

# Ambiguity in Election Games\*

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## **Abstract**

We construct a model in which the ambiguity of candidates allows them to increase the number of voters to whom they appeal when voters have intense preferences for one of the alternatives available. An ambiguous candidate may offer voters with different preferences the hope that their most preferred alternative will be implemented. We find conditions under which ambiguous strategies are chosen in equilibrium. These conditions include the case in which there is an outcome that is a majority winner against all other outcomes but is not the most preferred outcome for a majority of voters. It is shown that if the number of candidates or parties increases, ambiguity will not be possible in equilibrium, but a larger set of possible policies increases the chance that at least one candidate will choose to be ambiguous in equilibrium.

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*"Ambiguity thus increases the number of voters to whom a party may appeal. This fact encourages parties in a two party system to be as equivocal as possible about their stands on each controversial issue. And since both parties find it rational to be ambiguous, neither is forced by the other's clarity to take a more precise stand."*

Anthony Downs, *An Economic Theory of Democracy* (1957)

## 1 Introduction

The aim of this paper is to analyze the strategic use of ambiguity by candidates or parties in electoral competition. While there is general agreement that politicians choose sometimes to be ambiguous, it is not clear how such strategic ambiguity is to be reconciled with models based on rational voters with standard preferences. Given a set of alternatives or policies, the ambiguity of candidates will be represented by nondegenerate probability distributions or lotteries over the set of policies or pure alternatives. A lottery that assigns probability one to a certain policy (a degenerate lottery) reflects no ambiguity. These lotteries are interpreted as the voters' beliefs about the policy preferences of the candidates. The relevance of the presence of ambiguity with respect to a candidate during an electoral campaign is reflected in the vote: voters' beliefs about the candidates' preferences will determine how they vote. If the set of alternatives is an interval of the real line, only risk loving voters would prefer an ambiguous strategy to the degenerate lottery that assigned probability one to the expected value of that lottery. However, if the space of alternatives is multidimensional, the expected value of a lottery may have no meaning as an outcome in itself. In this case, even risk averse voters may prefer a nondegenerate lottery to any certain outcome, and candidates may choose ambiguous strategies in equilibrium even when there is an unambiguous strategy that is a majority winner among pure alternatives. Ambiguous strategies arise as equilibrium outcomes because they provide voters with "intense" preferences with the hope that their most preferred outcome might be chosen.

A simple example will illustrate this point. Consider the problem in which one of three equally populated cities is to become capital of a country. Suppose that the preferences of the people is that their first choice is their own city, and that they are almost indifferent between the other two cities but have a slight preference for the closer of the two. Suppose further that there is

an election with two candidates vying for the position to be the one to choose the capital city. If one of the candidates convinces voters that he strongly prefers one of the three cities, the second candidate's optimal strategy is to rule that city out, but try to convince the voters that he truly doesn't know which of the remaining two cities is his favorite. Voters in the identified city of choice of the first candidate will prefer that candidate, but the voters in the other two cities will all prefer the uncommitted candidate, since he offers at least some hope that their (strongly preferred) first choice will be the outcome. The outcome, of course, will be that the second candidate who refused to commit will be elected. Note that this is true even if the choice of the first candidate is the Condorcet winner among the three cities. In essence, the intensity of the voters' preferences for their first choice makes ambiguity strategically attractive for candidates.

We reiterate the point made above that the intensity of voters' preferences for their most preferred city has nothing to do with risk loving attitudes. Risk averse voters may naturally prefer a lottery that puts equal probability on the two unchosen cities to the certainty of the city identified by the unambiguous candidate. In this example, there is no alternative that would offer voters the "expected value" of the lottery as an alternative to the lottery; there is no such certain alternative here.

We will set out a model to investigate formally the circumstances in which ambiguity will arise in equilibrium in candidate elections. Our formal model is Downsian (that is, candidates do not care about implemented policy, but rather, care only about holding office). The insights it provides extend beyond this framework, however. In our model an election is a three stage game in which players, candidates and voters alike, care only about the policy that will be implemented after the election. Their preferences are thus defined on the set of policy alternatives, and their objective is to maximize their expected utility. Voters know that the winner of the election will decide which policy to implement depending on his private policy preferences. However, at the time voters have to cast their vote, they do not know the candidates' policy preferences with certainty; they only have beliefs about them. The first stage of the election game is the electoral campaign, in which candidates attempt to induce beliefs in voters to affect favorably the outcome of the election. In the second stage voters choose among the candidates according to their policy preferences and to the beliefs they have formed about the candidates' preferences. Finally, at the third stage the winner of the election chooses a policy.

A central feature of our analysis is the manner in which these three stages are linked together. We assume that candidates cannot credibly commit to implement arbitrary policies. We will assume that there is a more complicated process linking a candidate's strategy choice in the first stage and the beliefs voters have about the outcome that is likely to arise in the last stage should the candidate be elected. Specifically, consider the decision in the final stage of the candidate who has won the election. His objective is to implement a policy that is optimal according to his preferences, and he is the only player of the game. Our assumption that candidates cannot credibly commit to specific policies implies that the strategies available to the candidate at this point coincide with the set of policies. Thus, he will implement his most preferred policy.<sup>1</sup> The link between the second and third stage is the usual one for two candidate competition: the candidate that obtains a majority of votes wins the election.

Anticipating the candidates' behavior in the last stage, voters will cast their vote for the candidate who provides them with a higher expected utility, the expectation taken with respect to their beliefs about candidates' preferences. Voters are assumed to update their beliefs rationally during the campaign stage.

We assume that the objective of all players is to maximize expected utility of the policy ultimately chosen, hence, for any beliefs that voters may have about one candidate, the second candidate will maximize his expected utility by maximizing his chance of being elected. Thus at the campaign stage both candidates compete solely for the votes and they will try to induce in voters beliefs that maximize their chances of winning. It follows then that the solution to the three stage election game, in which all players have preferences on policies and care only about the policy that will be finally implemented, will be the same as the solution to the simpler game in which both candidates care only about winning the election.

The question, then, is how candidates can influence voters' beliefs. One possibility is to model the exchange of information between candidates and voters at the campaign stage as a cheaptalk game: candidates send costless signals to voters, and voters rationally update their beliefs about the candidates' preferences (types). Harrington (1992) showed that in the equilibria

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<sup>1</sup>We consider the situation in which there is a single election. When there is a sequence of elections that a candidate may be involved in, there may be a tradeoff between present and future elections. This is discussed in detail in the concluding section.

of such a game, all candidates send the same signal (pooling equilibrium), and therefore voters realize that the signals provide no information about the true preferences of the candidates, and do not update their beliefs at all. The intuition that candidates cannot favorably affect voters' beliefs is straightforward: any candidate who finds himself disadvantaged in the cheaptalk stage can exactly mimic the announcements of the other candidate.

Campaign statements may affect voters' beliefs only to the extent that they convey information about candidates' preferences, and hence, future policy choice. For campaign statements to convey information in equilibrium, voters must update their beliefs differently upon hearing a statement made by different candidates. One way that this can occur is if statements are supported by evidence regarding candidates' true preferences; in other words, the statements are not cheaptalk. For example, a candidate could make public his past tax records or document his past voting record (or some part of it).<sup>2</sup> This implies that a candidate can induce a restricted set of beliefs in voters, and that different candidates may be able to induce different sets of beliefs in voters (Bill Clinton stating that he is committed to a law that gay marriages may well have more impact than Bob Dole making the same statement). Furthermore, the changes in voters' beliefs that can be induced would naturally be biased towards beliefs that are closer to his true preferences.

Consequently, we can reduce the initial three stage election game to a one stage game with two Downsian candidates. The candidates care only about winning, and have (typically) different strategy sets that are the beliefs they may induce in voters. We will analyze this model to understand the conditions under which candidates may be ambiguous in equilibrium. Initially, we will deal with the case in which there are two candidates, or parties, and three alternatives, but these restrictions will then be relaxed.

The remainder of the paper is organized as follows. In section 2 we review related literature. Section 3 presents the formal description of the reduced form version of the election game. In section 4 we present and discuss the results on ambiguity in equilibrium for two-party competition. Section 5 presents an extension of the model with uncertainty on the voters' preferences and we replicate our previous findings. In section 6 we show that increasing

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<sup>2</sup>We don't suggest that such statements provide definitive proof of a candidate's preferences, only that they are examples of acts that candidates can take that other candidates cannot.

the number of candidates or parties decreases the possibility of ambiguity in equilibrium. Finally, section 7 offers some concluding remarks.

## 2 Related Literature

Fishburn (1972) provides one of the first formal results about ambiguity in electoral competition. He shows that for any set of alternatives, finite or infinite, and for any distribution of voters' preferences which satisfy relatively weak assumptions (conditions that are only necessary for expected utility representation), if we consider the set of all lotteries over alternatives as the choice set, the only possible Condorcet winner will be a degenerate lottery. Since the equilibrium outcome of a two-candidate competition à la Downs must coincide with the Condorcet winner, we cannot have ambiguity in equilibrium in such a model.

Since this result holds for any set of alternatives, and any reasonable distribution of voters' preferences, ambiguity can only arise as an equilibrium outcome if we depart from Fishburn's assumptions. In the game derived from Fishburn's model it is assumed that the candidates' strategy sets are the same: the set of all lotteries over alternatives or policies. A model that leads to pure strategy equilibria clearly must introduce differences in the candidates' strategy sets.

Shepsle (1972) considers a standard spatial model, where the set of alternatives is the real line and voters' preferences are single-peaked. Since this is a special case of Fishburn's model, the result discussed above applies if the strategy sets for the candidates are assumed to be the set of all lotteries. Shepsle introduces an asymmetry in the candidates' choice sets by restricting the incumbent to choose only degenerate lotteries, and the challenger is restricted to choose only nondegenerate lotteries. Shepsle finds that the challenger can only win the election if a majority of the voters' preferences exhibit risk loving, and these voters' ideal points lie in a neighborhood of the median voter's ideal point.

In this model, the defeat of the median voter's most preferred policy by an ambiguous strategy arises because a majority of voters derive utility from ambiguity. Thus, the choice of ambiguity by candidates during electoral campaigns is explained by the preferences of the decisive voters for ambiguity. This result certainly implies Downs' claim: "Ambiguity thus increases the number of voters to whom a party may appeal", but the explanation is less

than compelling since it rests on the questionable assumption that voters may be risk loving.

In this paper we set forth a Downsian model with an asymmetry in the candidates' strategy sets and provide conditions under which a candidate may choose an ambiguous strategy and win the election, even when his choice set contains degenerate lotteries. Hence, we also have candidates who choose ambiguous strategies in equilibrium in order to appeal to a larger number of voters, as Downs (1957) suggested, but, unlike Shepsle, with voters who do not have a direct preference for ambiguity. They derive utility only from the policy that is finally implemented; they prefer an ambiguous candidate only if he provides them with a higher expected utility. An ambiguous candidate provides different voters with the hope that their most preferred alternative will be chosen.

In the literature there are several approaches that offer alternative explanations for ambiguity that differ in spirit. These approaches assume that at the campaign stage candidates can credibly commit to implement certain policies should they win the election. The implication of this assumption is that at the last stage of the election game the winner of the election can only implement the policies to which he has committed during the campaign.

Under this assumption, the campaign stage may be represented as before as a signaling game, but in this case the signals that candidates send are costly, since they constrain the set of strategies candidates will have available once in office. Banks (1990) shows that if the cost of the signals is high enough, candidates with strong policy preferences decide to reveal their preferences to voters (commit to their most preferred policy) at the campaign stage: information revelation obtains in equilibrium. If the signaling costs are low, candidates' strategies in equilibrium are uninformative, and when the costs approach zero the model approaches Harrington's (discussed before). Notice that the reduced election game in this case differs from the one obtained when candidates cannot commit in a very relevant characteristic: a candidate's chances to win the election depend on the policy that he will implement in the last stage of the game. Therefore, at the campaign stage when candidates maximize their expected utility they face a tradeoff: committing to their most preferred policy increases their utility when in office but might decrease their chances of winning the election. Thus, an ambiguous strategy seems a natural way for candidates to optimize.

Aragones and Neeman (1998) explore this tradeoff in order to obtain ambiguity in equilibrium in two different frameworks: a spatial model, and a



model with a finite number of policies. In both cases voters are assumed to be risk averse, but when the indirect preference for ambiguity of the candidates is strong enough they will choose ambiguous strategies in equilibrium. Alesina and Cukierman (1990) analyze a model in which two candidates face the same tradeoff, but in a different setting. They construct a two period game of repeated elections, and assume that voters form beliefs on candidates' policy preferences from the observation of the outcome of the previous election game. In their model, the outcome of the game observed by the voters depends stochastically on the policy implemented and the incumbent is allowed to choose the precision of the signal the voters receive. The choice of a low precision, that will hide information on the policy implemented, optimally solves the tradeoff faced by the incumbent, and is the representation of ambiguity in this setting.

### 3 The model

There are two candidates who compete for votes in an election. We assume that there are three alternatives:  $Z = \{A, B, C\}$ . We denote by  $\Delta Z$  the set of probability distributions over the set of alternatives

$$\Delta Z = \{(p_A, p_B, p_C) : p_A, p_B, p_C \geq 0, p_A + p_B + p_C = 1\}.$$

We will abstract from the process by which politicians induce beliefs and model a politician's strategies as those beliefs that he induce in the voters. We discuss this reduced form aspect of our model and the type of processes that would justify it in the discussion section. We denote by  $W_i \subseteq \Delta Z$  the set of beliefs that candidate  $i$  can induce in voters;  $p^i \in W_i$  represents a strategy for candidate  $i$ . We will say that a strategy is *ambiguous* if it represents nondegenerate beliefs.

We focus on a particular class of strategy sets motivated by the assumptions discussed in the introduction. We will consider strategy sets for candidate  $i$  whose most preferred policy is alternative  $Z_i$  that contain all probability distributions over the set of alternatives  $Z$  that assign a minimum probability of  $a_i > 0$  to alternative  $Z_i$ . Formally  $W_i = \{p \in \Delta Z : p_{Z_i} \geq a_i\}$  for some  $a_i \in [0, 1]$ . We define similarly the strategy sets for candidates with other most preferred policies. Such a strategy set implies that a candidate is able to convince voters of his true preferred policy and, in addition, is able

to induce in voters beliefs that to a limited extent his most preferred policy choice is one of the other alternatives.  $\alpha$  can be interpreted as the extent to which the candidate is able to be ambiguous. The structure we place on the candidates' strategy sets is stronger than necessary for our results but eases exposition. We discuss below how the structure of the sets can be weakened without altering our results.

We assume that there are  $N$  voters, where  $N$  is a positive odd integer. Voters have preferences over alternatives that can be represented by von Neumann-Morgenstern utility functions. We normalize a voter's utility function so that it assigns one to the most preferred alternative and zero to the least preferred; we denote by  $x \in [0, 1]$  the utility of the intermediate alternative. Hence, each voter is characterized by an ordering of the alternatives represented by  $u : Z \rightarrow \{0, x, 1\}$  where  $x \in [0, 1]$  represents the intensity of the voter's preferences. It's worthwhile commenting on the interpretation of  $x$ . If the alternatives have a natural linear order (such as varying amounts of money), "intense" preferences (i.e.,  $x < .5$ ) correspond to risk loving preferences. But as the city example in the introduction illustrates, it may be the case that voters have intense preferences in problems unrelated to risk preferences.

We will restrict attention to the case in which voters who have the same preference order have the same preference intensity as well for the sake of clarity. Hence, there are six types of voters, each with one of the six possible ordinal (strict) rankings of the three alternatives. We will use  $G_k$  both to denote the name of one of these groups of voters and to denote the number of voters in that group; no confusion should arise from this abuse of notation. The preferences for each of the groups are given in the following table.

$G_1$	$A \succ B \succ C$
$G_2$	$A \succ C \succ B$
$G_3$	$B \succ A \succ C$
$G_4$	$B \succ C \succ A$
$G_5$	$C \succ A \succ B$
$G_6$	$C \succ B \succ A$

Given a voter's beliefs, he or she votes for the candidate that maximizes

his or her expected utility<sup>3</sup>. That is, voter  $j$  votes for candidate 1 if and only if  $E_{p_1}u_j > E_{p_2}u_j$  ( $E_{p_k}u_j$  is the expected utility for voter  $j$  with respect to his probability beliefs about candidate  $k$ 's most preferred policy), and votes for candidate 2 if and only if  $E_{p_1}u_j < E_{p_2}u_j$ , and does not vote if  $E_{p_1}u_j = E_{p_2}u_j$ .

We analyze first the case in which candidates know the distribution of voters' preferences and their intensities. We provide conditions under which pure strategy Nash equilibria exist, and further, when they involve one or the other candidate being ambiguous.

## 4 Ambiguity as an Equilibrium Outcome

Our aim is to demonstrate conditions under which candidates will choose non-degenerate lotteries in equilibrium, that is, choose to induce beliefs that leaves voters uncertain about the outcome that will be chosen if the candidate is elected. We begin by observing that if one of the pure alternatives is the first choice of a majority of voters, then no lottery can defeat it. In this case this alternative is the Condorcet winner (an alternative that defeats all other alternatives in pairwise elections using majority rule) in the set of pure alternatives  $Z$  as well as in the set of lotteries over pure alternatives,  $\Delta Z$ . A degenerate lottery that assigns probability one to this alternative will defeat any other lottery in  $\Delta Z$ . Therefore, if the lottery that assigns probability one to the Condorcet winner is available to one of the candidates (that is, in  $W_i$ , the set of beliefs candidate  $i$  can induce), then it will be chosen in equilibrium and it will defeat any lottery chosen by the opponent. Thus, there cannot be ambiguity in this case.

A more interesting situation arises when the preferences of the voters are such that no pure alternative is the first choice of a majority of voters but there is a Condorcet winner among the pure alternatives, that is, a pure alternative that defeats all other alternatives in  $Z$  in pairwise elections using majority rule. In this case, it is still possible that a lottery might defeat this alternative under majority rule. Thus, even if a candidate can convince the voters that he will choose the alternative that is most preferred by a majority to any other alternative, this candidate may be defeated by an opponent who

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<sup>3</sup>We assume that preferences over lotteries can be presented by expected utility for ease of exposition. Our results would still hold for a broader class of preferences satisfying some but not all of the axioms needed for expected utility.

chooses to be ambiguous.<sup>4</sup> Without loss of generality assume that alternative  $B$  is the Condorcet winner in the set of pure alternatives, but is not the first choice of a majority of voters. Suppose also that this is candidate 1's most preferred outcome; we will refer to this candidate as the Condorcet candidate. This candidate has available a strategy that induces beliefs in voters that put probability 1 on this outcome by assumption. Suppose the other candidate's most preferred outcome is  $A$ . If this candidate has limited ability to be ambiguous, voters will believe with probability close to 1 that  $A$  will be the outcome if he is elected. Since  $B$  is a Condorcet winner that defeats  $A$ , it will also defeat lotteries that put very high probability on  $A$ . Consequently, if the second candidate has sufficiently limited ability to be ambiguous, the first candidate can be guarantee to voters that he prefers  $B$  and win the election. Hence, again, the winning strategy will be unambiguous.

But suppose that the second candidate has greater scope for being ambiguous. Specifically, suppose he can induce in voters beliefs that his most preferred outcome is either  $A$  or  $C$ , with the probability on each being close to  $1/2$ . If each group of voters  $G_i$  has intense preferences (i.e.,  $x_i < .5$ ), all voters whose first choice is not  $B$  will prefer this lottery to the certainty of getting  $A$ . Hence, when voters have intense preferences and candidates have sufficient scope to be ambiguous, an ambiguous candidate will defeat one known to favor the Condorcet winner. Note that it may not be necessary for the ambiguous candidate to be able to induce a probability close to  $1/2$  on the outcome that is not his most preferred. While this degree of latitude to be ambiguous is sufficient to defeat the Condorcet winner when voters have intense preferences, the level of ambiguity necessary to be able to defeat the Condorcet winner is related to the intensity of preferences: the more intense the preferences (i.e., the smaller the  $x$ ), the lower will be the required degree of ambiguity. If there is any scope at all for ambiguity, for sufficiently small  $x$  the Condorcet winner will be defeated.

The argument above does not demonstrate that the candidate who can guarantee voters that the Condorcet winner is his most preferred outcome will lose, only that sufficiently ambiguous candidates will defeat this candidate if he makes known to voters his most preferred outcome. If the candidate associated with the Condorcet winner has limited scope to be ambiguous himself, he will be defeated. But, if he has sufficient scope himself to be ambiguous, he may still have a winning strategy. What this argument demonstrates is

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<sup>4</sup>An example along these lines with three voters was constructed by Zeckhauser (1969).

that if the non-Condorcet candidate has sufficient scope for ambiguity when voters have intense preferences, the winning strategy in any pure strategy equilibrium in the election will be ambiguous. Either the non-Condorcet candidate wins because the Condorcet candidate has insufficient flexibility to be ambiguous or the Condorcet candidate has sufficient flexibility to respond to any ambiguous strategy with a strategy that is itself ambiguous and appeals to a majority of the voters.

We will summarize this argument. Recall that candidate  $i$ 's strategy set is  $W_i = \{p \in \Delta Z : p_{Z_i} \geq a_i\}$

**Proposition 1:** *Consider a profile of preferences such that there is a Condorcet winner which is not the first choice of a majority of voters and voters have intense preferences (i.e.,  $x_i < .5, i = 1 \dots 6$ ). Suppose also that candidate 1 is a Condorcet candidate. Then there exist  $a_1, a_2 > 0$  such that a pure strategy equilibrium exists in the voting game, and the winning strategy is ambiguous.*

The proof is left to the appendix. Before going on, we will make several remarks about this result. The proof proceeds by showing that there will be a pure strategy equilibrium in which the winning strategy is ambiguous if and only if the  $a_i$ 's satisfy particular inequalities that depend on the given set of intensities  $(x_1, \dots, x_6)$ . From the inequalities that determine the set of  $a_i$ 's for which the winning strategy is necessarily ambiguous, we can say several things. First, suppose that for the levels of the ambiguity for the two candidates,  $a_1, a_2$ , there is a pure strategy equilibrium of the voting game in which the non-Condorcet candidate wins (necessarily with an ambiguous strategy). Then, if the voters preferences become more intense (i.e., the  $x_i$ 's decrease), that candidate will continue to have a winning strategy. If initially there is a pure strategy equilibrium with the Condorcet candidate winning with an ambiguous strategy, it may be the case that an increase in voters' intensities leads to nonexistence of pure strategy equilibria. However, if following an increase in voter's intensities there is a pure strategy equilibrium, it must be that the winning strategy is ambiguous. Roughly speaking, ambiguity is more likely when voters care more intensely about getting their most preferred outcome.

We will consider next the case of a Condorcet cycle in the set of pure alternatives  $Z$ , that is, every pure alternative is defeated by another pure alternative in a pairwise majority rule contest. Fishburn (1972) has shown that in this case there is no Condorcet winner in the set of lotteries over pure

alternatives  $\Delta Z$  either. Thus, if the strategies of the two candidates are all lotteries over pure alternatives, then there is no pure strategy equilibrium. In our framework, however, candidates are restricted in the set of beliefs that they can induce in voters, hence there is a possibility that pure strategy equilibria exist.

Suppose that the distribution of voters' preferences generates a Condorcet cycle  $A$  defeats  $B$ ,  $B$  defeats  $C$ , and  $C$  defeats  $A$ . Suppose further that candidate 1 and 2's most preferred outcomes are  $A$  and  $B$  respectively, and that each has very limited ability to be ambiguous. That is,  $W_1$  and  $W_2$  contain lotteries that respectively put probability close to 1 on the two candidates' most preferred outcomes  $A$  and  $B$ . Since the Condorcet cycle is such that  $A$  defeats  $B$ ,  $A$  will also defeat any lottery that puts probability sufficiently close to 1 on outcome  $B$ . Hence, it is straightforward that when there is sufficiently limited ability to be ambiguous, the candidate who can induce the beliefs in voters that his most preferred outcome is surely  $A$  will do so and, consequently, win the election. As in the case of a Condorcet winner discussed above, the winning strategy need not be ambiguous. But suppose now that the candidate who prefer outcome  $A$  has available a greater scope to be ambiguous, that is, he is able to induce beliefs in the voters that place greater probability on outcomes other than  $A$ . If voters have intense preferences, the outcome will be similar to that in the case above with a Condorcet winner. If candidate 2 has sufficient scope to be ambiguous, he will have available a strategy that defeats  $A$ . In particular, if he is able to induce in voters beliefs that he will never choose  $A$  but is indifferent over  $B$  and  $C$ , he will be able to defeat a candidate who is nearly perfectly identified with outcome  $A$ . For any level of intensities  $x_i$ , one could compute in a manner similar to that in the proof of theorem 1 the levels of ambiguity  $a_1$  and  $a_2$  for which there is a pure strategy equilibrium to the voting game in which the winning strategy is ambiguous. It is worthwhile pointing out that in the case that voters don't have intense preferences ( $x_i > .5$ ), there may be pure strategy equilibria in which the winning strategy is ambiguous. For example, if candidate 1 has minimal flexibility to be ambiguous while candidate 2 can induce beliefs in voters that his first choice is very likely  $C$ , candidate 2 will win, since in the voting cycle  $C$  defeats  $A$ , and consequently, any lottery putting probability close to 1 on  $A$ . However, the ability of candidate 2 to induce beliefs in voters that his most preferred outcome is almost certainly  $C$  when it is, in fact,  $B$  should probably be considered deception rather than ambiguity.

## More than three alternatives

We restricted our model to the case in which there are three alternatives primarily for ease of exposition. Our aim was to show that intense preferences for voters could lead naturally to ambiguity in equilibrium. In the discussion and proof of the result, we took the utility of a voter's middle ranked alternative for his normalized von Neumann-Morgenstern utility function to be a measure of the intensity of his preferences. When there are three alternatives, this single number completely characterizes the cardinal information about the relative strength of the voter's preferences. When there are four or more alternatives, there isn't a single number that characterizes the relative strength of preferences for any pair of outcomes. But from the logic of our result it can be seen that it is the relative strength of the preference for the most preferred outcome over all other outcomes that matters, and we will continue to use the utility of the second ranked alternative as a measure of the intensity of preferences of voters. Suppose that in the city example in the introduction there had been an arbitrary finite number of cities, and suppose that each voter has a most preferred city, but is close to indifferent over all other cities. The argument presented for the three alternative case would carry over to this more general case. If there is a Condorcet winner among the degenerate lotteries that is not the first choice of a majority of the voters, the ambiguous strategy that puts equal probability on all first choices other than the Condorcet winner will be preferred by a majority of voters to the Condorcet winner if the utility of the second ranked alternative is sufficiently low for all voters. Simply put, when a voter is almost indifferent between all outcomes but his most preferred, a small probability of getting that most preferred outcome is better than his second choice for sure. Thus, one could extend the model to the case of multiple alternatives by using as a measure of the intensity of voters' preferences the utility of the second ranked alternative for the normalized von Neumann-Morgenstern utility function. The extension of the arguments to this case follow those in this paper and we leave them to the reader.

## 5 Uncertainty about Voters' Intensities

The model of electoral competition we presented in the last section was highly simplified in order to make clear the role that strategic ambiguity could play. A consequence of the assumption that candidates know precisely the inten-

sity of voters' preferences is that after choosing strategies, one or the other of the candidates wins with probability 1. This made the expected payoffs to the candidates as a function of the joint choice of strategies highly discontinuous: any change in a candidate's strategy has either no effect whatsoever, or reverses the outcome completely. The difficulty in establishing the existence of pure strategy equilibria stemmed to a large extent from this discontinuity of the candidates' payoff functions. In this section we will extend the model above in a way that both makes the model more plausible from a positive point of view and increases the chance that pure strategy equilibria exist, thus enabling us to carry out several simple comparative statics exercises.

We maintain the assumption that voters with the same preference order have the same preference intensity, but assume that both candidates are uncertain about the intensities. We assume  $x_i$ ,  $i = 1, \dots, 6$  are independently and identically distributed, and that the candidates have common beliefs about the distribution function  $F$ . We also assume that  $F$  is strictly concave,<sup>5</sup> has density function  $f$  and is twice continuously differentiable, with positive density everywhere. The restriction to concave  $F$  is consonant with our interest in the case that voters have intense preferences; with a concave  $F$ , there is higher probability on lower values of  $x_i$  than on higher values.

The probability with which a candidate wins the election, that is, the probability with which a candidate gets at least a majority of the votes, depends on the strategies chosen by the two candidates  $(p^1, p^2)$ . Thus, the payoff function of candidate  $i$  (the probability that he wins) can be represented as a function  $P_i(p^1, p^2)$  which takes values in  $[0, 1]$ .

The assumptions on  $F$  lead to a richer interaction between the candidates, with both candidates having positive probability of winning for typical joint choices. The conditions are not, however, sufficient to guarantee existence of a pure strategy equilibrium. If we add an assumption that the candidates' strategy sets are disjoint, the payoff functions  $P_i$  are continuous and there will be a pure strategy equilibrium. We will maintain our assumed structure of candidates' strategy sets, namely that they contain all lotteries that put some minimum probability on the candidate's most preferred alternative. We add the assumption that this minimum probability is at least  $1/2$  in order

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<sup>5</sup>A strictly concave probability distribution is sufficient for the results that follow, but it is not necessary. They can be obtained under weaker assumptions, for example: probability distributions that are symmetric or have monotone likelihood ratio will satisfy the first order condition. The second order condition can also be satisfied by nonconcave probability distributions.



to guarantee that the candidates' strategy sets are disjoint.

Consider again the case in which the preference profile of the voters is such that there is a Condorcet winner in the set of pure alternatives but it is not the first choice of a majority of the voters. Suppose the Condorcet winner is the most preferred outcome of candidate 1 and that the second candidate's most preferred outcome is one of the other two outcomes. It is clear that the non-Condorcet candidate must choose an ambiguous strategy in equilibrium. If he were to play the unambiguous strategy that puts probability 1 on his most preferred outcome, the best response for the Condorcet candidate is to pick the degenerate lottery putting probability 1 on the Condorcet winner, which leads to the Condorcet candidate winning with probability 1. Trivially, this cannot be an equilibrium, since with any other strategy the non-Condorcet candidate will win with positive probability.

Hence, in a pure strategy equilibrium, the non-Condorcet candidate must choose an ambiguous strategy. But it can be shown that the best response of the Condorcet candidate to any ambiguous strategy is to be ambiguous himself, that is, in any pure strategy equilibrium, both candidates will choose ambiguous strategies. Further, the strategic use of ambiguity is "escalating": in equilibrium both candidates will be maximally ambiguous, in the sense that had they not been constrained by the exogenously given strategy sets, a further reduction of the probability placed on their most preferred outcome would increase their probability of winning. We summarize these findings in the following proposition whose proof is left to the appendix.

**Proposition 2:** *Consider a profile of preferences such that there is a Condorcet winner which is not the first choice of a majority of voters and voters' intensities are given by a distribution function  $F$  satisfying the conditions above. Suppose that candidates' strategy sets are as described above with  $a_i > \frac{1}{2}$  for  $i = 1, 2$  and that candidate 1 is a Condorcet candidate. Then there is a pure strategy equilibrium. Further, in any pure strategy equilibrium, both candidates put the minimal possible probability on their most preferred outcome given their strategy sets.*

This proposition demonstrates that for some political contests, the strategic use of ambiguity is a necessary component. The proof is similar to the proof of the first proposition in that establishes that a best response to an opponent's strategy is the solution to a set of linear inequalities. From these inequalities, we can see how the probability that a particular strategy will lead to being elected is affected by the parameters of the problem. First,

suppose we want to compare two problems with the voters in one having more intense preferences than in the other. One notion of an electorate having more intense preferences than a second would be that the distribution function of the second, say  $F$ , stochastically dominated the distribution function of the first, say  $G$ . In this case,  $G$  would put higher probability on lower intensities than  $F$ . In this case, the probability that any ambiguous strategy for the non-Condorcet candidate defeats the degenerate lottery that puts probability 1 on the Condorcet winner is higher for distribution function  $G$  than for  $F$ . An increase in the intensity of preferences as measured by stochastic dominance increases the benefit of any ambiguous strategy against the Condorcet winner.

We can also deduce from the inequalities determining the equilibrium how a change in the level of ambiguity available to a candidate, keeping his opponent's ambiguity level fixed, will affect the outcome. Specifically, the probability that a candidate will win increases when his allowed level of ambiguity alone is increased.

## 6 Increasing political competition eliminates ambiguity

In this section we relax the condition that there are two parties and consider the case in which there are  $I$  parties,  $I \geq 3$ . We assume that the objective function of each party is to maximize the proportion of votes it obtains in the election. We also relax the restriction on the set of alternatives and allow  $|Z| \geq 3$ . We assume that the set of strategies of each party is the set of all probability distributions over the set  $Z$ . We assume that voters have von Neumann-Morgenstern utility functions over  $Z$  and that they vote for the candidate who offers the highest expected utility. Lastly, we assume that if more than one party chooses the same lottery, the parties doing so get equal shares of the voters for whom this is the most preferred lottery on offer.

In the sections above, we provided conditions under which the election outcome was a pure strategy equilibrium in which the winning strategy was ambiguous. Roughly, the logic of the argument was that when voters had intense preferences, if the Condorcet outcome was not the most preferred outcome for a majority of the voters, it could be defeated by a lottery putting equal probability on the outcomes other than the Condorcet outcome. As

discussed above, the logic is not restricted to the case of three alternatives. Consider the set of voters whose most preferred outcome is not the Condorcet outcome. A lottery that places equal probability on all the outcomes that are the most preferred outcome for some voter in this set will be preferred by all of these voters to the Condorcet outcome if preferences are sufficiently intense.<sup>6</sup>

We will show that when there are sufficiently many candidates or parties, there cannot be ambiguity in equilibrium. The idea can be illustrated with a simple example. Think of the voters being divided into groups, with each group having precisely those voters with an identical most preferred outcome. Suppose that of these groups, the smallest nonempty group contained 10% of the voters. Lastly, suppose that there are at least 11 parties and that in equilibrium, at least one chooses an ambiguous strategy. If the most preferred outcome of each of the groups is the unambiguous strategy of some party, then clearly an ambiguous strategy will get no votes. Hence, if an ambiguous strategy is chosen in equilibrium, it must put positive probability on at least one outcome that is the most preferred outcome of some group and that is not offered as an unambiguous strategy by some party. But if there are at least 11 parties, at least one must get less than 10% of the votes. If this party chooses instead an unambiguous strategy that is the most preferred outcome of some group that is not targeted by any other party (in the sense that some party has chosen this group's most preferred outcome as an unambiguous strategy, it will get the votes of this group. But this must increase the number of votes it gets, contradicting the supposition that an ambiguous strategy could be an equilibrium choice when there were 11 parties.

The logic of this example is formalized in the following theorem.

**Proposition 3:** *Fix the distribution of voters and their utility functions over  $Z$ . Let  $N'$  be the smallest number of voters with a common most preferred outcome. There is a nonempty set of pure strategy equilibria and every party chooses an unambiguous strategy in every pure strategy equilibrium if the number of parties is greater than  $N/N'$ .*

There is a natural intuition behind the result: when the number of candidates is sufficiently large, competition for votes will provide incentives for candidates to target any group of voters that is not already targeted by another party. On the margin, a candidate will do better by specializing in

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<sup>6</sup>Recall that the utility of the second ranked alternative is the measure of intensity and that preferences are said to be more intense when this number is lower.

some subset of the untargeted voters than by competing with other ambiguous candidates for the entire pool of such voters.

The above result takes as given the number of alternatives. While this is a natural assumption to make, for many problems of interest, there is not a natural set of exogenously given alternatives; the alternatives are, in fact, determined within the political process. We presented in the introduction the example of the choice of a capital city, for which we assumed that there were three possible choices. In theory, one could imagine that any of the cities within the country is theoretically a potential capital of the country. The result above states that with sufficiently many parties, ambiguity will not be possible in equilibrium. The logic was that with sufficient number of candidates or parties, each subgroup of voters with a common most preferred alternative would be the target of some party that unambiguously chooses as a strategy the degenerate lottery that places probability 1 on that most preferred outcome. Once voters are targeted in this way, ambiguity offers no advantages.

However, consider reversing the order in which the number of alternatives and the number of parties are increased. For an initial set of alternatives, the fixed number of parties might be sufficiently large that the only pure strategy equilibria are in unambiguous strategies. But if the number of alternatives gets larger, there will be, in general, more groups of voters determined by most preferred outcomes. If the number of alternatives gets large enough, there won't be enough parties for each group to be targeted by at least one party. When the set of alternatives gets large enough, some party might find it advantageous to target the pool of disaffected voters with an ambiguous strategy that puts positive probability on precisely those alternatives that are some voter's most preferred outcome, but not the unambiguous strategy of any party. In the example of choosing a capital city, this would amount to a party choosing a strategy such as "We are not sure where the capital should be, but it should definitely *not* be any of those the other parties propose." In summary, the logic of our model suggests that when the set of potential alternatives is large relative to the set of parties, voters are relatively dispersed with respect to their most preferred outcomes and preferences are intense, ambiguity is more likely to arise as an equilibrium strategy.

## 7 Concluding remarks

*The possibility of commitment.* We have taken the approach in this paper that candidates cannot commit to post-election choices during an election. It is worthwhile discussing this modelling choice.

First, there are several disadvantages of assuming that commitment is possible. If candidates are assumed to be able to commit to their platform we often get cycles in pure strategies and we are forced to deal with mixed strategy equilibria (see for example, Myerson (1993) and Lizzeri (1997)). Mixed strategy equilibria are undesirable for several reasons. They do not offer clear qualitative predictions, and typically do not allow comparative statics. Additionally, in political models that assume commitment, it is not clear how commitment can be monitored if the platform committed to is represented by mixed strategy. An advantage of our approach is that without assuming commitment it yields pure strategy equilibria.

Besides the difficulties that arise if commitment is assumed to be possible, there is a very real question of the degree to which politician, in fact, can commit. Commitment is problematic in most models simply because the physical strategies that are available to a candidate who has won an election are almost always unchanged by actions taken prior to the election. Hence, commitment cannot be taken to mean that a candidate has eliminated the possibility that he will do something other than what he has promised.

Commitment is sometimes justified by an argument that in an environment with repeated elections, commitments can be sustained by a trigger strategy equilibrium in which voters will not reelect a candidate who does not fulfill his commitments. One can think of this as a reduced-form approach in which, observing that in the equilibria of the more complex "real" game, politicians in fact fulfill any commitments they make, an analyst writes down a one-shot game in which it is assumed that the politician can commit to policies.

There are several difficulties with this approach. First, in the actual repeated election game it may be true that, in equilibrium, politicians fulfill any commitments they make, but that does not mean they will fulfill all commitments they *could* make. It's possible that there are promises that a candidate could make that voters know will not be fulfilled, hence they would ignore. Consequently, a candidate wouldn't make such promises since they have no affect on voters' behavior. But a reduced-form approach to commitments would, in effect, assume that the candidate could *actually* make the

commitment, and voters would respond. In sum, a justification for assuming that candidates will fulfill commitments because of future consequences should they renege must model directly the costs and benefits of renegeing on commitments in order to determine which commitments will be made and have an effect on voters' behavior.

A second difficulty is that the trigger strategy may not be plausible for reasons similar to the reasons that strategies that are not subgame perfect are not plausible. Suppose that in a repeated election situation a candidate who has failed to fulfill a commitment he made previously is up for reelection. The strategy presumed in the repeated election reduced-form argument would prescribe that the voters vote against such a candidate. But what if the alternatives to this candidate are demonstrably worse than he is, as measured by the expected utility of future policies? There are difficulties in reconciling the credibility of voters' threats to punish candidates who renege on promises with the rational actor foundations of voting models.

A third problem with the repeated game trigger strategy justification for commitment stems from the particular simplifications inherent in typical voting models. Voting models typically posit two candidates who vie for office with the winner to choose a policy (or set of policies). This abstracts, for reasons of tractability, from aspects of the problem that are not central to many issues, but are important to the logic underlying the trigger strategy justification. Few interesting electoral competitions lead to a winner who unilaterally chooses the outcomes following the election. More typical is a situation such as in U.S. presidential elections in which the winner bargains with and cajoles a Congress to determine final outcomes. Here, the president cannot dictate outcomes ignoring Congressional sentiment, but rather, needs its cooperation or consent to enact many policies. But what is it that a presidential candidate can commit to in a world in which final outcomes are the result of some kind of bargaining or negotiating process? A candidate cannot commit, say, to spending more money on schools, since it isn't in his power to deliver on such promises. Instead the nature of promises must necessarily be of the form "I will do everything possible to increase spending on schools." But the enforcement of such promises or commitments via trigger strategies is problematical. How do voters verify that the candidate, upon being elected "did everything possible to increase school spending"? Certainly an elected candidate who voted against (or vetoed) such a bill would be understood to have reneged on his promise, but short of this, voters may find it difficult to determine the amount of energy an elected official puts

into effecting those policies to which he committed to in the election. When there is bargaining over a number of issues and the bargaining isn't public, it may be impossible for voters to know whether an elected official gave any particular issue top priority.

*Beliefs as strategy sets.* The discussion above details the difficulties with assuming that candidates can choose arbitrarily among the set of platforms, and commit to any of them. This is not to argue that they cannot make strong statements - even promises - and that these might affect voters' behavior through changes in their beliefs about what the candidate will do if elected. Casual observation makes it clear, though, that political candidates' promises often have little or no effect on voters.

The discussion of the role of commitment in voting models provides the basis for our approach - a weakening of commitment. There are actions that candidates can take and statements that they or others can make that can affect voters' beliefs. The choice of a minority member as a running mate may lead voters to revise their beliefs about a candidate's sincerity about civil rights, and endorsements from abortion rights groups may lead voters to revise their beliefs about a candidate's intentions if it's thought that the endorsing groups have superior information about a candidate's preferences. Our model starts from the premise that there *are* things candidates can do to affect voters' beliefs, but a candidate's ability to change beliefs is necessarily limited.

We could have modeled the political discourse underlying these ideas directly, but have chosen not to, instead modeling the belief formation process in reduced form in order to focus on ambiguity. We consider the process through which candidates alter voters' beliefs to be fundamentally interesting in its own right and plan in future work to model it specifically. We will make a few remarks about the relationship between our approach and one in which the belief formation process was modelled directly.

First, throughout our analysis we have assumed that a candidate's strategy choice affected voters' beliefs about that candidate only. From a positive perspective, much political discourse is aimed at affecting voters' beliefs about candidates' opponents. Past votes, acquaintances, and indiscretions are brought to the public's attention with the aim to make the other candidate less attractive. It might be possible to extend our model in a mechanical way to incorporate negative campaigning of this sort; a more satisfying approach would be to model directly the way in which the beliefs about the

two candidates arises nontrivially from candidates' choices.

*Restrictions on the beliefs that can be induced.* It is not so much the choice of beliefs as strategies that is important to our model as it is the *restrictions* on the beliefs available to candidates. The restriction to subsets of beliefs is important both from a positive modelling point of view (it is obvious that candidates are limited in this way), and because of the consequences of the restriction on the outcomes of the model (there are no pure strategy equilibria in the absence of restrictions).

We took a particularly simple tack in restricting the beliefs available to candidates, namely that they had available all beliefs that put some minimal probability on their most preferred alternative. Some aspects of this particular restriction are important while other aspects could be relaxed without changing qualitatively our results. First, it is important that the candidates' strategy sets be disjoint; it is often the case that there will only be mixed strategy equilibria when they intersect.

The specific restriction that the beliefs available to a candidate put minimal probability on his true most preferred outcome is largely for interpretation. As we said above, a candidate may be able to convince voters that the probability that he will choose a particular outcome is nearly one when he will in fact, choose another, but it seems to us that this should really be interpreted as deception, not the kind of ambiguity that motivated us in this work. The assumption that a candidate has available *all* beliefs that put minimal probability on his most preferred outcome isn't strictly necessary. To guarantee that there is a pure strategy equilibrium in the election game, it is enough that the candidates' strategy be subsets of the sets we considered. To assure that the winning strategy is ambiguous, the candidates have to have available to them sufficiently ambiguous strategies in the sets we considered.

*Welfare consequences of ambiguity.* We demonstrated conditions under which an ambiguous strategy could defeat a Condorcet winner. The ambiguous strategy that does this is a lottery that necessarily puts positive probability on a set of alternatives that by assumption a majority of people like less than the Condorcet outcome. It's worthwhile to note that this doesn't imply that ambiguity is necessarily welfare decreasing. Ex post, it's true that when the ambiguous candidate wins, there is an outcome that a majority of people would prefer to the outcome this candidate chose. But if we measure welfare ex ante, that is, before the uncertainty about candidates' true pol-



icy choices are known, welfare judgements are cloudier. Suppose a society could somehow ban ambiguity. When a Condorcet outcome is defeated by an ambiguous strategy, it's clear that a majority of voters would vote against outlawing ambiguity, since outlawing ambiguity is tantamount to voting for the Condorcet outcome in this case. If we were to make welfare judgements behind a veil of ignorance where one doesn't know if he will prefer the Condorcet outcome or the lottery, and not knowing whether one will win or lose if and when the ambiguous candidate chooses an outcome, it could be that in some cases, expected utility is higher when ambiguous strategies are allowed than when they are not.

## References

- [1] Alesina, A. and A. Cuckierman (1990) "The politics of ambiguity," *Quarterly Journal of Economics* 4: 829-850.
- [2] Aragones, E. and Z. Neeman (1996) "Strategic ambiguity in electoral competition," WP Universitat Pompeu Fabra. Revised: September 1998.
- [3] Banks, J (1990) "A Model of Electoral Competition with Incomplete Information" *Journal of Economic Theory* 50: 309-325.
- [4] Downs, A. (1957) "An Economic Theory of Democracy," New York: Harper and Row.
- [5] Fishburn, P. C. (1972) "Lotteries and Social Choice," *Journal of Economic Theory* 5: 189-207.
- [6] Harrington, J.E. (1992) "The Revelation of Information through the Electoral Process: An Exploratory Analysis," *Economics and Politics* 4: 255-275.
- [7] Lizzeri, A. (1997) "Budget Deficits and Redistributive Politics," mimeo.
- [8] Myerson, R. B. (1993) "Incentives to cultivate special interest groups under alternative electoral systems," *American Political Science Review* 87(4): 856-869.
- [9] Shepsle, K. A. (1970) "A note on Zeckhauser's majority rule with lotteries on alternatives," *Quarterly Journal of Economics* 84: 705-709.
- [10] Shepsle, K. A. (1972) "The strategy of ambiguity: uncertainty and electoral competition," *American Political Science Review* 66: 555-568.
- [11] Zeckhauser, R. (1969) "Majority rule with lotteries on alternatives," *Quarterly Journal of Economics* 83: 696-703.

## 8 Appendix

### Proof of Proposition 1:

First we find conditions for existence of a pure strategy equilibrium if the sets of strategies for the two candidates are given by  $W_1 = \{p \in \Delta Z : p_B \geq a_1\}$  and  $W_2 = \{p \in \Delta Z : p_A \geq a_2\}$ , where alternative  $B$  is the Condorcet winner but it is not the first choice of a majority of voters. Notice that in equilibrium there must be one party that has a winning strategy, that is, a strategy that defeats any strategy available to the opponent. We will prove that a pure strategy equilibrium exists if and only if either one of the following two conditions is satisfied:

- i)  $a_1 > \max \left\{ \frac{1}{1-x_1+1-x_6}, \frac{a_2}{(1-x_6)} \right\}$
- ii)  $a_2 > \max \left\{ \frac{1-x_6}{1+(1-x_3)(1-x_6)}, a_1(1-x_6) \right\}$

If the first condition holds all winning strategies are nondegenerate lotteries. When the second condition holds all winning strategies are nondegenerate lotteries if and only if  $a_2 < 1 - x_6$ .

For a given set of lotteries  $W \subseteq \Delta Z$ , let  $D(W)$  denote the set of lotteries that defeat all lotteries in  $W$  in pairwise contests. Candidate 2 has a winning strategy if and only if  $W_2 \cap D(W_1) \neq \emptyset$ , that is, his strategy set contains at least one lottery that defeats all strategies available to his opponent. A lottery that defeats all lotteries in  $W_1$  must be  $p' \in W_1^c$  such that  $p' \in \bigcap_{p \in W_1} D(p)$ . In particular, a winning strategy for candidate 2 must defeat the degenerate lottery that assigns probability one to the Condorcet winner. In order to defeat the lottery  $(0, 1, 0)$  it is necessary and sufficient to find a lottery that is preferred by all voters in groups  $G_1$  and  $G_6$ : that is all voters whose second choice is the Condorcet winner. Thus we need  $p \in \Delta Z$  such that  $E_p u_1 = p_A + p_B x_1 > x_1 = u_1(B)$  and  $E_p u_6 = p_C + p_B x_6 > x_6 = u_6(B)$ . Using these conditions we find

$$D(0, 1, 0) = \left\{ (p_A, p_B, p_C) \in \Delta Z : 1 - \frac{p_A}{x_1} < p_B < 1 - \frac{p_A}{1-x_6} \right\}.$$

This set is non empty if and only if  $x_1 + x_6 < 1$ , which is satisfied if the voters in groups  $G_1$  and  $G_6$  have intense preferences.

With a similar argument, for any lottery  $p \in W_1$  we can find the set of lotteries that defeat it:

$$D(p) = \left\{ p' \in \Delta Z : \frac{p_A}{x_1} + p_B - \frac{p_{A'}}{x_1} < p_{B'} < \frac{p_A}{1-x_6} + p_B - \frac{p_{A'}}{1-x_6} \right\}$$

$$\begin{aligned} & \cup \{p' \in \Delta Z : p_A x_3 + p_B - p_{A'} x_3 < p_{B'} < b\} \\ & \cup \{p' \in \Delta Z : p_C x_4 + p_B - p_{C'} x_4 < p_{B'} < b\} \end{aligned}$$

The first subset is nonempty for all  $p \in W_1$  as long as  $x_1 + x_6 < 1$ , and the other two subsets are empty for some  $p \in W_1$  (for instance for  $p$  with  $p_A = 0$  or  $p_C = 0$ ). Since we are interested in the intersection of such sets for all  $p \in W_1$  we should focus our attention in the intersection of the following sets:

$$D_{\sim}(p) = \left\{ p' \in \Delta Z : \frac{p_A}{x_1} + p_B - \frac{p_{A'}}{x_1} < p_{B'} < \frac{p_A}{1-x_6} + p_B - \frac{p_{A'}}{1-x_6} \right\}.$$

Observe that  $D(p) \subset D(p')$  for all  $p$  and  $p'$  such that

$$\frac{p_A}{1-x_3} + p_B - \frac{p_{A'}}{1-x_3} < p_{B'} < \frac{p_A}{x_1} + p_B - \frac{p_{A'}}{x_1}.$$

Therefore  $D(W_1) = \bigcap_{p \in W_1} D(p) = \bigcap_{p \in \{p \in W_1 : p_B = a_1\}} D(p)$ , that is, there exists a lottery  $p \in \Delta Z$  that defeats all lotteries in  $W_1$  if and only if  $p \in \Delta Z$  defeats all lotteries in  $\{p \in \Delta Z : p_B = a_1\}$ .

Finally, observe that  $p \in \Delta Z$  will defeat all lotteries in  $\{p \in \Delta Z : p_B = a_1\}$  if and only if it defeats  $\{(0, a_1, 1 - a_1), (1 - a_1, a_1, 0)\}$ . Thus, we have that

$$\begin{aligned} D(W_1) &= D(0, a_1, 1 - a_1) \cap D(1 - a_1, a_1, 0) \\ &= \left\{ p \in \Delta Z : \frac{1 - a_1}{x_1} + a_1 - \frac{p_A}{x_1} < p_B < a_1 - \frac{p_A}{1 - x_6} \right\}. \end{aligned}$$

and this set is nonempty if and only if  $x_1 + x_6 < 2 - \frac{1}{a_1}$ . A condition similar to the one that guarantees existence of a lottery that defeats the Condorcet winner is also sufficient to guarantee that the set of lotteries that defeats any lottery in  $W_1$  is not empty: intensities of preferences for voters in groups  $G_1$  and  $G_6$  need to be high enough (see figure 1).

Thus  $W_2 \cap D(W_1) \neq \emptyset$  if and only if  $a_1 > \frac{1}{1-x_1+1-x_6}$  and  $a_1 > \frac{a_2}{(1-x_6)}$ . The first condition guarantees that the set  $D(W_1)$  is not empty and the second one guarantees that the two sets have a nonempty intersection. Observe that all lotteries in  $D(W_1)$  are nondegenerate lotteries. Since the winning strategies must be contained in  $W_2 \cap D(W_1)$ , all of them must be nondegenerate lotteries.

Suppose that candidate 1's strategy set is  $S_1$ , a proper subset of  $W_1$ , then we have that  $D(S_1) \supseteq D(W_1)$  thus the conditions that guarantee a winning strategy for candidate 2 are even weaker than the ones we find when the strategy set of candidate 1 is  $W_1$ . On the other hand, any proper subset of  $W_2$  that intersects with  $D(W_1)$  provides candidate 2 with a winning strategy.

Similarly, candidate 1 has a winning strategy if and only if  $W_1 \cap D(W_2) \neq \emptyset$ , that is, his strategy set contains at least one lottery that defeats all strategies available to his opponent. For any lottery  $p \in W_2$  the set of lotteries that defeats it is given by:

$$D(p) = \left\{ p' \in \Delta Z : (p_A - p'_A) x_3 < p'_B - p_B < \frac{p_A - p'_A}{1 - x_6} \right\} \\ \cup \left\{ p' \in \Delta Z : p'_B - p_B > \max \left\{ \frac{p_A - p'_A}{x_1}, (p_A - p'_A) \frac{x_4}{1 - x_4} \right\} \right\}.$$

The first subset is nonempty for all  $p \in W_2$ , and the other subset is empty for some  $p \in W_2$  (for instance for  $p$  with  $p_C = 0$ ). Since we are interested in the intersection of such sets for all  $p \in W_2$  we should focus our attention in the intersection of the following sets:

$$\tilde{D}(p) = \left\{ p' \in \Delta Z : p_B + (p_A - p'_A) x_3 < p'_B < p_B + \frac{p_A - p'_A}{1 - x_6} \right\}$$

Notice that the lottery that assigns probability one to the Condorcet winner is contained in this set for all  $p$  with  $\frac{p_A + p_C}{p_A} < \frac{1}{1 - x_6}$ .

Using the argument developed for the previous case we can conclude that  $D(W_2) = \bigcap_{p \in W_2} D(p) = \bigcap_{p \in \{p \in W_2 : p_B = a_2\}} D(p)$ , that is, there exists a lottery  $p \in \Delta Z$  that defeats all lotteries in  $W_2$  if and only if  $p \in \Delta Z$  defeats all lotteries in  $\{p \in \Delta Z : p_A = a_2\}$ , and  $p \in \Delta Z$  will defeat all lotteries in  $\{p \in \Delta Z : p_A = a_2\}$  if and only if it defeats  $\{(a_2, 0, 1 - a_2), (a_2, 1 - a_2, 0)\}$ . Thus, we have that

$$D(W_2) = D(a_2, 0, 1 - a_2) \cap D(a_2, 1 - a_2, 0) \\ = \left\{ p \in \Delta Z : \frac{1 - a_2 - p_B}{x_3} + a_2 < p_A < a_2 - p_B (1 - x_6) \right\}.$$

This set is not empty if and only if  $a_2 > \frac{1 - x_6}{1 + (1 - x_3)(1 - x_6)}$  (see Figure 2). Thus  $W_1 \cap D(W_2) \neq \emptyset$  if and only if  $a_2 > \frac{1 - x_6}{1 + (1 - x_3)(1 - x_6)}$  and  $a_2 > a_1(1 - x_6)$ . The

first condition guarantees that the set  $D(W_2)$  is not empty and the second one guarantees that the two sets have a non empty intersection. Observe that all lotteries in  $D(W_2)$  are nondegenerate lotteries as long as  $a_2 < 1 - x_6$ , that is, the set of strategies available to candidate 2 must be large enough. Otherwise  $(0, 1, 0)$  is a winning strategy. Since the winning strategies must be contained in  $W_1 \cap D(W_2)$ , when  $a_2 < 1 - x_6$  all winning strategies must be nondegenerate lotteries.

Suppose that candidate 2's strategy set is  $S_2$ , a proper subset of  $W_2$ , then we have that  $D(S_2) \supseteq D(W_2)$  thus the conditions that guarantee a winning strategy for candidate 1 are even weaker than the ones we find when the strategy set of candidate 2 is  $W_2$ . On the other hand, any proper subset of  $W_1$  that intersects with  $D(W_2)$  provides candidate 1 with a winning strategy.

The conditions for existence of equilibrium involve only strict inequalities. If any of them is satisfied with equality the candidates tie in equilibrium (at least one group of voters is indifferent). Finally, when these conditions are not met, no candidate has a winning strategy, and therefore, there is no pure strategy equilibrium.<sup>7</sup> Thus, we have that there is no pure strategy equilibrium if and only if  $a_1(1 - x_6) < a_2 < \frac{1-x_6}{1+(1-x_3)(1-x_6)}$  or  $\frac{a_2}{1-x_6} < a_1 < \frac{1}{1-x_1+1-x_6}$ . These conditions characterize situations in which either both sets  $D(W_1)$  and  $D(W_2)$  are empty or, if one of them is not empty, its intersection with the opponents set is empty.

**Proof of Proposition 2:**

Suppose that  $W_1 = \{p \in \Delta Z : p_B \geq a_1\}$  and  $W_2 = \{p \in \Delta Z : p_A \geq a_2\}$ . We consider again the case in which the preference profiles of the voters is such that there is a Condorcet winner in the set of pure alternatives but it is not the first choice of a majority of the voters. As before, we assume that alternative  $B$  is the Condorcet winner.

First, notice that the probability with which a lottery  $p$  defeats  $(0, 1, 0)$  is given by  $F\left(\frac{p_A}{1-p_B}\right)F\left(1 - \frac{p_A}{1-p_B}\right)$ , that is, as discussed in the previous section, we need that the proposed lottery is preferred to alternative  $B$  by voters in groups  $G_1$  and  $G_6$ . If we define  $x(p_A, p_B) = \frac{p_A}{1-p_B}$  and maximize  $F(x(p_A, p_B))F(1 - x(p_A, p_B))$ , we find the set of lotteries that defeat  $(0, 1, 0)$  with maximal probability. The first and second order conditions only depend on the value of  $x(p_A, p_B)$ :

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<sup>7</sup>We have not investigated the possibility of mixed strategy equilibria in which a candidate mixes, choosing to induce different beliefs with positive probability on more than one belief vector.

$$\frac{f(x)}{F(x)} = \frac{f(1-x)}{F(1-x)} \text{ and } \frac{f'(x)}{f(x)} + \frac{f'(1-x)}{f(1-x)} \leq \frac{f(x)}{F(x)} + \frac{f(1-x)}{F(1-x)}$$

Since we assume that  $F(x)$  is a strictly concave function, we have that  $\ln[F(x)]$  is also strictly concave, and thus its first derivative,  $\frac{f(x)}{F(x)}$ , is strictly decreasing. Therefore  $\frac{f(x)}{F(x)} \neq \frac{f(y)}{F(y)}$  for all  $x \neq y$  and  $x = \frac{1}{2}$  is the only value that satisfies the first order condition. Furthermore, the strict concavity of  $F(x)$  guarantees that  $F''(x) = f'(x) \leq 0$ ; consequently, at  $x = \frac{1}{2}$ ,  $f'(x) < \frac{[f(x)]^2}{F(x)}$ , and the second order conditions is satisfied at  $x = \frac{1}{2}$ . Thus, there is a local maximum at  $x = \frac{1}{2}$ . At the extreme points of the domain (0 and 1) the function  $F(x)F(1-x)$  has value zero. Since  $x = \frac{1}{2}$  is the only critical point, we have that  $x = \frac{1}{2}$  is the only maximizer and the lotteries that satisfy  $p_B = 1 - 2p_A$  defeat  $(0, 1, 0)$  with maximal probability:  $[F(1/2)]^2$ .

Similarly, the probability with which a lottery  $p'$  defeats a lottery  $p$  in the set  $W_1 = \{p \in \Delta Z : p_B \geq a_1\}$  is given by  $F\left(\frac{p_{A'} - p_A}{p_B - p_{B'}}\right) F\left(1 - \frac{p_{A'} - p_A}{p_B - p_{B'}}\right)$ . Again, as before, this lottery must be preferred to all lotteries in  $W_1$  by all voters in groups  $G_1$  and  $G_6$ . We consider the problem of maximizing  $F\left(\frac{p_{A'} - p_A}{p_B - p_{B'}}\right) F\left(1 - \frac{p_{A'} - p_A}{p_B - p_{B'}}\right)$  and we find that the first and second order conditions of this problem are exactly as the previous ones. Thus,  $\frac{p_{A'} - p_A}{p_B - p_{B'}} = \frac{1}{2}$  is the unique maximizer and the lotteries that satisfy  $p_{B'} = p_B - 2(p_{A'} - p_A)$  defeat  $p \in W_1 = \{p \in \Delta Z : p_B \geq a_1\}$  with maximal probability.

Observe that the maximal probability with which any lottery can defeat a strategy available to candidate 1,  $F(x^*)F(1-x^*)$ , depends only on the probability distribution of the intensity of the voters preferences. Given our assumption of concavity on the probability distribution this maximal probability is simply  $m = [F(1/2)]^2$ .

Now we will show that a pure strategy equilibrium exists if and only if  $a_2 \geq \frac{1}{2}$  and  $a_1 + a_2 > 1$ , and that it is unique: in equilibrium candidate 1 chooses  $(0, a_1, 1 - a_1)$  and candidate 2 chooses  $(a_2, 0, 1 - a_2)$ .

i) Suppose that  $a_2 \geq \frac{1}{2}$  and  $a_1 + a_2 > 1$ . If  $a_2 > 1 - \frac{a_1}{2}$ , both candidates have dominant strategies: there is a unique equilibrium in which candidate 1 will choose  $(0, a_1, 1 - a_1)$  and candidate 2 will choose  $(a_2, 0, 1 - a_2)$ . Candidate 2 cannot defeat any strategy of candidate 1 with probability  $m$ , and  $(a_2, 0, 1 - a_2)$  is a best response for candidate 2 to any strategy available to candidate 1 since it is the closest lottery to a lottery that can defeat any strategy

of candidate 1 with probability  $m$ . On the other hand,  $(0, a_1, 1 - a_1)$  is a dominant strategy for candidate 1 because the distance between the lotteries that defeat it with probability  $m$  and the lotteries available to candidate 2 is maximal.

If  $a_2 \leq 1 - \frac{a_1}{2}$ , there is a subset of strategies of candidate 1 that candidate 2 cannot defeat with probability  $m$  (all lotteries  $p \in W_1$  such that  $p_B < 2(a_2 - p_A)$ ) and the best response of candidate 2 to any of these strategies is  $(a_2, 0, 1 - a_2)$ . If candidate 2 chooses  $(a_2, 0, 1 - a_2)$ , and  $a_2 \geq \frac{1}{2}$  the best response for candidate 1 is  $(0, a_1, 1 - a_1)$ . Thus, there is a unique equilibrium in which candidate 1 chooses  $(0, a_1, 1 - a_1)$  and candidate 2 chooses  $(a_2, 0, 1 - a_2)$ .

ii) If  $a_1 + a_2 \leq 1$ , that is, the strategy sets of the two candidates have a nonempty intersection, there is a subset of strategies for candidate 1 that candidate 2 cannot defeat with maximal probability:  $p \in W_1$  such that  $p_B < 2(a_2 - p_A)$ . The best response of candidate 2 to any of these is  $(a_2, 0, 1 - a_2)$ . The best response of candidate 1 to  $(a_2, 0, 1 - a_2)$  is now  $(a_1, 1 - a_1, 0)$ , but candidate 2 can choose a lottery that defeats it with probability  $m$ . Thus there is no pure strategy equilibrium in this case.

iii) Finally, suppose that  $a_2 < \frac{1}{2}$  and  $a_1 + a_2 > 1$ . If  $a_2 > \frac{a_1}{2}$  there is a subset of strategies of candidate 1 that candidate 2 cannot defeat with probability  $m$ : all lotteries  $p \in W_1$  such that  $p_B < 2(a_2 - p_A)$ . The best response of candidate 2 to any of these strategies is  $(a_2, 0, 1 - a_2)$ . If candidate 2 chooses  $(a_2, 0, 1 - a_2)$ , the best response for candidate 1 is  $(1 - a_1, a_1, 0)$ . In this case there is no equilibrium, because candidate 2 will respond with  $(1 - \frac{a_1}{2}, 0, \frac{a_1}{2})$ , the strategy that maximizes his probability of winning against  $(1 - a_1, a_1, 0)$ , and candidate 1 will want to move back to  $(0, a_1, 1 - a_1)$ . If  $a_2 \leq \frac{a_1}{2}$ , for every strategy available to candidate 1 there is a strategy available to candidate 2 that allows him to win with probability  $m$ , therefore, there is no pure strategy equilibrium. ♦

**Proof of Proposition 3:**

Let  $I$  denote the number of candidates competing in the election and  $N'$  denote the size of the smallest group of voters that have the same alternative as their first choice. First notice that if each alternative is chosen with probability one by at least one candidate, then clearly a candidate that uses an ambiguous strategy will get no votes. Hence, if an ambiguous strategy is chosen in equilibrium, it must put positive probability on at least one alternative that is not chosen with probability one by any candidate. But if  $N' \geq \frac{N}{I}$ , there must be at least one candidate that gets less than  $\frac{N}{I}$ . If



this candidate chooses instead an unambiguous strategy that is the most preferred outcome of some group that is not chosen by any other candidate, it will get the votes of this group. Since  $N_z \geq N' \geq \frac{N}{T}$  for all  $z$ , this must increase his payoff.

We will show that for  $I \geq \frac{N}{N'}$  there exists a set of pure strategy equilibria in which all candidates choose degenerate lotteries, and all alternatives are chosen with probability one by at least one candidate.<sup>8</sup>

If all candidates use only degenerate lotteries, that is,  $P = (p_1, \dots, p_I)$  with  $p_i \in \Delta Z_d$  for all  $i = 1, \dots, I$ , let  $k_z$  denote the number of candidates that assign probability one to alternative  $z$ . Then the payoff for candidate  $i$  will be given by:  $\pi_i(P) = \frac{N_z}{k_z}$  if candidate  $i$  has decided to assign probability one to alternative  $z$ , where  $N_z$  represents the number of voters that have  $z$  as their first choice. We will show now that for each  $I$  we can find values for  $k_z$ , with  $\sum_{z \in Z} k_z = I$  and  $k_z \geq 1$  for all  $z \in Z$  such that the corresponding strategies (degenerate lotteries) form a Nash equilibrium.

Observe that if  $k_z \geq 1$  for all  $z \in Z$ , deviations to ambiguous strategies are not profitable since a voter will always vote for a candidate that chooses her most preferred alternative with probability one, a candidate that chooses an ambiguous strategy will receive zero votes.

Let  $\text{int}[q]$  denote the integer part of  $q$ , where  $q$  is any positive number and let  $t^* = I - \sum_{z \in Z} \text{int}\left[\frac{N_z I}{N}\right]$ . Observe that  $0 \leq t^* \leq |Z| - 1$ .

First suppose that  $t^* = 0$ , and consider  $\tilde{k}_z = \text{int}\left[\frac{N_z I}{N}\right]$  for all  $z \in Z$ .

Notice that since  $I \geq \frac{N}{N'}$  then we must have  $\tilde{k}_z \geq 1$  for all  $z \in Z$ .

Suppose that  $P$  represents the profile of strategies in which all candidates choose degenerate lotteries and such that for each alternative  $z$  there are exactly  $\tilde{k}_z$  candidates that assign probability one to it.

Since for each  $i$  we have that  $\pi_i(P) = \frac{N_z}{\text{int}\left[\frac{N_z I}{N}\right]}$  for some  $z \in Z$ , we must have that  $\pi_i(P) \geq \frac{N}{I}$ , and this can only be true if  $\pi_i(P) = \frac{N}{I}$  for all  $i$ . Any deviation of  $i$  from assigning probability one to  $z$  to assigning probability one to  $z'$  is not profitable since as a result the payoff corresponding to  $z$  increases and the one corresponding to  $z'$  decreases. Thus, the candidates' strategies corresponding to these values  $\tilde{k}$  form an equilibrium.

Now suppose that  $t^* > 0$ . Let  $Z_{t^*}$  be the set of alternatives corresponding to the most preferred ones by each one of the  $t^*$  largest groups of voters.

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<sup>8</sup>This condition implies that  $I \geq |Z|$ , since we always must have  $N' \leq \frac{N}{|Z|}$ .

Define  $\tilde{k}_z = \text{int} \left[ \frac{N_z I}{N} \right] + 1$  for all  $z \in Z_{t^*}$  and  $\tilde{k}_z = \text{int} \left[ \frac{N_z I}{N} \right]$  for all  $z \in Z \setminus Z_{t^*}$ , and let  $P'$  represent the profile of strategies corresponding to these values. We will show that these strategies form an equilibrium.

Again we have that  $\tilde{k}_z \geq 1$  for all  $z \in Z$ , and therefore deviations to ambiguous strategies are not profitable in this case either.

Given  $P'$ , for all  $z \in Z_{t^*}$  we must have that  $\pi_z(P') < \frac{N}{I-t^*}$  which is the payoff that all candidates that assign probability one to alternatives in  $Z \setminus Z_{t^*}$  obtain. Thus, to prove that  $P'$  is an equilibrium we have to show that the candidates that obtain the smallest payoff (candidates that assign probability one to the alternative in  $Z_{t^*}$  which is most preferred by the smallest group of voters) do not want to deviate to assigning probability one to the alternative which guarantees the largest payoff (which is the alternative in  $Z \setminus Z_{t^*}$  most preferred by the largest group of voters). That is, the following condition must hold:

$$\frac{N_z}{\text{int} \left[ \frac{N_z I}{N} \right] + 1} > \frac{N_{z'}}{\text{int} \left[ \frac{N_{z'} I}{N} \right] + 1}$$

where  $z$  is the alternative in  $Z_{t^*}$  which is most preferred by the smallest group of voters and  $z'$  is the alternative in  $Z \setminus Z_{t^*}$  most preferred by the largest group of voters. And this condition always holds since  $N_z > N_{z'}$ . ♦

