

Adaptive learning in models with lagged variables

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Abstract: In this work I study the stability of the dynamics generated by adaptive learning processes in intertemporal economies with lagged variables. I prove that determinacy of the steady state is a necessary condition for the convergence of the learning dynamics and I show that the reciprocal is not true characterizing the economies where convergence holds. In the case of existence of cycles I show that there is not, in general, a relationship between determinacy and convergence of the learning process to the cycle. I also analyze the expectational stability of these equilibria.

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1. Introduction

Recently, in the intertemporal equilibrium literature, attention was centered in dynamics generated by some learning process. Some of these works proved the convergence to some type of rational expectations equilibrium: Steady states (Grandmont [7]), cycles (Guesnerie and Woodford [10], Evans and Honkapohja[5], Grandmont and Laroque [8]) or sunspots equilibria (Evans and Honkapohja [4], Woodford[16]). Others authors showed that the dynamics generated by simple learning rules can be complicated (Chatterji [3], Hommes [11], Bullard [2]). Some of these papers analyze intertemporal economies where the dynamics is of the one-step forward looking type and others use structures with lagged variables.

In this work I analyze the dynamics generated by adaptive learning processes in economies with lagged variables in their excess demand functions or the functions that determines the dynamics. I show the cases where the process allows to “learn” the stationary equilibria (steady state or cycles). Also I analyze the expectational stability (E-stability) of these equilibria in such models.

For learning the interior steady state Grandmont [7] showed that the convergence of the resulting learning dynamics depend on the interaction between the eigenvalues of the perfect foresight dynamics and the eigenvalues of the prevision rule so that the stability of the system can be obtained if the influence of the expectations is weak enough and traders have enough confidence on the stability of the equilibrium. On the other hand, Guesnerie and Woodford [10] proved that the cycles can be learned by adaptive processes if and only if the derivative of the k -step backward looking map at the cycle is lower than one and greater than a critical value which depend on the order of the cycle and the parameter of the adaptive process; in particular it is easy to prove that determinacy of the steady state (cycle of order 1) is a necessary and sufficient condition for local stability of the dynamics of the learning process; but this was made for one-step forward looking economies. Evans and Honkapohja [5] found that the expectational stability of cycles in stochastic one-step forward looking models is closely related with the determinacy of them.

The goal of this work is: 1) To characterize the economies with lagged variables where the adaptive learning processes converge to the stationary steady state, 2) To verify that in the case of existence of cycles there is no a direct equivalence between determinacy and the possibility of learning the cycle and 3) To analyze the expectational stability of cycles in these types of models. The first point is in order to specify the Grandmont’s result in terms of the parameters of the economy. The second point shows that adaptive learning processes are not good criteria for testing the rational expectations hypothesis as claimed by Guesnerie [9] for models without lagged variables. Finally the third point claims that determinacy and E-stability are not completely related for models with memory.

The paper is divided in five sections. In section 2 I define the framework I will work and I give the general hypotheses I will use. In section 3 I present the results of convergence to the steady state under the learning dynamics and I show how indeterminate cycles can be learned with adaptive learning. In section 4 I analyze the E-stability of cycles in the general framework considered. I give the conclusions in section 5 and the proofs are given in the appendix.

2. The framework, adaptive learning processes and the general hypotheses

Let us consider an economy with one good and with the dynamics governed by the following function:

$$Z : X \times X \times X \rightarrow \Re$$

where $X \subset \Re$ is the state variable set. Usually this function is interpreted in the following way: If x_{t-1} is the value of the state variable in the previous period and x_{t+1}^e is the forecast for the next period of the same variable then an optimal decision that equilibrates x_{t-1} and x_{t+1}^e is a value x_t such that $Z(x_{t-1}, x_t, x_{t+1}^e) = 0$.

Models with memory (lagged variables) were frequently studied in the intertemporal equilibrium literature for proving existence of different types of equilibria (indeterminate steady states, cycles and sunspots) by Reichlin [13], Farmer [6], Woodford [14], de Vilder [14] and Michel and Venditti [12] among others. Here I am doing the same simplification used by Chatterji [3] and Grandmont [7] which is to consider that the agents only look at the expected value of the future state variable (x_{t+1}^e) for taking their current decisions and they do not take in account other moments of this random variable.

A perfect foresight equilibrium is a sequence $(x_t)_{t \geq 0}$ such that:

$$Z(x_{t-1}, x_t, x_{t+1}) = 0 \text{ for all } t \geq 1. \quad (1)$$

Here I will suppose the existence of steady state and/or cycles that I define at once.

Definitions:

1) An (interior) *steady state* for the model Z is an interior point of X such that the sequence $x_t = \bar{x}$ for all $t \geq 0$ is a perfect foresight equilibrium.

2) A *k-cycle* ($k \geq 2$) for the model Z is a k -tuple (x^1, x^2, \dots, x^k) such that the sequence defined by $x_t = x^l$ if $t \bmod(k) = l$ and $x_t = x^k$ if $t \bmod(k) = 0$ is a perfect foresight equilibrium and k is the lowest integer with this property.

Regularity conditions with respect to the stationary steady state and cycles are the following,

$$(RSS) Z_i(\bar{x}, \bar{x}, \bar{x}) \neq 0 \text{ for } i = -1, 1.^1$$

(RC) $Z_i(x^{l-1}, x^l, x^{l+1}) \neq 0$; $Z_i(x^{k-1}, x^k, x^1) \neq 0$; $Z_i(x^k, x^1, x^2) \neq 0$, for $l = 2, \dots, k-1$ and $i = -1, 0, 1$.

The learning process I will consider will be an adaptive learning process, but this will depend on the type of equilibrium we are studying. In general, agents will formulate expectations about the future expected value of the state variable as an average between observed value and the expected value in some past period, it means:

$$x_{t+1}^e = \alpha x_{t-p} + (1 - \alpha)x_{t-p}^e, \quad \alpha \in (0, 1) \quad (2)$$

where $p \in \mathcal{Z}_+$ is the lag that agents will select in order to learn some equilibrium.

Indeterminacy of some of these equilibria is related with the existence of infinitely many equilibria close to the equilibrium in consideration (for example, when we are taking the *Sup* norm).

¹ Z_i for $i = -1, 0, 1$ represents the partial derivative of Z with respect to the first, second and third variable respectively.

Definition: The steady state (or cycle) \bar{x} ((x^1, x^2, \dots, x^k)) is indeterminate if for all neighborhood of $(\bar{x}, \bar{x}, \dots)$ ($(x^1, x^2, \dots, x^k, x^1, \dots)$) there exists a continuum of sequences $(x_t)_{t \geq 0}$ in this neighborhood which are a perfect foresight equilibria.

In the next section I will characterize the indeterminacy of these equilibria and I will relate it with the instability of the dynamics generated by the learning process. In such a case I will say that the process does not allow to learn the equilibrium.

Definition: The process (2) allows to learn the steady state \bar{x} (or the cycle (x^1, x^2, \dots, x^k)) if the dynamics defined by (1) and (2) is locally stable in \bar{x} ((x^1, x^2, \dots, x^k)).

3. Learning and Determinacy of the Equilibria

In this section we will see in what extent the concepts of determinacy and learning of some equilibrium are related. I will characterize the parameters of the model which allow the stability of the equilibrium under the learning process. The following lemma will be useful for characterizing the indeterminacy of the steady state.

Lemma 3.1. *The roots of the equation $r^2 + mr + n = 0$ have modulus lower than one if and only if $|n| < 1$ and $|m| < 1 + n$.*

As an application of this lemma we can conclude the equivalence between determinacy and learnability of the steady state in one-step forward looking economies. For these economies the dynamics is given by the backward looking map

$$x_t = \phi(x_{t+1}^e) \quad (3)$$

In this case is easy to see that for a steady state being indeterminate it is necessary and sufficient that $|\phi'(\bar{x})| > 1$.

Theorem 3.2. *In the one-step forward looking economies (3) an steady state \bar{x} is determinate if and only if the learning process (2) with $p = 1$ allows to learn it for all $\alpha \in (0, 1)$.*

In order to characterize the dynamics with perfect prevision in models with memory we can use the fact that in most of these models it is easy to find the backward looking map, it means that the equation (1) can be written as:

$$x_{t-1} = \phi(x_t, x_{t+1}^e). \quad (1')$$

Under the perfect prevision hypothesis it results the following bidimensional dynamics:

$$X_{t-1} = (x_{t-1}, x_t) = (\phi(x_t, x_{t+1}^e), x_t) = \Phi(X_t).$$

In models with lagged variables the indeterminacy of the steady state is equivalent to the stability of the linearized model around the steady state, i.e. the equation:

$$(x_{t-1} - \bar{x})Z_{-1}(\bar{x}, \bar{x}, \bar{x}) + (x_t - \bar{x})Z_0(\bar{x}, \bar{x}, \bar{x}) + (x_{t+1} - \bar{x})Z_1(\bar{x}, \bar{x}, \bar{x}) = 0$$

must have a convergent dynamics to the steady state. This is equivalent to say that the roots of the following polynomial have modulus lower than one (Woodford[15], Boldrin and Rustichini[1]):

$$Z_1(\bar{x}, \bar{x}, \bar{x})r^2 + Z_0(\bar{x}, \bar{x}, \bar{x})r + Z_{-1}(\bar{x}, \bar{x}, \bar{x}) = 0. \quad (4)$$

Lemma 3.3. *If (RSS) holds then the following statements are equivalent:*

- i) \bar{x} is an indeterminate steady state.*
- ii) Equation (4) has roots with modulus lower than one.*
- iii) The matrix $\Phi'(\bar{x}, \bar{x})$ has eigenvalues greater than one.*

In order to simplify notation I will define

$$a = \frac{Z_{-1}(\bar{x}, \bar{x}, \bar{x})}{Z_1(\bar{x}, \bar{x}, \bar{x})} ; b = \frac{Z_0(\bar{x}, \bar{x}, \bar{x})}{Z_1(\bar{x}, \bar{x}, \bar{x})},$$

with this notation and using lemma 3.1 we can show the set of parameters of the model where there is indeterminacy of the steady state in figure 1 (i.e. the set of all $(b, a) \in \mathfrak{R}$ such that $|a| < 1$ and $|b| < 1 + a$).

Fig. 1

Now I will present two types of adaptive rules for learning the steady state \bar{x} . The first one is based on the correction of expectation- observation in the current period.

Theorem 3.4. *Under the hypothesis (RSS) the learning process (2) with $p = 0$ allows to learn the steady state \bar{x} if and only if $\alpha \in (0, 1)$ satisfies:*

$$\left| \frac{(1 - \alpha)a}{b + \alpha} \right| < 1 \quad \text{and} \quad \left| \frac{a - (1 - \alpha)b}{b + \alpha} \right| < 1 - \frac{(1 - \alpha)a}{b + \alpha}$$

The figure 2 shows (for $\alpha \in (0, 1)$) the values of (a, b) which allow to learn \bar{x} .

Fig. 2

We can observe that for each $\alpha \in (0, 1)$ there exist economies (values of a and b) such that there is an indeterminate steady state which can be learned and reciprocally a determinate steady state which can not be learned with the adaptive rule proposed. Another adaptive rule which can relate the concepts of determinacy and convergence of the dynamics of the learning rule is when we consider $p = 1$, it means that the formulation of expected values is made from the expectation- observation in the previous period. The following theorems characterize the economies where the stationary steady state can be learned (or not) by the adaptive process under consideration. Let us define the following sets:

$$\mathcal{C}_1 = \{(b, a) \in \mathfrak{R}^2 / |a + 1| < |b|\}$$

$$\mathcal{C}_2 = \{(b, a) \in \mathcal{C}_1 / |a| < |b|\}$$

It is easy to see that if $(b, a) \in \mathcal{C}_1$ then \bar{x} is determinate. Figure 3 shows these sets.

Fig. 3

Theorem 3.5. *If (RSS) holds then:*

- 1) If $(b, a) \in \mathcal{C}_1$ then there exists $\alpha_m \in (0, 1)$ such that for each $\alpha \in (\alpha_m, 1)$ the process (2) with $p = 1$ and α allows to learn \bar{x} .*
- 2) If $(b, a) \in \mathcal{C}_2$ then the process (2) with $p = 1$ and α allows to learn \bar{x} for all $\alpha \in (0, 1)$.*

Theorem 3.6 claims that for economies with (b, a) in the set \mathcal{C}_1 the adaptive processes putting enough gain in the observations will allow to learn the equilibrium. Furthermore,

if the economy parameters (b, a) are in \mathcal{C}_2 the adaptive process will always generate a dynamics which converges to the equilibrium. This theorem completes the result found by Grandmont [7] when he analyzes sufficient conditions for local stability from error learning.

It is easy to see that economies in $\mathcal{C}_1 - \mathcal{C}_2$ do not exhibit local stability under adaptive learning for all α ; we can see it in the following example: Consider an economy where $(b, a) \in \mathcal{C}_1 - \mathcal{C}_2$ and pick any $\alpha < 1 - \frac{|b|}{|a|}$; in this case the adaptive rule does not allow to learn the steady state although it is determinate. So it is possible the non-convergence of the learning rule to the determinate equilibrium.

Theorem 3.6. *Suppose that hypothesis (RSS) holds, $Z_0(\bar{x}, \bar{x}, \bar{x}) \neq 0$ and (b, a) belongs to the interior of \mathcal{C}_1^c then for all $\alpha \in (0, 1)$ the adaptive learning process (2) with $p = 1$ and α is locally unstable at \bar{x} . In particular, if \bar{x} is indeterminate then it is local unstable for all $\alpha \in (0, 1)$*

We can observe that the set \mathcal{C}_1^c includes economies with determinate equilibrium.

Finally I will apply adaptive learning processes to economies with lagged variables and with deterministic cycles. The existence of cycles was largely proved in different models (Farmer [6], Reichlin [13], Michel and Venditti [12]) and the possibility of learning these equilibria was proved by Guesnerie and Woodford [10] and Evans and Honkapohja [5] but for one-step forward looking economies. As we saw for the case of steady states (cycles of order 1) in models with memory the determinacy of this equilibrium is not a sufficient condition for learning it with adaptive processes. For the case of cycles I will show that this condition is not necessary.

Consider the cycle $C = \{x^1, x^2, \dots, x^k\}$ of the economy Z . As I claimed before in most of these models with lagged variables the intertemporal equilibrium equation can be written as a functional relationship between the expectations of the future state and the current optimal value of the state variable with the past value of the state variable, so the equation $Z(x_{t-1}, x_t, x_{t+1}^e) = 0$ can be expressed as:²

$$x_{t-1} = \phi(x_t, x_{t+1}^e).$$

From this we can define the following dynamical system which describes the evolution of the state variable under the hypothesis of perfect prevision:

$$X_{t-1} = (x_{t-1}, x_t) = (\phi(x_t, x_{t+1}^e), x_t) = \Phi(X_t)$$

So, if C is a cycle for Z then the set $\{X_1 = (x^1, x^2), X^2 = (x^2, x^3), \dots, X^k = (x^k, x^1)\}$ is a cycle for Φ or what is the same X^l is a fixed point of Φ^k for $l = 1, \dots, k$. In this case the indeterminacy of the cycle can be characterized by the eigenvalues of $(\Phi^k)'$.

Proposition 3.7. *C is a determinate cycle of Z if and only if $(\Phi^k)'(X^1)$ has at least one eigenvalue lower than one.*

Let us define:

$$a_1 = -\frac{Z_0(x^k, x^1, x^2)}{Z_{-1}(x^k, x^1, x^2)}, \quad a_l = -\frac{Z_0(x^{l-1}, x^l, x^{l+1})}{Z_{-1}(x^{l-1}, x^l, x^{l+1})}, \quad l = 2, \dots, k-1; \quad a_k = -\frac{Z_0(x^{k-1}, x^k, x^1)}{Z_{-1}(x^{k-1}, x^k, x^1)}$$

²This relationship can also be obtained from the regularity of the cycle in a neighborhood of it

$$b_1 = -\frac{Z_1(x^k, x^1, x^2)}{Z_{-1}(x^k, x^1, x^2)}, \quad b_l = -\frac{Z_1(x^{l-1}, x^l, x^{l+1})}{Z_{-1}(x^{l-1}, x^l, x^{l+1})}, \quad l = 2, \dots, k-1; \quad b_k = -\frac{Z_1(x^{k-1}, x^k, x^1)}{Z_{-1}(x^{k-1}, x^k, x^1)}.$$

With this notation the Jacobian matrix of Φ^k results:

$$A = \begin{pmatrix} a_1 & b_1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_k & b_k \\ 1 & 0 \end{pmatrix}.$$

So, a cycle is indeterminate if and only if the eigenvalues of A have modulus greater than one and applying the lemma 3.1 we have the following

Proposition 3.8. *C is an indeterminate cycle if and only if $|\det(A)| > 1$ and $|\operatorname{tr}(A)| < |\det(A)| + \operatorname{sign}(\det(A))$.*

Next I will define the learning process for the cycle. With the hypothesis (RC) I can define the following functions: $x_{t+1}^e = \varphi(x_{t-1}, x_t)$ and $x_t = \psi(x_{t-1}, x_{t+1}^e)$. Guesnerie and Woodford [10] present the following rule as a natural adaptive learning process for learning cycles:

$$x_{t+1}^e = \alpha x_{t+1-k} + (1 - \alpha)x_{t+1-k}^e,$$

it means, the process (2) with $p = k - 1$. The actual dynamics generated by this process is given by:

$$\begin{aligned} x_t &= \psi(x_{t-1}, \alpha x_{t+1-k} + (1 - \alpha)\varphi(x_{t-1-k}, x_{t-k})) \\ x_{t-1} &= x_{t-1} \\ &\vdots \\ x_{t-k} &= x_{t-k}. \end{aligned}$$

If we define $Y_t = (x_t, \dots, x_{t-k})$ the system above can be written as $Y_t = F(Y_{t-1})$ where $F : \mathfrak{R}^{k+1} \rightarrow \mathfrak{R}^{k+1}$ is a C^1 function with cycle $\{Y^1, Y^2, \dots, Y^k\}$ where $Y^1 = (x^1, x^k, x^{k-1}, \dots, x^1)$, $Y^2 = F(Y^1) \dots Y^k = F(Y_{k-1})$. Therefore the process (2) with $p = k - 1$ allows to learn the cycle C if and only if the cycle $\{Y^1, Y^2, \dots, Y^k\}$ is locally stable for F or what is the same, if the Jacobian matrix of $F^{(k)}$ at Y^1 has eigenvalues with modulus lower than one. Let us introduce the following notation: For $k \geq 3$ let $w^l = (a_l^{-1}, 0, \dots, 0, -\alpha a_l^{-1} b_l, (1 - \alpha), -(1 - \alpha)a_l^{-1}) \in \mathfrak{R}^{k+1}$ and for the case $k = 2$ let $w^l = (a_l^{-1}(1 - \alpha b_l), (1 - \alpha), -(1 - \alpha)a_l^{-1}) \in \mathfrak{R}^3$ for $l = 1, \dots, k$. Therefore we have the following lemma.

Lemma 3.9. *Let $C = \{x^1, x^2, \dots, x^k\}$ be a cycle of Z which satisfies (RC). The process (2) with $p = k - 1$ allows to learn the cycle C if and only if the eigenvalues of the following matrix have modulus lower than one:*

$$M(\alpha) = \begin{pmatrix} w^1 \\ I & 0 \end{pmatrix} \begin{pmatrix} w^2 \\ I & 0 \end{pmatrix} \cdots \begin{pmatrix} w^k \\ I & 0 \end{pmatrix}$$

where I is the $k \times k$ identity matrix and 0 is the null vector of $\mathfrak{R}^{k \times 1}$

This adaptive learning is not a good method for obtaining convergence to these types of stationary equilibria. In theorem 3.6 we saw that determinacy of the equilibrium is a necessary condition for learning; here I will show that it is not the case. The following theorem shows that for $k = 2$ determinacy of the equilibrium is not a necessary condition for local stability of the dynamics generated by the learning process around the cycle.

Theorem 3.10. *Let $C = \{x^1, x^2\}$ be a cycle for Z which holds (RC). If the cycle is indeterminate and $\text{sign}(a_1a_2)=\text{sign}(b_1b_2)$ then the process (2) with $p = 1$ does not allow to learn the cycle.*

The condition $\text{sign}(a_1a_2)=\text{sign}(b_1b_2)$ is necessary because if it does not hold we can find indeterminate cycles which can be learned by adaptive processes. For example if $a_1a_2 = -0.5$, $b_1 = b_2 = 1.1$ then the cycle is indeterminate, but if we take $\alpha = 0.95$ then the eigenvalues of the learning dynamics are: $r_1 = 0.0726$ and $r_{2,3} = 0.0858 \pm 0.0992i$ and it is easy to see that their modulus are lower than one, so the learning dynamics converges to the cycle.

4. E-stability

In this section I analyze the expectational stability of regular cycles in the framework considered above. Expectational stability (E-stability) of cycles was studied by Evans and Honkapohja [5] in stochastic one-step forward looking models and they proved that it amounts to the determinacy of the cycle (the strong E-stability). Here I show that these two concepts are not equivalent.

To analyze if a cycle $C = \{x^1, \dots, x^k\}$ is E-stable we proceed as follow: Suppose that the perceived cycle (probably non-rational) is $(\theta^1, \dots, \theta^k)$ so the actual law of motion of the economy is $x_1 = \psi(\theta^k, \theta^2)$, $x_2 = \psi(\theta^1, \theta^3)$, ..., $x_k = \psi(\theta^{k-1}, \theta^1)$. Therefore the E-stability of the cycle is related with the local stability of (x^1, \dots, x^k) in the following differential equation:

$$\frac{d\theta}{d\tau} = T(\theta) - \theta \quad (5)$$

where $T(\theta) = (\psi(\theta^k, \theta^2), \psi(\theta^1, \theta^3), \dots, \psi(\theta^{k-1}, \theta^1))$ and τ is called *notional* or *virtual* time. Then the cycle C is E-stable if and only if is local stable for the differential equation (5).

Lemma 4.1. *The cycle $C = \{x^1, \dots, x^k\}$ is E-stable if and only if $T'(x^1, \dots, x^k)$ has all its eigenvalues with real part lower than one. For $k \geq 3$ this matrix is given by:*

$$\begin{pmatrix} 0 & -a_1^{-1}b_1 & 0\dots & 0 & 0 & a_1^{-1} \\ a_2^{-1} & 0 & -a_2^{-1}b_2\dots & 0 & 0 & 0 \\ \vdots & & & & & \\ 0 & 0 & 0\dots & a_{k-1}^{-1} & 0 & -a_{k-1}^{-1}b_{k-1} \\ -a_k^{-1}b_k & 0 & 0 & 0\dots & a_k^{-1} & 0 \end{pmatrix}$$

and for $k = 2$:

$$\begin{pmatrix} 0 & a_1^{-1}(1 - b_1) \\ a_2^{-1}(1 - b_2) & 0 \end{pmatrix}$$

Firstly I will characterize the economies where the steady state is strongly E-stable. Strong stability of some equilibrium says that this equilibrium is E-stable with respect to any overparametrization of it. An interesting fact we can find is that these economies are just the economies where the steady state can be learned by adaptive learning processes (theorem 3.5 part 1)).

Theorem 4.2. *If the steady state of Z holds (RSS) and $Z_0(\bar{x}, \bar{x}, \bar{x}) \neq 0$ then it is strongly E-stable if and only if $(b, a) \in C_1$.*

Finally I will enunciate a theorem which shows that E-stability is not equivalent to determinacy of cycles of order 2.

Theorem 4.3. *Let $C = \{x^1, x^2\}$ be an indeterminate cycle of Z which satisfies (RC). Then C is E-stable if and only if $\text{sign}(a_1 a_2) = -\text{sign}(b_1 b_2)$.*

5. Conclusions

In this work I study the behavior of the local dynamics generated by adaptive learning processes and the expectational stability around cycles in dynamical models with lagged variables. When we want to learn the steady state it is shown the parameter set where it is possible to learn the equilibrium from different lags in the adaptive learning. It is also shown that there exist economies where the determinate equilibrium can not be learned adaptively, although it is proved that the determinacy of the steady state is a necessary condition for convergence of the dynamics generated by the learning process.

For economies with cycles I give conditions for local stability under adaptive learning. In this case it is shown that determinacy is neither necessary nor sufficient condition for learning the equilibria. This is a contrast with the results found by Guesnerie and Woodford [10].

With respect to expectational stability I find that the economies where the steady state is strongly E-stable are the same where this steady state can be learned by adaptive rules. In the case of cycles I prove that E-stability and determinacy are not related as in models without lagged variables as proved by Evans and Honkapohja [5].

APPENDIX

Proof of Lemma 3.1. Let $\Delta = m^2 - 4n$, r_1 and r_2 roots of the quadratic equation.

(\Rightarrow) If $\Delta \geq 0$ then $m-2 < \pm\sqrt{\Delta} < m+2$. For “+” we have $\sqrt{\Delta} < m+2 \Rightarrow m > -1-n$. For “-” we have $m-2 < -\sqrt{\Delta} \Rightarrow m < 1+n$. Since $n = r_1 r_2$ then $|n| < 1$. So $|m| < 1+n$.

If $\Delta < 0$ then $m^2 < 4n < n^2 + 2n + 1 \Rightarrow |m| < 1+n$ and $n = \bar{r}_1 r_1 = |r|^2 < 1$.

(\Leftarrow) Let us consider $\Delta \geq 0$. Since $m > -1-n$ then $(m+2)^2 > m^2 - 4n$; but $|m| < 2$ so $r_1 < 1$. Furthermore $m+2 > -\sqrt{\Delta}$ then $r_2 < 1$. Also from $m < 1+n$ we can conclude $r_2 > -1$ and because of $2-m > -\sqrt{\Delta}$ then $r_1 > -1$. Therefore $|r_1| < 1$ and $|r_2| < 1$.

If $\Delta < 0$ then $|r_i|^2 = n$, therefore $|r_1| < 1$ and $|r_2| < 1$.

Proof of theorem 3.2. The dynamics under the learning process is given by the following equations:

$$x_t = \phi(x_{t+1}^e) \text{ and } x_{t+1}^e = \alpha x_{t-1} + (1-\alpha)x_{t-1}^e.$$

Then the actual dynamics is given by $x_t = \phi(\alpha x_{t-1} + (1-\alpha)\phi^{-1}(x_{t-2}))$. Analyzing the local dynamics in \bar{x} and using lemma 3.1 we can obtain the equivalence.

Proof of lemma 3.3. The equivalence between *i)* and *ii)* is not difficult to prove and is given in the references. Let us check the equivalence with *iii)*. The matrix:

$$\Phi'(\bar{x}, \bar{x}) = \begin{pmatrix} -Z_0 Z_{-1}^{-1} & -Z_1 Z_{-1}^{-1} \\ 1 & 0 \end{pmatrix} (\bar{x})$$

has its characteristic equation equivalent to $Z_{-1}r^2 + Z_0r + Z_1 = 0$ which has as its roots the inverses of the roots of (4), so the equivalence results.

Proof of theorem 3.4. The local dynamics is given by:

$$ax_{t-1} + bx_t + x_{t+1}^e = 0 \text{ and } x_{t+1}^e = \alpha x_t + (1 - \alpha)x_t^e$$

so we have $(b + \alpha)x_t + (a - (1 - \alpha)b)x_{t-1} - a(1 - \alpha)x_{t-2} = 0$. Applying lemma 3.1 to the correspondent characteristic equation the theorem results.

Proof of theorem 3.5. The characteristic polynomial for the dynamics is:

$$P(\alpha, r) = r^3 + \left(\frac{a + \alpha}{b}\right)r^2 - (1 - \alpha)r - (1 - \alpha)\frac{a}{b}.$$

1) Since $P(1, r) = 0$ has roots $\{0, -\frac{a+1}{b}\}$, I will prove that for α close to one $P(\alpha, r) = 0$ has roots with modulus lower than one. I will divide the proof in two parts:

Part I: The root $P(1, -\frac{a+1}{b}) = 0$. Since³ $P_r(1, -\frac{a+1}{b}) = (\frac{a+1}{b})^2$ we have two cases:

Case i): $a = -1$. In this case we have that $P(\alpha, r) = r^3 - \frac{(1-\alpha)}{b}r^2 - (1-\alpha)r + \frac{(1-\alpha)}{b}$. So $P(\alpha, -1) = -\alpha$, $P(\alpha, 1) = \alpha$ and then there exists a real root of $P(\alpha, r) = 0$ in $(-1, 1)$. The polynomial $P_r(\alpha, r) = 3[r^2 - \frac{2(1-\alpha)}{3b}r - \frac{1-\alpha}{3}]$ has real roots with modulus lower than one because of lemma 3.1. For checking this let us observe that $|\frac{1-\alpha}{3}| < 1$ and $1 + (-\frac{1-\alpha}{3}) = \frac{2}{3}(1 + \frac{\alpha}{2})$; so for α close to one this expression is greater than $|\frac{2(1-\alpha)}{3b}|$. Therefore if the other two roots of $P(\alpha, r)$ are real then they must be in $(-1, 1)$ and the graphic of $P(\alpha, r)$ is as shown in figure 4.

Fig. 4

If the other roots of $P(\alpha, r)$ are complex they will have modulus lower than one if $|r_1| > (1 - \alpha)/|b|$ (because the product of all roots must be equal to $(1 - \alpha)/b$). For seeing this we can calculate $P(\alpha, (1 - \alpha)/b) = \frac{1-\alpha}{b}f(\alpha)$ where $f(\alpha) = \frac{2}{b^2}\alpha^2 + (1 - \frac{4}{b^2})\alpha + 2(\frac{1}{b^2} - 1)$. Since $f(1) = -1$, for α close to one $Sign(P(\alpha, (1 - \alpha)/b)) = -Sign(b)$, then for α close to one $|r_1| \in (\frac{1-\alpha}{|b|}, 1)$ showing that all the roots have modulus lower than one for α close to one.

Case ii): $a \neq -1$. Then there exist A_1 (neighborhood of $\alpha = 1$) and a function $r : A_1 \rightarrow \mathfrak{R}$, $r \in C^1$ such that $r(1) = -\frac{a+1}{b}$ and for all $\alpha \in A_1$, $P(\alpha, r(\alpha)) = 0$. Since $(b, a) \in \mathcal{C}_1$ and r is continuous then $|r(\alpha)| < 1$ for α close to 1.

Part II: The root $P(1, 0) = 0$. Because of $P_r(1, 0) = 0$ let us consider $P_\alpha(1, 0) = \frac{a}{b}$.

Case i): $a \neq 0$. Then there exists V_0 (neighborhood of $r = 0$) and a function $\alpha : V_0 \rightarrow \mathfrak{R}$, $\alpha \in C^1$ such that $\alpha(0) = 1$ and $P(\alpha(r), r) = 0$ for all $r \in V_0$. Since $\alpha''(0) = -2(1 + \frac{1}{a})$

³The index means the partial derivative of the polynomial with respect to the indicated variable

we have three possibilities: $a = -1$, $a \in (-1, 0)$, $a \notin [-1, 0]$; the first one was analyzed in case i) of part I, let us see the others two.

Subcase ia): $a \in (-1, 0)$. From case ii) in Part I we know that for α close to one there exists a real root of $P(\alpha, r) = 0$ which is close to $-\frac{a+1}{b}$, let us call it $r(\alpha)$, then:

$$P(\alpha, r) = (r - r(\alpha))\left[r^2 + \left(\frac{a + \alpha}{b} + r(\alpha)\right)r + \frac{(1 - \alpha)a}{r(\alpha)b}\right].$$

Using lemma 3.1 we can observe that the equation:

$$r^2 + \left(\frac{a + \alpha}{b} + r(\alpha)\right)r + \frac{(1 - \alpha)a}{r(\alpha)b} = 0 \quad (*)$$

has roots with modulus lower than one for α close to one. For checking it we can see that $(1 - \alpha)$ is close to zero and $r(\alpha)$ is close to $-\frac{a+1}{b}$, so $|\frac{(1-\alpha)a}{r(\alpha)b}| < 1$. Also we can see that:

$$\left|\frac{a + \alpha}{b} + r(\alpha)\right| \leq \left|\frac{a + 1}{b} + r(\alpha)\right| + |1 - \alpha|,$$

so the lefthandside is close to zero when α is close to one and therefore lower than $1 + \frac{(1-\alpha)a}{r(\alpha)b}$. From this the roots of (*) have modulus lower than one for α close to one.

Subcase ib): $a \notin [-1, 0]$. In this case $\alpha''(0) < 0$ and the graphic of α is as shown in figure 5. Therefore $P(\alpha, r) = 0$ has two real roots (r_1, r_2) with modulus lower than one (in fact they are close to zero) for α close to one.

Fig. 5

Case ii): $a = 0$. In this case the characteristic polynomial is:

$$P(\alpha, r) = r(r^2 + \frac{\alpha}{b}r - (1 - \alpha)).$$

Since $(b, a) \in \mathcal{C}_1$ then $|b| > 1$ and therefore (using lemma 3.1) $P(\alpha, r) = 0$ has all its roots with modulus lower than one.

From Parts I and II we can conclude that there exists $\alpha_m \in (0, 1)$ such that if $\alpha \in (\alpha_m, 1)$ then the polynomial $P(\alpha, r)$ has all its roots with modulus lower than one.

2) From $(b, a) \in \mathcal{C}_2$ we have that $P(\alpha, 1) = \frac{\alpha}{b}(1+a+b) > 0$ and $P(\alpha, -1) = \frac{\alpha}{b}(1+a-b) < 0$, Then there exists a real root of $P(\alpha, r)$ in $(-1, 1)$. Let $r_1 \in \mathcal{R}$ be such a root.

Using lemma 3.1 we can prove that $P_r(\alpha, r) = 3r^2 + 2(\frac{a+\alpha}{b})r - (1 - \alpha)$ have two real roots with modulus lower than one, because for $(b, a) \in \mathcal{C}_2$ we have that $|\frac{a+\alpha}{b}| < 1$ for all $\alpha \in (0, 1)$ and then $|\frac{2}{3}(\frac{a+\alpha}{b})| < 1 - \frac{1-\alpha}{3}$ and obviously $|\frac{1-\alpha}{3}| < 1$. Let m and n the roots of $P_r(\alpha, r)$. If $P(\alpha, m)P(\alpha, n) < 0$ then all roots of $P(\alpha, r)$ are in $(-1, 1)$. If $P(\alpha, m)P(\alpha, n) > 0$ then $P(\alpha, r)$ have two complex roots $\rho e^{\pm i\theta}$ and $r_1 \rho^2 = (1 - \alpha)\frac{a}{b}$, so if $|r_1| > (1 - \alpha)|\frac{a}{b}|$ then $\rho < 1$. Let us prove this in two cases:

Case a > 0: $P(\alpha, -\frac{a}{b}) = \frac{\alpha}{b}(\frac{a}{b})^2$ then $Sign(P(\alpha, -\frac{a}{b})) = Sign(b)$; since $P(\alpha, 1) > 0$ and $P(\alpha, -1) < 0$ we have that $|r_1| > |\frac{a}{b}| > (1 - \alpha)|\frac{a}{b}|$.

Case $a < 0$: $P(\alpha, (1 - \alpha)\frac{a}{b}) = (1 - \alpha)(\frac{a}{b})f(\alpha)$, where $f(\alpha) = ((\frac{a}{b})^2 - \frac{a}{b^2})\alpha^2 + (\frac{a}{b^2} - 3(\frac{a}{b})^2 + 1)\alpha + 2((\frac{a}{b})^2 - 1)$ is a parabola with a positive coefficient of α^2 . From $f(0) = 2((\frac{a}{b})^2 - 1) < 0$ and $f(1) = -1$ we can conclude that $f(\alpha) < 0$ for all α in $(0, 1)$, so $Sign(P(\alpha, (1 - \alpha)\frac{a}{b})) = Sign(b)$, then $|r_1| > (1 - \alpha)|\frac{a}{b}|$.

Proof of theorem 3.6. Let us consider the following sets:

$$R_1 = \{(b, a) \in \mathfrak{R}^2 / a > b - 1, b > 0\}, \quad R_2 = \{(b, a) \in \mathfrak{R}^2 / a > -b - 1, b < 0\}$$

$$R_3 = \{(b, a) \in \mathfrak{R}^2 / a < b - 1, b < 0\}, \quad R_4 = \{(b, a) \in \mathfrak{R}^2 / a < -b - 1, b > 0\}$$

We have that $P(\alpha, -1) = \frac{\alpha}{b}(a - b + 1)$; $P(\alpha, 1) = \frac{\alpha}{b}(a + b + 1)$. In regions R_1 and R_3 we have $P(\alpha, -1) > 0$; since $P(\alpha, -\infty) = -\infty$ then there exists at least one root with modulus greater than one, so the local dynamics does not converge to \bar{x} . Analogously in regions R_2 and R_4 we have $P(\alpha, 1) < 0$; since $P(\alpha, +\infty) = +\infty$ then there exists at least one root with modulus greater than one, so the local dynamics does not converge to \bar{x} .

Proof of theorem 3.10. A 2-cycle is indeterminate if and only if:

$$|b_1 b_2| > 1 \text{ and } |a_1 a_2 + b_1 + b_2| < |b_1 b_2| + sign(b_1 b_2),$$

it results from proposition 3.9. The characteristic polynomial of the local dynamics generated by the learning rule (2) with $p = 1$ is:

$$P(\alpha, r) = r^3 - [a_1^{-1} a_2^{-1} (1 - \alpha b_1)(1 - \alpha b_2) + 2(1 - \alpha)]r^2 +$$

$$[a_1^{-1} a_2^{-1} (1 - \alpha)(1 - \alpha b_1)(1 - \alpha b_2) + (1 - \alpha)^2 + a_1^{-1} a_2^{-1} (1 - \alpha)(1 - \alpha b_1) + a_1^{-1} a_2^{-1} \alpha (1 - \alpha)(1 - \alpha b_2) b_1]r -$$

$$(1 - \alpha)^2 a_1^{-1} a_2^{-1}$$

$$\Rightarrow P(\alpha, 1) = \alpha^2 \left[1 - \frac{(1 - b_1)(1 - b_2)}{a_1 a_2} \right].$$

If $sign(a_1 a_2) = sign(b_1 b_2)$ then indeterminacy implies:

$$\frac{(1 - b_1)(1 - b_2)}{a_1 a_2} > 1,$$

therefore $P(\alpha, 1) < 0$ then there exists at least a root with modulus greater than one.

Proof of lemma 4.1. It is straightforward from the stability condition of the system (5) using the fact that the negativeness of the real part of the eigenvalue of $T'(x^1, \dots, x^k) - I$ is equivalent to say that the real part of the eigenvalue of $T'(x^1, \dots, x^k)$ must be lower than one.

Proof of theorem 4.2. Suppose that we consider the steady state \bar{x} as a degenerated n -cycle, so the stability condition of this cycle is (using lemma 4.1) that the following $n \times n$ matrix has eigenvalues with real part lower than one:

$$A_n = \begin{pmatrix} 0 & -b^{-1} & 0 \dots & 0 & 0 & -ab^{-1} \\ -ab^{-1} & 0 & -b^{-1} \dots & 0 & 0 & 0 \\ \vdots & & & & & \\ 0 & 0 & 0 \dots & -ab^{-1} & 0 & -b^{-1} \\ -b^{-1} & 0 & 0 & 0 \dots & -ab^{-1} & 0 \end{pmatrix}, \quad n \geq 3$$

and for $n = 2$:

$$A_2 = \begin{pmatrix} 0 & -b^{-1}(1+a) \\ -b^{-1}(1+a) & 0 \end{pmatrix}.$$

For $n = 2$ the condition for stability of the cycle is that $|a + 1|/|b| < 1$, it means that $(b, a) \in \mathcal{C}_1$. For $n \geq 3$, λ is eigenvalue of A_n if and only if $\det((b\lambda)I - M_n) = 0$ where

$$M_n = \begin{pmatrix} 0 & -1 & 0 & 0\dots & 0 & 0 & -a \\ -a & 0 & -1 & 0\dots & 0 & 0 & 0 \\ 0 & -a & 0 & -1\dots & 0 & 0 & 0 \\ \vdots & & & & & & \\ 0 & 0 & 0 & 0\dots & -a & 0 & -1 \\ -1 & 0 & 0 & 0\dots & 0 & -a & 0 \end{pmatrix}$$

It can be verified that the eigenvalues of M_n are $\{aw_k + \bar{w}_k; k = 0, \dots, n-1\}$ if n is odd and $\{a\bar{v}_k + v_k; k = 0, \dots, n-1\}$ if n is even, where w_k and v_k are the n -th roots of 1 and -1 respectively. In any case:

$$\lambda = \frac{a+1}{b} \cos(\theta_k) \pm \frac{a-1}{b} \sin(\theta_k) i$$

So the eigenvalues have its real part lower than one if and only if $(b, a) \in \mathcal{C}_1$

Proof of theorem 4.3. It is easy to see that the condition for E-stability of two cycles is that:

$$\frac{(1-b_1)(1-b_2)}{a_1a_2} < 1.$$

From the indeterminacy of the cycle we have that: $|a_1a_2 + b_1 + b_2| < |b_1b_2| + \text{sign}(b_1b_2)$ and $|b_1b_2| > 1$.

(\Rightarrow) If $b_1b_2 > 1$ then indeterminacy implies $a_1a_2 < (1-b_1)(1-b_2)$ so from E-stability it results $\text{sign}(a_1a_2) = -1$.

If $b_1b_2 < -1$ then indeterminacy implies $a_1a_2 > (1-b_1)(1-b_2)$ so from E-stability it results $\text{sign}(a_1a_2) = 1$.

(\Leftarrow) If $b_1b_2 > 1$ then $a_1a_2 < 0$ and indeterminacy implies that $(1-b_1)(1-b_2)/(a_1a_2) < 1$ so the cycle is E-stable.

If $b_1b_2 < -1$ then $a_1a_2 > 0$ and indeterminacy implies that $(1-b_1)(1-b_2)/(a_1a_2) < 1$ so the cycle is E-stable.

REFERENCES

1. Blodrin M. and Rustichini A., *Growth and indeterminacy in dynamic models with externalities*, *Econometrica* **62-2** (1994), 323-342.
2. Bullard J., *Learning equilibria*, *Journal of Economic Theory* **64** (1994), 468-485.
3. Chatterji S., *Temporary equilibrium dynamics with Bayesian learning*, *Journal of Economic Theory* **67** (1995), 590-598.
4. Evans G. and Honkapohja S., *On the local stability of sunspot equilibria under adaptive learning rules*, *Journal of Economic Theory* **64** (1994), 142-161.

5. Evans G. and Honkapohja S., *Local convergence of recursive learning to steady state and cycles in stochastic non-linear models*, *Econometrica* **63-1** (1995), 195-206.
6. Farmer R., *Deficit and cycles*, *Journal of Economic Theory* **40** (1986), 77-88.
7. Grandmont J.-M., *Expectations formation and stability of large socioeconomic systems*, *Econometrica* **66-4** (1998), 741-781.
8. Grandmont J.-M. and Laroque G., *Stability of cycles and expectations*, *Journal of Economic Theory* **40** (1986), 138-151.
9. Guesnerie R., *Theoretical tests of the Rational Expectations hypothesis in economic dynamical models*, *Journal of Economic Dynamics and Control* **17** (1993), 847-864.
10. Guesnerie R. and Woodford M., *Stability of cycles with adaptive learning rules*, *Equilibrium Theory and Applications*, ed. by W. Barnett, B. Cornet, C. d'Aspremont, J. Gabszewicz and A. Mas-Colell. Cambridge U.K.: Cambridge University Press (1991).
11. Hommes C., *Adaptive learning and road to chaos*, *Economic Letters* **36** (1991), 127-132.
12. Michel P. and Venditti A., *Optimal growth and cycles in overlapping generations models*, *Economic Theory* **9-3** (1996), 511-528.
13. Reichlin P., *Equilibrium cycles in an overlapping generations economy with production*, *Journal of Economic Theory* **40** (1986), 89-102.
14. de Vilder R., *Complicated endogenous business cycles under gross substitutability*, *Journal of Economic Theory* **71** (1996), 416-442.
15. Woodford M., *Stationary sunspot equilibrium in a finance constrained economy*, *Journal of Economic Theory* **40** (1986), 128-137.
16. Woodford M., *Learning to believe in sunspots*, *Econometrica* **58** (1990), 277-307.