A Simple Model of Multiple Equilibria Based on Risk[1]

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Abstract

This paper shows how risk may aggravate fluctuations in economies with imperfect insurance and multiple assets. A two period job matching model is studied, in which risk averse agents act both as workers and as entrepreneurs. They choose between two types of investment: one type is riskless, while the other is a risky activity that creates jobs.

Equilibrium is unique under full insurance. If investment is fully insured but unemployment risk is uninsured, then precautionary saving behavior dampens output fluctuations. However, if both investment and employment are uninsured, then an increase in unemployment gives agents an incentive to shift investment away from the risky asset, further increasing unemployment. This positive feedback may lead to multiple Pareto ranked equilibria. An overlapping generations version of the model may exhibit poverty traps or persistent multiplicity. Greater insurance is doubly beneficial in this context since it can both prevent multiplicity and promote risky investment.

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1 Introduction

This paper proposes a business cycle propagation mechanism which may arise in economies with imperfect insurance and multiple assets. Precautionary behavior will cause changes in output to be self-reinforcing if we make the following three assumptions about the economy. First, suppose that a decrease in output leads to an increase in uninsured risk in the economy— a reasonable assumption, since recessions greatly increase the probabilities of unemployment and of bankruptcy.[1] Suppose further that an increase in the risks individuals face causes them to shift their investment away from risky assets and towards riskless assets, as has been argued recently in the literature on precautionary behavior.[2] Finally, suppose that a fall in risky asset investment leads to a fall in output, as might be the case if risky investment were interpreted as the foundation of firms or expansion of their productive capacity. Then we see that the economy exhibits a positive feedback: a fall in output increases risk, which decreases risky investment, further decreasing output. Hence, the impact of exogenous shocks is likely to be amplified by asset choice under imperfect insurance. Moreover, this amplification, if sufficiently strong, could lead to multiple equilibria at different levels of output and utility.

A simple model is constructed in this paper to illustrate this mechanism. We study a two-period job matching model with risk averse agents who act both as workers and as entrepreneurs. When output falls, their probability of job finding decreases, making their labor income both lower in mean and more variable. They take this increased risk into account when choosing how to split their investment between a safe asset, which can be interpreted as storage, and a risky asset, which can be interpreted as small business formation. The rise in unemployment probability resulting from a fall in output causes them to decrease their risky asset investment, which then, through the matching function, further decreases output and employment.

[1] Two recent papers studying how consumption risk changes during recessions are Pistaferri (1998) and Storesletten, Telmer, and Yaron (1998).

[2] See Gollier and Pratt (1996) for a summary of the literature.

To understand the role of risk in our model, we must distinguish between two aspects of precautionary behavior under imperfect insurance: the *precautionary saving* decision and the *portfolio choice* decision. By precautionary saving, we mean the impact of future risk on the *total amount saved*, whereas portfolio choice refers to the distribution of that saving *across different assets*. When there is just a single, riskless asset, we are in a situation of pure precautionary saving. In this case, uninsured unemployment risk implies a *negative* feedback effect which dampens fluctuations: a higher individual probability of unemployment leads to greater investment, which in turn tends to decrease unemployment. However, when there is a portfolio decision to be made between a riskless asset and an imperfectly insured risky asset, then a *positive* feedback effect is also present, for a rise in unemployment risk causes agents to shift away from the risky asset, further increasing unemployment.[3]

Under full insurance, our model has a unique equilibrium. If investment is fully insured but unemployment risk is incompletely insured, then uniqueness holds *a fortiori* because of the negative feedback from precautionary saving. However, when both unemployment and investment are less than perfectly insured, the positive feedback from portfolio choice may lead to multiple equilibria at high and low levels of output. Multiplicity may remain important even in the long run when the two period model is extended to allow for overlapping generations; moreover, the OLG version of the model displays other interesting risk-based phenomena, such as poverty traps.

The distinction between precautionary saving and portfolio choice is also crucial to the model's policy implications. We usually suppose that greater insurance comes at a cost in terms of decreased investment, since insurance offsets precautionary saving. However, as emphasized by Elmendorf and Kimball (1991), when there is a nontrivial portfolio choice, greater insurance encourages risky investment. If risky investment is undersupplied in equilibrium, as it will often be in this model, then there is no tradeoff between insurance and investment.

[3]While these claims depend on the utility function and on the roles of risky and riskless investment in job creation, we will see that they are true in our model.

In the next subsection, we provide a brief review of related literature. We state the baseline model in Section 2. The numerical examples of multiple equilibria based on risk, which constitute the main results of the paper, are described in Section 3. Next, we consider how the level of investment and uniqueness of equilibrium are affected by different insurance environments. Section 5 considers the effects of risk in an overlapping generations version of the model. Section 6 concludes.

1.1 Related literature

There is abundant evidence of precautionary behavior in consumption and asset choice. To begin with, idiosyncratic consumption fluctuation is much greater than aggregate fluctuation, suggesting imperfect insurance. Under intuitively reasonable forms of risk aversion such as Kimball's (1993) *standard risk aversion*[4], this incompletely insured risk implies a marginal propensity to consume higher than that of the permanent income model, and also implies that individuals with higher levels of asset holdings will be willing to hold riskier portfolios. Both of these predictions appear to be supported by the data. The fact that firms finance a large part of their investment out of cash flow suggests that firms may exhibit precautionary behavior also.

It is not our intention here to explain these microeconomic observations, nor to document further the evidence for precautionary behavior, which is summarized more completely by Aiyagari (1994), Carroll (1992,1997), and Deaton (1992) and, for the case of firms, by Bernanke, Gertler, and Gilchrist (1996). Instead, this paper will explore some implications of the microeconomics of precautionary behavior in a macroeconomic context. For this purpose, we should discuss three additional areas of literature.

Since the seminal contribution of Diamond (1982), many papers on business cycles, unemployment, and growth have proposed that inefficiently low levels of economic activity might occur if the macroeconomy has more than one equilibrium. A recent example, structurally similar to standard neoclassical

[4]See footnote 6 for a definition.

economies, is Farmer and Guo (1994). Cooper and John (1988) showed that such arguments all shared a feature they called *strategic complementarities*—that is, positive feedbacks among agents' choice variables, as in Diamond's paper, where an increase in the average level of production gives an incentive for each individual agent to raise production as well. The present paper augments this literature by proposing a new and different source of strategic complementarities. Most previous papers, like Farmer and Guo, have founded their positive feedbacks on aggregate increasing returns to scale or on market power, that is, on changes in the *level* of the return to productive activity when more agents are active. Here, instead, we focus on changes in the *riskiness* of the return to productive activity. The only similar argument was made by Chatterjee (1988), who pointed out that economic activity may be less risky when more agents are active, by the law of large numbers. The law of large numbers plays no role here.

The second area of relevant literature is that on precautionary behavior under imperfect insurance. Early work on risk aversion included Pratt (1964) and Arrow (1965). Precautionary saving was studied by Leland (1968), Sandmo (1970), Rothschild and Stiglitz (1970,1971), and by Kimball (1990), who showed the mathematical link between measures of risk aversion and of the precautionary saving incentive. Pratt and Zeckhauser (1987), Kimball (1993), and Gollier and Pratt (1996) consider the effects of risk on portfolio choice. Barsky, Mankiw, and Zeldes (1986) point out that proportional or progressive income taxes act as a form of insurance on labor income, so that a postponement of labor income taxes raises consumption and lowers saving by counteracting the precautionary motive. Elmendorf and Kimball (1991) revisit this issue in the context of multiple assets, and argue that the effect may be reversed for *risky* assets: in their model, any postponement of labor income taxes which raises consumption must also *raise* risky asset investment, because greater insurance of labor income risk makes individuals more willing to confront asset risk. The individual choice problem in this paper is closely related to Elmendorf and Kimball's partial equilibrium model. The present paper shows how strategic complementarities arise very naturally when Elmendorf and Kimball's model is extended to a general equilibrium environment.

Finally, this paper is related in motivation to the growing literature on the macroeconomic implications of imperfect corporate financial markets. The "credit channel" of monetary transmission is discussed by Bernanke and Gertler (1989), Kiyotaki and Moore (1997), Bernanke, Gertler, and Gilchrist (1998), and den Haan, Ramey, and Watson (1999). Other papers emphasize empirical differences in the investment behavior of different classes of firms, which may be evidence of financial market imperfections; see Fazzari, Hubbard, and Peterson (1988), Gertler and Gilchrist (1994), Cooley and Quadrini (1998), and Pratap and Rendon (1998). One of the most closely related papers is Greenwald and Stiglitz (1993), which argues that if firms are risk averse, then shocks to their balance sheets will have effects on output, which then further influences balance sheets. This paper shows the other side of the coin: if investors are risk averse, then changes in the riskiness of the economy will have effects on output, which then further influences the riskiness of the economy. Some papers in growth theory have also stressed the role of financial factors, notably Acemoglu and Zilibotti (1997), who also study positive feedbacks based on risky investment.

2 The model

An agent in this two-period job matching economy plays two roles: she is both a worker and an entrepreneur. As a worker, she supplies labor inelastically, but can do so only if she finds a job. As an entrepreneur, she chooses between two types of assets: a riskless one which can be thought of as a storage technology, and a risky one which can be interpreted as small business formation. Two types of risk thus arise naturally for all agents: risk of unemployment, and risky returns to investment. The feedbacks in our model result from the fact that changes in unemployment risk affect agents' willingness to invest in the risky asset, together with the fact that risky investment is an input to the matching function.

2.1 The individual's problem

Each agent chooses initial consumption c_1, riskless investment b, and risky investment h, which will together imply a random level of second-period consumption c_2^*. The utility function is

$$\gamma u(c_1) \; + \; Ev(c_2^*)$$

The constant γ governs the relative importance of time 1 and time 2 utility; we will focus on the cases $\gamma = 0$ and $\gamma = 1$. The functions u and v are assumed to have the same constant relative risk aversion (CRRA) form $u(x) = v(x) = \frac{1}{1-\alpha}x^{1-\alpha}$. As is well known, this utility function has several strong properties: it satisfies an Inada condition at zero, its derivatives alternate in sign, with derivatives of odd order positive and of even order negative, and the ratios $-v^{(n)}/v^{(n-1)}$ of its derivatives are decreasing and convex functions.[5] Many authors have argued recently that properties like these are needed for intuitively reasonable consumption and asset choice behavior, such as a greater willingness to tolerate risk by the wealthy, and have tried to determine precisely which properties of CRRA are essential. Most of the results in this paper follow from the postulates of Kimball's (1993) concept of standard risk aversion.[6,7]

Agents begin the initial period with a stock of goods W. What remains of this wealth after investment constitutes time 1 consumption:

$$c_1 \;\; = \;\; W - b - h \tag{1}$$

Time 2 consumption is financed out of labor income and investment income. Both of these forms of income may be risky, and both of these risks may be insured.

[5] The quantity $-v''(x)/v'(x)$ is called absolute risk aversion, absolute prudence is $-v'''(x)/v''(x)$, and absolute temperance is $-v''''(x)/v'''(x)$. For CRRA utility, these are all decreasing and convex functions of x.

[6] Standard risk aversion is equivalent to increasing and concave utility together with decreasing absolute risk aversion and decreasing absolute prudence.

[7] To clarify the assumptions underlying our results, we will not argue directly from the CRRA functional form, but instead will prove our propositions on the basis of the properties of its derivatives.

Labor income, prior to insurance transfers, is given by the random variable $\tilde{w}$, which varies according to employment status. The individual is employed at time 2 with endogenous probability n, and unemployed with probability $1-n$. When she is employed, the realization of $\tilde{w}$ is the wage $\overline{w}$, and when she is unemployed, $\tilde{w}$ is the value of home production, $\underline{w}$, where $0 \leq \underline{w} < \overline{w}$. Her mean labor income will be written as $\mu_w(n) \equiv \overline{w}n + \underline{w}(1-n)$. To compare the effects of different insurance environments, it is convenient to introduce the variable $\phi \in [0,1]$ to represent the degree of unemployment insurance, where $\phi = 1$ represents full insurance and $\phi = 0$ is no insurance. Insurance is financed by fair transfers between the employed and the unemployed at time two. Thus post-transfer labor income is given by the random variable $w^* \equiv (1-\phi)\tilde{w} + \phi\mu_w(n)$ which takes the values $\overline{w}^* \equiv (1-\phi)\overline{w} + \phi\mu_w(n)$ with probability n and $\underline{w}^* \equiv (1-\phi)\underline{w} + \phi\mu_w(n)$ with probability $1-n$, so that a decrease in ϕ is a mean-preserving spread of the labor income distribution.[8] Note that ϕ is to be regarded as an exogenous parameter characterizing the types of information problems limiting insurance in the economy; we will not treat insurance coverage as a choice variable.

The second period payoff of riskless investment b is the non-random quantity Rb where $R \geq 0$. The variable h can be seen as hiring of workers for the agent's small business, or equivalently, as purchases of machines necessary for one's employees to use. The endogenous, deterministic variable q represents the number of workers hired per unit of hiring expenditure. Entrepreneurial income per worker is a random variable $\tilde{\pi}$, taking the value $\overline{\pi}$ with exogenous probability p and $\underline{\pi}$ with probability $1-p$; its mean is denoted as $\mu_\pi \equiv \overline{\pi}p + \underline{\pi}(1-p)$. The random entrepreneurial income received by a given individual is assumed independent of the random labor income that individual receives. As with unemployment risk, we allow an exogenous degree ψ of insurance (or diversification) of risky investment. Post-transfer risky income per worker hired is the weighted average $\pi^* \equiv (1-\psi)\tilde{\pi} + \psi\mu_\pi$ of the underlying

[8] All random variables are indicated by asterisks or tildes. An asterisk is used to represent any variable which may include an insurance component.

random variable $\tilde{\pi}$ and its mean μ_π. Total income received at time two from risky investment, after transfers, can be written as $qh\pi^*$.

If we call the consumption that is achieved after insurance transfers c_2^*, then the worker's problem can be stated as

$$\max_{b,\,h} U \equiv \gamma u(c_1) + Ev(c_2^*)$$

$$\text{subject to:} \quad c_1 + b + h = W$$

$$\text{and} \quad c_2^* = Rb + qh\pi^* + w^*$$

The expectation is taken with respect to the random variables π^* and w^*.

2.2 General equilibrium.

The employment rate n and the hiring rate q at time two are linked by a matching function. Without loss of generality, we can normalize the population to one. Since this is only a two-period model, we can assume that the entire population is involved in job search in period one; the number employed in the second period will equal the number of matches formed. If we denote the aggregate level of hiring by H, then since the number of workers hired by firms must equal the number of jobs found by workers,[9] we require

$$qH = N(1, H) \equiv N(H) = n \tag{2}$$

Note that we can suppress the trivial dependence of the matching function $N(1, H)$ on the fixed unit mass of workers available for time 2 employment. We assume that $N(H)$ is between 0 and 1 for all $H \geq 0$, and that $N'(H) \geq 0$ and $N''(H) \leq 0$, which is to say that there are (weakly) decreasing returns to scale in the matching function. Equilibrium also implicitly defines a hiring

[9] We assume that the entrepreneur's hiring process is deterministic for simplicity only. Implicitly, this means workers who "find jobs" are shared across firms. It would undoubtedly seem more natural to assume that the number of workers actually hired by any given entrepreneur were an exponential random variable with mean qh. However, this extra element of risk in the entrepreneur's problem would not fundamentally alter the precautionary saving and portfolio choice incentives which underlie the feedbacks studied here.

rate function resulting from the level of aggregate hiring, $Q(H) \equiv N(H)/H$; the assumptions on N imply that $Q'(H) \leq 0$.

The firm operated by an entrepreneur experiences an idiosyncratic technology shock $\tilde{y}$ which is uncorrelated across entrepreneurs, and which is unknown at the time of hiring. The shock $\tilde{y}$ represents the marginal product of labor at the entrepreneur's firm, and takes the values $\overline{y}$ with exogenous probability p and $\underline{y}$ with probability $1-p$. We abstract from optimal contracting issues and make the assumption that workers are paid a fraction β of their expected marginal product; that is, we assume a non-random wage $\overline{w} \equiv \beta\mu_y$, where $\mu_y \equiv \overline{y}p + \underline{y}(1-p)$.[10] The random entrepreneurial income per employee that is received by a given employer (prior to any insurance transfers) is thus $\tilde{\pi} \equiv \tilde{y} - \overline{w}$.

Finally, we define equilibrium. Note that conditional on aggregate H, the individual's problem consists of the maximization of a concave function over a bounded, convex set, implying the existence of a unique optimal choice of c_1, b, and h. Thus we only need to consider symmetric equilibria: there cannot be equilibria in which different individuals choose different levels of h. Since the population is 1, aggregate consistency requires that $h = H$. Hence an equilibrium is a level of aggregate hiring H such that an individual chooses $h = H$ when faced with the probability of employment $n = N(H)$ and the hiring rate $q = Q(H)$.

3 Computing equilibria

We begin this section by showing how to compute equilibria of our economy, which also allows us to prove existence of a stable equilibrium. Then, in subsections 3.2 and 3.3, we compute several numerical examples in which the economy exhibits multiple equilibria.

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[10] Results would be similar if we assumed that the value of home production $\underline{w}$ also affected the wage.

3.1 Existence, uniqueness, and stability

Consider first the case $\gamma = 0$, so that consumption gives no utility in the first period. This is a pure model of portfolio choice, since all initial wealth will be saved for the second period, making the precautionary saving decision trivial. After eliminating b, the necessary conditions for the $\gamma = 0$ case are:[11]

$$r(h,H) \equiv \frac{\partial}{\partial h} Ev(c_2^*) = E[Q(H)\pi^* v'(c_2^*)] \leq s(h,H) \text{ and } h \geq 0 \tag{3}$$

with at least one equality,

$$s(h,H) \equiv \frac{\partial}{\partial b} Ev(c_2^*) = RE[v'(c_2^*)] \leq r(h,H) \text{ and } h \leq W \tag{4}$$

with at least one equality, and

$$h = H \tag{5}$$

Here $r(h,H)$ is the expected marginal utility from a unit of investment in the risky asset, as a function of individual investment h and aggregate investment H, and $s(h,H)$ is the expected marginal utility from investment in the safe asset. If there is any expenditure on the safe asset at a given H, then we must have $r(h,H) \leq s(h,H)$; the inequality may be strict only if the optimal choice is the corner $h = 0$. If there is any expenditure on the risky asset at a given H, we must have $s(h,H) \leq r(h,H)$, with equality if $h < W$.

To compute an equilibrium, it helps to define the functions $\mathcal{R}(H) \equiv r(H,H)$ and $\mathcal{S}(H) \equiv s(H,H)$ which are the expected marginal utilies of investment under *symmetric behavior*, that is, when both individual and aggregate investment in the risky asset are assumed to be H. Graphing these functions, we can characterize equilibria as follows:

Lemma 1. Interior equilibria satisfy $\mathcal{S}(H) = \mathcal{R}(H)$ for $H \in (0,W)$; $H = 0$ is an equilibrium if $\mathcal{S}(0) \geq \mathcal{R}(0)$, and $H = W$ is an equilibrium if $\mathcal{S}(W) \leq \mathcal{R}(W)$.

[11] The functions r and s depend on h and H through the probability of employment $N(H)$ and through consumption $c_2^* = R(W-h) + Q(H)\pi^* h + w^*$.

Figures 3 and 11 show numerical examples of the functions $\mathcal{R}$ and $\mathcal{S}$ in which we observe three interior equilibria. The remarkable nonmonotonicity of these curves reflects the fact that a symmetric increase in H implies higher but riskier investment income, lower unemployment risk, and lower returns to risky investment.

Note that as long as Q and v' are continuous, $\mathcal{R}$ and $\mathcal{S}$ are continuous functions. Q is continuous everywhere under our assumptions, but the Inada condition implies that marginal utility v' goes to infinity whenever c_2^* goes to zero (which would occur for some h if $\underline{\pi}^* < 0$). At any H where c_2^* approaches zero under symmetric behavior, $\mathcal{S}(H)$ goes to $+\infty$ and $\mathcal{R}(H)$ goes to $-\infty$. This is what happens at the right hand side of Figure 11, where as H increases symmetrically, c_2^* eventually becomes negative with positive probability and $\mathcal{R}$ and $\mathcal{S}$ become infinite. The fact that the curves must become infinite with opposite sign wherever they are discontinuous implies the following conclusion.

Lemma 2. There exist an odd number of equilibria.

Proof. If $\mathcal{R}$ and $\mathcal{S}$ never cross, then there is a unique equilibrium at a corner, satisfying either $\mathcal{R}(0) \leq \mathcal{S}(0)$ or $\mathcal{R}(W) \geq \mathcal{S}(W)$. If $\mathcal{R}$ crosses $\mathcal{S}$ once from above, then there is a unique interior equilibrium, while if $\mathcal{R}$ crosses $\mathcal{S}$ once from below, then there are equilibria at both corners plus an interior equilibrium. Multiple crossings can only add an even number of equilibria to those already described.

These conclusions are not altered by the discontinuities in $\mathcal{R}$ and $\mathcal{S}$ where consumption is zero with positive probability. We have seen that $\mathcal{S} > \mathcal{R}$ near any discontinuity, which implies an odd number of equilibria to the left of any discontinuity and an even number of equilibria to its right. **QED.**

For some purposes it will be helpful to consider the best response function $h(H)$, which describes the utility maximizing choice of individual hiring h for any level of aggregate hiring H. In terms of the best response function, an equilibrium is any H satisfying $H = h(H)$. It is natural to call a given equilibrium $\hat{H}$ stable if for all sufficiently small $\epsilon > 0$ we have $h(\hat{H}+\epsilon) < \hat{H}+\epsilon$, and the reverse for $\epsilon < 0$. Since $h(H) > H$ if and only if $\mathcal{R}(H) > \mathcal{S}(H)$, stability corresponds to equilibria where $\mathcal{R}$ crosses $\mathcal{S}$ from above, or to corner equilibria. Hence the arguments used in the proof of Lemma 2 imply:

Proposition 1. There exists a stable equilibrium. The equilibria at highest and lowest H are stable, and any other equilibria alternate in stability.

So far we have confined our discussion to the one period case, that is, $\gamma = 0$. However, the arguments already made can be easily extended to the two period case $\gamma > 0$, in which there is a nontrivial precautionary saving decision as well as the portfolio choice decision. When $\gamma > 0$, our assumption of an Inada condition on u implies that there must be an interior choice of c_1; only the portfolio decision between h and b may involve corners.

For the two period version of the model, we can locate equilibria by a graphical method similar to that we used previously if we proceed as follows. For each H, consider the optimal choice of consumption and riskless investment *conditional* on symmetric risky investment $h = H$. Individuals will then solve:

$$\underset{b}{max} \; \gamma u(c_1) + Ev(c_2^*) \quad \text{s.t.: } c_1 = W - H - b \text{ and } c_2^* = Rb + \pi^* Q(H)H + w^*$$

Let the solution of this problem be $b(H)$, which is necessarily less than $W - H$ but may be zero. Then define $\mathcal{S}(H) \equiv u'(W - H - b(H))$, which is the marginal value of initial wealth conditional on $h = H$. Similarly, define the expected marginal utility of risky asset holding as $\mathcal{R}(H) \equiv Q(H)E\pi^* v'(Rb(H) + Q(H)H\pi^* + w^*)$. Under this definition, an equilibrium is still of the form $\mathcal{R}(H) = \mathcal{S}(H)$, or is a corner satisfying $\mathcal{R}(0) \leq \mathcal{S}(0)$. These functions are continuous under the same conditions we specified above, so we can conclude:

Remark. The conclusions of Lemmas 1 and 2 and Proposition 1 remain true when $\gamma > 0$. However, it will never be the case, for $\gamma > 0$, that $\mathcal{R}(W) \geq \mathcal{S}(W)$.

3.2 Example 1: Portfolio choice only, and kinked matching technology

We now construct an example of multiple equilibria under pure portfolio choice, that is, for the case $\gamma = 0$. A particularly easy and transparent way to obtain multiple equilibria is to use a Leontief matching technology of the form $N(1,H) \equiv \min(H/k,1)$, or equivalently $Q(H) = 1/k$ for $H \leq k$ and $Q(H) = 1/H$ for $H > k$.[12] Under this "kinked" matching function, the individual probability of unemployment goes to zero if there is sufficient aggregate hiring. Coupled with the Inada condition implied by CRRA utility, this matching technology can make individual investment behavior very sensitive to aggregate investment. If we consider a level of individual hiring h which yields consumption below or even near zero for the unemployed, and a level of aggregate hiring H close to the kink, then a small change in aggregate hiring may imply a substantial change in risk, so that individual hiring may respond strongly.

Multiple equilibria result under the parameter values in Table 1. The matching technology is shown in Figure 1; we see that the probability of unemployment hits zero at $H = 1$. Figure 2 shows consumption under symmetric behavior in the four states an individual may face (unemployment or employment, combined with bad or good idiosyncratic productivity shocks). That is, consumption is calculated for both possible values of π^* and of w^*, imposing $h = H$:

$$c_2^* \; = \; R(W-H) \; + \; Q(H)H\pi^* \; + \; w^*$$

The starred curve shows expected consumption. Note that although consumption falls severely in the worst state, it is never negative for these parameters.

Figures 3 and 4 show two ways of looking at the multiple equilibria arising in this example. Figure 3 graphs the expected marginal utilities of safe investment, $\mathcal{S}(H)$, and of risky investment, $\mathcal{R}(H)$. At low levels of H, $\mathcal{R}(H)$ is high because $Q(H)$ is high (it is equal to 1) and because H is small enough that consumption remains quite high even under a bad idiosyncratic technology shock. Gradually $\mathcal{R}(H)$ decreases towards $\mathcal{S}(H)$, as higher levels of H cause consumption in the worst state, c_{UL},[13] to fall ever lower, leading to a stable

[12] We can interpret this as a technology in which exactly one machine must be purchased for each worker, at cost k; there are no matching frictions, but if the number of machines is not equal to the number of workers, then the scarcer resource is rationed.

[13] See the introduction to Section 4 for notational conventions.

Table 1: Parameters for Example 1

Worker characteristics	
Weight on utility when young (γ)	0
Coefficient of relative risk aversion (α)	2
Initial wealth (W)	1.3
Firm's technology	
Cost per machine (k)	1
MPL under good shock ($\overline{y}$)	4
MPL under bad shock ($\underline{y}$)	2
Probability of good shock (p)	0.8
Wage of the unemployed ($\underline{w}$)	0
Return to riskless asset (R)	1.0
Worker's bargaining share (β)	0.5

equilibrium at $h = H = 0.7067$. However, as the probability of employment approaches 1, individuals again become willing to invest in the risky asset, for the probability of consuming c_{UL} approaches zero, and thus there is an unstable equilibrium at $h = H = 0.9358$. Finally, above $H = 1$, the marginal value of the risky asset decreases steadily relative to the safe asset, both because $Q(H)$ is falling and because c_{EL} is falling. There is a third stable equilibrium at $h = H = 1.2195$. Figure 4 shows the best response function $h(H)$, in which the same three equilibria are visible as multiple crossings of the 45^o line.

Table 2 documents the values of the endogenous variables at the three equilibria. As mentioned above, the probability of unemployment goes to zero at the highest equilibrium, and the rate of hiring $Q(H)$ falls below one. Equilibrium consumption is reported for all four possible idiosyncratic outcomes. The worst consumption outcome, c_{UL}, falls from 0.7347 at the lowest equilibrium to 0.2805 at the highest equilibrium, but at the highest equilibrium this outcome occurs with probability zero. Expected consumption is also shown; it is highest at the unstable equilibrium because matching externalities decrease expected consumption at the highest stable equilibrium. Nonetheless, expected utility is highest at the upper equilibrium, since the risk of the worst idiosyncratic state is eliminated. The risky consumption distibution at the lowest equilibrium

Table 2: Example 1: Equilibria

	Lower stable eqm	*Unstable eqm*	*Higher stable eqm*
Hiring (H)	0.7067	0.9358	1.2195
Probability of employment (n)	0.7067	0.9358	1.0000
Rate of hiring (q)	1.0000	1.0000	0.8200
c_{UL}	0.7347	0.5514	0.2805
c_{EL}	2.5347	2.3514	2.0805
c_{UH}	2.1480	2.4229	2.2805
c_{EH}	3.9480	4.2229	4.0805
Expected consumption	3.1374	3.7330	3.6805
Return to safe asset	1.0000	1.0000	1.0000
Expected return to risky asset	1.8000	1.8000	1.4760
$\mathcal{S}(H)$	0.2178	0.1268	0.0943
$\mathcal{R}(H)$	0.2178	0.1268	0.0943
Expected utility	-0.3881	-0.3013	-0.2922
Certainty equivalent	2.5770	3.3182	3.4225

is equivalent in expected utility to the consumption of 2.5770 units with certainty; at the highest equilibrium, the risky consumption distribution is worth 3.4225 units with certainty. One should note, however, that the rankings of utility and expected consumption across equilibria need not have this order. It is well known that search models may have equilibria with too much employment, as well as too little employment; both cases can occur in this model for some parameter values.

In Figures 5-8, we show how equilibrium hiring varies with changes in some exogenous parameters. Figure 5 shows the equilibrium correspondence resulting from variation in initial wealth W. At low levels of wealth, only an equilibrium with low hiring is observed; agents are born too poor to support an equilibrium with high employment. However, as soon as W passes one, a new pair of equilibria appears at higher H; in fact, for approximately $W \in (1, 1.2)$, the upper equilibrium is at a corner, in which all wealth is invested in the risky asset. Note that the middle equilibrium is necessarily unstable. For $W > 1.2$, some wealth is invested in the riskless asset too, as $Q(H)$ is driven down by aggregate hiring. Finally, the lower two equilibria disappear around $W = 1.4$.

One interesting interpretation of this graph is that it shows a large effect of internal cash holdings on investment, as in the credit channel literature. For $W \approx 1$, a small increase in wealth available could lead to a very large jump in investment, as new, higher equilibria arise; similarly, around $W \approx 1.4$, the low equilibria disappear. Figure 6 shows the response of equilibrium hiring to p, the probability of a good idiosyncratic technology shock, which has a similar shape.

Figures 7 and 8 show the equilibrium correspondences for $\underline{y}$ and $\overline{y}$, respectively. Note that increases in $\underline{y}$ and $\overline{y}$ have very different implications for equilibrium hiring— almost opposite implications, in fact. This illustrates an important fact about this economy's response to shocks: both the effects on the expected return to investment and on its risk are relevant. Raising $\underline{y}$ brings the two possible values of $\tilde{y}$ closer together, raising the mean return of the risky investment while lowering its risk. Since both of these effects encourage hiring, every stable equilibrium rises when $\underline{y}$ increases; moreover, equilibria with low H may vanish or higher equilibria may appear. Raising $\overline{y}$, on the other hand, raises the expected return to hiring but also *increases* its risk. These conflicting effects mean that equilibrium hiring reacts nonmonotonically to $\overline{y}$; moreover, for sufficiently large $\overline{y}$, the effect of increased risk dominates, causing stable equilibrium levels of H to fall with $\overline{y}$.

Considering how changes in $\overline{y}$ and $\underline{y}$ affect equilibrium, it is also interesting to study the effects of proportional changes to both values of $\tilde{y}$, which might be more analogous to the usual productivity shocks of real business cycle theory. In the current example, proportional increases in the distribution of $\tilde{y}$ have an effect much like that seen in Figure 7 for shocks to $\underline{y}$, leading to increases in equilibrium hiring, but this is not always the case. In this model, profits $\tilde{y} - \overline{w}$ per unit of hiring may be either positive or negative under the bad shock $\tilde{y} = \underline{y}$. In the current example they are positive, which means that an increase in h raises consumption in all states, holding b fixed. However, under parameterizations such that $\underline{y} - \overline{w} < 0$, an increase in h lowers consumption in the worst state (fixing b again), which means that very high levels of hiring are extremely risky. Under such a parameterization, then, the equilibrium correspondence arising from proportional shifts to the distribution of $\tilde{y}$ has a shape more like that shown in Figure 8 for shocks to $\overline{y}$, nonmonotonic and eventually sloping down.

The example we have just studied spells out the main point of the paper: portfolio choice under risk may lead to positive feedbacks and hence to economic instability. We can summarize the findings as follows:

> **Proposition 2.** If both unemployment risk and investment risk are less than fully insured ($\phi < 1$ and $\psi < 1$), then there may be multiple equilibria even when the matching function has decreasing returns to scale.

In the sections which follow, we will show that neither the kinked matching technology nor the assumption that consumption occurs only in the second period is crucial for our example. Imperfect insurance, however, is necessary for multiplicity.

3.3 Extensions.

Two potentially objectionable assumptions were made in the preceding multiple equilibrium construction. First, we assumed a kinked matching technology implying zero unemployment at a finite level of aggregate hiring H. Second, we considered a model of pure portfolio choice between b and h, ignoring the choice of first period consumption c_1. We will now relax both assumptions.

Example 2 uses the smooth matching function

$$N(H) \equiv (1 - \exp(-H/k))$$

The total number of matches is zero when H is zero, and converges to 1 (the total population) as H approaches infinity. We will use the parameters $W = 1.7$, $\overline{y} = 6.6$, $k = 0.1$, and otherwise as in Table 1. The matching function is shown in Figure 9. Consumption, under symmetric behavior $h = H$, is shown in Figure 10, for all four idiosyncratic states and in expected value.

Figure 11 shows the functions $\mathcal{S}(H)$ and $\mathcal{R}(H)$, and Figure 12 illustrates the best response function. As in our previous example, $\mathcal{R}(H)$ initially declines,

Table 3: Example 2: Equilibria

	Lower stable eqm	*Unstable eqm*	*Higher stable eqm*
Hiring (H)	0.1698	0.3393	0.6621
Probability of employment (n)	0.8170	0.9664	0.9987
Rate of hiring (q)	4.8105	2.8481	1.5084
c_{UL}	0.8439	0.5489	0.1991
c_{EL}	3.6839	3.3889	3.0390
c_{UH}	4.6022	4.9943	4.7929
c_{EH}	7.4422	7.8343	7.6329
Expected consumption	6.1709	6.8498	6.7103
Return to safe asset	1.0000	1.0000	1.0000
Expected return to risky asset	13.6618	8.0885	4.2837
$\mathcal{S}(H)$	0.0821	0.0528	0.0421
$\mathcal{R}(H)$	0.0821	0.0528	0.0421
Expected utility	-0.2073	-0.1733	-0.1719
Certainty equivalent	4.8226	5.7690	5.8155

as symmetric increases in h and H cause entrepreneurs experiencing both unemployment and a bad idiosyncratic technology shock to suffer ever lower consumption c_{UL}. However, increases in H also mean a lower probability of unemployment, so that $\mathcal{R}(H)$ can be nonmonotonic. Here, unlike the previous example with a kinked matching function, the probability of unemployment in always positive, so that $\mathcal{R}(H)$ and $\mathcal{S}(H)$ diverge as c_{UL} becomes negative. The three resulting equilibria are analyzed numerically in Table 3.

The equilibrium correspondences resulting from changes in underlying parameters are shown in Figures 13-16. Except for the effects of p, they are qualitatively similar in shape to those arising from the previous example. However, without the kink in the matching technology, multiplicity occurs over a substantially smaller parameter range.

A second objection to our previous examples is that they have assumed only one period of consumption utility. If we allow for $\gamma > 0$, so that there is a desire for consumption in both periods, then there is a precautionary saving decision as well as a portfolio choice: first an agent must decide his total level of saving for the second period, and then he must choose how to allocate this saving across different assets. While imperfect insurance *discourages* allocation of saving to risky assets, imperfect insurance *encourages* precautionary saving, so it is by no means obvious ex ante what the overall effect on risky asset holding of a rise in unemployment risk should be.[14] Hence it is important to compute a two period model as well, to see whether the interaction of precautionary saving and portfolio choice still allows positive feedbacks strong enough for multiple equilibria.

Example 3, illustrated in Figure 17 and Table 4, is a two-period model with a kinked matching technology. Parameters are the same as in Example 1, except that initial wealth is set at $W = 4.05$.[15] Note that in this example there is less second period consumption at the upper equilibrium than at the lower equilibrium. The risk associated with the lowest equilibrium causes substantial precautionary saving (in the riskless asset) and hence high expected consumption at time 2. Nonetheless, lifetime utility rises from a level equivalent to a constant riskless stream of 2.9183 units at the lower equilibrium to a level equivalent to 3.1482 units at the highest equilibrium. In fact, even time 2 consumption is worth more (3.3333 units) in terms of riskless consumption at the upper equilibrium than at the lower equilibrium (3.1914 units), in spite of the fact that time 2 consumption is lower in expected value. We conclude that multiplicity may still arise even with the offsetting influence of precautionary saving in the initial period.

[14] Elmendorf and Kimball (1991) demonstrate for their model that any increase in risk sufficient to decrease consumption must also decrease risky asset holding, an insight which will carry over to this paper.

[15] Notice that the unstable equilibrium in this example has h and b fairly close to their levels in Example 1. Initial wealth of $W = 4.05$ was chosen deliberately to ensure the existence of a solution to the first-order conditions yielding second period behavior similar to that in Example 1. Unfortunately, this method of constructing a multiple equilibrium example with two periods of consumption from an equilibrium with one period of consumption does not always work, since there is no guarantee that the second-order conditions will be satisfied nor that the stability properties of the equilibrium will be the same.

Table 4: Example 3: Equilibria

	Lower stable eqm	*Unstable eqm*	*Higher stable eqm*
Hiring (H)	0.8837	0.9688	1.0675
Riskless investment (b)	0.4801	0.2304	0.0000
Probability of employment (n)	0.8837	0.9688	1.0000
Rate of hiring (q)	1.0000	1.0000	0.9368
Initial consumption (c_1)	2.6861	2.8508	2.9825
c_{UL}	0.6569	0.4242	0.2000
c_{EL}	2.4569	2.2242	2.0000
c_{UH}	2.4244	2.3618	2.2000
c_{EH}	4.2244	4.1618	4.0000
Expected consumption (per. 2)	3.6617	3.7182	3.6000
Return to safe asset	1.0000	1.0000	1.0000
Expected return to risky asset	1.8000	1.8000	1.6862
$\mathcal{S}(H)$	0.1386	0.1230	0.1124
$\mathcal{R}(H)$	0.1386	0.1230	0.1124
Expected utility (per. 2)	-0.3131	-0.2986	-0.3000
Certainty equivalent (per. 2)	3.1944	3.3488	3.3333
Lifetime utility	-0.6853	-0.6493	-0.6353
Equivalent constant consumption	2.9183	3.0798	3.1482

4 Implications of insurance for uniqueness and investment

We have ruled out the kinked matching technology and zero first period consumption as explanations for our multiple equilibrium examples. However, we have not yet demonstrated that imperfect insurance is critical for our results. We would also like to see how the effects of insurance differ depending on whether the precautionary saving or portfolio choice motive is dominant. Hence we now compare a pure precautionary saving economy and a pure portfolio choice economy to demonstrate that insurance does indeed play a critical role in our examples. We will also show that the policy implications of the portfolio choice and precautionary saving decisions are very different with respect to insurance.

To avoid excessive notation, we will write the value function as U, with subscripts to indicate its derivatives. We will use E and U in subscripts or conditional expectations to denote the employed and unemployed idiosyncratic states, and H and L to denote the high and low idiosyncratic technology shocks ($\overline{y}$ and $\underline{y}$), so for instance c_{UL} denotes the worst possible consumption outcome. Single subscripts will occasionally denote conditional expectations. The excess return on the risky asset will be written as $x^* \equiv \pi^* Q - R$. Hence some equivalent ways of writing the first order condition on hiring when $\gamma = 0$ are:

$$U_h \;\equiv\; Ex^*v' \;\equiv\; (1-p)E(x^*v'|H) + pE(x^*v'|L) \;\equiv\; (1-p)\overline{x}^*v'_H + p\underline{x}^*v'_L \;=\; 0 \tag{6}$$

4.1 Fully insured investment: implications of precautionary saving

We found in Section 3 that this model economy may exhibit multiple equilibria. Imperfect insurance is necessary for multiplicity, as we now show.

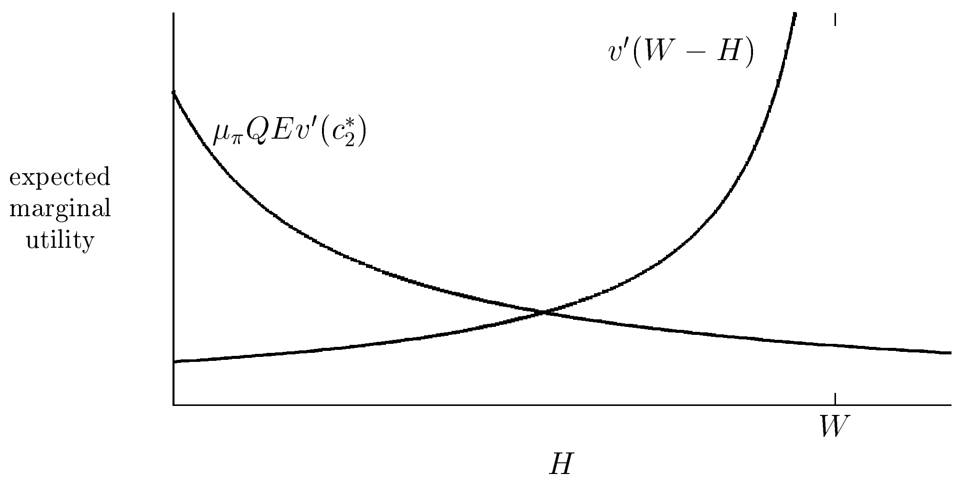
Proposition 3. If investment risk is fully insured ($\psi = 1$), then the economy has a unique equilibrium.

Proof. When $\psi = 1$, profits $\pi^* = \mu_\pi$ are nonstochastic, so they factor out of the expectation. If equilibrium involves an interior choice of h and b, a necessary condition will be $R = \mu_\pi Q(H)$, from which we calculate the unique equilibrium level of hiring $H = Q^{-1}(R/\mu_\pi)$.

If there is a corner equilibrium $b = 0$, then equilibrium requires

$$u'(W - H) \;=\; \mu_\pi Q(H) E v'(c_2^*) \;=\; \mu_\pi Q(H) E v'(\mu_\pi Q(H) H + w^*) \qquad (7)$$

where the probability of employment is $N(H)$. A diagram is helpful:



Since $u'' < 0$, the left hand side of (7) is strictly increasing in H, and it approaches infinity at $H = W$. The right hand side is decreasing for three reasons. First, $Q(H)$ is decreasing. Second, $Q(H)H = N(H)$ is increasing in H, so c_2^* rises under a symmetric increase in H, decreasing v'. Finally, the employed state, which has lower marginal utility, becomes more probable as H increases. Since the two curves have slopes of opposite sign, there is either exactly one crossing, or a corner equilibrium $H = 0$. **QED.**

The preceding diagram helps us relate this paper to previous literature, since it shows the role of decreasing returns in our uniqueness proof. If instead there were increasing returns to aggregate investment, that is, $Q'(H) > 0$, then both curves in the diagram might slope up, allowing multiple crossings. Thus,

Remark. There may be multiple equilibria even when investment risk is fully insured, if the matching function has increasing returns to scale.

We will not construct an example here, since this source of multiplicity is familiar, and is not the focus of this paper; see for example Diamond (1982).

Notice that our uniqueness result requires only full insurance of investment ($\psi = 1$); employment may be less than fully insured ($\phi < 1$). In fact, when $\phi < 1$, the factors which lead to uniqueness are strengthened, because the precautionary saving incentive implies a negative feedback arising from risk.

Consider the first order condition for hiring when investment is fully insured:[16]

$$u'(W-h) \;=\; \mu_\pi Q(H) E v'(c_2^*) \;=\; \mu_\pi Q(H) E v'(\mu_\pi Q(H) h + w^*) \qquad (8)$$

If an individual's probability of employment n increases, holding fixed the other endogenous variables that influence the individual's decision, then that individual will spend (weakly) less on hiring. The change in individual hiring as a function of the probability of employment can be written as

$$\frac{dh}{dn} \;=\; \frac{\mu_\pi Q(v'_U - v'_E)}{u'' + \mu_\pi^2 Q^2 E v''} \;<\; 0 \qquad (9)$$

where $v'_E = v'(\mu_\pi Q h + \overline{w}^*)$ and $v'_U = v'(\mu_\pi Q h + \underline{w}^*)$. To put it another way, if an individual faces a higher probability of unemployment, he will hire more, which then helps to lower unemployment.

The negative feedback term also appears when we consider how individual hiring responds to changes in aggregate hiring.

Remark. When investment is fully insured, imperfect unemployment insurance adds a negative term to the slope of the best response function.

[16]We will restrict the discussion in the text to the case where $b = 0$ in equilibrium. With full investment insurance, b and h are perfect substitutes, so an individual invests only in hiring h as long as aggregate hiring is low enough that its return is greater than the return on the riskless asset, that is, $\mu_\pi Q(H) > R$. At higher levels of aggregate hiring, all marginal investment is in the riskless asset b; the claim that individual hiring is weakly decreasing in employment remains true.

Consider again the first-order condition (7). By differentiating, we find that the slope of the best response function is:

$$h'(H) \quad = \quad \frac{\mu_\pi Q' E v' + (\mu_\pi)^2 Q' Q E v'' + \mu_\pi N' Q [v'_E - v'_U]}{-u'' - (\mu_\pi Q)^2 E v''} \tag{10}$$

Here the first two terms in the numerator represent the substitution and wealth effects from changes in aggregate hiring, which are present regardless of the nature of insurance. The third term in the numerator is the "extra term" which is necessarily zero if there is complete unemployment insurance, but is negative as long as $\phi < 1$.

A related effect arises if we consider the degree of unemployment insurance instead of the probability of employment. Since greater unemployment insurance makes precautionary saving less necessary, higher unemployment insurance leads to less individual hiring, as the following proposition shows.

Proposition 4. If $\psi = 1$, then $\frac{\partial}{\partial \phi} h(H) \leq 0$, with equality only if hiring is at its minimum or maximum levels $h(H) = 0$ or $\mu_\pi Q(H) = R$.

Proof. The left-hand side of the first order condition (8) is strictly increasing in h and asymptotic at W, while the right-hand side is strictly decreasing in h. By assumption, a decrease in unemployment insurance ϕ is a mean preserving spread of w^* and hence of c_2^*, which raises $Ev'(c_2^*)$ since we have assumed prudence (that is, $v''' > 0$.) Hence a decrease in unemployment insurance makes the two sides of (8) cross at a higher h (except when hiring is at its minimum or maximum levels $h = 0$ or $\mu_\pi Q(H) = R$ so that all marginal resources are spent on c or b.) **QED.**

An immediate corollary of this proposition is that the equilibrium level of hiring H falls (weakly) when insurance rises, since the crossing of $h(H)$ with the 45^o line must rise. Hence we conclude that in a model where only the precautionary saving incentive is active, there is a tradeoff between insurance and investment. While a rise in unemployment insurance ϕ tends to improve welfare by lowering consumption risk, it also decreases the incentive to invest. This can lower output and the probability of unemployment and hence partially or totally offset the initial welfare gains.

4.2 Incompletely insured investment: implications of portfolio choice

We have seen that full investment insurance implies uniqueness, as well as a negative feedback from uninsured unemployment risk. However, we know from Proposition 2 that when we allow for incomplete investment insurance, multiplicity may arise. We will now study the necessary conditions for multiplicity, focussing on the case $\gamma = 0$ to see the implications of portfolio choice alone. We start, though, with two lemmas which use the properties of CRRA utility to sign some frequently encountered expectations, because the imperfect insurance case yields unpleasant algebra in spite of the deliberate simplicity of the model.[17]

The first order condition states that the expected marginal utility premium Ex^*v' of the risky asset is zero along the best response function. Hence the following result is useful.

Lemma 3. (Adapted from Elmendorf and Kimball (1991).)
Assume v exhibits decreasing absolute risk aversion, and that $E(x^*v') \leq 0$. Then the following inequality holds:

$$E(x^*v'') \quad \geq \quad 0 \tag{11}$$

Now consider $n \geq 3$. If for derivatives of all orders up to n, $-v^{(n)}/v^{(n-1)}$ is a decreasing function, then

$$(-1)^n E(x^*v^{(n)}) \quad \geq \quad 0 \tag{12}$$

Proof. See appendix.

The first inequality in this lemma implies that the expected marginal utility premium Ex^*v' is increased (along the best response function) by an additional unit of wealth. An immediate corrollary is that individual hiring is increasing in wealth:[18]

$$\partial h/\partial W \quad = \quad -R(Ex^*v'')/(E(x^*)^2v'') \quad > \quad 0$$

---

[17]The reader may notice that the proofs of Lemmas 3 and 4 hold for any distribution of the firm's idiosyncratic productivity shock, and not only to the two-point distribution assumed in the model.

[18]Using the fact that the overall objective function is concave, it is easy to show that $\partial h/\partial W > 0$ in the two period case ($\gamma > 0$) as well.

The next lemma will help us sign expressions involving the effects of changes in the unemployment rate. Note that $Ex^*v' = 0$ and $Ex^* > 0$ along the best response function as long as there is an interior choice of h and b.

> **Lemma 4.** Assume v exhibits decreasing absolute risk aversion, and that $Ex^*v' = 0$ and $Ex^* > 0$. Then
>
> $$Ex^*v'|U - Ex^*v'|E \ < \ 0 \tag{13}$$
>
> Now consider $n \geq 2$. If furthermore, for derivatives of all orders up to n, $-v^{(n)}/v^{(n-1)}$ is a decreasing and convex function, then
>
> $$(-1)^n(Ex^*v^{(n)}|U - Ex^*v^{(n)}|E) \ > \ 0 \tag{14}$$

Proof. See appendix.

Lemma 4 implies a positive feedback from portfolio choice analogous to the negative feedback we previously found from precautionary saving. If we increase an individual's probability of employment, holding other endogenous variables fixed, then that individual will increase risky investment, since she now feels more willing to take risks. We have

$$\frac{dh}{dn} \ = \ \frac{Ex^*v'|U - Ex^*v'|E}{E(x^*)^2v''} \ > \ 0$$

The numerator is negative by Lemma 4, and the denominator is obviously negative as well. Equivalently, an increase in the risk of unemployment causes individuals to hire less, amplifying the rise in unemployment.

This term also enters if we look at the overall effect of aggregate hiring on individual hiring.

> **Remark.** If investment is less than fully insured and is sufficiently risky, there are positive extra terms in the response of individual hiring to aggregate hiring.

The slope of the best response function can be written as

$$\frac{\partial h}{\partial H} \ = \ \frac{U_{hH}}{-U_{hh}} \ = \ \frac{E\left[Q'\pi^*v'\right] + E\left[x^*Q'\pi^*hv''\right] + \phi N'(\overline{w} - \underline{w})Ex^*v'' + N'E\left[Ex^*v'|E - Ex^*v'|U\right]}{-E\left[(x^*)^2v''\right]}$$

Full insurance means excess returns to the risky asset must be zero, that is, $x^* = 0$. The denominator is positive, since $v'' < 0$, though it approaches zero at full insurance. The first term in the numerator is the substitution effect arising as greater aggregate hiring makes individual hiring less effective, which is negative since $Q' > 0$ and at equilibrium $E\pi^* v' > 0$. Note that all the other terms are zero if investment is fully insured, since they all contain x^*. The second term is necessarily positive as long as there is sufficient investment risk, that is, $\underline{\pi}^* < 0$; the last two terms are positive by Lemmas 3 and 4. Hence when hiring is nearly fully insured, individual hiring responds very negatively to aggregate hiring; but when hiring is less than fully insured, there are a number of terms going the opposite direction to encourage individual hiring.

Lemmas 3 and 4 also help us interpret the following necessary and sufficient condition for multiplicity. We know from Proposition 1 that multiple equilibria exist if and only if there exists an unstable equilibrium. Using our previous notation, an unstable equilibrium is characterized by the condition $d\mathcal{R}(H)/dH > d\mathcal{S}(H)/dH$. Algebraically, in the pure portfolio choice case $\gamma = 0$, this condition is equivalent to the following:

Remark. Multiple equilibria exist if and only if at some equilibrium the following condition is satisfied:

$$N'[Ex^*v'|E - Ex^*v'|U] + Q'E(\pi^*v') + [N'E(x^*\pi^*v'') - RE(x^*v'')] > 0$$

Examining this expression, we find that the first term is positive; it is the increase in the expected marginal utility premium Ex^*v' caused by the fall in unemployment resulting from an increase in aggregate hiring. The second, which is negative, is the decrease in Ex^*v' as the rate of hiring $Q(H)$ decreases due to an increase in aggregate hiring. The last term is the decrease in Ex^*v' which occurs as one unit of expenditure is reallocated from riskless assets to risky assets; it is unambiguously negative if $\underline{\pi}^* < 0$. In other words, if, at a given equilibrium, an aggregate reallocation of investment towards the risky asset diminishes the risk from unemployment enough to offset both the effects of decreasing returns and the direct utility impact of risky investment, then that equilibrium is unstable. Note also that the first term is zero if unemployment is fully insured. Hence we find that, at least when $\gamma = 0$ and $\underline{\pi}^* < 0$, uninsured unemployment risk is *also* necessary for multiple equilibria in this model, as is uninsured investment risk.

Finally, we consider how insurance affects the level of risky asset investment in this model. The next proposition shows that, in contrast to the case of precautionary saving, there is *not* any tradeoff between insurance and risky investment in a model of portfolio choice. The level of individual investment, at any level of aggregate investment, is increasing both in investment insurance and in unemployment insurance.

Proposition 5. If $\gamma = 0$, $\phi < 1$, and $\psi < 1$, then $\frac{\partial h(H)}{\partial \phi} \geq 0$ and $\frac{\partial h(H)}{\partial \psi} \geq 0$, both with strict inequality at interior equilibria.

Proof. This is a statement about $\frac{\partial h}{\partial \psi} = \frac{U_{h\psi}}{-U_{hh}}$ and $\frac{\partial h}{\partial \phi} = \frac{U_{h\phi}}{-U_{hh}}$ at each level of aggregate hiring H. The denominator $-U_{hh} = -E\left[(x^*)^2 v''\right] > 0$ is positive. Now note that at an interior equilibrium, by (6), $\underline{x}^* < 0$ and $\overline{x}^* > 0$. Hence

$$U_{h\psi} \;=\; E\left[Q(\mu_\pi - \tilde{\pi})v'\right] \;+\; E\left[Qh(\mu_\pi - \tilde{\pi})x^* v''\right]$$

$$= \;(1-p)(\mu_\pi - \underline{\pi})Q\left[Ev'|L - Ev'|H + h\underline{x}^* Ev''|L - h\overline{x}^* Ev''|H\right]$$

is evidently positive. Next, consider

$$U_{h\phi} \;=\; E\left[(\mu_w - \tilde{w})x^* v''\right] \;=\; (1-n)(\mu_w - \underline{w})\left[Ex^* v''|U - Ex^* v''|E\right]$$

which is positive by Lemma 4 if the absolute risk aversion of v is both decreasing and convex. **QED**.

A corollary of Proposition 5 is that a decrease in insurance lowers every stable equilibrium. Moreover, it may cause high equilibria to disappear or new low equilibria to appear.

We should now emphasize the clear policy conclusion arising from these calculations. The desirability of insurance depends greatly on whether the precautionary saving or portfolio choice incentive is quantitatively more important. It is frequently noted that public policies designed to increase insurance may have the unfortunate side effect of decreasing saving and hence investment. However, if savers choose between risky and riskless assets and we are more concerned with the former, then insurance may be beneficial by raising risky investment.[19] Moreover, insurance may stabilize the economy by damping positive feedbacks and eliminating multiplicity.

5 An overlapping generations extension

Although we have seen that portfolio choice under imperfect insurance may lead to positive feedbacks and multiple equilibria, the numerical examples show that multiplicity only occurs for some parameter values. One crucial parameter is the initial wealth W, which would be better regarded as a state variable determined by previous periods' choices. While a fully dynamic model is beyond the scope of this paper, we now develop an overlapping generations (OLG) extension which partially endogenizes the wealth of the young, and provides some evidence in favor of the dynamic robustness of multiplicity.

The simplest OLG baseline to consider is a two-period lifespan without altruism, so that assets are not passed from generation to generation. In spite of the absence of altruism, wealth is indeed linked across generations, for it is natural to assume that the young face random labor income— that is, like the old, they face a risk of unemployment. The probability of employment in a given period will be determined by the level of risky asset investment which was chosen in the *previous* period. As in the static version of the model, individuals of a given generation impose externalities on one another as they choose how much risky investment to undertake. Here, however, they likewise impose externalities on the next generation.

We spell out the model by considering generation one, which is young at time one, and chooses consumption and risky and safe investment at that time. The only change in the individual's problem relative to our previous model is that initial income W_1^* is now an idiosyncratic random variable taking values $\overline{W}$ with probability n_1 and $\underline{W}$ with probability $1-n_1$. The realization of W_1^* is assumed *known* at time one when generation one's decisions are made; young agents learn whether they are currently employed before choosing consumption

[19] This point was made by Elmendorf and Kimball (1991); the observation that insurance may help stabilize the economy is a contribution of this paper's general equilibrium analysis.

and investment. Conditional on the realization of W_1^*, their choice problem is exactly that we have analyzed earlier:

$$\max_{b,h} \; u(c_1) \; + \; Ev(c_2^*)$$

$$\text{subject to:} \quad c_1 \; + \; b \; + \; h \; = \; W_1^*$$

$$\text{and} \quad c_2^* \; = \; Rb \; + \; q_2 h \pi_2^* \; + \; w_2^*$$

The expectation is taken with respect to the random variables π_2^* and w_2^* only. Once again, we can allow for insurance levels ϕ and ψ on unemployment and investment.

The only important difference from our previous model is in the aggregate consistency conditions. Assuming a population of mass one in each generation, the probability of employment at time one is

$$n_1 \;\; = \;\; \frac{1}{2} N(H_0) \tag{15}$$

where, as before, the function N gives the total number of matches formed at time one, and H_0 is aggregate hiring chosen by the young (generation zero) *at time zero.* When the young of generation one, at time one, choose in turn their safe and risky investment, they must take into account the rates of hiring and employment which will prevail at time two, q_2 and n_2. These rates are determined by the aggregate hiring activity of generation one:

$$n_2 \;\; = \;\; \frac{1}{2} N(H_1) \quad \text{and} \quad q_2 \;\; = \;\; Q(H_1) \;\; \equiv \;\; N(H_1)/H_1 \tag{16}$$

so that in equilibrium $n_2 = \frac{1}{2} q_2 H_1$.

The aggregate consistency condition for the labor market gives time one aggregate hiring as a weighted average of the behavior of those who are employed and unemployed at time one. Since a given agent's hiring decision depends on his initial wealth, and on n_2 and q_2 which are both known functions of H_1, we can write aggregate consistency as follows:

$$H_1 \;\; = \;\; n_1 h(\overline{W}, H_1) \; + \; (1 - n_1) h(\underline{W}, H_1) \tag{17}$$

Finally, the random variable W_1^* is some combination of wages and other income received by the young in the first period of life. We write $W_1^* = W^y + w_1^*$, where W^y is an exogenous endowment and w_1^* is the wage (after insurance transfers), which has the same distribution as in the static version of the model.

As it happens, making our model dynamic does not complicate its solution at all. The simplicity of the solution arises from the fact that in generation one's decision problem, the future matters only through q_2 and n_2, both of which are determined only by generation one's *own* aggregate hiring H_1. Furthermore, the past only matters through the employment rate at time 1, which is determined by aggregate hiring of generation zero, H_0. The changing fractions of employed and unemployed individuals in the aggregate consistency condition (17) generate a first-order difference equation in aggregate hiring:

$$H_1 \;=\; \frac{1}{2}N(H_0)h(\overline{W},H_1) \;+\; (1-\frac{1}{2}N(H_0))h(\underline{W},H_1) \;\equiv\; H(H_1,H_0) \quad (18)$$

Hence the dynamics of the OLG model can be fully characterized by calculating the two *static* best response functions $h(\overline{W},H)$ and $h(\underline{W},H)$. The role of expectations and the possibility of multiple equilibria are captured by allowing idiosyncratic hiring at time one to depend on aggregate hiring *at time one*, just as in the static model, and equilibrium is again given diagrammatically as a point where the aggregate best response (18) crosses the 45^o line. Intergenerational dynamics arise only from changes in the fraction employed in each generation. For some generations t, the best response function (18) may cross the 45^o line more than once; in this case we will assume that a sunspot variable causes agents to select one of the stable equilibria. We assume the sunspot causes each stable equilibrium to be selected with positive probability; there is no need to specify more precisely this aspect of the model.

5.1 Long run behavior of the OLG model

To analyze the behavior of the difference equation (18), the basic point to remember is that, from Lemma 3, an increase in an agent's initial wealth raises his best response function. In the OLG context, the best response function is higher for those born employed: for all H, $h(\overline{W}, H) \geq h(\underline{W}, H)$, which means that the aggregate best response function (18) shifts up when the employment rate rises. Thus as long as equilibrium is unique, the dynamics of H_t are monotonic.

Lemma 5. Consider a time T_0 with given employment n_0 and a time interval $I \equiv \{t : T_0 \leq t \leq T_1\}$. If, given n_0, there exists a unique equilibrium at each $t \in I$, then aggregate hiring H_t changes monotonically during the time interval I.

Proof. As a corollary of Lemma 3, $\frac{\partial}{\partial H_t} H(H_{t+1}, H_t) > 0$. Now suppose that given initial aggregate hiring H_0, the next period's unique equilibrium $H_1 = H(H_1, H_0)$ is higher: $H_1 > H_0$. It follows that $H_2 = H(H_2, H_1) > H_1$, since the time 2 reaction function will cross the 45^o line at a higher point. By induction, H_t is strictly increasing for all time. The analysis for the cases $H_1 = H_0$ and $H_1 < H_0$ is analogous. **QED.**

Making use of Lemma 5, we now show that full insurance implies monotonic convergence to a unique steady state, as long as exogenous initial wealth W^y is positive. Unsurprisingly, the economy may have more than one steady state if $W^y \leq 0$, since in this case an economy which started with very low employment would be unable to muster enough resources to get the hiring process started.

Proposition 6. Under full insurance, the OLG model displays a unique equilibrium each period. Moreover, aggregate hiring converges monotonically to a steady state H^* which is unique if $W^y > 0$.

Proof. See appendix.

When there is imperfect insurance, there may be multiple equilibria $H_{t+1} = H(H_{t+1}, H_t)$ for some values of previous hiring H_t. Let $\mathcal{M}$ be the set of previous hiring levels H_t at which multiple equilibria exist, and assume $\mathcal{M}$ is an interval (this was always the case in the numerical experiments conducted for this paper). When there exist multiple equilibria, hiring is no longer necessarily monotonic in time, but it is still true that $h(W_t, H_{t+1})$ is monotonic in W_t and hence that $H(H_{t+1}, H_t)$ is monotonic in H_t at any H_{t+1}.

These monotonicity properties will still help us characterize the dynamics. It is possible that multiplicity eventually disappears as the economy reaches a region in which there is monotonic convergence to a unique steady state. But it is also possible that there are "poverty traps", in the sense that the economy can converge to different steady states depending on its history. Alternatively, for some parameters and initial conditions, the dynamics outside the region of multiplicity may converge into the region of multiplicity, so that eventually the economy *always* exhibits multiplicity. We summarize these possibilities in the following proposition.

Proposition 7. Assume the region of multiplicity $\mathcal{M}$, if nonempty, is an interval. Then, from a given initial condition H_0, two types of long run behavior are possible:

(1) With probability 1, the economy converges to some steady state $H^* = H(H^*, H^*)$ which is not an element of $\mathcal{M}$.

(2) With probability 1, the economy eventually enters and remains within set $\mathcal{M}$.

For given parameters, different initial conditions may determine whether (1) or (2) applies, or may cause a switch from one steady state H^* to another.

Proof. Consider the following positive probability events:

Event G: there is an arbitrarily long series of sunspot realizations such that the *highest* equilibrium is selected each time.

Event B: there is an arbitrarily long series of sunspot realizations such that the *lowest* equilibrium is selected each time.

By an argument like that used in Lemma 5, the economy converges to a fixed point G^* under event G, and to a fixed point B^* under event B.

If, starting within $\mathcal{M}$, event G eventually leads to a level of hiring $H > \max(\mathcal{M})$, then the economy *may* converge to an equilibrium above set $\mathcal{M}$. If event B eventually results in a level of hiring $H < \min(\mathcal{M})$, then the economy *may* converge to an equilibrium below set $\mathcal{M}$. If either form of convergence *can* occur, then hiring converges out of $\mathcal{M}$ with probability 1 as $t \to \infty$. Otherwise equilibrium remains within $\mathcal{M}$ if it ever enters.

We construct examples below to show that there exist parameters for which these outcomes actually occur. **QED.**

An example which yields persistent multiplicity is depicted in Figure 18. The parameters are $\gamma = 0$, $\alpha = 2$, $R = 1$, $\overline{y} = 3$, $\underline{y} = 1$, $p = 0.99$, $k = 0.5$, $\beta = 0.5$, $W^y = 0.5$, $\underline{w} = 0$, and $\phi = \psi = 0$. and we assume the matching function has the kinked functional form $N = \min\{H/k, 2\}$ where 2 is the total population at any time. The figure shows the aggregate best response functions at a variety of different unemployment rates. The lowest curve is the aggregate best response function when all agents are unemployed; equivalently, it is the individual best response function of an unemployed agent. Note that there is only one equilibrium when all agents are unemployed. Higher curves correspond to higher employment; the highest is the aggregate best response function under full employment. We see that there are multiple equilibria at full employment. In this example, as long as the employment rate is greater than 0.5 (approximately), the aggregate best response function crosses the 45^o line three times, implying three equilibria. For these parameters, employment increases monotonically each period until the region of multiplicity is reached, and thereafter the economy always exhibits multiplicity.

Proposition 7 makes clear that a variety of other types of behavior are possible in this model. By changing the parameters of the previous example to $p = 0.9$ and $\underline{y} = 1.2$, we obtain an economy which exhibits poverty traps. Figure 19 shows how the aggregate best response function changes as employment rises from zero to one. Note in this case that there is a unique equilibrium under full unemployment, and also under full employment. Nonetheless, for employment rates around 80% to 90%, the economy exhibits multiplicity. The multiplicity does not last long– usually after a period or two, a shock takes the economy up or down out of the region of multiplicity. If the escape is in the upward direction, the economy converges monotonically up to a steady state at $n = 1$ and $H = 1.26$. If the escape is downward, the economy converges monotonically down to a steady state with $n = H = 0.42$.

It should be noted that the types of equilibria identified in Proposition 7 assume no real shocks to the economy. In an economy with real shocks, there is the additional possibility that an economy which has converged to a neighborhood of a steady state could jump back into a region of multiplicity at a later time. If the underlying dynamics involve poverty traps, then the economy could spend long periods caught near the high or low steady states, with occasional random transitions between the two basins of attraction.

6 Discussion

This paper has shown that precautionary behavior under imperfect insurance may lead to a particularly strong form of volatility: an economy which has a unique equilibrium under full insurance may have multiple equilibria when insurance is imperfect. This contributes to the literature on precautionary behavior by moving beyond the partial equilibrium analysis of previous studies to a general equilibrium model which can address the strategic interactions between agents. Moreover, we have found that different aspects of precautionary behavior, namely precautionary saving and portfolio choice, have very different implications for those strategic interactions and for public policy regarding insurance. The paper also contributes to the literature on strategic complementarities and multiple equilibria by identifying a new source of positive feedbacks— portfolio choice under risk— which is intuitively appealing and may be quantitatively important.

6.1 Robustness

Why might we expect the effects of portfolio choice under imperfectly insured risk to be quantitatively significant? First, because risk of bad shocks varies enormously over the business cycle— the probability of unemployment or of bankruptcy can easily double when the economy goes into recession. Models which base positive feedbacks on market power or on aggregate increasing returns to scale have to contend with the fact that the *level* of output does not change very much over the cycle. Hence the average return to economic activity is unlikely to vary nearly as much as the riskiness of economic activity over the business cycle, giving less scope for strong feedbacks.

Secondly, portfolio choice is plausibly volatile. Even when the total amount of saving is tied down tightly by the desire to smooth consumption over time, small changes in the attractiveness of different assets may cause large portfolio reallocations. Moreover, although precautionary saving and portfolio choice effects go in opposite directions, Elmendorf and Kimball (1991) show that standard risk aversion suffices to make the portfolio choice effects stronger.

On the other hand, this model is by no means unambiguous in its support of the claim that portfolio choice under risk makes the economy fluctuate more. This mechanism may be said to make the economy more volatile if it actually results in multiplicity, but this usually occurs over a limited parameter range. We may also say the mechanism increases volatility when underlying parameters change so as to move the economy past a region of multiplicity, resulting in hysteresis, or even if the economy moves past a region where there is a large jump in equilibrium hiring, due to risk effects, without actual multiplicity. However, portfolio choice under risk does not always amplify the economy's response to parameter changes. A good example is the case of proportional increases in the distribution of the idiosyncratic technology shock $\tilde{y}$, as discussed in subsection 3.2. Since such a change increases both the mean and the variance of the return to hiring, its effects are very ambiguous and may be larger or smaller when investment insurance increases. Thus the type of shocks to which the economy is exposed— and in particular the type of idiosyncratic variation in those shocks— will greatly affect the relevance of the model.

Another source of doubt is that it is difficult to draw conclusions about dynamics from the static two period model we have considered. While we have loosely treated the existence of multiple equilibria as evidence of volatility, it is unclear whether a dynamic version of the model would lead to permanent traps at low or high equilibria, or to continuing fluctuation between low and high levels of output, or to yet another type of time series. We need a better understanding of the types of dynamics arising in the model in order to judge its relevance.

6.2 Extensions

The overlapping generations version analyzed in section 5 is a first step towards understanding the dynamics of the model, but it is rigged to avoid the issue of asset accumulation. A more convincing model would consider agents' ability to accumulate assets over time in order to protect themselves from risk. To study such a model we would have to calculate the dynamics of the distribution of safe and risky asset holdings. This would be a complicated computational problem requiring an approximation to the aggregate state, like the method of Krusell and Smith (1995).

One possible outcome of such a model is that individuals might accumulate riskless assets to the point where they would no longer behave in a risk averse manner; this has been a frequent finding in previous dynamic precautionary saving models, such as Aiyagari (1994) and Krusell and Smith (1995). But another possibility is that changes in unemployment risk could provoke substantial portfolio reallocations, which would then feed back on unemployment and other aggregate states. We should note that the previous literature on dynamic precautionary saving has typically treated the risk of bad shocks as *exogenous*, which makes such feedbacks impossible. Moreover, most such studies have considered only one type of assets. It is still an open question, then, whether a multiperiod OLG version of the model of this paper might exhibit substantial long run volatility in spite of individuals' desires to build up a buffer stock of riskless assets.

An additional weakness of the current model is its focus on individual worker/entrepreneurs as the main risk takers in the economy. A model of risky investment cannot be realistic without considering firms; and for firms the appropriate type of risk to consider is not unemployment but bankruptcy. A model of endogenous bankruptcy risk capable of delivering positive feedbacks like those in this paper is an important step for future work.

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Appendix

Lemma 3. (Based on Elmendorf and Kimball (1991), Lemmas 3 and 4.) Assume v exhibits decreasing absolute risk aversion, and that $E(x^* v') \leq 0$. Then the following inequality holds:

$$E(x^* v'') \;\geq\; 0 \tag{19}$$

Now consider $n \geq 3$. If for derivatives of all orders up to n, $-v^{(n)}/v^{(n-1)}$ is a decreasing function, then

$$(-1)^n E(x^* v^{(n)}) \;\geq\; 0 \tag{20}$$

Proof. Define $\xi \equiv RW$; then $c_2^* = \xi + x^* h + w^*$. Let $\hat{v}(\xi + xh)$ be expected utility conditional on the realization x of x^*, that is,

$$\hat{v}(\xi + xh) \;\equiv\; E\left[v(\xi + x^* h + w^*) | x^* = x\right] \;=\; (1-n)v(\xi + xh + \underline{w}^*) + nv(\xi + xh + \overline{w}^*)$$

By the assumption of DARA, v' is a log-convex function. Log-convexity is preserved under expectations (Artin's theorem), so $\hat{v}'$ is also a log-convex function, that is, $\hat{v}$ inherits DARA from v.

Now since $\hat{v}$ exhibits DARA, $-\hat{v}''(\xi + xh)/\hat{v}'(\xi + xh)$ is decreasing in x (if $h > 0$), so that for any x,

$$x \left(\frac{\hat{v}''(\xi + xh)}{\hat{v}'(\xi + xh)} - \frac{\hat{v}''(\xi)}{\hat{v}'(\xi)} \right) \;\geq\; 0$$

Therefore we have

$$\begin{aligned} Ex^* v''(c_2^*) \;&=\; Ex^* v''(\xi + x^* h + w^*) \;=\; Ex^* \hat{v}''(\xi + x^* h) \\ &\geq\; Ex^* \hat{v}''(\xi + x^* h) \;-\; \frac{\hat{v}''(\xi)}{\hat{v}'(\xi)} Ex^* \hat{v}'(\xi + x^* h) \\ &=\; Ex^* \left[\frac{\hat{v}''(\xi + x^* h)}{\hat{v}'(\xi + x^* h)} \;-\; \frac{\hat{v}''(\xi)}{\hat{v}'(\xi)} \right] \hat{v}'(\xi + x^* h) \;\geq\; 0 \end{aligned}$$

This proves the first part of the theorem. Having used DARA and $Ex^* v' \leq 0$ to prove that $Ex^* v'' \geq 0$, it is now straightforward to use the assumption of decreasing absolute prudence to prove that $Ex^* v''' \leq 0$, and so on by induction. **QED.**

We will prove Lemma 4 by mathematical induction. The proofs of the first claim of Lemma 4 and of the inductive step are stated here as separate results, Lemmas 4A and 4B.

Lemma 4A. Assume v exhibits decreasing absolute risk aversion, and that $Ex^*v' = 0$ and $Ex^* > 0$. Then

$$Ex^*v'|U \ < \ 0 \ < \ Ex^*v'|E \qquad (21)$$

Proof. Since $Ex^*v' = 0$ is a weighted average of $Ex^*v'|U$ and $Ex^*v'|E$, these are of opposite signs. Define

$$\xi_u \equiv RW + \underline{w}^* \quad \text{and} \quad \xi_e \equiv RW + \overline{w}^*$$

and also

$$\eta_u(x^*) \equiv x^* \left(\frac{v'(\xi_u + x^*h) - v'(\xi_u)}{v'(\xi_u)} \right) \quad \text{and} \quad \eta_e(x^*) \equiv x^* \left(\frac{v'(\xi_e + x^*h) - v'(\xi_e)}{v'(\xi_e)} \right)$$

We can now write

$$Ex^*v'|U \ = \ v'(\xi_u)(E\eta_u(x^*) + Ex^*) \quad \text{and} \quad Ex^*v'|E \ = \ v'(\xi_e)(E\eta_e(x^*) + Ex^*)$$

Note that in the expressions above, v' and Ex^* are positive. Because $v'' < 0$, $\eta_u(x) \leq 0$ and $\eta_e(x) \leq 0$ for all x. Moreover, we will show below that $\eta_u(x) < \eta_e(x)$ for all nonzero x. This will imply that $E\eta_u(x^*) < E\eta_e(x^*)$ which suffices to show that $Ex^*v'|U \ < \ 0 \ < \ Ex^*v'|E$ since we already know that the two conditional expectations are of opposite signs.

To complete the proof we must show that $\eta_u(x) < \eta_e(x)$ when x is nonzero. Consider any $x > 0$. Then we have $\eta_u(x) < \eta_e(x)$ as long as

$$\frac{v'(\xi_u + xh)}{v'(\xi_u)} \ < \ \frac{v'(\xi_e + xh)}{v'(\xi_e)}$$

or equivalently,

$$v'(\xi_u + xh)v'(\xi_e) < v'(\xi_e + xh)v'(\xi_u)$$

This is equivalent to

$$\log v'(RW + \underline{w}^* + xh) + \log v'(RW + \overline{w}^*) \ < \ \log v'(RW + \overline{w}^* + xh) + \log v'(RW + \underline{w}^*)$$

which is an implication of log-convexity of v', which is equivalent to DARA.

An analogous argument applies for all $x < 0$. **QED.**

Lemma 4B. For $n \geq 2$, assume that $-v^{(n)}/v^{(n-1)}$ is a decreasing and convex function. Assume that

$$(-1)^{n-1}Ex^*v^{(n-1)}|U \geq 0 \text{ and } (-1)^{n-1}[Ex^*v^{(n-1)}|U - Ex^*v^{(n-1)}|E] \geq 0$$

Then

$$(-1)^{n}Ex^*v^{(n)}|U \geq 0 \text{ and } (-1)^{n}[Ex^*v^{(n)}|U - Ex^*v^{(n)}|E] \geq 0$$

Proof. Define ξ_u and ξ_e as in Lemma 4A, and also

$$\zeta_u(x) \equiv x\left[\frac{v^{(n)}(\xi_u + xh)}{v^{(n-1)}(\xi_u + xh)} - \frac{v^{(n)}(\xi_u)}{v^{(n-1)}(\xi_u)}\right]$$

with an analogous definition for $\zeta_e(x)$.

Note that if $-v^{(n)}/v^{(n-1)}$ is decreasing, then $\zeta_u(x) \geq 0$ and $\zeta_e(x) \geq 0$. Furthermore, if $-v^{(n)}/v^{(n-1)}$ is convex, then $\zeta_u(x) > \zeta_e(x)$ for all nonzero x.

Now we can write

$$\begin{aligned} Ex^*v^{(n)}|U &= E[\zeta_u(x^*)v^{(n-1)}(\xi_u + x^*h)] + \frac{v^{(n)}(\xi_u)}{v^{(n-1)}(\xi_u)}Ex^*v^{(n-1)}(\xi_u + x^*h) \\ &= E[\zeta_u(x^*)v^{(n-1)}|U] + \frac{v^{(n)}(\xi_u)}{v^{(n-1)}(\xi_u)}Ex^*v^{(n-1)}|U \end{aligned}$$

with an analogous expression for $Ex^*v^{(n)}|E$.

To prove the lemma we must compare $Ex^*v^{(n)}|U$ with $Ex^*v^{(n)}|E$. The following observations suffice for a direct comparison of the two:

$$\zeta_u(x^*) > \zeta_e(x^*) > 0 \quad \text{for each nonzero } x^*,$$

$$(-1)^n v^{(n-1)}(\xi_u + x^*h) > (-1)^n v^{(n-1)}(\xi_e + x^*h) > 0 \quad \text{for all } x^*,$$

$$v^{(n)}(\xi_u)/v^{(n-1)}(\xi_u) < v^{(n)}(\xi_e)/v^{(n-1)}(\xi_e) < 0, \quad \text{and}$$

$$(-1)^{n-1}Ex^*v^{(n-1)}|U \geq 0 \quad \text{and} \quad (-1)^{n-1}[Ex^*v^{(n-1)}|U - Ex^*v^{(n-1)}|E] \geq 0.$$

QED.

Proposition 6. Under full insurance, the OLG model displays a unique equilibrium each period. Moreover, aggregate hiring converges monotonically to a steady state H^* which is unique if $W^y > 0$.

Proof. For simplicity, we will confine ourselves to the case $R < \mu_\pi Q(\overline{W})$, which implies that under full insurance there will be no investment in the riskless asset b. Also, we will assume $\underline{w} = 0$, and we will make explicit use of the CRRA functional form. Note that the example illustrated in Figure 19 satisfies these restrictions; hence the poverty traps arising in that example can be attributed to incomplete insurance, and not to some other feature of the model.

Under full insurance, labor income at time t is $\mu_w(H_{t-1}) = \frac{1}{2}\overline{w}N(H_{t-1})$ and profit per worker is μ_π. Hence consumption of the young at t is $c_t = \tilde{W}_t - h_t = W^y + \frac{1}{2}\overline{w}N(H_{t-1}) - h_t$ and the consumption of the same individuals when they are old at time $t+1$ is $c_{t+1} = Q(H_t)\mu_\pi h_t + \frac{1}{2}\overline{w}N(H_t)$. In equilibrium we must have $h_t = H_t$, so $Q(H_t)h_t = N(H_t)$. Thus equilibrium at time t requires

$$u'(W^y + \frac{1}{2}\overline{w}N(H_{t-1}) - H_t) \quad = \quad Q(H_t)\mu_\pi v'(N(H_t)\mu_\pi + \frac{1}{2}\overline{w}N(H_t)) \qquad (22)$$

The left hand side of the above equation is strictly increasing in H_t while the right hand side is decreasing in H_t, so there is a unique equilibrium H_t associated with each initial H_{t-1}.

By Lemma 5, the uniqueness of aggregate hiring at each point in time implies that dynamics are always monotonic. We must still show, however, that the dynamics necessarily converge to the same steady state H^* regardless of initial conditions. We will prove this by showing that the graph of the difference equation $H_t(H_{t-1})$ necessarily crosses the 45^o line with a slope less than one, so that in fact it cannot cross more than once.

By implicitly differentiating the necessary condition (22), we find that the slope of the graph of the difference equation describing the equilibrium dynamics is

$$\frac{dH_t}{dH_{t-1}} \quad = \quad \frac{\frac{1}{2}\overline{w}N'(H_{t-1})u''(c_t)}{u''(c_t) + \mu_\pi Q'(H_t)v'(c_{t+1}) + Q(H_t)\mu_\pi(\mu_\pi + \frac{1}{2}\overline{w})N'(H_t)v''(c_{t+1})}$$

Since the first-order condition is $u'(c_t) = Q(H_t)\mu_\pi v'(c_{t+1})$, we notice that

$$\frac{u''(c_t)}{Q(H_t)\mu_\pi v''(c_{t+1})} \quad = \quad \frac{v'u''}{u'v''} \quad = \quad \frac{c_{t+1}}{c_t}$$

where the last equality makes use of the assumption that both u and v are the same CRRA function. Hence we can simplify the expression for the slope, at a fixed point $H_{t-1} = H_t$, as follows:

$$\frac{dH_t}{dH_{t-1}} \quad = \quad \frac{\frac{1}{2}\overline{w}N'c_{t+1}}{(Q'/Q)(v'/v'')c_t + c_{t+1} + (\mu_\pi + \frac{1}{2}\overline{w})N'c_t} \quad < \quad \frac{\frac{1}{2}\overline{w}N'c_{t+1}}{c_{t+1} + (\mu_\pi + \frac{1}{2}\overline{w})N'c_t}$$

We now use $\overline{w} = \beta\mu_y$, $\mu_\pi = (1-\beta)\mu_y$, $c_t = W^y - H + \frac{1}{2}\beta\mu_y N$, and $c_{t+1} = (1 - \frac{1}{2}\beta)\mu_y N$ to simplify further:

$$\frac{dH_t}{dH_{t-1}} < \frac{\frac{1}{2}\beta\mu_y N'[(1-\frac{1}{2}\beta)\mu_y N]}{(1-\frac{1}{2}\beta)\mu_y N + (1-\frac{1}{2}\beta)\mu_y N'(W^y - H + \frac{1}{2}\beta\mu_y N)} = \frac{\beta\mu_y N' N}{2N + N'(2(W^y - H) + \beta\mu_y N)}$$

This latter fraction is evidently less than one as long as $N + N'(W^y - H)$ is positive. Notice then that

$$\frac{N}{N'} = \frac{N}{Q + HQ'} > \frac{N}{Q} = H$$

Thus $N + N'(W^y - H) > N'W^y$ which is greater than or equal to zero as long as $W^y \geq 0$. Hence $W^y \geq 0$ suffices for $\frac{dH_t}{dH_{t-1}} < 1$, which is our desired result. **QED.**

Fig.1: Aggregate matching function

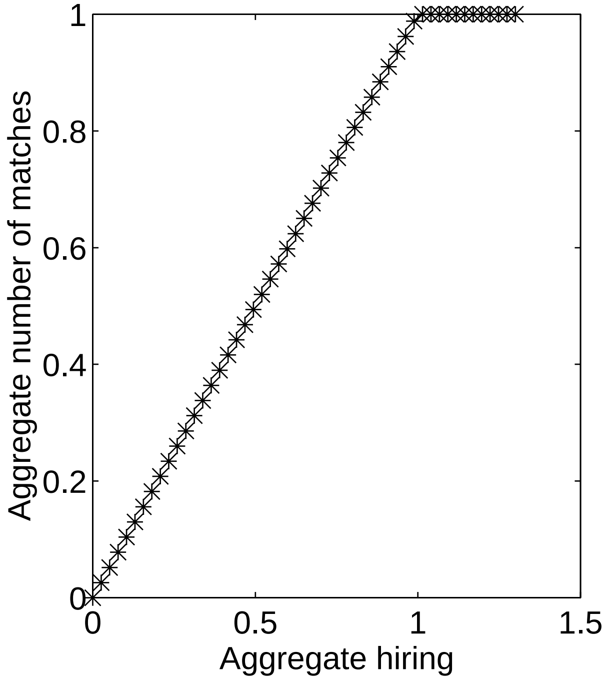


Fig.2: Distribution of consumption

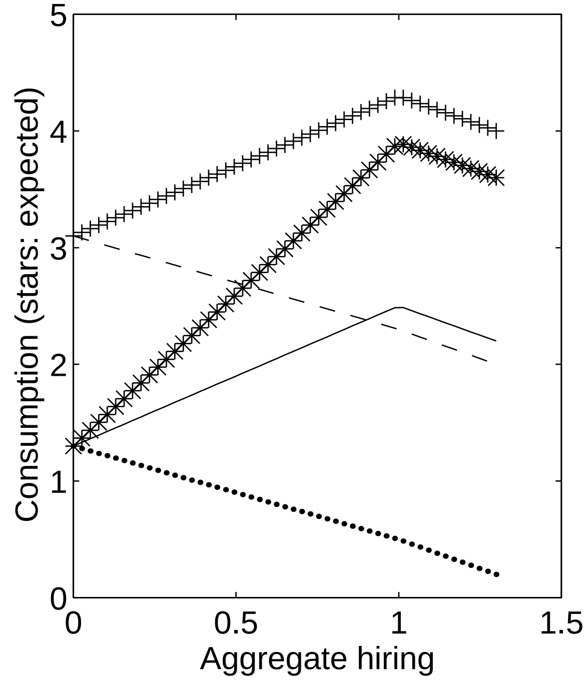


Fig.3: S(H) solid, R(H) dash

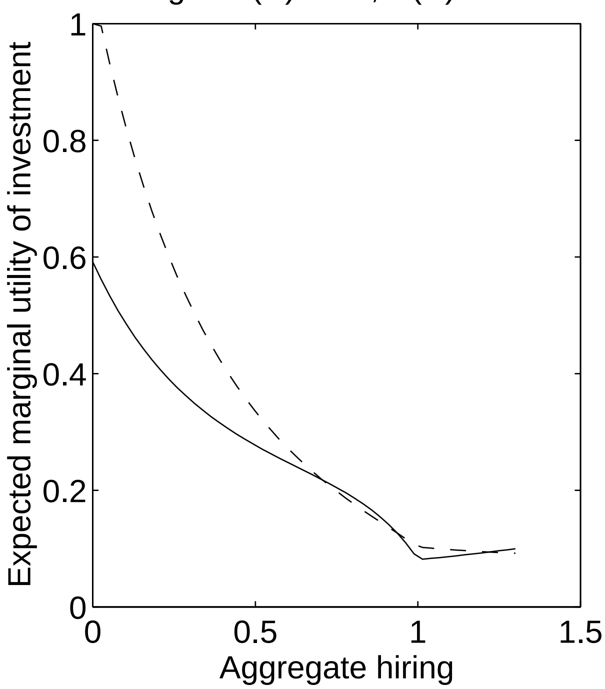


Fig.4: Reaction function (solid)

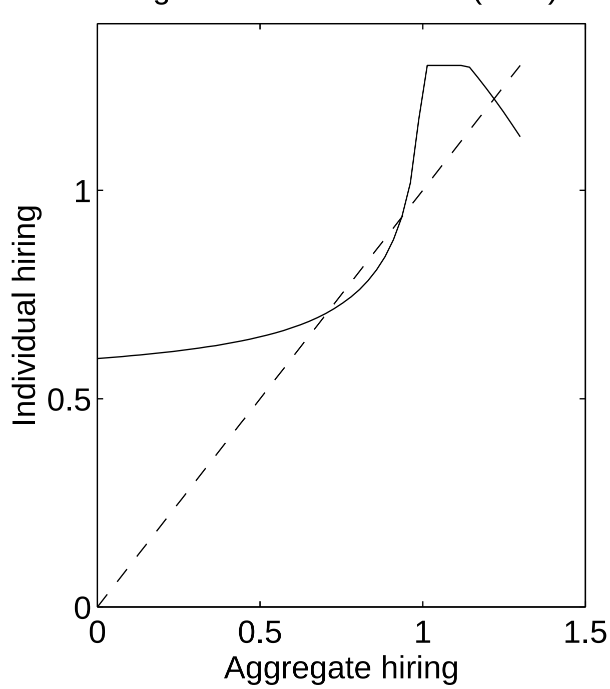


Fig.5: Equilibrium correspondence

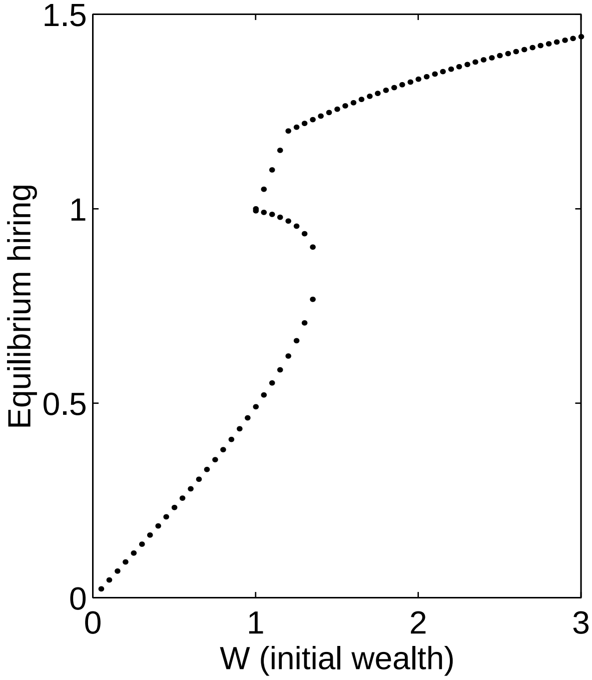


Fig.6: Equilibrium correspondence

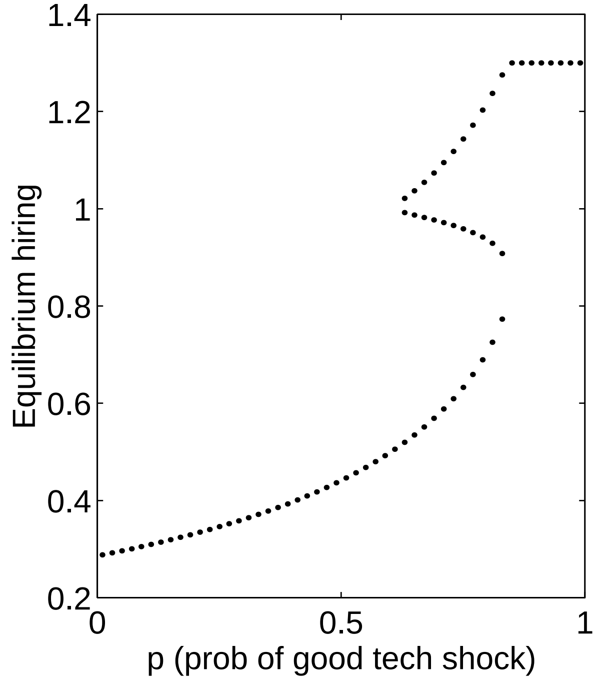


Fig.7: Equilibrium correspondence

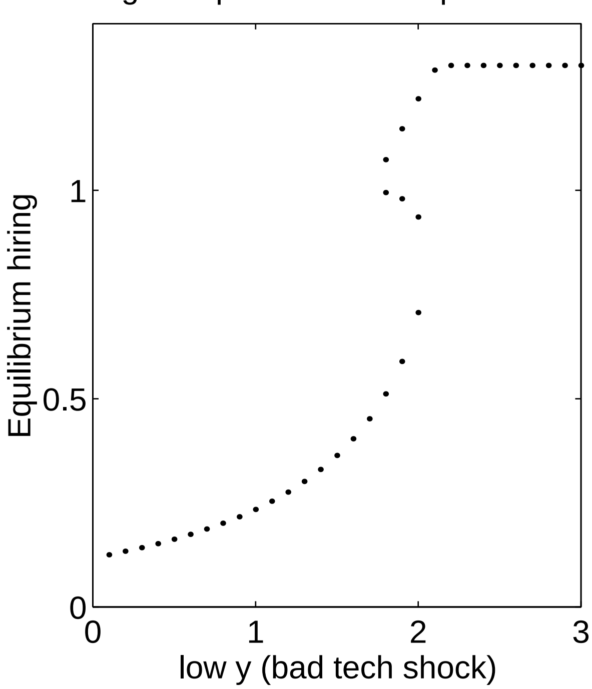


Fig.8: Equilibrium correspondence

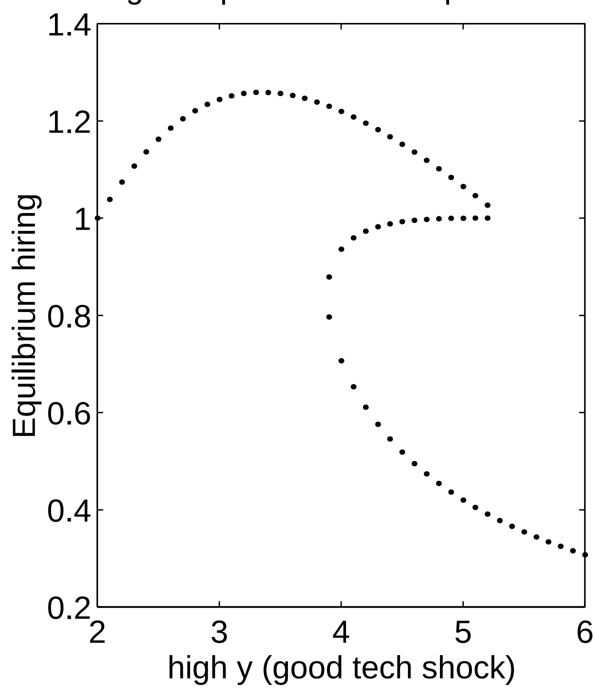


Fig.9: Aggregate matching function

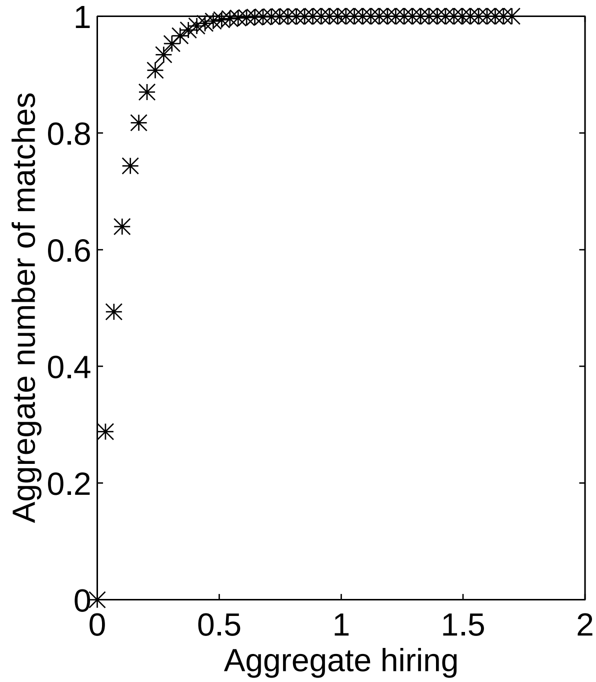


Fig.10: Distribution of consumption

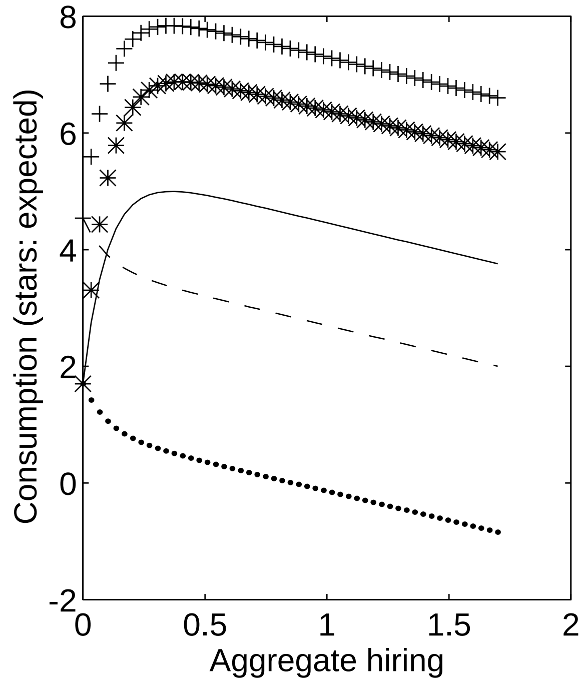


Fig.11: S(H) solid, R(H) dash

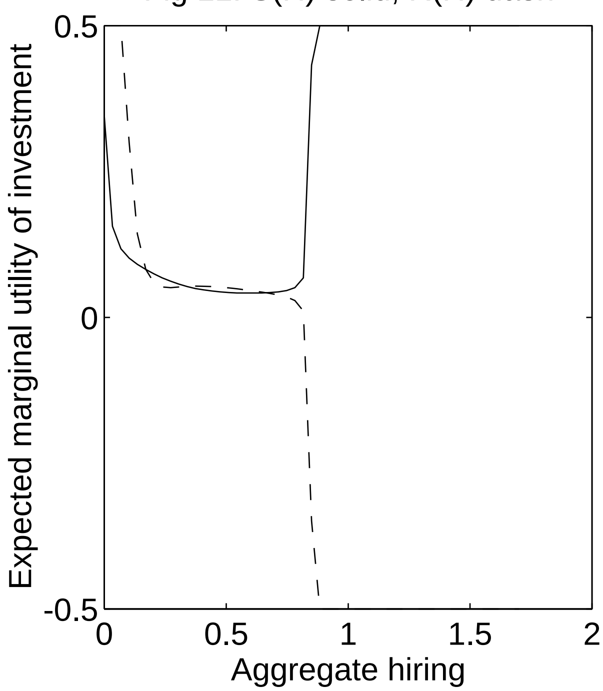
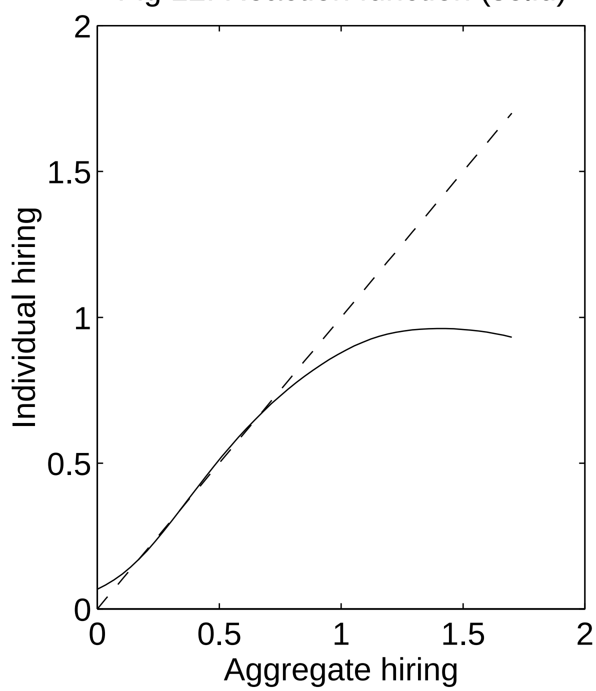


Fig.12: Reaction function (solid)



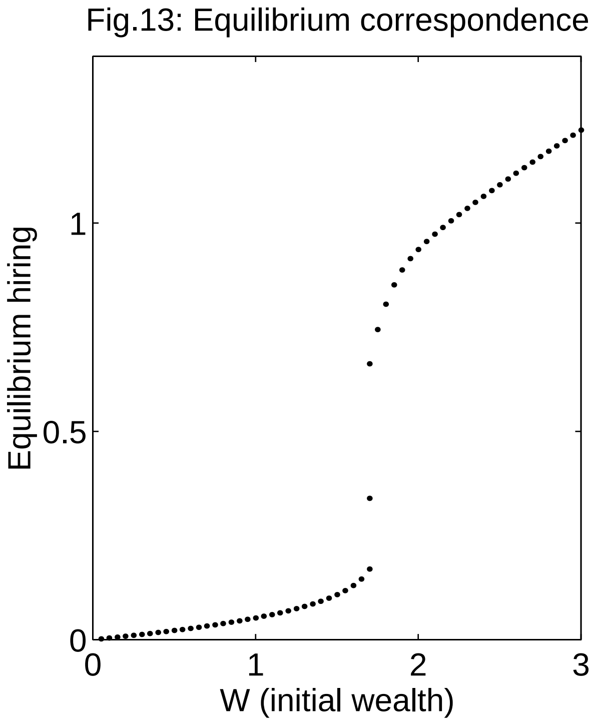


Fig.13: Equilibrium correspondence
Equilibrium hiring
W (initial wealth)
0
0.5
1
0
1
2
3

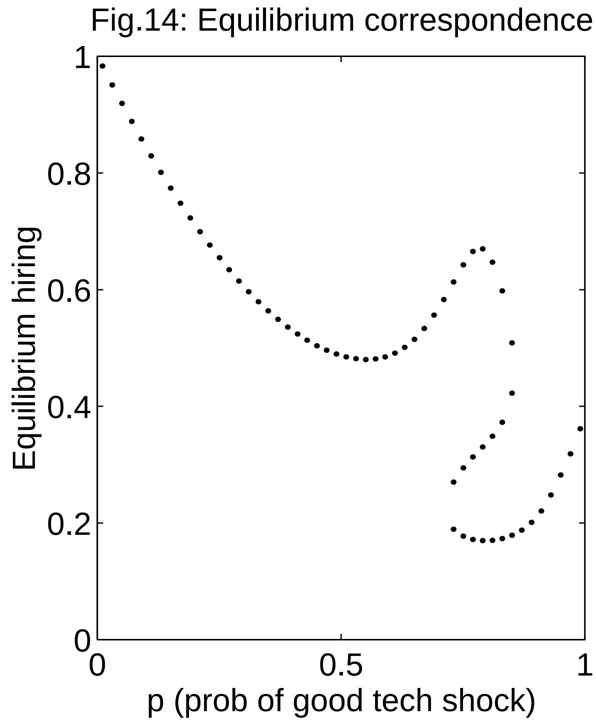


Fig.14: Equilibrium correspondence
Equilibrium hiring
p (prob of good tech shock)
0
0.2
0.4
0.6
0.8
1
0
0.5
1

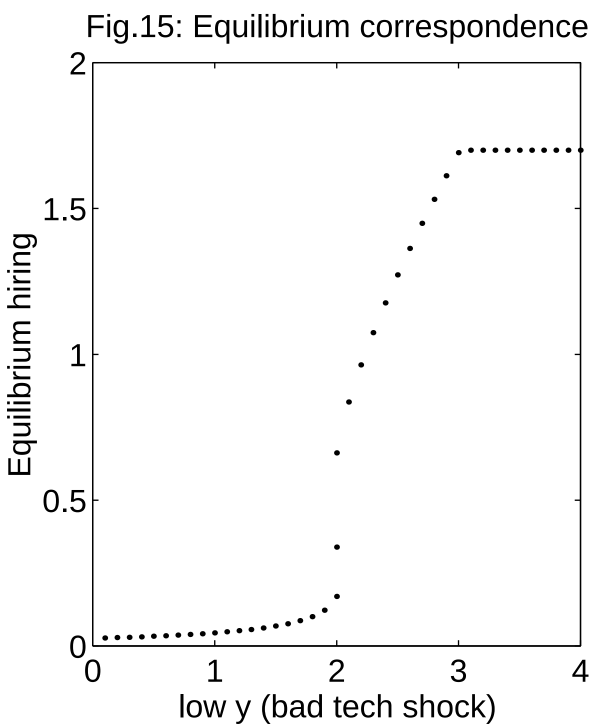


Fig.15: Equilibrium correspondence
Equilibrium hiring
low y (bad tech shock)
0
0.5
1
1.5
2
0
1
2
3
4

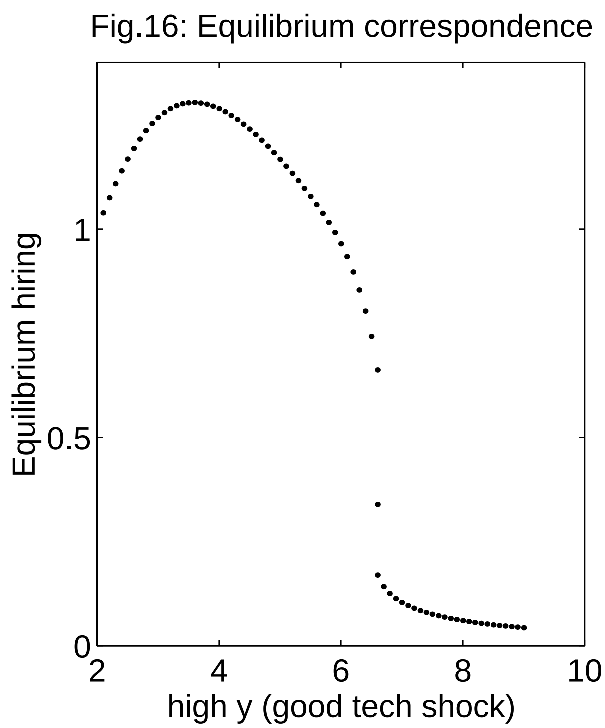


Fig.16: Equilibrium correspondence
Equilibrium hiring
high y (good tech shock)
0
0.5
1
2
4
6
8
10

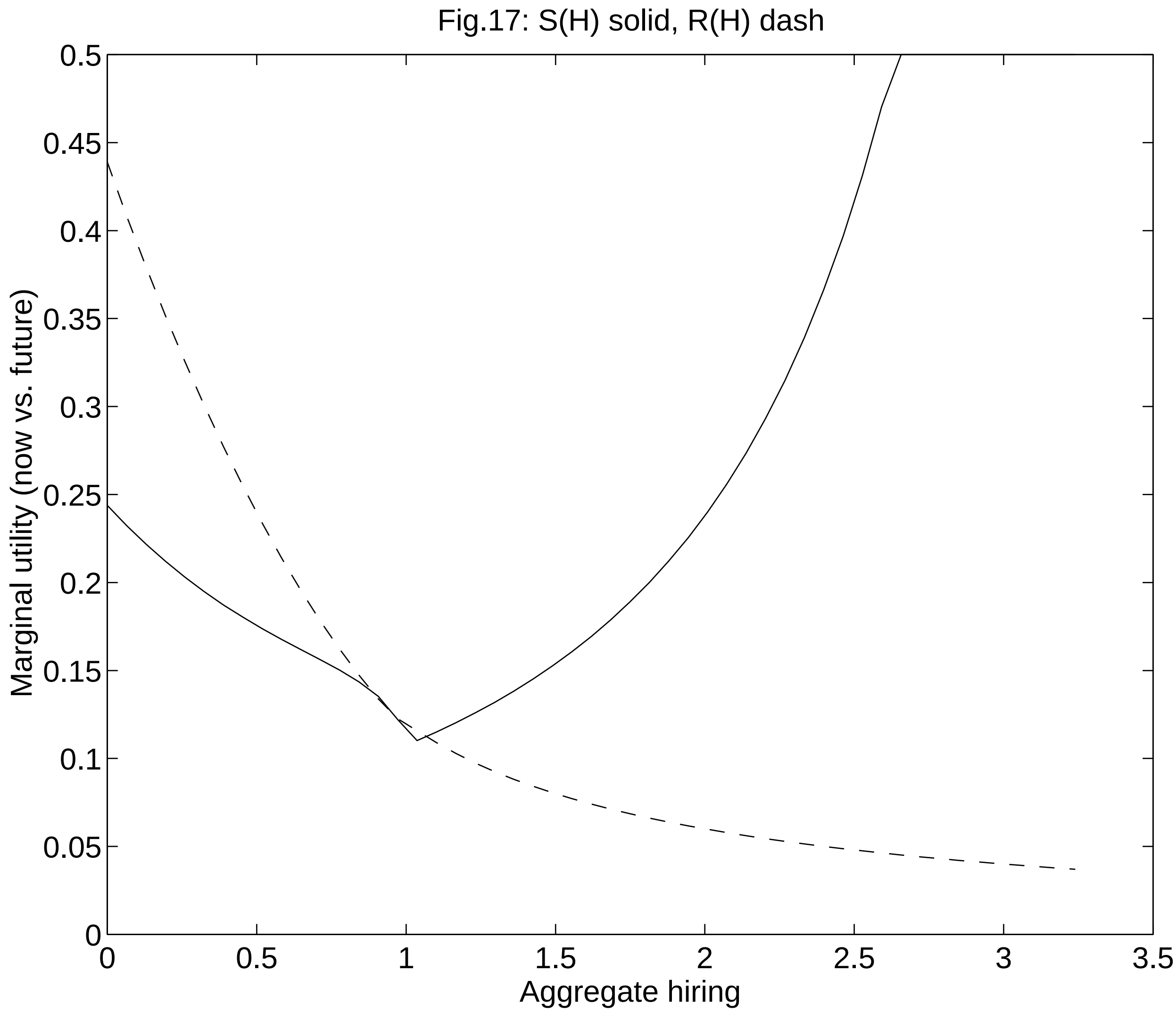


Fig.17: S(H) solid, R(H) dash
Marginal utility (now vs. future)
Aggregate hiring
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5
0
0.5
1
1.5
2
2.5
3
3.5

Fig.18: Reaction fn: u=0 to 1

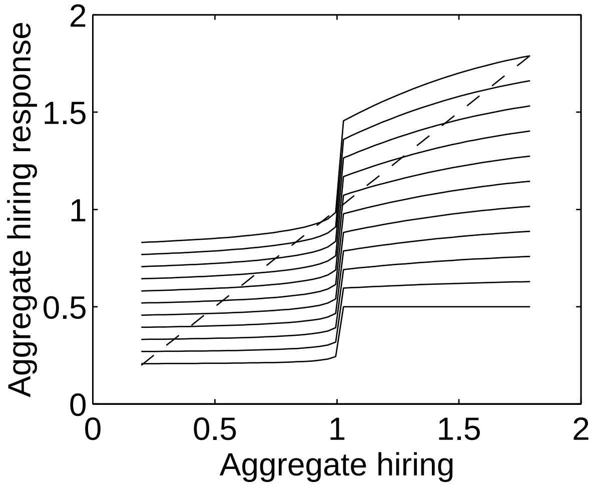


Fig.19: Reaction fn: u=0 to 1