A Singular Function and its relation with the Number Systems involved in its Definition

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Abstract

Minkowski's ?(x) function can be seen as the confrontation of two number systems: regular continued fractions and the alternated dyadic system. This way of looking at it permits us to prove that its derivative, as it also happens for many other non-decreasing singular functions from [0,1] to [0,1], when it exists can only attain two values: zero and infinity. It is also proved that if the average of the partial quotients in the continued fraction expansion of x is greater than k* =5.31972, and ?'(x) exists then ?'(x)=0. In the same way, if the same average is less than $k^{**}=2\log_2(\Phi)$, where Φ is the golden ratio, then ?'(x)=∞. Finally some results are presented concerning metric properties of continued fraction and alternated dyadic expansions.

1. Introduction

In 1904 Minkowski, see [1], builds the function $?(x)$ with the idea of matching all quadratic irrationals in $[0,1]$ to the periodic dyadic rationals. In 1938, Denjoy, see [3], proves that $?(x)$ is a singular function in the sense that its derivative is zero almost everywhere in $[0,1]$ and exhibits an analytic expression for $?(x)$: if $x=[0;a_1,\dots,a_n,\dots]$ denotes the regular continued fraction expansion of x, then

$$?(x) = \frac{1}{2^{a_1-1}} - \frac{1}{2^{a_1+a_2-1}} + \cdots + \frac{(-1)^{n+1}}{2^{a_1+\cdots+a_n-1}} + \cdots .$$

Salem, in 1943, see [8], proves the singularity of $?(x)$ using a metric property of continued fractions:

> *The set S of $x \in [0,1]$ for whom the continued fraction expansion has unbounded partial quotients is such that if $?'(x)$ exists and is finite then it vanishes.*

As the Lebesgue measure of S is 1, see [4, p. 69], the singularity of $?(x)$ is proved.

In 1960, Kinney, in [5], following the original scheme of Minkowski and working with the partitions of $[0,1]$ determined by the different stages of the Farey tree (also known as the Stern–Brocot tree) proved that if $x = ?^{-1}(y)$, and y is a normal number under the dyadic system (in the sense of Borel) then given any $\epsilon > 0$ there exists $n(\epsilon)$ such that for any stage of the Farey tree F_n posterior to $F_{n(\epsilon)}$ we have:

$$\text{If } x \in J(n,x) \quad |?(J(n,x))| \geq |J(n,x)|^{\alpha+\epsilon},$$

where the interval $J(n,x)$ has as endpoints two consecutive fractions of the n–th Farey tree stage.

This result allows Kinney to find a Lipschitz condition constant α, that, in its turn determines the Hausdorff dimension of the set formed by the inverse images of normal numbers in the dyadic system. The value of α is:

$$\alpha = \left[2 \int_0^1 \log_2(1+x)\, d\,?(x) \right]^{-1} .$$

Recently, in 1995, Tichy and Uitz, see [9], following Kinney's ideas exhibit a family of singular functions g_λ, which generalize Minkowski's function. In this last paper, there is a numerical approximation for Kinney's constant, $\alpha \approx 0.875$.

Our purpose in the present paper is to study more closely the links between the metric properties of the number systems involved in Denjoy's definition of $?(x)$, continued fractions and alternated dyadic (the dyadic system with alternating signs in the expansion) , and the singularity of $?(x)$. We intend to show that some of the previous results admit a re–interpretation in a number system context.

2. Salem's approach

Salem, in [8], proves the singularity of $?(x)$ in the following way. Let

$$S = \{x = [0; a_1, a_2, \ldots, a_n, \ldots] \limsup_{n\to\infty} a_n = \infty\},$$

and let $r_n = p_n/q_n = [0; a_1, \ldots, a_n]$ be the n–th convergent and $\rho_n = ?(r_n)$. If $a'_{n+1} = [a_{n+1}; a_{n+2}, \ldots]$, we have

$$x = \frac{a'_{n+1}p_n + p_{n-1}}{a'_{n+1}q_n + q_{n-1}}, \quad \left| x - \frac{p_n}{q_n} \right| = \frac{1}{(a'_{n+1}q_n + q_{n-1})q_n}, \tag{2.1}$$

and, consequently, we have the double inequality

$$\frac{1}{(a_{n+1}+2)q_n^2} < \left| x - \frac{p_n}{q_n} \right| < \frac{1}{a_{n+1}q_n^2}. \tag{2.2}$$

On the other hand, if $S_n = a_1 + \cdots + a_n$, we have

$$y - \rho_n = (-1)^n \left(\frac{1}{2^{S_{n+1}-1}} - \frac{1}{2^{S_{n+2}-1}} + \cdots \right), \tag{2.3}$$

and hence

$$\frac{1}{2^{S_{n+1}}} < |y - \rho_n| < \frac{1}{2^{S_{n+1}-1}}. \tag{2.4}$$

If δ_n denotes the differential quotient,

$$\delta_n = \left| \frac{y - \rho_n}{x - r_n} \right|,$$

we can write the double inequality:

$$\frac{a_{n+1}q_n^2}{2^{S_{n+1}}} < \delta_n < \frac{2(a_{n+1}+2)q_n^2}{2^{S_{n+1}}}. \tag{2.5}$$

Now, considering the sequence δ_n,

$$\begin{aligned} \frac{\delta_n}{\delta_{n-1}} &< \frac{2}{2^{a_{n+1}}} \left(\frac{a_{n+1}+2}{a_n} \right) \left(\frac{q_n}{q_{n-1}} \right)^2 \\ &< \frac{2}{2^{a_{n+1}}} \left(\frac{a_{n+1}+2}{a_n} \right) (a_n+1)^2 < C \frac{a_n a_{n+1}}{2^{a_{n+1}}}, \end{aligned} \tag{2.6}$$

where C is an absolute constant. A simple calculation leads to $C = 24$.

Salem's reasoning is the following: if $x \in S$, there exists a strictly increasing subsequence of partial quotients $\{a_{n_k}\}$, with $a_{n_k} \to \infty$. Thus:

$$\liminf_{n\to\infty} \frac{\delta_n}{\delta_{n-1}} = 0. \tag{2.7}$$

If $?'(x)$ existed and were finite and different from 0, then δ_n/δ_{n-1} should tend necessarily to 1. Thus, if $?'(x)$ exists and is finite then $?'(x) = 0$. Finally as the derivative exists almost everywhere in $[0, 1]$, the singularity of $?(x)$ is proved.

Salem's proof suggests that the essential metric property of the continued fraction expansion of x for which $?'(x) = 0$ is the unboundedness of its partial quotients. This is not the case as we shall see refining somewhat Salem's proof.

Theorem 2.1. *Let* $\overline{S} = \{x : \limsup a_n(x) \geq 12\}$*. For* $x \in \overline{S}$*, if* $?'(x)$ *exists then* $?'(x) = 0$*.*

Proof. From inequality (2.6):

$$\frac{\delta_n}{\delta_{n-1}} < 24\frac{a_n a_{n+1}}{2^{a_{n+1}}}.$$

As long as a subsequence $\{a_{n_k}\}$ can be found for which:

$$\frac{\delta_{n_k}}{\delta_{n_{k-1}}} < 24\frac{a_{n_k} a_{n_{k1}}}{2^{a_{n_{k1}}}} \leq C < 1, \qquad (C \text{ constant}), \tag{2.8}$$

we can ensure that if $?'(x)$ exists and is finite $\lim_{n\to\infty} \delta_n = 0$.

Now, if $x \in \overline{S}$, let us consider a subsequence $\{a_{n_k}\}$ such that $a_{n_k} \leq a_{n_{k+1}}$, with $a_{n_k} \geq 12$ for all k. For this subsequence we have:

$$\frac{\delta_{n_k}}{\delta_{n_{k-1}}} < 24\frac{a^2_{n_{k+1}}}{2^{a_{n_{k+1}}}} \leq C < 1, \tag{2.9}$$

as for $h \geq 12$ we have $24h^2 < 2^h$. □

Remark 2.2. The set $\overline{S}$ includes all the quadratic irrationals with a continued fraction expansion such that any of the periodic terms is greater or equal than 12.

It is convenient to notice that both, S and $\overline{S}$ have Lebesgue measure one and their images, $?(S)$ and $?(\overline{S})$ have also measure one. This remark makes us conjecture that both sets contain x for which $?'(x) = \infty$, as we shall presently prove rigorously in the next sections.

3. Close study of $?'(x)$

Following the steps of Kinney in [5], but using directly the continued fraction expansions instead of Farey fractions we are going to find an analytical expression for $?'(x)$ better suited for our purposes than Salem's.
Let $x = [0; a_1, a_2, \cdots, a_n, \cdots]$ be the expansion of $x \in [0, 1]$ as a regular continued fraction. We denote by $R_n(x)$ its n–th convergent, $R_n(x) = [0; a_1, \cdots, a_n] = p_n/q_n$. If $?'(x)$ exists then it has to coincide with the following limit:

$$?'(x) = \lim_{n\to\infty} \frac{?(R_n(x)) - ?(R_{n-1}(x))}{R_n(x) - R_{n-1}(x)}, \tag{3.1}$$

for the terms of the sequence $\{R_n(x)\}$ are the endpoints of a sequence of nested intervals with limit x:

$$R_0 < R_2 < R_4 < \cdots < x < \cdots < R_5 < R_3 < R_1$$

and $\lim R_n = x$. Both, numerator and denominator in (3.1) have the same sign.

We have:

$$|R_n(x) - R_{n-1}(x)| = \frac{1}{q_n q_{n-1}}, \quad |?(R_n(x)) - ?(R_{n-1}(x))| = \frac{1}{2^{a_1+\cdots+a_n-1}}.$$

Calling $S_n = a_1 + \cdots + a_n$, (3.1) can be written as

$$?'(x) = \lim_{n\to\infty} \frac{2q_n q_{n-1}}{2^{S_n}}. \tag{3.2}$$

Using (3.2), our next theorem extends Salem's result freeing it from any metric consideration:

Theorem 3.1. *If $?'(x)$ exists and is finite then $?'(x) = 0$.*

Proof. Let $\delta_n = (2q_n q_{n-1})/2^{S_n}$, and let us prove that $\lim_{n\to\infty} \delta_n \neq k$, for any positive constant k. In fact, we shall prove that δ_n/δ_{n-1} can never tend to 1 when n tends to infinity.

We have

$$\frac{\delta_n}{\delta_{n-1}} = \frac{q_n}{q_{n-2}2^{a_n}}, \tag{3.3}$$

and from the recursive definition of q_n, we have, for all n:

$$q_n = a_n q_{n-1} + q_{n-2} = q_{n-1}\left(a_n + \frac{q_{n-2}}{q_{n-1}}\right) = q_{n-1}\left(a_n + \frac{1}{a_{n-1} + \frac{q_{n-3}}{q_{n-2}}}\right),$$

and, consequently

$$\begin{aligned} q_n &= q_{n-2}\left(a_n + \frac{1}{a_{n-1} + \frac{q_{n-3}}{q_{n-2}}}\right)\left(a_{n-1} + \frac{q_{n-3}}{q_{n-2}}\right) \\ &= q_{n-2}\left(a_n\left(a_{n-1} + \frac{q_{n-3}}{q_{n-2}}\right) + 1\right). \end{aligned}$$

Notice that $q_{n-3}/q_{n-2} = [0; a_{n-2}, a_{n-3}, \ldots, a_1] < 1$. If we call $\frac{q_{n-3}}{q_{n-2}}S = x_{n-1}$, replacing in (3.3) and simplifying we finally get

$$\frac{\delta_n}{\delta_{n-1}} = \frac{a_n(a_{n-1} + x_{n-1}) + 1}{2^{a_n}}. \tag{3.4}$$

For values of δ_n/δ_{n-1} very near 1 we have:

$$a_{n-1} + x_{n-1} \approx \frac{2^{a_n} - 1}{a_n}.$$

And from the fact that $\forall x \in \mathbb{R},\ x > 4.9\ldots, \quad x + 1 < (2^x - 1)/x$ we infer that if $a_n \geq 5, a_n < a_{n-1}$.

In this way, if $\delta_n/\delta_{n-1} \approx 1$ from some position onwards, it will be necessary that the partial quotients, a_n, decrease till they reach, at the least, the value 5. Let us examine carefully the situation that arises when we reach a certain n for which $a_n = 5$. We shall have $a_n + x_n \in (5, 6)$, and thus the values of δ_{n+1}/δ_n will be within:

$$\frac{5a_{n+1}+1}{2^{a_{n+1}}} < \frac{\delta_{n+1}}{\delta_n} < \frac{6a_{n+1}+1}{2^{a_{n+1}}},$$

getting the best approximation to 1 for $a_{n+1} = 5$:

$$\frac{26}{32} < \frac{\delta_{n+1}}{\delta_n} < \frac{31}{32} \Longrightarrow \left|\frac{\delta_{n+1}}{\delta_n} - 1\right| > \frac{1}{32}.$$

For the values of a_n less than 5 we can establish the following. For $a_n = 1, 2, 3$, the best approximations of δ_{n+1}/δ_n to 1 are those for $a_{n+1} = 2, 3, 4$, and when we reach a position n for which $a_n = 4$, we shall have $a_n + x_n \in (4, 5)$, getting the inequalities:

$$\frac{4a_{n+1}+1}{2^{a_{n+1}}} < \frac{\delta_{n+1}}{\delta_n} < \frac{5a_{n+1}+1}{2^{a_{n+1}}},$$

which, examined for the values $a_{n+1} = 4$ and $a_{n+1} = 5$ lead to the bounds:

$$\frac{17}{16} < \frac{\delta_{n+1}}{\delta_n} < \frac{21}{16} \Longrightarrow \left|\frac{\delta_{n+1}}{\delta_n} - 1\right| > \frac{1}{16},$$
$$\frac{31}{32} < \frac{\delta_{n+1}}{\delta_n} < \frac{26}{32} \Longrightarrow \left|\frac{\delta_{n+1}}{\delta_n} - 1\right| > \frac{3}{16}.$$

After this analysis the conclusion we reach is that it is impossible for the terms in the sequence δ_n/δ_{n-1} to differ from 1 less 1/32 from some place onwards. □

The theorem we have just established, shows that Minkowski's function $?(x)$, in the points where the derivative exists (in a wide sense), can only take two values: 0 or ∞.

This behaviour is also presented by the more well–known Cantor's devil's staircase, for which is rather simple to prove that $C'(x) = 0$ if x does not belong to Cantor's ternary set and in those points of Cantor's set where $C'(x)$ exists in a wide sense, $C'(x) = \infty$. These are all the possibilities concerning the value of the derivative. The same can be proved for the family of singular functions of Riesz–Nagy ([7]), or the wider family found in Wimp and Gho ([10]): for these functions, the derivative cannot be finite and different from zero. The proof can follow the same scheme as the proof of the previous theorem.

There exist though families of singular functions for which there are points in which the derivative is finite and different from zero, see ([2]).

In the next section we establish a few metric theorems that will provide us with information about different sets for which, if the derivative exists at their points, it has to be zero or infinity. These theorems will enable us to exhibit sets of Lebesgue measure one whose image by $?(x)$ has measure zero and viceversa, sets of measure zero with image of measure one. The metric properties that define these sets discriminate the points at which the derivative is zero from those at which the derivative is infinite, aspect that Salem's approach did not contemplate.

4. New metric results for Minkowski's function

Theorem 4.1. *Given $x = [0; a_1, \dots, a_n, \dots]$, and assuming that $?'(x)$ exists, its value must be zero if the following property is verified:*

$$\liminf_{n\to\infty} \frac{S_n(x)}{n} \geq k,$$

where the constant k, is the solution of the equation

$$2\log_2(1+x) - x = 0.$$

The value of k is, approximately, 5.31972.

Proof. Let us consider expression (3.2) for $?'(x)$:

$$?'(x) = \lim_{n\to\infty} \delta_n = \lim_{n\to\infty} \frac{2q_n q_{n-1}}{2^{S_n}}.$$

By the arithmetic–geometric mean inequality we have:

$$q_n < \Pi_{j=1}^{n}(a_j+1) \leq \left(1+\frac{S_n}{n}\right)^n,$$

and, consequently,

$$\delta_n = \frac{2q_n q_{n-1}}{2^{S_n}} < \frac{2q_n^2}{2^{S_n}} < 2\left[\frac{\left(1+\frac{S_n}{n}\right)^2}{2^{\frac{S_n}{n}}}\right]^n.$$

A sufficient condition for $\delta_n \to 0$ is to have, from some place n_0 onwards, the following inequality:

$$\forall n \geq n_0, \quad \frac{\left(1+\frac{S_n}{n}\right)^2}{2^{\frac{S_n}{n}}} \leq C < 1,$$

which is equivalent to:

$$2\log_2\left(1+\frac{S_n}{n}\right) - \frac{S_n}{n} \leq k < 0.$$

The only root of $2\log_2(1+x) - x = 0$ is $k = 5.31972\dots.$ □

In a similar way, we can state a weaker result that ensures $?'(x) = \infty$ whenever the average of the partial quotients is asymptotically bounded

Theorem 4.2. *If $?'(x)$ exists and we have*

$$\limsup_{n\to\infty} \frac{S_n(x)}{n} < 2\log_2\Phi \approx 1.388483\dots,$$

where $\Phi = (1+\sqrt{5})/2$, then $?'(x) = \infty$.

Proof. The golden ratio $\Phi = [1; 1, 1, \ldots, 1, \ldots]$ presents the slowest growth possible in the denominators q_n of the convergents of a regular continued fraction. These denominators constitute exactly the Fibonacci sequence and it is seen at once that

$$\forall x \in (0,1), \quad q_n(x) \geq C\Phi^n,$$

which implies:

$$2q_nq_{n-1} \geq \frac{2C^2}{\Phi}\Phi^{2n},$$

and, replacing these inequalities in (3.2) we get

$$\delta_n \geq \frac{2C^2}{\Phi}\left(\frac{\Phi^2}{2^{\frac{S_n}{n}}}\right)^n.$$

Taking logarithms, $\Phi^2/2^{S_n/n} > 1$ if

$$\frac{S_n}{n} < 2\log_2(\Phi).$$

□

Let us examine an example. For the following irrational quadratics

$$\alpha = [0; \overline{\underbrace{1, 1, \cdots, 1}_{30}, 12}], \quad \beta = [0; \overline{1, 12}],$$

according to theorems (4.1) and (4.2), we have the following limits:

$$\frac{S_n(\alpha)}{n} \to 1.354\ldots, \quad \frac{S_n(\beta)}{n} \to 6.5,$$

and, consequently:

$$\text{If } ?'(\alpha) \text{ exists, then } ?'(\alpha) = \infty$$
$$\text{If } ?'(\beta) \text{ exists, then } ?'(\beta) = 0.$$

In this way, it is easy to construct points belonging to Salem's set S, such that if the derivative exists it takes the value infinity. For instance, for the number whose continued fraction expansion is:

$$x = [0; 2, 1, 1, 1, 3, \overbrace{1, \ldots, 1}^{7}, 4, \overbrace{1, \ldots, 1}^{11}, \ldots, n, \overbrace{1, \ldots, 1}^{4n-5}, \ldots],$$

we have

$$\lim_{n\to\infty} \frac{S_n(x)}{n} = \frac{5}{4} < 2\log_2(\Phi),$$

and, according to theorem (4.2) if the derivative exists at x, it must be ∞, in spite of the fact that $x \in S$.

5. Metric sets for whose elements $?'$ is either 0 or ∞

Theorem 5.1. *If x, under the shift transformation of continued fractions, $Tx = 1/x - \lfloor 1/x \rfloor$, has an orbit whose asymptotic distribution function is Gauss measure, $dg = \log_2(1+x)$, then if $?'(x)$ exists its value is 0.*

Proof. It is seen at once that if under the shift transformation x has an orbit whose a.d.f. is $\log_2(1+x)$ we have:

$$\lim_{n\to\infty} \frac{S_n(x)}{n} = \infty.$$

and, according to theorem (4.1) we have that, if $?'(x)$ exists it is zero. Let us denote by N_{cf} the set of real numbers such that their orbit under the shift transformation has $\log_2(1+x)$ as its a.d.f. We have the following result, see [6]:

$$\lambda(N_{\mathrm{cf}}) = 1, \quad \lambda(?(N_{\mathrm{cf}})) = 0,$$

where λ is Lebesgue's measure. The result is a consequence of the fact that the images of N_{cf} are not normal, in the sense of Borel, in the alternated dyadic system. We are going presently to prove that the inverse images of normal numbers in the alternated dyadic system constitute a set where, if the derivative exists at one of its points, it is infinite. □

Let N_2 denote the set of normal numbers under the alternated dyadic system. It is clear that we have

$$\lambda(?^{-1}(N_2)) = 0, \quad \lambda(N_2) = 1,$$

and, moreover, we have the following result:

Theorem 5.2. *If $?'(x)$ exists (in a wide sense) for $x \in ?^{-1}(N_2)$, then $?'(x) = \infty$.*

Proof. The normality of $y = ?(x)$ under the alternated dyadic system is equivalent to say that its orbit under the corresponding shift transformation, $Ty = 1 - 2y$, is uniformly distributed in $(0, 1]$. If this orbit is $\{y_n\}$, then the orbit of x generated by the residue function of the continued fraction system will be $\{?^{-1}(y_n)\}$, which has as a.d.f. Minkowsk's function $?(x)$ as we have

$$\#\{i : ?^{-1}(y_i) \le z,\ i = 1, 2, \dots, n\} = \#\{i : y_i \le ?(z),\ i = 1, 2, \dots, n\}$$

and, as $\{y_n\}$ is uniformly distributed, we have:

$$\lim_{n\to\infty} \frac{\#\{i : ?^{-1}(y_i) \le z,\ i = 1, 2, \dots, n\}}{n} = ?(z).$$

Taking logarithms in $\delta_n = (2q_nq_{n-1})/2^{S_n}$ we have

$$\log_2(\delta_n) = 1 + \log_2(q_n) + \log_2(q_{n-1}) - S_n,$$

$$\log_2(\delta_n) = \left(\frac{1}{n} + \frac{\log_2(q_n)}{n} + \frac{\log_2(q_{n-1})}{n} - \frac{S_n}{n}\right) n.$$

Let us study the asymptotic behavior of $\log_2(q_n)/n$, knowing that the orbit of x has $?(x)$ as a.d.f. We have

$$q_n = q_{n-1}\left(a_n + \frac{1}{q_{n-1}/q_{n-2}}\right) = q_{n-1}\underbrace{[a_n; a_{n-1}, \cdots, a_1]}_{1/\theta_n}.$$

Therefore we have

$$\frac{\log_2(q_n)}{n} = -\frac{1}{n}\sum_{i=1}^{n}\log_2(\theta_i), \tag{5.1}$$

and taking limits

$$\lim_{n\to\infty}\frac{\log_2(q_n)}{n} = -\int_0^1 \log_2(x)\ d?(x).$$

Separating the integral in two parts

$$\int_0^1 \log_2(x)\ d?(x) = \int_0^{\frac{1}{2}} \log_2(x)\ d?(x) + \int_{\frac{1}{2}}^1 \log_2(x)\ d?(x),$$

and making the change $x = y/(1+y)$ in the first part and the change $x = 1/(1+y)$ in the second part, we have:

$$-\int_0^1 \log_2(x)\ d?(x) = 2\int_0^1 \log_2(1+y)\ d?(y),$$

which, according to the value found by Tichy in [9] is exactly $1/0.875 \approx 1.1428$.

Replacing this value in the expression

$$\lim_{n\to\infty}\left(2\frac{\log_2(q_n)}{n} - \frac{S_n}{n}\right),$$

and considering that that $\lim S_n/n = 2$, as $?(x)$ is normal, we have as a numerical approximation for the whole expression within the parenthesis 0.2857. Consequently, for the elements of the set $?^{-1}(N_2)$ we have

$$\lim_{n\to\infty}\delta_n(x) = \infty,$$

completing thus the proof of theorem (5.2). □

An alternative calculation for $\log_2(q_n)/n$ can be carried out using directly equation (5.1), taking as θ_i the first terms of a sequence whose a.d.f. be $?(x)$. Thus, taking the first 1000 terms of the sequence presented in [6], which enumerates all the positive rationals in $(0,1]$ in such a way that the enumeration has $?(x)$ as its a.d.f., we get the approximate value 1.143076, giving an alternative way of computing $\alpha = 0.8748325$

Theorem 5.3. *The Hausdorff dimensions of* $?(N_{cf})$ *and* $?^{-1}(N_2)$ *are 0 and* $\left(2\int_0^1 \log_2(1+x)d?(x)\right)^{-1}$*, respectively.*

Proof. For each positive integer n let us consider the following partitions of $(0,1)$. On the one hand,

$$S_2(n) =$$
$$\left\{\frac{1}{2^{k_1}} - \frac{1}{2^{k_2}} + \cdots + \frac{(-1)^{n+1}}{2^{k_n}} : 0 \leq k_1 < \cdots < k_{n-1} < k_n - 1; k_j \in \mathbb{Z}^+\right\}$$

(that is, all the n–term finite expansions), and on the other hand,

$$S_{cf}(n) = \left\{[0; a_1, a_2, \ldots, a_n] : a_j \in \mathbb{Z}^+; a_n > 1\right\}$$

(that is, all n–term finite regular continued fractions). The function $?(x)$ maps each interval of partition $S_{cf}(n)$

$$J_{cf}(a_1, \ldots, a_n) = ([0; a_1, a_2, \ldots, a_n], [0; a_1, a_2, \ldots, a_n + 1])$$

onto each interval of partition $S_2(n)$:

$$J_2(S_1, \ldots, S_n) = \left(\frac{1}{2^{S_1-1}} - \cdots \pm \frac{1}{2^{S_n-1}}, \frac{1}{2^{S_1-1}} - \cdots \pm \frac{1}{2^{S_n}}\right)$$

where, as before, $S_j = a_1 + \cdots + a_j$. The lengths of both intervals are

$$|J_{cf}(a_1, \ldots, a_n)| = \frac{1}{q_n^2 + q_n q_{n-1}}; \quad |J_2(S_1, \ldots, S_n)| = \frac{1}{2^{S_n}}.$$

This means that the ratio between both lengths is:

$$\overline{\delta}(a_1, \ldots, a_n) = \frac{|J_2(S_1, \ldots, S_n)|}{|J_{cf}(a_1, \ldots, a_n)|} = \frac{q_n^2 + q_n q_{n-1}}{2^{S_n}}.$$

Now, let $x = [0; a_1, a_2, \ldots, a_n] \in ?^{-1}(N_2)$. According to Theorem 5.2,

$$\lim_{n\to\infty} \frac{S_n(x)}{n} = 2; \lim_{n\to\infty} \frac{\log_2(q_n(x))}{n} = 2\int_0^1 \log_2(1+y)d?(y).$$

Hausdorff's dimension of $?^{-1}(N_2)$ is

$$\sup\left\{\alpha \in \mathbb{R}^+ : \lim_{n\to\infty} \frac{|J_2(S_1, \ldots, S_n)|}{|J_{cf}(a_1, \ldots, a_n)|^\alpha}\right\} < \infty.$$

If we denote the expression in the limit by $\delta_n(\alpha)$,

$$\delta_n(\alpha) = \frac{(q_n^2 + q_n q_{n-1})^\alpha}{2^{S_n}}.$$

The condition $\delta_n(\alpha) < \infty$ is equivalent to

$$\lim_{n\to\infty} \delta_n(\alpha) = \left(\frac{\alpha \log_2(q_n^2 + q_n q_{n-1})}{n} - \frac{S_n}{n}\right) \cdot n \neq \infty.$$

Finally, α must verify

$$\lim_{n\to\infty} \frac{\log_2(q_n^2)}{n} - \frac{S_n}{n} = 0.$$

That is, the value of α is

$$\alpha = \lim_{n\to\infty} \frac{S_n/n}{\log_2(q_n^2)/n} = \frac{1}{2\int_0^1 \log_2(1+x)d?(x)}.$$

The same ideas can be used to compute the Hausdorff dimension of $?(N_{cf})$:

$$\sup\left\{\beta \in \mathbb{R}^+ : \lim_{n\to\infty} \frac{|?(J_{cf}(a_1,\dots,a_n))|^\beta}{|J_{cf}(a_1,\dots,a_n)|} \neq 0.\right\} \tag{5.2}$$

Now $[0; a_1, \dots, a_n] = y \in ?(N_{cf})$. For these elements we have

$$\lim_{n\to\infty} \frac{S_n(y)}{n} = \infty;\ \lim_{n\to\infty} \frac{\log_2(q_n(y))}{n} = \frac{\pi^2}{12\ln^2(2)}.$$

Condition (5.2) is equivalent to

$$\lim_{n\to\infty} \left(\frac{\log_2(q_n^2 + q_n q_{n-1})}{n} - \beta\frac{S_n}{n}\right)\cdot n \neq -\infty,$$

which is only possible if β takes the value

$$\lim_{n\to\infty} \frac{\log_2(q_n^2 + q_n q_{n-1})/n}{S_n/n} = 0.$$

□

6. Conclusions

Minkowski's singular function derivative $?'(x)$ can only take two values when it exists in a broad sense: 0 and ∞. The same happens for the well–known family of Riesz–Nagy.

If $\liminf_{n\to\infty} S_n/n > 5.31972$, and $?'(x)$ exists then necessarily $?'(x) = 0$. This implies that for the set of real numbers x whose regular continued fraction expansion follows Gauss law $\log_2(1+x)$, the derivative $?'(x)$ can only be zero.

We also prove a weak converse: if $\limsup_{n\to\infty} S_n/n < 2\log_2(\Phi)$, then $?'(x) = \infty$. As a complement to this result we prove that for the inverse images of normal numbers y under the alternated dyadic system, for which the derivative exists we have $?'(?^{-1}(y)) = \infty$, or, equivalently $(?^{-1})'(y) = 0$

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