_T)$; then \tilde{s}_{T-1} from $\mathcal{N}(\tilde{s}_{T-1} | Y_T; F_T; \tilde{A}_{T-1}; \tilde{s}_T)$ and so on. Let $\tilde{s}^s = (\tilde{s}_1; \dots; \tilde{s}_T)$ and $Y^s = (Y_1; \dots; Y_T)$ for $s = 1, \dots, T$. The density of the typical term in (21) is

$$\begin{aligned}
 & \mathcal{N}(\tilde{s}_t | Y_T; F_T; \tilde{A}_t; \tilde{s}^{t+1}) \\
 & / \mathcal{N}(\tilde{s}_t | Y^t; F_t; \tilde{A}_t; \tilde{s}^{t+1}) \mathcal{N}(\tilde{s}_{t+1} | Y_t; F_t; \tilde{A}_{t-1}; \tilde{s}_t) f(Y^{t+1}; \tilde{s}^{t+1} | Y_t; F_t; \tilde{s}_{t+1}) \\
 & / \mathcal{N}(\tilde{s}_t | Y^t; F_t; \tilde{A}_t) \mathcal{N}(\tilde{s}_{t+1} | F_t; \tilde{A}_{t-1}; \tilde{s}_t)
 \end{aligned} \tag{22}$$

The last row follows from the fact that, conditional on \tilde{s}_{t+1} , the joint density of $(Y^{t+1}; \tilde{s}^{t+1})$ is independent of \tilde{s}_t and, conditional on $\tilde{s}_t, \tilde{s}_{t+1}$ is independent of Y_t .

The second density of (22) is Gaussian with moments $\mathbb{E}_{\tilde{s}_t}$ and $\text{S}_{\tilde{s}_t}$. The first was derived in section 3, and it is Gaussian with mean $\hat{\tilde{s}}_t | t = \hat{\tilde{s}}_t | t-1 + \hat{\tilde{s}}_t | t-1 \text{L} \hat{A}_{22}^{-1} Y_t | \text{L} \hat{t}^{-1} | \text{L} \hat{\tilde{s}}_t | t-1$ and variance $\hat{\tilde{s}}_t | t = \hat{\tilde{s}}_t | t-1 | \hat{\tilde{s}}_t | t-1 \text{L} \hat{A}_{22}^{-1} \text{L} \hat{\tilde{s}}_t | t-1$. Hence, $\mathcal{N}(\tilde{s}_t | Y_T; F_t; \tilde{A}_t; \tilde{s}^{t+1}) \gg \text{N}(\hat{\tilde{s}}_t; \hat{\tilde{s}}_t)$ where $\hat{\tilde{s}}_t = \hat{\tilde{s}}_t | t + \frac{1}{2} \text{M}_t^{-1} \text{M}_t \tilde{s}_{t+1} | \frac{1}{2} \hat{\tilde{s}}_t | t$; $\hat{\tilde{s}}_t = \hat{\tilde{s}}_t | t | \text{M}_t^{-1} \text{M}_t \tilde{s}_{t+1} | \text{M}_t$ and $\text{M}_t = \frac{1}{2} \hat{\tilde{s}}_t | t^{-1} | \hat{\tilde{s}}_t | t$.

To be concrete the following algorithm can be used to sample \tilde{s}_t : first, starting from given initial conditions, we run the Kalman filter to recursively get $\hat{\tilde{s}}_t$ and $\hat{\tilde{s}}_t$; then we simulate \tilde{s}_T from a normal with mean $\hat{\tilde{s}}_T | T$ and variance $\hat{\tilde{s}}_T | T$; \tilde{s}_{T-1} from $\text{N}(\hat{\tilde{s}}_{T-1}; \hat{\tilde{s}}_{T-1})$, and so on until \tilde{s}_0 is simulated from $\text{N}(\hat{\tilde{s}}_0; \hat{\tilde{s}}_0)$ where, for each t , $\hat{\tilde{s}}_t = \hat{\tilde{s}}_t | t + \frac{1}{2} \text{M}_t^{-1} \text{M}_t \tilde{s}_{t+1} | \frac{1}{2} \hat{\tilde{s}}_t | t$ and $\hat{\tilde{s}}_t = \hat{\tilde{s}}_t | t | \text{M}_t^{-1} \text{M}_t \tilde{s}_{t+1} | \text{M}_t$.

One special case of the setup described in this subsection deserve some attention. Suppose informative priors on all the parameters except that on H , whose prior is now diffuse, so that the prior for S_u is diffuse as well. Then the setup resembles the normal-inverse prior of Kadiyala and Karlsson (1997) and implies that posterior dependence among the coefficients of different equations obtains even when there is prior independence. Hence, the major difference of our prior with the specification used by these authors is that we use a three stage hierarchy, so that both the mean and the variance of \tilde{s}_t are random variables, while they take the mean and the variance of \tilde{s}_t to be

fixed. Note also that our specification does not restrict S_u to be diagonal and therefore permits complicated interactions among variables within and across countries.

Finally, it is worth mentioning that in all the setups we have considered in this section, our prior specification maintains a Kronecker structure for the statistical model. Such a specification is useful since, on one hand, it allows to handle the computations for relatively large systems in a simple fashion and, on the other, imposes symmetry restrictions which appear to be desirable in an unrestricted VAR system of the type examined here. Clearly these restrictions may be inappropriate for structural or restricted VAR systems and alternative specifications, along the lines of Sims and Zha (1998), should be used.

5 Forecasting

Once posterior estimates are obtained, forecasts can be computed. In order to obtain multistep forecasting formulas for a panel VAR and to compute turning points probabilities, it is convenient to rewrite (1) in a companion VAR(1) form

$$Y_{it} = \sum_{j=1}^K B_{it}^j Y_{jt-1} + D_{it} Z_t + U_{it} \quad (23)$$

where Y_{it} and U_{it} are $G \times 1$ vectors, B_{it}^j is a $G \times G \times G$ matrix and D_{it} is a $G \times E \times q$ matrix. Stacking for i , and repeatedly substituting we have

$$Y_t = \sum_{r=0}^{h-1} B_{t-r} Y_{t-h} + \sum_{s=0}^{h-1} B_{t-r} D_{t-s} Z_{t-s} + \sum_{s=0}^{h-1} B_{t-r} U_{t-s} \quad (24)$$

or

$$y_t = J \sum_{r=0}^{h-1} B_{t-r} Y_{t-h} + \sum_{s=0}^{h-1} \odot_{st} D_{t-s} Z_{t-s} + \sum_{s=0}^{h-1} \odot_{st} U_{t-s} \quad (25)$$

where $\odot_{st} = \sum_{r=0}^{s-1} B_{t-r}$ and $J = I_N - J_1$, $J_1 = [I_G \ 0]$ and J is a selection matrix such that $JY_t = y_t$, $JU_t = u_t$ and $J'JU_t = U_t$. The expression in (25) can be used to compute the h steps ahead forecast of the NG dimensional vector Y_t .

First, we compute a "point" forecast for y_{t+h} . The forecast function is given by

$$y_t(h) = J \sum_{r=0}^{h-1} B_{t+h-r} Y_t + \sum_{s=0}^{h-1} \odot_{st+h} D_{t+h-s} Z_{t+h-s} \quad (26)$$

or, recursively

$$y_t(h) = JB_{t+h} y_t(h-1) + D_{t+h} Z_{t+h}$$

where D_{t+h} is the $NG \times E \times q$ matrix $[d_{1t+h} \ d_{2t+h} \ \dots \ d_{Nt+h}]$ and $B_{t+h} = \text{diag}(B_{1t+h}, B_{2t+h}, \dots, B_{Nt+h})$ with $B_{it+h} = [B_{it+h}^1, B_{it+h}^2, \dots, B_{it+h}^N]$. A new way to obtain a h step ahead forecasts is to use the posterior mean of B_{t+h} and D_{t+h} and the mean of the predictive density for Z_{t+h} , conditional on the information at time

Estimates for the posterior mean of the coefficients can be obtained from the recursive formulas for β_{st} (and, consequently, for α_t) using expressions like (9) or by drawing from distributions like (20) and (21) in a recursive fashion. Call these estimates $\hat{\beta}_{t+h|t}$ and $\hat{\alpha}_{t+h|t}$. The forecast error is $y_{t+h} - \hat{y}_t(h) = \sum_{s=0}^{h-1} \alpha_{st+h} u_{t+h-s} + [y_t(h) - \hat{y}_t(h)]$. To measure the forecasting performance it is useful to compute the Mean Square Error (MSE) or the Mean Absolute Error (MAD) of the estimated forecast which are given by

$$\begin{aligned} \text{MSE}(\hat{y}_t(h)) &= \sum_{s=0}^{h-1} \alpha_{st+h}^2 \text{SE}[u_{t+h-s}] + \text{MSE}[y_t(h) - \hat{y}_t(h)] \\ \text{MAD}(\hat{y}_t(h)) &= \sum_{s=0}^{h-1} |u_{t+h-s}| + \text{MAD}[y_t(h) - \hat{y}_t(h)] \end{aligned}$$

The first term on the RHS of each equation can be obtained using posterior mean estimates of β_{t+h-r} and of u_t , conditional on the information at time t , while for the second term an approximation can be computed along the lines of Lutkepohl (1991, p.86(89)). Clearly, if a researcher is interested in point forecasts using the average value of the parameters, then the previous formulas apply using for $\hat{\beta}_{t+h|t}$ and $\hat{\alpha}_{t+h|t}$ the posteriors derived in section 3.2.

In many situations, it may be more appealing to compute "average" forecasts h (step ahead) using the predictive density $f(y_{t+h} | F_t) = \int f(y_{t+h} | F_t; \#) p(\# | F_t)$ where $f(y_{t+h} | F_t; \#)$ is the conditional density of the future observation vector given $\#$, and $p(\# | F_t)$ is the posterior pdf of $\#$ at time t . To compute forecasts for y_{t+h} we can sample from the predictive density numerically. For each $i = 1, \dots, M$ we draw $\#^{(i)}$ from the posterior distribution and simulate the vector $y_{t+h}^{(i)}$ from the density $f(y_{t+h} | F_t; \#^{(i)})$. $\{y_{t+h}^{(i)}\}_{i=1}^M$ constitutes a sample, from which we can compute the necessary moments. The value of the forecast is then the ergodic average $\hat{y}_{t+h} = M^{-1} \sum_{i=1}^M y_{t+h}^{(i)}$ and its numerical variance can be estimated using $\text{var}(\hat{y}_{t+h}) = M^{-1} \sum_{s=1}^r \frac{1}{r+1-s} (Q_s + Q_s')$ where $Q_s = M^{-1} \sum_{i=s+1}^M (y_{t+h}^{(i)} - \hat{y}_{t+h})(y_{t+h}^{(i)} - \hat{y}_{t+h})'$.

Note that since the computation of the impulse response function for orthogonalized shocks is a simple corollary of the calculation of forecasts, the approach we provide here to calculate point and average forecasts can also be used to compute impulse responses. In fact, given the information up to time t , computing impulse response at $t+h$ is equivalent to calculating the difference between the conditional forecasts at $t+h$, given that at $t+1$ there has been a one unit impulse in one of the orthogonal shocks, and the unconditional forecast, i.e. with the value of the vector that would have occurred without shocks (see Koop (1992) for an application to structural VAR models). This idea is exploited in a recent paper by Waggoner and Zha (1998). The authors, using a version of (25), develop two Bayesian methods for computing probability distributions of conditional forecasts. The last term in (25) represents the dynamic impact of structural shocks which affect future realizations of variables through the impulse response matrix α_{st} . With conditions or constraints imposed on this last term we can produce what they call conditional forecasts.

In order to compute structural impulse responses and their error bands we must work with a structural VAR, e.g. impose some restrictions on the contemporaneous coefficient matrix A prior

(° at or informative) can then be assigned to the non-zero elements of this matrix, as suggested by Sims and Zha (1998). The extension of their approach to panel data is however not straightforward and we postpone this issue to future work.

Turning point predictions can also be computed from the predictive density of future observations (see in Zellner, Hong and Min (1991)). Let us define turning points as follows:

Definition 5.1 A downward turn for unit i at time $t+h+1$ occurs if S_{it+h} the growth rate of the reference variable (typically, GNP) satisfies for all h $S_{it+h-2}; S_{it+h-1} < S_{it+h} > S_{it+h+1}$. An upward turn for unit i at time $t+h+1$ occurs if the growth rate of the reference variable satisfies $S_{it+h-2}; S_{it+h-1} > S_{it+h} < S_{it+h+1}$.

Similarly, we define a non-downward turn and a non-upward turn:

Definition 5.2 A non-downward turn for unit i at time $t+h+1$ occurs if S_{it+h} satisfies for all h $S_{it+h-2}; S_{it+h-1} < S_{it+h} < S_{it+h+1}$. A non-upward turn for unit i at time $t+h+1$ occurs if the growth rate of the reference variable satisfies $S_{it+h-2}; S_{it+h-1} > S_{it+h} > S_{it+h+1}$.

Although there are other definitions in the literature (see e.g. Lippi and Moore (1991)) this is the most used one and it suits for our purposes. Let $f(Y_{i,t+h} | F_t) = \int_{Y_{p,t+h}} f(Y_{t+h} | F_t) dY_{p,t+h}$ be the marginal predictive density for the variables of unit i after integrating the remaining p variables and let $K(S_{it+h}^1 | F_t) = \int_{S_{it+h}^2} \dots \int_{S_{it+h}^G} f(S_{it+h}^1, \dots, S_{it+h}^G | F_t) dS_{it+h}^2 \dots dS_{it+h}^G$ be the marginal predictive density for the growth rate of the reference variable, which we order to be the first in the list, in unit i .

Take now the simplest case of $h=0$. To compute the probability of a turning point we have to calculate S_{it+1}^1 . Given the marginal predictive density K , the probability of a downturn in unit i is

$$P_{Dt} = \Pr(S_{it+1}^1 < S_{it}^1 | S_{it-2}^1; S_{it-1}^1 < S_{it}^1; F_t) = \int_{-\infty}^{S_{it}^1} K(S_{it+1}^1 | S_{it-2}^1; S_{it-1}^1; S_{it}^1; F_t) dS_{it}^1 \quad (27)$$

and the probability of an upturn is

$$P_{Ut} = \Pr(S_{it+1}^1 > S_{it}^1 | S_{it-2}^1; S_{it-1}^1 > S_{it}^1; F_t) = \int_{S_{it}^1}^{\infty} K(S_{it+1}^1 | S_{it-2}^1; S_{it-1}^1; S_{it}^1; F_t) dS_{it}^1 \quad (28)$$

Using a numerical sample from the predictive density satisfying $S_{it-2}^1; S_{it-1}^1 < S_{it}^1$, we can approximate these probabilities using the frequencies of realizations which are less than or greater than S_{it}^1 . With a symmetric loss function, minimization of the expected loss leads to predict the occurrence of turning point at $t+1$ if $P_{Dt} > 0.5$ or $P_{Ut} > 0.5$.

For $h \neq 0$ the probability of a turning point can be computed using the joint predictive density for all future observations, i.e. in the case of a downturn,

$$P_{Dt+h} = \Pr(S_{it+h+1}^1 < S_{it+h}^1 > S_{it+h-2}^1; S_{it+h-1}^1 | F_t) = \int_{-\infty}^{S_{it}^1} \int_{S_{it}^1}^{\infty} \int_{-\infty}^{S_{it}^1} K(S_{it+h+1}^1 < S_{it+h}^1 > S_{it+h-2}^1; S_{it+h-1}^1 | F_t) dS_{it+h}^1 dS_{it+h-1}^1 dS_{it+h-2}^1 \quad (29)$$

Given the available panel data structure we may also be interested in computing the probability that a turning point occurs jointly for $m \cdot N$ units of panel. For example, we would like to compute the probability that at $t+1$ there will be a recession in European countries. Let $K(S_{t+h}^1 | F_t)$ be the joint predictive density of the reference variable for the m units of interest. Then the probability of a downturn is:

$$P_{Dt}^m = \Pr(S_{it+1}^1 < S_{it}^1 \quad i = 1; \dots; m | S_{it-2}^1; S_{it-1}^1 < S_{it}^1; F_t) = \int_{-\infty}^{S_{it}^1} \int_{-\infty}^{S_{mt}^1} \dots \int_{-\infty}^{S_{t+1}^1} K(S_{t+1}^1 | S_{t-2}^1; S_{t-1}^1 < S_t^1; F_t) dS_{1t}^1 \dots dS_{mt}^1 \quad (30)$$

6 An application

In this section we apply the methodology to the problem of forecasting growth rates and predicting turning points in the G-7 countries. For each country we consider three national variables (GDP, real stock returns and real money growth) and a world one (the median real stock return in OECD countries) which is assumed to be exogenous in each equation. Hence there are 21 variables in the panel VAR. These variables are chosen after a rough specification search over about 10 variables because they appear to have the highest in-sample pairwise and multiple correlation with output growth. Among the variables we tried are the nominal interest rate, the slope of the term structure and inflation. Data is sampled quarterly from 1973,1 to 1993,4 and taken from IMF statistics. Data from 1973,1 to 1988,4 is used to estimate the parameters and data from 1989,1 to 1993,4 to evaluate the forecasting performance and to predict turning points.

We compare the forecasting performance of our panel VAR specifications with those obtained with other models suggested in the literature. As a benchmark we first run two versions of a tri-variable VAR (2) model for each country separately. The first one is an unrestricted (VAR). The second a weakly restricted VAR (BVAR) where we use a standard Litterman prior with a mean of one on the first lag, a general tightness of 0.15, no decay in the lags and a weight of 0.5 on the lags of other variables. Since these two models do not exploit cross sectional information nor do they allow for time variation, they can be used as a benchmark to measure the improvements obtained by specifications which allow any of these two features in the model.

Also for comparison, we run a single equation AR (3) model for GDP growth for each single country, augmented with two lags of real stock returns, 1 lag of real money balances and one lag of the median world real stock return. This is the specification used by Garcia Ferrer et al (1987), Zellner and Hong (1989) and Zellner, Hong and Min (1991) to forecast annual growth rates of output in 18 countries. With the extended sample and the higher frequency of the data we have available we confirm their results for all of the G-7 countries. This model represents a restricted version of the previous unrestricted VAR where insignificant lags are purged from the specification. The forecasting power of this model is measured when parameters are estimated with OLS (OLS) and with the three shrinkage procedures: a ridge estimator (RIDGE), an estimator obtained assuming an exchangeable prior on the coefficients (as in Garcia Ferrer et al. (1987)) (EXCHANGEABLE) and an estimator obtained using a g prior (as in Zellner and Hong (1989)) (G-PRIOR). The two latter estimators attempt to improve upon OLS by combining the information coming from each unit

with the one from the pooled sample. They differ in the way they combine individual and pooled information. Notice that none of these estimators allows for time variations in the coefficients.

Finally, as a term of comparison, we use a version of the panel VAR specification suggested by Ballabriga et al (1998) (PBVAR). This model specification does not use the information coming from the cross section - every variable is treated in the same way regardless of the country where it is from - but allows for time variations in the coefficients of the model. The model has the same structure as Doan, Litterman and Sims (1984) and assumes that the coefficient vector β_t for the entire system has an AR(1) structure of the form $\beta_t = \mathbb{M} \beta_{t-1} + u_t$ where u_t , conditional on the information available, is normal with mean zero and variance S_{u_t} . The matrices β_0 , \mathbb{M} , and S_{u_t} depend on 7 hyperparameters: five parameters controlling the structure of S_{u_0} (a general tightness (μ_1), a tightness on variables of the same country (μ_3), a tightness on the variables of other countries (μ_4), a geometric lag decay with parameter (μ_2), and a tightness on world variables (μ_5)); a parameter describing the structure of \mathbb{M} (μ_6); and a parameter controlling the prior mean on the first lag of β_0 (μ_7). Table 1 reports the optimal values selected by maximizing the in-sample predictive density of the model with a simplex algorithm.

We produce forecasts from two versions of our panel VAR model: one with a modified Minnesota prior (PA NEL 1), and one with a fully hierarchical specification (PA NEL 2). In the former, the nine prior parameters are selected to maximize the predictive density using a simplex method. Their optimal values are reported in table 2. For both PBVAR and PA NEL 1 forecasts are computed using the posterior mean of the coefficients, after we have plugged in the estimates of the prior parameters in the formula. For PA NEL 2 posterior estimates of the coefficients are computed numerically using MCMC methods and forecasts are directly obtained from these estimates.

In setting up the panel VAR models we assume that $H = V$, where V is known. For the PA NEL 1 specification we compute the scale factors V and the matrix S_u as follows. We estimate a trivariate VAR for each country and take the average of the estimated variance/covariance matrix of the residuals across countries as a measure of V . Furthermore, for each of the three variables we estimate a λ variable VAR (the same variable across countries) and store the variance/covariance matrices of the residuals. An estimate of S_u is obtained as:

$$\hat{S}_u = \begin{matrix} & \begin{matrix} 0 & & & & 1 \\ \begin{matrix} \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{matrix} & \begin{matrix} \frac{1}{4} & 0 & \dots & 0 \\ 0 & \frac{1}{4} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \frac{1}{4} \end{matrix} & \begin{matrix} 1 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{matrix} \\ \begin{matrix} \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{matrix} & & & & \begin{matrix} 0 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{matrix} \end{matrix} - \begin{matrix} \mathbb{B} & 0 & 0 & 0 \\ 0 & v_j & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix} \mathbb{A}$$

where the first matrix contains on the diagonal the estimated standard deviations obtained by running the three λ variate VARs; while the second matrix contains just one element different from zero, the $(j;j)$ element, which is obtained from the diagonal of the matrix V . For the PA NEL 2 specification we need to choose the scale and the degrees of freedom in the various Wishart distribution. We still set $H = V$ with V estimated as before. Following Kadiyala and Karlsson (1997) we set the degrees of freedom $\nu_0 = N + 2 + (T_j - p) \times G$; $\nu_1 = k + 2 + N + g$; $\nu_2 = k + 2 + (t_j - p) \times G$ while the scale matrices \mathbb{M}_0 , \mathbb{W}_1 and \mathbb{W}_2 are such that $S_{u_t}; S_e$ have the same structure as in the PA NEL 1 specification.

We compare the forecasting ability of various models using both the Theil-U statistics and the Mean Absolute Deviation (MAD) at 1 and 4 periods ahead. These statistics are reported in table 3. Note that the various specifications we use are in increasing order of complexity and flexibility. Therefore, at each stage we can assess the forecasting improvements obtained adding one extra feature to the model.

To examine the performance of various models as business cycle indicators we compute turning points predictions one period ahead. Following Zilner et al. (1991), we compute the total number of turning points, the number of downturns and no-downturns, and the number of upturns and no-upturns in the sample (across all countries) and for each procedure we report the number of correct cases in table 4.

Finally, for each model, we compute the probability that there will be a downward turn in the growth rate of US output in 1989:1-1993:4, given the information available in 1988:4. According to the official NBER classification the long expansion of the 1980's terminated in 1990:3 and it was followed by a brief and shallow recession. The probabilities for the nine models for each of the 16 periods we consider are presented in table 5.

The forecasting performances of univariate OLS, ridge and exchangeable procedures are very similar. The minimum and maximum values of the Theil-U across countries at one and four steps for the latter two are slightly smaller, but the mean and the median at both steps are practically identical. On the other hand, a univariate model where the parameters are shrunk with a gprior is somewhat better than OLS in all the dimensions: the maximum, the median, the mean and the minimum value across countries of the Theil-U at both steps are significantly lower than those obtained with OLS.

Unrestricted VAR models are not very successful in forecasting growth rates of output, given the large number of parameters to be estimated. This noticeable in particular in the case of Japan, Germany and the UK where the Theil-U are significantly worse than those obtained with univariate specifications at the one step horizon. However, unrestricted VAR model outperform all univariate specifications at the four step horizon. Hence, the presence of interdependencies across variables helps in predicting the evolution of the growth rate of output in the medium run. BVAR are significantly better than VAR and univariate approaches at the one step horizon. In terms of the median value the gains are of the order of 5-6% over univariate specifications and of more than 10% over the unrestricted VAR. However, the performance at the four step horizon turns out to be inferior to the one of unrestricted VAR, and comparable to the one of univariate shrinkage procedures. This is to be expected since to improve the performance at short horizons BVAR tend to reduce both the memory and the interdependencies of the system, which we have seen are useful exactly when medium-long run forecasts have to be made.

Adding time variation in the coefficients and interdependencies across countries substantially improves the forecasting performance both at short and at medium horizons. For example, the median Theil-U at one step goes from 0.85 with a simple BVAR to 0.82 with the panel version of this model and for 5 countries the Theil-U is lower by as much as 10%. Similarly, the mean across countries drops by about 3% with the PBVAR specification. The improvement is noticeable also at longer horizons. The distribution of the Theil-U across countries at the four step horizon is similar to the one obtained with a unrestricted VAR, which is the best among the benchmark models.

Our refinement of the Litterman's prior, which allow for both cross sectional and time series a-priori restrictions, gives a performance which is essentially similar to the one of the PBVAR model both at the one and at the four step horizons. Few features of the optimally estimated parameters are worth discussing. First, while μ_6 , the time variation parameter in the variance of ϵ_t is different from zero, it does not appear to add much to the performance of the model. Hence, at least with quarterly data, allowing for heteroskedasticity does not help in improving the quality of the forecasts. Second, while in the PBVAR, the coefficient vector evolves with a persistence of 0.95 but with very small variance, in our PA-IEL1 specification the time varying component of the coefficients is close to be a white noise. Note that this difference is inconsequential for forecasting and can be explained by examining the role of the parameters regulating the cross sectional prior (i.e. the tightness α^* and β^*). These parameters force a high degree of coherence across countries in the time invariant component and leave the time varying component to randomly evolve. In the PBVAR this distinction is not possible and to produce coefficients which are almost constant over time it is necessary to have close to a random walk dynamics coupled with a small variance. Using equation (9), one can see in fact that coefficients of the PA-IEL1 model are approximately constant over time and are tightly linked to each other because of the restrictions imposed on α^*_i . The omission of the fixed effect component, which is precisely what the PBVAR does, biases upward estimates of the persistence parameter and this may explain why the two estimated specifications are so different. Third, the maximized value of predictive density of the PA-IEL1 model is significantly higher than the one of the PBVAR model (-36.90 vs. -985.35) suggesting that the our specification fits the data for the in-sample period better. However, this superior in-sample fit appears to be unimportant for forecasting out-of-sample. That is, the (wrong) restrictions that the PBVAR imposes and which biases the persistence parameter of the time varying coefficients do not translate in poor forecasts at the horizons we consider. We conjecture that this may have to do with the peculiarity of the forecasting sample more than with true similarities between the two specifications.

The performance of the PA-IEL2 specification is also comparable to the one obtained with PBVAR at the one step horizon. However, while the ranking of the Theil-U across countries in PBVAR and PA-IEL1 were identical, there is some reshuffling with the PA-IEL2 specification. That is, the model is somewhat better for Japan and France and somewhat worse for the US and Italy. At the four step horizon the performance of the model is significantly worse than any other model. While we have not been able to find a reason for this result, we conjecture that this has to do with the fact that the presence of a large amount of randomness in the specification of the model compounds at long horizons and worsens significantly its performance. In fact, the difference between PA-IEL1 and PA-IEL2 specifications, apart from problem of precision of estimates is only in the fact that there is an additional layer of uncertainty in the prior of the model.

The relative performance of the various models with the MAD is somewhat similar to the one obtained with the Theil-U at both horizons. However, four features deserve a comment. First, all univariate shrinkage procedures appear to be better than OLS at the one step horizon. The same is true at four steps horizons except for the case of gprior, which is now significantly worse. Second, unrestricted and simple BVAR display a somewhat mediocre performance both at one and four steps horizons. In general, the distribution of the MAD across countries is more concentrated but the mean and the median are above those obtained with univariate shrinkage approaches. Third,

the improvements obtained with panel VAR approaches are significant and our refinement of the Litterman's prior produces the best distribution of MAD at the one step horizon. The improvements are primarily concentrated for those countries which are in the central part of the distribution and this is reflected in the lower median value we obtain. Fourth, the Panel EL 2 specification is better than any other when we use the mean MAD across country to measure the forecasting specifications at both horizons. That is, Panel EL 2 produces a distribution of MAD across countries which is centered below the one obtained with other models and more concentrated. Notice also that the significant forecasting differences produced by Panel EL 2 for the Theil-U and the MAD at the four step horizon probably have to do with the different way the two criteria treat forecasting outliers.

In sum, using interdependencies, adding time variation in the coefficients and using cross sectional restrictions in the prior for the coefficients helps in improving forecasts at short-medium horizon. Nevertheless, it should be pointed out that the distribution of forecasting statistics across countries is very wide; for example, the MAD for Italy is 6 times the one of the U.S. This differences indicate that the process for the growth rate of GDP in some countries does not share much features with the growth rate of GDP of other G-7 countries and that significant improvements on the results we present can be obtained by restricting attention to the subset of the countries which are more similar. Also notice that the forecasting performance for U.S and Canada GDP growth is very similar across specifications and jointly improves with the complexity of the model, confirming that there are forecasting externalities which can be obtained by cross-sectionally linking the national models for the two countries.

How good are various approaches in predicting turning points? Out of 96 total actual turning points in the sample, univariate approaches recognize between 72 and 75. Differences primarily emerge when we try to predict upturns and non-upturns and for this type of turning points, Zellner's-g approach is better than the others. Unrestricted VAR models are very poor in this dimension and recognize about 10% less turning points than Zellner's-g approach. The performance of the BVAR model is comparable to the one of univariate Ridge and Exchangeable approaches but, contrary to them, it predicts upturns and non-upturns better than downturns and non-downturns. The performance of the PBVAR model is surprisingly poor: it is the second worst in recognizing the total number of turning points and is comparable to unrestricted VARs in predicting downturns and non-downturns. Finally, our two Panel approaches produce 73 and 74 turning point forecasts and recognize the same number of upturns and non-upturns. Comparatively speaking they substantially improve over PBVAR and are competitive with the best approaches.

Three further conclusions can be drawn from table 4. First, different models are better in recognizing different types of turning points. If predicting downturns (and non-downturns) is more important than predicting upturns (and non-upturns) our results suggest that VAR, BVAR and PBVAR should not be used. Second, while in terms of linear forecasting statistics there was a clear ranking of procedures, with more complicated ones doing a better job, when we look at nonlinear forecasting statistics, simple univariate approaches, and OLS in particular, are as good as other more refined approaches. Third, Panel VAR models of the type we have proposed do a better job than any other procedure when we jointly use linear and nonlinear statistics to measure forecasting performance.

Given that our suggested specifications are good in forecasting on average, we would like to

know if they are also good in predicting a specific episode of interest, i.e., the downward turn in real activity occurred in the US in 1990:3. This is interesting because alternative approaches, which were forecasting pretty well in the sample 1970-1980, failed to find any relevant signs in the data that would predict that a downturn and a short recession were forthcoming (see e.g. Stock and Watson (1993)). Interestingly enough, and contrary to most forecasting models, all procedures predict that there is a significant probability that a peak in economic activity will occur at 1990:3. For univariate procedures this probability is much larger than the threshold of 0.5 which we use to consider the date a downward turn. In fact all four univariate approaches predict the existence of a peak with probability above 0.64. Single country VAR, with and without a Bayesian prior are worse than univariate procedures (probability 0.32 and 0.36 respectively) but this may be due to the larger number of parameters to be estimated with the information available at 1988:4. The PBVAR specification is overwhelmingly predicting a downward turn in 1990:3 (probability is 0.82) and does not produce any false alarm in the neighborhood of this date. The second Panel VAR approach improves over single country VAR substantially and produces probability of a downturn in 1990:3 which is comparable with those of univariate approaches. The performance of the first Panel approach is poor and fails to produce a probability in excess of 0.5 in 1990:3. Note also that while univariate approaches have the tendency to produce a false alarm in 1989:4, probably due to the stock market crash of the fall of 1989, the probabilities produced by VAR and BVAR at dates other than 1990:3 are small and never exceed 0.5. The PBVAR model, on the other hand, produces a high probability of a downturn in 1991:3 a date where a downturn materialized. The second panel specification also produces a high probability of a downward turn in 1991:3 while the probabilities at other dates are small. Finally notice that the peak in 1989:2 is missed by all approaches: the ones which give highest probability to this event are the PBVAR (0.42) and the first Panel VAR approach (0.41).

In conclusion, our proposed Bayesian Panel VAR approach is at least as good as any other approach we have examined and in many cases improves the forecasting performance of existing specifications. This is true when we compare procedures using linear and non-linear forecasting statistics and when we look at specific historical episodes.

7 Conclusions

The task of this paper was to describe the issues of specification, estimation and forecasting in a macro-panel VAR model with interdependencies. The point of view used is Bayesian. Such an approach has been widely used in the VAR literature since the works of Doan, Litterman and Sims (1984), Litterman (1986), and Sims and Zha (1998) and provides a convenient framework where one can allow for both interdependencies and meaningful time variations in the coefficients. We decompose the parameter vector into two components, one which is unit specific and the other which is time specific. We specify a flexible prior on these two components which parsimoniously takes into account possible interdependencies in the cross section and allows for time variations in the evolution of the parameters over time. The prior shares features with those of Lindley and Smith (1972), Doan, Litterman and Sims (1984) and Hsiao et al. (1998) and it is specified to have a hierarchical structure, which allows for various degrees of ignorance in the researcher's information

about the parameters.

Bayesian VARs are known to produce better forecasts than unrestricted VAR and, in many situations, ARIMA or structural models (Canova (1995) for references). By allowing interdependencies and some degree of information pooling across units in the model specification we introduce an additional level of flexibility which may improve the forecasting ability of these models.

We analyze several special cases of our specification and compute Bayesian estimators for the mean parameter in the cross section and for the individual coefficients. In some cases analytical formulas for the posterior mean are available using standard formulas. Whenever the parameters of the prior are unknown, we employ the predictive density of the model to estimate them and plug in our estimates in the relevant formulas in an empirical Bayesian fashion.

In the case of fully hierarchical priors, a Markov Chain Monte Carlo method (the Gibbs sampler) is employed to calculate posterior distributions. Such an approach is particularly useful in our setup since it exploits the recursive features of the posterior distribution. Recursive formulas for multistep, multiunit forecasts, consistent with the information available at each point in time, are provided using the posterior of the parameters or the predictive density of future observations. The predictive density of future observation is also used to compute turning point probabilities.

To illustrate the performance of the proposed approach, we apply the methodology to the problem of predicting output growth, of forecasting turning points in output growth and computing the probability of a recession in the G-7 using a three variables (output growth, real stock returns and real money growth) for each country in the panel. To evaluate the model we also provide a forecasting comparison with other specifications suggested in the literature. We show that our panel VAR approach improves over existing univariate and simple BVAR models when we measure the forecasting performance using the Theil-U and the MAD criteria both at the one step and at the four steps horizons. The improvements are of the order of 5-10% with the Theil-U and about 2-4% with the MAD. The forecasting performance of our specification is also slightly better than the one of a BVAR model which mechanically extends the Litterman prior to the panel case. In terms of turning point prediction, the two versions of our panel approach are able to recognize about 80% of turning points in the sample and they turn out to be the best for this task, along with Zellner's g-prior shrinkage approach. The simple extension of the Litterman's prior to the panel case does poorly along this dimension and, among all the procedures employed is the second worst. Finally, all the procedures produce a high probability of a downturn at 90:3, the date selected by the BER committee to terminate the long expansion of the 80's. In this instance, our approach is competitive with the best and avoids the false alarms that other approaches produce at other dates.

We consider the work presented in this paper as the first step in developing a coherent theory for Bayesian Panel VAR models which take into consideration both the nature of interdependencies, the similarities in the statistical model across units and the existence of time variation in the coefficients. Extensions of the theory outlined here include the formulation of interesting hypothesis on the nature of the interdependencies, on the similarities across units and on time variations and the development of tools to undertake structural identification in these models. The work of Sims and Zha (1998) is the starting point for extensions in this latter case.

Appendix

Proof of proposition 1

(i) Notice that (14) and (15) can be written as

$$\theta^* = \alpha^* \left[Z_t' Z_t \hat{\Sigma}_{t|t-1}^{-1} Z_t' + S_u + W_t \Sigma W_t' \right]^{-1} Y_t \mid Z_{t \leq t|t-1} + \alpha^{-1} 1^{\otimes} \quad (31)$$

$$\alpha^* = \alpha^{-1} + Z_t' Z_t \hat{\Sigma}_{t|t-1}^{-1} Z_t' + S_u + W_t \Sigma W_t' \quad (32)$$

Setting $\alpha^{-1} = 0$, the result follows.

(ii) The posterior distribution of θ^* is normal with mean $\theta^* = S_N^{-1} + \hat{A}_{12} \hat{A}_{22}^{-1} [Y_t \mid Z_t (1 + \hat{\Sigma}_{t|t-1})]$ and variance $V_\alpha^* = \hat{A}_{11} \mid \hat{A}_{12} \hat{A}_{22}^{-1} \hat{A}_{21}$ where $\hat{A}_{11} = (S_N \alpha S_N' + \Sigma)$; $\hat{A}_{12} = (S_N \alpha S_N' + \Sigma) W_t'$; $\hat{A}_{21} = W_t \hat{A}_{11}$; $\hat{A}_{22} = W_t \hat{A}_{11} W_t' + Z_t \hat{\Sigma}_{t|t-1}^{-1} Z_t' + S_u$. In a more compact way V_α^* can be written as

$$V_\alpha^* = W_t' \left[S_u + Z_t \hat{\Sigma}_{t|t-1}^{-1} Z_t' \right]^{-1} W_t + \mathbf{i} S_N \alpha S_N' + \Sigma \quad \Sigma^{-1} \quad (33)$$

Notice that $(S_N \alpha S_N' + \Sigma)^{-1} = \Sigma^{-1} \mid \Sigma^{-1} S_N \mathbf{i} S_N' \Sigma^{-1} S_N + \alpha^{-1} \Sigma^{-1} S_N' \Sigma^{-1}$. Hence, for $\alpha^{-1} = 0$, this expression is equal to Σ^{-1} . Moreover the posterior mean can be written as

$$\theta^* = V_\alpha^* \left[W_t' \left[S_u + Z_t \hat{\Sigma}_{t|t-1}^{-1} Z_t' \right]^{-1} Y_t \mid Z_{t \leq t|t-1} + \mathbf{i} S_N \alpha S_N' + \Sigma \quad \Sigma^{-1} S_N 1^{\otimes} \right] \quad (34)$$

Using the previous result and the fact that $\mathbf{F} S_N = 0$, the result follows.

(iii) The posterior mean and variance of $\theta_{s,t}$ can be written as

$$\theta_{s,t}^* = - \mathbf{F} Z_t' W_t \mathbf{i} S_N \alpha S_N' + \Sigma \quad \Sigma W_t' + S_u \quad \Sigma^{-1} (Y_t \mid Z_t 1^{\otimes}) + \hat{\Sigma}_{t|t-1}^{-1} \hat{\Sigma}_{s,t|t-1} \quad (35)$$

$$\Sigma_{s,t}^{*-1} = Z_t' W_t \mathbf{i} S_N \alpha S_N' + \Sigma \quad \Sigma W_t' + S_u \quad \Sigma^{-1} Z_t + \hat{\Sigma}_{t|t-1}^{-1} \quad (36)$$

The matrix $W_t (S_N \alpha S_N' + \Sigma) W_t' + S_u$ can be written as $Z_t \alpha Z_t' + (W_t \Sigma W_t' + S_u)$ and its inverse is equal to $\mathbf{M}^{-1} \mid \mathbf{M}^{-1} Z_t \mathbf{i} Z_t' \mathbf{M}^{-1} Z_t \mathbf{F} \alpha^{-1} \Sigma^{-1} Z_t' \mathbf{M}^{-1}$ where $\mathbf{M} = (W_t \Sigma W_t' + S_u)$. Setting $\alpha^{-1} = 0$; the last expression reduces to $\mathbf{M}^{-1} \mid Z_t \mathbf{i} Z_t' \mathbf{M}^{-1} \Sigma^{-1} Z_t' \mathbf{M}^{-1}$. Premultiplying this matrix by Z_t' , we get a zero matrix. Hence, from (35) and (36) $\Sigma_{s,t}^{*-1} = \hat{\Sigma}_{t|t-1}^{-1}$ and the posterior distribution for $\theta_{s,t}$ is just equal to the prior.

Proof of proposition 2

Recall that $S_u^{-1} = 0$ implies $\hat{\alpha}_{t|t-1}^{-1} = 0$:

(i) Consider the posterior moments (31) and (32). $[(W_t \zeta W_t' + S_u) + L_t \hat{\alpha}_{t|t-1} L_t']^{-1} = M^{-1}$; $M^{-1} L_t L_t' M^{-1} L_t + \hat{\alpha}_{t|t-1}^{-1} L_t' M^{-1}$ where M was previously defined. When $\hat{\alpha}_{t|t-1}^{-1} = 0$; this reduces to M^{-1} ; $M^{-1} L_t L_t' M^{-1} L_t + \hat{\alpha}_{t|t-1}^{-1} L_t' M^{-1}$ which gives a zero matrix if premultiplied by L_t' . Consequently $a^* = a$ and $\alpha^* = 1$ and the posterior distribution of α is just equal to its prior.

(ii) Consider the posterior moments (33) and (34). When $\hat{\alpha}_{t|t-1}^{-1} = 0$, $S_u + L_t \hat{\alpha}_{t|t-1} L_t' = S_u$; $S_u^{-1} L_t L_t' S_u^{-1} L_t + \hat{\alpha}_{t|t-1}^{-1} L_t' S_u^{-1}$. Substituting into (33) and (34), gives the result

(iii) The proof of this statement comes straight from (35) and (36).

Definition of the matrices for the Gibbs sampler

$$\begin{aligned} \hat{W}_1 &= W_1 + \sum_i A_i E_i A_i' V^{-1} A_i E_i A_i'; \\ \hat{W}_2 &= W_2 + \sum_i (\alpha_i - \frac{1}{2} \alpha_{t-1})' V_1^{-1} (\alpha_i - \frac{1}{2} \alpha_{t-1}); \\ \hat{M}_o &= M_o + \sum_t Y_{ti} B_t W_t' H^{-1} Y_{ti} B_t W_t'; \\ \hat{P}_o &= P_o + \sum_t Y_{ti} B_t W_t' S^{-1} Y_{ti} B_t W_t'; \\ \hat{\alpha} &= \hat{V}_\alpha \tilde{A} X' W_t' (S - H)^{-1} (Y_{ti} L_{t,t}) + \zeta^{-1} S_N \alpha; \\ \hat{V}_\alpha &= \tilde{A} X' W_t' (S - H)^{-1} W_t + \zeta^{-1}; \\ \alpha^* &= \hat{V}^* (V - -1)^{-1} \sum_i R_i \alpha_i + a^{-1}; \\ \hat{V}^* &= N (V - -1)^{-1} + a^{-1} \end{aligned}$$

where Y_t is $N \times G$, B_t is $N \times Gk$ and $W_t = (I_G - X_t')$; Model (1) is just a row vectorization of $Y_t = B_t W_t' + U_t$, where $B_t = [\text{vec}(B_{1t}); \dots; \text{vec}(B_{Nt})]$; Here $\text{vec}()$ is the row vectorization of a matrix; $B_{it} = A_i + \alpha_t E_i$ is a $G \times k$ matrix and the parameter vectors α_i and α_t in (4) and (6) are the row vectorizations of A_i and α_t respectively.

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Table 1: Estimated Hyperparameters: PBVAR

| | |
|---|---------|
| General tightness (μ_1) | 0.01 |
| Lag decay (μ_2) | 13.96 |
| Own country tightness (μ_3) | 3.5e-05 |
| Other countries tightness (μ_4) | 7.3e-04 |
| World variable tightness (μ_5) | 5.0e-07 |
| AR coefficient (μ_6) | 0.95 |
| Prior mean on the first lag (μ_7) | 0.11048 |

Table 2: Estimated Hyperparameters: PANEL 1

| | |
|--|--------|
| Tightness for α ($\mu_{1\alpha}$) | 0.1207 |
| Tightness for λ ($\mu_{1\lambda}$) | 0.1300 |
| Tightness for α (μ_{α}) | 0.0004 |
| Lag decay (μ_2) | 1.9156 |
| Tightness on other countries (μ_3) | 0.0046 |
| Tightness on world variables (μ_4) | 4.7804 |
| Law of motion of λ (μ_5) | 0.1211 |
| Time variation (μ_6) | 0.4295 |
| Prior mean on first lag (μ_7) | 0.0754 |

Table 3
Theil-U Statistics

| Method | Step | US | Japan | Germany | UK | France | Italy | Canada | Median | Mean |
|----------------|------|------|-------|---------|------|--------|-------|--------|--------|------|
| VAR | 1 | 1.06 | 0.88 | 0.91 | 0.94 | 1.00 | 0.73 | 0.95 | 0.94 | 0.92 |
| | 4 | 0.73 | 0.95 | 0.56 | 0.81 | 1.32 | 0.96 | 0.72 | 0.81 | 0.86 |
| BVAR | 1 | 0.83 | 0.89 | 0.69 | 0.91 | 0.90 | 0.80 | 0.85 | 0.85 | 0.84 |
| | 4 | 0.75 | 0.89 | 0.65 | 0.79 | 1.16 | 1.00 | 0.70 | 0.89 | 0.85 |
| OLS | 1 | 1.21 | 0.86 | 0.88 | 0.86 | 0.90 | 0.79 | 0.91 | 0.88 | 0.90 |
| | 4 | 0.77 | 0.90 | 1.07 | 0.76 | 0.98 | 1.03 | 0.67 | 0.90 | 0.88 |
| Ridge | 1 | 1.17 | 0.83 | 0.89 | 0.85 | 0.89 | 0.79 | 0.89 | 0.89 | 0.90 |
| | 4 | 0.76 | 0.88 | 1.06 | 0.75 | 0.99 | 1.01 | 0.68 | 0.88 | 0.87 |
| Exchangeable | 1 | 1.18 | 0.84 | 0.90 | 0.85 | 0.89 | 0.78 | 0.89 | 0.89 | 0.90 |
| | 4 | 0.76 | 0.90 | 1.09 | 0.75 | 0.99 | 1.01 | 0.68 | 0.90 | 0.88 |
| gprior | 1 | 1.06 | 0.86 | 0.69 | 0.78 | 1.00 | 0.72 | 0.92 | 0.86 | 0.86 |
| | 4 | 0.83 | 1.07 | 0.77 | 0.75 | 1.12 | 1.02 | 0.70 | 0.83 | 0.89 |
| PBVAR | 1 | 0.82 | 0.85 | 0.68 | 0.76 | 0.98 | 0.73 | 0.85 | 0.82 | 0.81 |
| | 4 | 0.86 | 0.91 | 0.77 | 0.75 | 1.08 | 1.03 | 0.66 | 0.86 | 0.87 |
| Panel 1 | 1 | 0.81 | 0.88 | 0.67 | 0.75 | 1.02 | 0.70 | 0.88 | 0.81 | 0.81 |
| | 4 | 0.86 | 0.90 | 0.76 | 0.74 | 1.07 | 1.03 | 0.66 | 0.86 | 0.86 |
| Panel 2 | 1 | 0.93 | 0.81 | 0.69 | 0.78 | 0.99 | 0.78 | 0.85 | 0.81 | 0.82 |
| | 4 | 0.83 | 1.59 | 1.62 | 1.55 | 1.47 | 1.93 | 0.90 | 1.55 | 1.41 |
| MAD Statistics | | | | | | | | | | |
| VAR | 1 | 0.46 | 1.71 | 1.74 | 1.35 | 1.26 | 2.91 | 0.65 | 1.35 | 1.44 |
| | 4 | 0.35 | 1.55 | 1.18 | 1.33 | 1.66 | 2.74 | 0.56 | 1.33 | 1.34 |
| BVAR | 1 | 0.46 | 1.62 | 1.48 | 1.32 | 1.15 | 3.22 | 0.58 | 1.32 | 1.40 |
| | 4 | 0.40 | 1.39 | 1.25 | 1.28 | 1.42 | 2.98 | 0.51 | 1.28 | 1.40 |
| OLS | 1 | 0.56 | 1.59 | 1.51 | 1.37 | 1.06 | 3.17 | 0.57 | 1.37 | 1.40 |
| | 4 | 0.34 | 1.54 | 1.58 | 1.28 | 1.14 | 3.19 | 0.54 | 1.28 | 1.37 |
| Ridge | 1 | 0.54 | 1.50 | 1.68 | 1.31 | 1.07 | 3.14 | 0.56 | 1.31 | 1.40 |
| | 4 | 0.36 | 1.46 | 1.72 | 1.25 | 1.17 | 3.09 | 0.53 | 1.25 | 1.37 |
| Exchangeable | 1 | 0.54 | 1.52 | 1.68 | 1.32 | 1.06 | 3.14 | 0.56 | 1.32 | 1.40 |
| | 4 | 0.35 | 1.48 | 1.73 | 1.26 | 1.17 | 3.09 | 0.53 | 1.26 | 1.37 |
| gprior | 1 | 0.53 | 1.63 | 1.33 | 1.18 | 1.26 | 2.89 | 0.54 | 1.26 | 1.34 |
| | 4 | 0.41 | 1.60 | 1.35 | 1.18 | 1.34 | 3.12 | 0.51 | 1.34 | 1.36 |
| PBVAR | 1 | 0.46 | 1.47 | 1.29 | 1.17 | 1.27 | 2.85 | 0.53 | 1.27 | 1.29 |
| | 4 | 0.44 | 1.48 | 1.27 | 1.12 | 1.31 | 3.14 | 0.51 | 1.27 | 1.32 |
| Panel 1 | 1 | 0.46 | 1.53 | 1.24 | 1.08 | 1.37 | 2.82 | 0.54 | 1.24 | 1.29 |
| | 4 | 0.44 | 1.48 | 1.27 | 1.11 | 1.31 | 3.14 | 0.50 | 1.27 | 1.32 |
| Panel 2 | 1 | 0.49 | 1.45 | 1.27 | 1.18 | 1.32 | 3.09 | 0.60 | 1.27 | 1.34 |
| | 4 | 0.55 | 1.40 | 1.25 | 1.11 | 1.43 | 2.96 | 0.65 | 1.25 | 1.33 |

Notes: VAR is a VAR (2) model for output growth, real stock returns and real money growth, BVAR is the same model with a Minnesota prior. OLS refer to a model where the parameters are estimated with OLS, Ridge to a Ridge correction, Exchangeable to a model with an exchangeable prior and gprior to Zellner's gprior specification. PBVAR is a 21 VAR model with a Minnesota prior and time variations, Panel 1 is a panel VAR model with all 7 countries with a modified Minnesota prior and Panel 2 is the same model with a hierarchical prior.

Table 4: Turning points forecasts

| Method | Turning Points | DT & NDT | UT & NUT |
|--------------|----------------|----------|----------|
| TRUE | 96 | 47 | 49 |
| VAR | 65 | 32 | 33 |
| BVAR | 72 | 34 | 38 |
| OLS | 74 | 37 | 37 |
| Ridge | 72 | 37 | 35 |
| Exchangeable | 72 | 37 | 35 |
| gprior | 75 | 37 | 38 |
| PBVAR | 68 | 32 | 36 |
| Panel 1 | 73 | 36 | 37 |
| Panel 2 | 74 | 37 | 37 |

Notes: VAR is a VAR (2) model for output growth, real stock returns and real money growth, BVAR is the same model with a Minnesota prior. OLS refer to a model where the parameters are estimated with OLS, Ridge to a Ridge correction, Exchangeable to a model with an exchangeable prior and gprior to Zellner's gprior specification. PBVAR is a 21 VAR model with a Minnesota prior and time variations Panel 1 is a panel VAR model with all 7 countries with a modified Minnesota prior and Panel 2 is the same model with a hierarchical prior. DT means downturn, NDT means non-downturn, UT means upturn and NUT means a non-upturn.

Table 5: Probabilities of a downturn in US GDP growth

| quarter | VAR | BVAR | OLS | RIDGE | EXCHANGEABLE | gprior | PBVAR | PANEL 1 | PANEL 2 |
|-------------------|-------|-------|-------|-------|--------------|--------|-------|---------|---------|
| 89:1 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| 89:2 ^d | 0.000 | 0.005 | 0.005 | 0.010 | 0.000 | 0.270 | 0.420 | 0.410 | 0.160 |
| 89:3 | 0.020 | 0.010 | 0.005 | 0.010 | 0.200 | 0.250 | 0.010 | 0.250 | 0.230 |
| 89:4 | 0.780 | 0.590 | 0.625 | 0.815 | 0.370 | 0.280 | 0.070 | 0.210 | 0.470 |
| 90:1 | 0.200 | 0.375 | 0.365 | 0.160 | 0.070 | 0.050 | 0.070 | 0.230 | 0.040 |
| 90:2 | 0.000 | 0.005 | 0.000 | 0.000 | 0.070 | 0.080 | 0.040 | 0.220 | 0.030 |
| 90:3 ^d | 0.645 | 0.660 | 0.700 | 0.660 | 0.320 | 0.360 | 0.820 | 0.300 | 0.550 |
| 90:4 | 0.005 | 0.010 | 0.030 | 0.015 | 0.280 | 0.380 | 0.040 | 0.250 | 0.210 |
| 91:1 | 0.000 | 0.005 | 0.000 | 0.003 | 0.230 | 0.050 | 0.130 | 0.240 | 0.020 |
| 91:2 | 0.000 | 0.000 | 0.000 | 0.000 | 0.170 | 0.060 | 0.000 | 0.250 | 0.000 |
| 91:3 ^d | 0.005 | 0.015 | 0.000 | 0.000 | 0.180 | 0.490 | 0.790 | 0.230 | 0.630 |
| 91:4 | 0.015 | 0.005 | 0.005 | 0.035 | 0.250 | 0.350 | 0.080 | 0.240 | 0.320 |

Notes: A ^d indicates that a downturn occurred in output growth at that date