Weak approximations. A Malliavin Calculus approach *

by

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February 15, 1999

Abstract

We introduce a variation of the proof for weak approximations that is suitable for studying the densities of stochastic processes which are evaluations of the flow generated by a stochastic differential equation on a random variable that maybe anticipating. Our main assumption is that the process and the initial random variable have to be smooth in the Malliavin sense. Furthermore if the inverse of the Malliavin covariance matrix associated with the process under consideration is sufficiently integrable then approximations for densities and distributions can also be achieved. We apply these ideas to the case of stochastic differential equations with boundary conditions and the composition of two diffusions.

Key words: stochastic differential equations, boundary conditions, weak approximation, numerical analysis.

Mathematics Subject Classification (1991): 60H99, 34B10, 34B15, 65Nxx

*Journal of Economic Literature Classification System:*C15

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1 Introduction

Starting as early as Milshtein [22] and due to its many applications the area of weak approximations for stochastic differential equations (sde's) has been growing rapidly (e.g. see Kloeden and Platen [16]).

To explain what are the issues in this area of study, let's consider a simple case first. Define $\Phi_t(x)$ as the solution of the following one dimensional stochastic differential equation:

$$\Phi_t(x) = x + \int_0^t \sigma(\Phi_s(x)) \circ dW_s + \int_0^t b(\Phi_s(u,x))ds, \quad t \in [0,1]. \tag{1}$$

Here W is a one dimensional Wiener process and b, $\sigma : \mathbb{R} \to \mathbb{R}$ are smooth with bounded derivatives. The above stochastic integral is the Stratonovich integral.

The main problem of study in the area of strong and weak approximation theory is how to approximate Φ and what is the error of approximation. For this we define the Euler approximation. Let $\pi = \{0 = t_0 < ... < t_N = 1\}$ be a partition of $[0,1]$ such that $\|\pi\| := \max\{t_{k+1}-t_k; \ k = 0,...,N-1\} \leq \delta$. Define the shift operator

$$\eta(t) = t_k \qquad \text{if } t_k < t \leq t_{k+1},$$

where t_k and t_{k+1} are in π. We set $\eta(0) = 0$. Let $\overline{\Phi}$ denote the Euler-Maruyama scheme defined by

$$\overline{\Phi}_t(x) = x + \int_0^t \sigma(\overline{\Phi}_{\eta(s)}(x))dW_s + \int_0^t m(\overline{\Phi}_{\eta(s)}(x))ds. \tag{2}$$

Here $m = b + \frac{1}{2}\sigma'\sigma$.

A well known result (see e.g. Theorem 10.2.2 in [16]) establishes that

$$E\left(\sup_{t \in [0,1]} |\Phi_t(x) - \overline{\Phi}_t(x)|^p\right) \leq C(x)\delta^{p/2}. \tag{3}$$

Here, $p > 1$ and $C(x)$ is a positive function that has polynomial growth in x and is independent of δ and the partition π.

This type of result measures the path-by-path difference between the solution of (1) and its Euler-Maruyama approximation (2). For this reason this type of result is usually called strong approximation theorem. Its method of proof is based on Gronwall's lemma.

A different way of measuring the difference between Φ and $\overline{\Phi}$ is through their laws and in particular the following result holds (see, e.g. Theorem 14.1.5 in [16]):

$$|E(f(\Phi_t(x))) - E(f(\overline{\Phi}_t(x)))| \leq C(x)\delta, \tag{4}$$

where $f \in C_p^\infty(\mathbb{R})$ (that is, f is a real valued, smooth function with polynomial growth at infinity). This type of result is different from (3). In fact, if f above were Lipschitz (with Lipschitz constant K) using (3) we could only obtain that

$$|E(f(\Phi_t(x))) - E(f(\overline{\Phi}_t(x)))| \leq KE(|\Phi_t(x) - \overline{\Phi}_t(x)|) \leq KC(x)\delta^{1/2},$$

while (4) establishes that the rate of convergence is of order δ. Results like (4) are known as weak approximation theorems. The method of proof is centered on the fact that if we define $u(t,x) =$

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$E(f(\Phi_t(x)))$ then u satisfies the following PDE:

$$\frac{\partial u}{\partial s}(s,x) + m(x)\frac{\partial u}{\partial x}(s,x) + \frac{1}{2}\sigma^2(x)\frac{\partial^2 u}{\partial x^2}(s,x) = 0$$
$$u(0,x) = f(x).$$

Having defined u, one rewrites (4) in terms of u and applies Itô's formula. At some point during the proof the Markov property of the process $\Phi_t(x)$ is used.

Due to its connections with partial differential equations the area of weak approximations has recently been of increasing interest. Also this type of results give information about many functionals of the solution process. In particular, the moments and the law of the process $\Phi_t(x)$. For example, Bally and Talay [3], [4], widened the class where f belongs to have a result like (4) hold. In particular, one can obtain approximations for the distribution and density functions of Φ. Other related results were developed by Hu and Watanabe [14] and Kohatsu-Higa [17].

Another direction of development has been to consider other stochastic differential equations of a type different from (1). For example, Bossy and Talay [5] and [6] considered stochastic differential equations related to the Burgers and McKean-Vlasov equations. These stochastic equations are of a type different from (1). In particular the law of the solution process is also part of the coefficients in the equation.

In all these variants the essential technique to obtain a result like (4) is to find an appropiate modification of the basic argument that uses u (which also needs to be modified), the adaptedness and the Markov property of the underlying stochastic process.

Until recently, it was common to believe that anticipating stochastic differential equations were not amenable to this type of argument to study the numerical approximations due to the lack of adapted properties and the Markov property. In Ahn and Kohatsu-Higa [2], we defined and analyzed the weak and strong rate of convergence for a Euler type scheme in the case of $\Phi_t(X_0)$ where X_0 is a smooth random variable (in the Malliavin sense) but not necessarily adapted to $\mathcal{F}_0$. We assumed that the joint distribution of the vector $(X_0, W_{t_1}, ..., W_{t_N})$ is known and therefore possible to simulate. These results proved that one could use simulations to study the path and probabilistic properties of such anticipating processes.

Nevertheless, the proof of weak approximation was complicated and required stringent conditions on the random variable X_0. Here the problem comes from the fact that although $\Phi_t(X_0)$ satisfies an stochastic differential equation of anticipative type, there is no partial differential equation associated to it. Furthermore the Markov property is not satisfied.

In this article we propose to continue this study. We want to concentrate our efforts in ordinary stochastic differential equations with boundary conditions. As many high order sde's with boundary conditions can be reduced to first order ones, we will start studying the weak approximations for first order stochastic differential equations with boundary conditions. In the future we will handle the second order equations.

Here, we consider approximations for the density of the solution to a sde with boundary conditions. These equations arise naturally as extensions of ordinary differential equations with boundary condi-

tions. The solutions of such equations can be written as $\Phi_t(X_0)$. There are various added complications to the definition and the analysis of approximations in this case.

First, one has to approximate X_0, through a procedure that resembles the classical shooting method. This method is not well defined in the whole sample space therefore we will need a localization procedure. Secondly, the approximation to X_0 does not satisfy the requirements of the weak approximation results in [2]. Third, we are interested in approximating the density of a process with the possible complication that the approximating process may not have a density in itself, although the limit may have one.

To solve these problems, we propose a variant of the classical proof of weak rate of convergence.

In order to give a clear proof of our final goal (see Theorem 4.3), we will gradually introduce this modification to the classical proof. The final goal is to find rates of convergence for approximations to the densities of random variables of the type $\Phi_t(X_0)$.

First, we study the case when X_0 is a random variable such that the joint law of $(X_0, W_{t_1}, ..., W_{t_N})$ is known beforehand. Although this is not the case for sde's with boundary conditions this will be an important step towards our final goal. In the second step, we study the case when X_0 also has to be approximated. First, in the case X_0 is generated by another diffusion and then in the case of stochastic differential equations with boundary conditions.

Instead of using the Markov property and $u(t, x)$ we use the integration by parts of Malliavin calculus. Therefore the variant we introduce to prove these results requires the use of flow properties, techniques of Malliavin calculus and the smoothness of the processes involved.

With these tools we will prove weak approximation results of the type (4) for the cases mentioned above. Furthermore one can also obtain extensions when f belongs to a wider class that includes the indicator function and the delta function. This type of results will provide a way to approximate the distribution and density functions of the processes involved through the use of appropiate Monte Carlo methods.

The study of approximations for density functions has the added difficulty that one has to show that the Malliavin covariance matrix of the approximating process is uniformly bounded with respect to the step size. We prove in the Appendix that something close to this happens (see Lemma 7.2) which will be enough for our approximation result.

After some preliminaries we will discuss in Section 3 our method of proof for weak approximations in the case of diffusions composed with an anticipating random variable. In this section we suppose that X_0 is a smooth random variable in the Malliavin sense. Then we apply this result to the case when X_0 is the final point of another diffusion. This example provides a first case where the initial random variable also needs to be approximated.

Then we start to consider our approach in a more difficult anticipative setting such as in the case of stochastic differential equations with boundary conditions. This study is carried out in Section 4 and is divided into two parts. First for the one dimensional case (Section 4.1) and then in the general multidimensional case (Section 4.2).

In the first, the approximation can be considered as the natural generalization of the shooting method for ordinary differential equations with boundary conditions. Here, we have the added complication

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that a localization technique is needed. We show that our method of proof also works under the appropiate localization. Therefore, the main difference with Section 3 is the fact that X_0 is only locally smooth and that it also has to be approximated. Finally we consider the approximation of densities for the one dimensional case. In this case besides the localization procedure we also need to start considering when the density of the solution to the sde with boundary conditions exists. This involves further calculations related to the Malliavin covariance matrix and the added problem of considering a non-degenerate approximation.

At the beginning of Section 4 we also give a brief introduction to stochastic differential equations with boundary conditions.

Then at Section 5 we give a short description of a possible real application of these equations together with a general result of diffusion approximation. We finish with some conclusions and possible generalizations of our method. At the end of this article in Section 7, we have collected a series of auxiliary results that are used throughout the text.

In this article, C will denote positive constants that may change from one line to the next. Furthermore these constants are always assumed independent of δ and the partition π unless it is explicitly stated otherwise.

2 Preliminaries

Now we introduce some basic tools from Malliavin calculus that will be used throughout the text. For further reference see [23]. Let $(\Omega, \mathcal{F}, P)$ be the canonical Wiener space which supports a d-dimensional Wiener process W. Let $C_b^\infty(\mathbb{R}^{dn})$ be the set of C^∞ functions $f : \mathbb{R}^{nd} \to \mathbb{R}$ which are bounded and have bounded derivatives of all orders. The class of real random variables of the form $f(W_{t_1}, \ldots, W_{t_n})$, $f \in C_b^\infty(\mathbb{R}^{nd})$, $t_1, \ldots, t_n \geq 0$, (the class of infinitely differentiable functions with bounded derivatives) is denoted by $\mathcal{S}$. $\mathbb{D}^{1,p}$ designates the Banach space which is the completion of $\mathcal{S}$ with respect to the norm:

$$\|F\|_{1,p} = \{E|F|^p\}^{1/p} + \left(\sum_{j=1}^d E[\{\int_0^1 |D_s^j F|^2 \, ds\}^{p/2}]\right)^{1/p},$$

where ($\mathbb{I}_A$ denotes the indicator function of the set A)

$$D_s^j F = \sum_{i=1}^n \frac{\partial f}{\partial x_{ji}}(W_{t_1}, \ldots, W_{t_d}) \ \mathbb{I}_{[0,t_i]}(s).$$

$\mathbb{D}^{\alpha,p}$ is defined analogously and its associated norm is denoted by $\|\cdot\|_{\alpha,p}$. Also, let $\mathbb{D}^{\alpha,\infty} = \cap_{p\geq 1}\mathbb{D}^{\alpha,p}$ and $\mathbb{D}^\infty = \cap_{p\geq 1} \cap_{\alpha\geq 1} \mathbb{D}^{\alpha,p}$. The localization of $\mathbb{D}^{\alpha,p}$ is denoted by $\mathbb{D}_{loc}^{\alpha,p}$. That is, $\mathbb{D}_{loc}^{\alpha,p}$ is the set of random variables F such that there exists a sequence $\{(\Omega^\epsilon, F^\epsilon), \ 0 < \epsilon < 1\} \subset \mathcal{F} \times \mathbb{D}^{\alpha,p}$ such that

(i) $\Omega^\epsilon \uparrow \Omega, \ a.s.$

(ii) $F = F^\epsilon \ a.s.$ on Ω^ϵ.

Then one defines $D_{s_1}^{j_1}...D_{s_k}^{j_k}F = D_{s_1}^{j_1}...D_{s_k}^{j_k}F^\epsilon$ on Ω^ϵ for $k \leq \alpha$, $j_1, ..., j_k \in \{1, ..., d\}$ and we say that $(\Omega^\epsilon, F^\epsilon)$ localizes F.

The adjoint of the closed unbounded operator $D : \mathbb{D}^{1,2} \to L^2([0,1] \times \Omega)$ is usually denoted by δ and is called the Skorohod integral. Its domain can be characterized as the set of measurable processes $u \in L^2([0,1] \times \Omega)$ such that there exists a positive constant C that may depend on u such that

$$|E(\int_0^1 D_t F u_t dt)| \leq C\|F\|_2,$$

for all $F \in \mathbb{D}^{1,2}$. Then the Skorohod integral for u an element of its domain, is the square integrable random variable determined by the duality relation

$$E(\delta(u)F) = E(\int_0^1 D_t F u_t dt), \tag{5}$$

for all $F \in \mathbb{D}^{1,2}$. The Skorohod integral turns out to be an extension of the classical Itô integral and it allows the integration of processes that are not necessarily adapted.

In order to avoid confusion we will use D for the derivative defined above and ∇ or the $'$ notation for classical derivatives of functions.

When considering densities of random variables we will use the concept of Malliavin covariance matrix. For this, define for $F \in (\mathbb{D}^{1,1})_{loc}^m$ the Malliavin covariance matrix of F as $\Delta_F^{ij} = \langle DF^i, DF^j \rangle_{L^2[0,1]}$. If $F \in \mathbb{D}^\infty$ and $\det \Delta_F^{-1} \in \cap_{p>1} L^p(\Omega)$, then F has a smooth density.

A measurable process u with integrable paths $a.s.$ is said to be Stratonovich integrable with respect to W^j if for any increasing sequence of partitions $\{\Pi_n = \{0 = t_0^n < ... < t_n^n = 1\}\}$ of $[0, T]$, such that $|\Pi_n| \to 0$, the sequence

$$\sum_{i=0}^{n-1} \frac{1}{t_{i+1}^n - t_i^n} (\int_{t_i^n}^{t_{i+1}^n} u_s ds)(W_{t_{i+1}^n}^j - W_{t_i^n}^j),$$

converges in probability. In such a case we will denote the limit by $\int_0^1 u_t \circ dW_t^j$. It is well-known that if $u \in \mathbb{L}_{C,loc}^{1,2}$ (for a definition of this space, see [23]), then u is Stratonovich integrable with respect to W^j, for $j \in \{1, ..., k\}$.

Now we give some notation related to stochastic differential equations. From here onwards we reduce our study to one dimensional sde's for convenience.

Let $\Phi_t(s, x)$ be the stochastic flow (sometimes also denoted by $\Phi(t, x)$ or $\Phi_t(x)$ in the case $s = 0$) defined as the solution of

$$\Phi_t(s,x) = x + \int_s^t \sigma(\Phi_u(s,x)) \circ dW_u + \int_s^t b(\Phi_u(s,x)) \, du, \; s \leq t \in [0,1],$$

where $b, \sigma : \mathbb{R} \to \mathbb{R}$ are smooth functions with bounded derivatives. Now we define the Euler approximation for Φ. For this, let $\pi = \{0 = t_0 < ... < t_N = 1\}$ be a partition of $[0, 1]$. As before we define the shift operator $\eta(t)$ and we take $\|\pi\| \leq \delta$. Let $\overline{\Phi}$ denote the Euler-Maruyama scheme defined by

$$\overline{\Phi}_t(s,x) = x + \int_s^t \sigma(\overline{\Phi}_{\eta(u)}(s,x)) dW_u + \int_s^t m(\overline{\Phi}_{\eta(u)}(s,x)) du. \tag{6}$$

Here $m = b + \frac{1}{2}\sigma'\sigma$. In the particular case that $s = 0$ we simplify the notation and use $\overline{\Phi}_t(x)$ and $\Phi_t(x)$ instead of $\overline{\Phi}_t(0, x)$ and $\Phi_t(0, x)$.

We will also use some terminology related to the high order Itô-Taylor formula as stated in [16]. That is, let $\mathcal{M} = \{(j_1, ..., j_l); \ j_i \in \{0, 1\}, \ i \in \{1, ..., l\}, \text{ for } l = 1, ...\} \cup \{v\}$ where v denotes the multi-index of length 0. For a multi-index $\alpha = (j_1, ..., j_l)$ define the length of α as $l(\alpha) = l$, also define $n(\alpha)$ as the number of zeros in α, $-\alpha = (j_2, ..., j_l)$ and $\alpha- = (j_1, ..., j_{l-1})$. Then for $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ define the following operators:

$$L^1 f(s, x) = \sigma(x) \frac{\partial f}{\partial x}(s, x),$$

$$L^0 f(s, x) = \frac{\partial f}{\partial s}(s, x) + m(x) \frac{\partial f}{\partial x}(s, x) + \frac{1}{2} \sigma^2(x) \frac{\partial^2 f}{\partial x^2}(s, x).$$

For $\alpha = (j_1, ..., j_l)$ define by induction

$$g_\alpha = \begin{cases} g & ; l = 0 \\ L^{j_1} g_{-\alpha} & ; l \geq 1. \end{cases} \tag{7}$$

In the case that the function g is not explicitely stated we shall always take it to be the identity function $g(t, x) = x$. Also define the following Wiener functionals for an adapted process f:

$$I_\alpha[f(\cdot)]_{s,t} = \begin{cases} f(t) & ; l = 0 \\ \int_s^t I_{\alpha-}[f(\cdot)]_{s,u} dW_u^{j_l} & ; l \geq 1. \end{cases}$$

Here $dW_u^0 = du$ and $I_{\alpha,s,t} = I_\alpha[1]_{s,t}$. Also let $_{,\beta} = \{\alpha \in \mathcal{M}; \ l(\alpha) \leq \beta\}$ and $B(_{,\beta}) = \{\alpha \in \mathcal{M} -_{,\beta}; \ -\alpha \in_{,\beta}\}$.

Also the space of differentiable functions with polynomial growth at infinity is defined for $k = 0, 1, ...$ as

$$C_p^k(\mathbb{R}^d) \equiv C_p^k(\mathbb{R}^d; \mathbb{R}) = \{f \in C^k(\mathbb{R}^d; \mathbb{R}); \ \exists K \geq 0, \ r \in \mathbb{N} \text{ such that } |\partial_y^j f(y)| \leq K(1 + |y|^r),$$

$$\text{for any } l(j) \leq k, \ y \in \mathbb{R}\}.$$

Here, j denotes a multi-index from $\{1, ..., d\}^{l(j)}$, $l(j)$ denotes the length of the multi-index and $\partial_y^j f(y)$ denotes the high order partial derivative of f with respect to the indices in j. In the case $d = 1$ we write $C_p^k \equiv C_p^k(\mathbb{R})$.

3   Weak approximation for the composition of two diffusions

In this section we will study the weak approximation of $\Phi_t(X_0)$ by $\overline{\Phi}_t(X_0)$, where X_0 is an appropiate smooth random variable. Therefore we will be assuming that one knows how to simulate $(X_0, W_{t_1}, ..., W_{t_N})$. Although we work with one dimensional sde's, the results in this section have straightforward generalizations to the multidimensional case.

We start with a preliminary result on the generalized strong rate of convergence of the Euler scheme. This is a necessary step in order to prove the weak rate of convergence.

PROPOSITION 3.1 *Assume that for* $k \in \mathbb{N}$, $X_0 \in \mathbb{D}^{k,\infty}$ *with* $\sup_{s \in [0,1]} E|D_s X_0|^p < \infty$ *for all* $p > 1$. *Then* $\Phi_t(X_0)$, $\overline{\Phi}_t(X_0) \in \mathbb{D}^{k,\infty}$ *and furthermore there exists a positive constant depending on* p *and* k, $C(k, p)$ *such that*

$$\|\Phi_t(X_0) - \overline{\Phi}_t(X_0)\|_{k,p} \leq C(k, p) \delta^{\frac{1}{2}}.$$

Proof: $\Phi_t(X_0)$, $\overline{\Phi}_t(X_0) \in \mathbb{D}^{k,\infty}$ are proved using Lemmas 7.3, 7.4 in the Appendix and the chain rule for Malliavin calculus.

We assume in the rest of the proof that $k = 1$, the general case is left for the reader. In this case, one needs to estimate

$$\|\Phi_t(X_0) - \overline{\Phi}_t(X_0)\|_{1,p} \quad = \quad \{E|\Phi_t(X_0) - \overline{\Phi}_t(X_0)|^p\}^{1/p} + (E[\{\int_0^1 |D_s(\Phi_t(X_0) - \overline{\Phi}_t(X_0))|^2 \, ds\}^{p/2}])^{1/p}.$$

First, we have that due to Lemma 7.3 in the Appendix, that there exists a positive constant $C(p)$ such that

$$E|\Phi_t(X_0) - \overline{\Phi}_t(X_0)|^p \leq C(p)\delta^{\frac{p}{2}}.$$

It is not difficult to compute $D_s\Phi_t(X_0)$ and $D_s\overline{\Phi}_t(X_0)$. This gives for $s \leq t$

$$\begin{aligned}
D_s(\Phi_t(X_0)) \quad &= \quad D_s(\Phi_t)(X_0) + \nabla(\Phi_t)(X_0)D_sX_0 \\
&= \quad \nabla(\Phi_t)(s, \Phi_s(X_0))\sigma(\Phi_s(X_0)) + \nabla(\Phi_t)(X_0)D_sX_0.
\end{aligned}$$

Analogously, we have for $s \leq t$

$$D_s(\overline{\Phi}_t(X_0)) = \nabla(\overline{\Phi}_t)(\eta_1(s), \overline{\Phi}_{\eta_1(s)}(X_0))\sigma(\overline{\Phi}_{\eta(s)}(X_0)) + \nabla(\overline{\Phi}_t)(X_0)D_sX_0.$$

Here, $\eta_1(s) := \min\{t_k; \ t_k \geq s\}$ and $\nabla(\overline{\Phi}_t)(\eta_1(s), x) = 1$ if $s \leq t < \eta_1(s)$. Now consider

$$\begin{aligned}
E(|D_s(\Phi_t(X_0) - \overline{\Phi}_t(X_0))|^p) \ \leq \ &C(p)\{E(|\nabla(\Phi_t)(s, \Phi_s(X_0))\sigma(\Phi_s(X_0)) - \nabla(\overline{\Phi}_t)(s, \overline{\Phi}_s(X_0))\sigma(\overline{\Phi}_s(X_0))|^p) \\
&+E(|\nabla(\overline{\Phi}_t)(s, \overline{\Phi}_s(X_0))\sigma(\overline{\Phi}_s(X_0)) - \nabla(\overline{\Phi}_t)(s, \overline{\Phi}_s(X_0))\sigma(\overline{\Phi}_{\eta(s)}(X_0))|^p) \\
&+E(|\{\nabla(\overline{\Phi}_t)(s, \overline{\Phi}_s(X_0)) - \nabla(\overline{\Phi}_t)(\eta_1(s), \overline{\Phi}_s(X_0))\}\sigma(\overline{\Phi}_{\eta(s)}(X_0))|^p) \\
&+E(|\{\nabla(\overline{\Phi}_t)(\eta_1(s), \overline{\Phi}_s(X_0)) - \nabla(\overline{\Phi}_t)(\eta_1(s), \overline{\Phi}_{\eta_1(s)}(X_0))\}\sigma(\overline{\Phi}_{\eta(s)}(X_0))|^p) \\
&+E(|\nabla(\Phi_t)(X_0) - \nabla(\overline{\Phi}_t)(X_0)|^p|D_sX_0|^p)\} = e_1 + e_2 + e_3 + e_4 + e_5.
\end{aligned}$$

We have that $\sup_{s \in [0,t]}(e_1 + e_2 + e_4 + e_5) \leq C(p)\delta^{p/2}$ due to Lemma 7.4 in the Appendix.

To find the rate of convergence to 0 of e_3 it is enough to note that the processes $\alpha_t = \overline{\Phi}_t(s, x) - \overline{\Phi}_t(\eta_1(s), x)$ and $\beta_t = \nabla\overline{\Phi}_t(s, x) - \nabla\overline{\Phi}_t(\eta_1(s), x)$ are the solution of linear stochastic differential equations. For example, α_t solves the following equation

$$\begin{aligned}
\alpha_t \quad = \quad &\int_s^{\eta_1(s)} m(\overline{\Phi}_{\eta(u)}(s, x))du + \int_s^{\eta_1(s)} \sigma(\overline{\Phi}_{\eta(u)}(s, x))dW_u \\
&+ \int_{\eta_1(s)}^t \int_0^1 \nabla m((1-v)\overline{\Phi}_{\eta(u)}(\eta_1(s), x) + v\overline{\Phi}_{\eta(u)}(s, x))dv\alpha_u du \\
&+ \int_{\eta_1(s)}^t \int_0^1 \nabla\sigma((1-v)\overline{\Phi}_{\eta(u)}(\eta_1(s), x) + v\overline{\Phi}_{\eta(u)}(s, x))dv\alpha_u dW_u.
\end{aligned}$$

From here it follows using Gronwall's Lemma that $E(\sup_{0 \leq t \leq 1} |\alpha_t|^p) \leq C(p)\delta^{p/2}$. In the same manner one proves that β_t satisfies same property. Then one can conclude that $e_3 \leq C(p)\delta^{p/2}$. $\qquad\square$

This proof can actually be used to prove even stronger statements than the ones proven here (see Lemma 7.4 in the Appendix).

THEOREM 3.1 *Let π be a partition of size δ and $\overline{\Phi}$ be defined as in (6). Suppose that $f \in C_p^4(\mathbb{R})$. Also assume that b and σ are smooth functions with bounded derivatives. Finally, suppose that $X_0 \in \mathbb{D}^{2,\infty}$ and $\sup_s E|D_s X_0|^4 < \infty$, $\sup_{s,u} E|D_s D_u X_0|^2 < \infty$. Then for any $t \in [0,1]$, there exists a positive constant C that depends on f such that*

$$|E[f(\Phi_t(X_0)) - f(\overline{\Phi}_t(X_0))]| \leq C\delta.$$

Proof: First assume without loss of generality that $t \in \pi$. We then have

$$
\begin{aligned}
Ef(\overline{\Phi}_t(X_0)) - Ef(\Phi_t(X_0)) &= \sum_{i=1}^{n(t)} E[f(\Phi_t(t_i, \overline{\Phi}_{t_i}(X_0))) - f(\Phi_t(t_{i-1}, \overline{\Phi}_{t_{i-1}}(X_0)))] \\
&= \sum_{i=1}^{n(t)} E\left[f(\Phi_t(t_i, \overline{\Phi}_{t_i}(X_0))) - f(\Phi_t(t_i, \overline{\Phi}_{t_{i-1}}(X_0))) \right. \\
&\qquad \left. + f(\Phi_t(t_i, \overline{\Phi}_{t_{i-1}}(X_0))) - f(\Phi_t(t_i, \Phi_{t_i}(t_{i-1}, \overline{\Phi}_{t_{i-1}}(X_0)))) \right] \\
&= \sum_{i=1}^{n(t)} E\left[\frac{\partial}{\partial x} f(\Phi_t(t_i, x)) \Big|_{x = \overline{\Phi}_{t_{i-1}}(X_0)} \left\{ \overline{\Phi}_{t_i}(X_0) - \overline{\Phi}_{t_{i-1}}(X_0) \right. \right. \\
&\qquad \left. - (\Phi_{t_i}(t_{i-1}, \overline{\Phi}_{t_{i-1}}(X_0)) - \overline{\Phi}_{t_{i-1}}(X_0)) \right\} + R_i(\overline{\Phi}_{t_i}(X_0)) \\
&\qquad \left. - R_i(\Phi_{t_i}(t_{i-1}, \overline{\Phi}_{t_{i-1}}(X_0))) \right] \tag{8}
\end{aligned}
$$

where $n(t) = \max\{j;\ t_j \leq t\}$ and

$$R_i(U) = \int_0^1 \int_0^{s_2} \frac{\partial^2}{\partial x^2} f(\Phi_t(t_i, \cdot))(U + s_1(U - \overline{\Phi}_{t_{i-1}}(X_0)))(U - \overline{\Phi}_{t_{i-1}}(X_0))^2 \, ds_1 ds_2. \tag{9}$$

To shorten the proof we will select some terms to show how the proof is done.

For example in (8) one has to analyze the term (this is a particular form of the Itô-Taylor formula, see Theorem 5.5.1 in [16])

$$
\left(\overline{\Phi}_{t_i}(X_0) - \overline{\Phi}_{t_{i-1}}(X_0) - (\Phi_{t_i}(t_{i-1}, \overline{\Phi}_{t_{i-1}}(X_0)) - \overline{\Phi}_{t_{i-1}}(X_0)) \right) =
$$

$$
\sum_{\alpha \in B(\Gamma_1)} I_{\alpha, t_{i-1}, t_i} \left(g_\alpha(\cdot, \Phi_\cdot(t_{i-1}, \overline{\Phi}_{t_{i-1}}(x))) \right) \Big|_{x = X_0}. \tag{10}
$$

Here $g(x) = x$ and $g_{(0,1)}$ is defined in (7). Some of these terms will involve stochastic integrals that have to be plugged into (8). For example, the expectation of one of these terms will generate for $\alpha = (0,1)$

$$
E\left\{ \frac{\partial}{\partial x} f(\Phi_t(t_i, x)) \Big|_{x = \overline{\Phi}_{t_{i-1}}(X_0)} I_{(0,1), t_{i-1}, t_i}(g_{(0,1)}(\cdot, \Phi_\cdot(t_{i-1}, \overline{\Phi}_{t_{i-1}}(x)))) \Big|_{x = X_0} \right\}.
$$

Next, this equals

$$
\int_{t_{i-1}}^{t_i} E\left(\frac{\partial}{\partial x} f(\Phi_t(t_i, x)) \Big|_{x = \overline{\Phi}_{t_{i-1}}(X_0)} \int_{t_{i-1}}^s G(\Phi_u(t_{i-1}, \overline{\Phi}_{t_{i-1}}(x))) dW_u \Big|_{x = X_0} \right) ds, \tag{11}
$$

where $G(x) = m(x)\sigma'(x) + \frac{1}{2}\sigma^2(x)\sigma''(x)$. Then applying the formula (51) one has that the above expression becomes

$$\int_{t_{i-1}}^{t_i} \int_{t_{i-1}}^{s} E \left\{ \frac{\partial}{\partial x} f(\Phi_t(t_i, x)) \bigg|_{x = \overline{\Phi}_{t_{i-1}}(X_0)} D_u X_0 \frac{\partial G(\Phi_u(t_{i-1}, \overline{\Phi}_{t_{i-1}}(\cdot)))}{\partial x}(X_0) \right.$$

$$\left. + D_u \frac{\partial}{\partial x} f(\Phi_t(t_i, x)) \bigg|_{x = \overline{\Phi}_{t_{i-1}}(X_0)} G(\Phi_u(t_{i-1}, \overline{\Phi}_{t_{i-1}}(X_0))) \right\} du\, ds. \tag{12}$$

To finish one obtains (using Lemmas 7.3 and 7.4) that the above integrand is smaller than $C\delta^2$ uniformly in $\{(s, u) \in [t_{i-1}, t_i]^2; \quad u \leq s\}$. Here C is a positive constant independent of the partition π, δ, s and t. The treatment of the other terms in (10) is similar.

For example, we will bound in the first term in (12). That is, we will prove that there exists a positive constant independent of the partition π and δ such that

$$\sup_{u \in [t_{i-1}, t_i]} \left| E \left(D_u X_0 \frac{\partial}{\partial x} f(\Phi_t(t_i, x)) \bigg|_{x = \overline{\Phi}_{t_{i-1}}(X_0)} \frac{\partial G(\Phi_u(t_{i-1}, \overline{\Phi}_{t_{i-1}}(\cdot)))}{\partial x}(X_0) \right) \right| < C. \tag{13}$$

In fact, we have $D_u X_0 \in L^p(\Omega)$, for any $p \leq 2$, uniformly in u by hypothesis. As f has polynomial growth at infinity and using flow properties together with Lemma 7.3, we also have that $\frac{d}{dx} f(\Phi_t(t_i, x)) \bigg|_{x = \overline{\Phi}_{t_{i-1}}(X_0)} \in L^p(\Omega)$, for any $p > 1$ uniformly in $\{(t_i, t_{i-1}) \in [0, t]^2; \quad t_{i-1} < t_i\}$.

Now note that $G(x)$ is smooth with polynomial growth at infinity. Therefore we have as before that $\frac{\partial}{\partial x} G(\Phi_u(t_{i-1}, \overline{\Phi}_{t_{i-1}}(X_0))) \in L^p(\Omega)$, for any $p > 1$ uniformly in $t_{i-1} \in [0, t]$. Therefore the result follows.
$\square$

Remark 3.1 *1. When dealing with the residues in (9) it is necessary to repeat some of the steps shown in the proof above. In fact, for example we will have a term of the type*

$$(\overline{\Phi}_{t_i} - \overline{\Phi}_{t_{i-1}}(X_0))^2 = \left(m(\overline{\Phi}_{t_{i-1}}(X_0))(t_i - t_{i-1}) + \sigma(\overline{\Phi}_{t_{i-1}}(X_0))(W_{t_i} - W_{t_{i-1}}) \right)^2.$$

Here one expands the square and uses again (51) whenever increments of the Wiener process appear.
2. The condition $\sup_{s,u} E|D_s D_u X_0|^2 < \infty$. is used in (13) in the case $\alpha = (1, 1)$.
3. Note that the conditions on the derivatives of X_0 can be relaxed if one uses the appropiate Hölder inequalities in (13). This weaker condition is of the form: There exists $\epsilon > 0$ such that

$$\sup_{s \in [0,1]} E|D_s X_0|^{2+\epsilon} + \sup_{s,u \in [0,1]} E|D_s D_u X_0|^{1+\epsilon} < \infty.$$

4. Theorem 3.1 is an improvement of Theorem 4.2 in [2]. The method of proof is different. Theorem 4.2 in [2] is strongly based in some generator of a highly complex process which in spirit resembles the classical proofs that one can find in e.g. [16], Chapter 14.

We will now give a first application of this theorem. Consider the weak approximation problem for $f(\Phi_t(X_0))$ where X_0 is generated by a diffusion. Up to the previous theorem it was assumed that the

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vector $(X_0, W_{t_1}, ..., W_{t_N})$ had a known joint law that can be simulated. Now we consider the case when X_0 also needs to be approximated through an Euler-Maruyama approximation. For this, let Z be the diffusion defined by

$$Z_t = Z_0 + \int_0^t B(Z_s)dW_s + \int_0^t A(Z_s)ds. \tag{14}$$

Here Z_0 is any $\mathcal{F}_0$-measurable random variable in $L^p(\Omega)$ for any $p > 1$. Analogously define the Euler approximation scheme $\overline{Z}$ for Z using a partition $\pi' := \{0 = s_0 < ... < s_M = 1\}$ with $|\pi'| \leq \delta'$. Note that the noise that generates Z is the same to the one that generates X. Therefore, in general, X and Z are not independent.

Assume that A and B are smooth with bounded derivatives then $Z_1 \in \mathbb{D}^{2,\infty}$ and furthermore

$$\sup_{s_1,...,s_k} E[\sup_t |D_{s_1}...D_{s_k}(Z_t - \overline{Z}_t)|^p] \leq C_k(\delta')^{\frac{p}{2}},$$

for any $k \in \mathbb{N}$ (this is done using essentially the same proof as in Proposition 3.1). Therefore by Theorem 3.1 we obtain that if $f \in C_p^3$ there exists a positive constant C that may depend on f and t but it is independent of π, π', δ and δ' such that

$$|Ef(\Phi_t(Z_1)) - Ef(\overline{\Phi}_t(Z_1))| \leq C\delta.$$

A realistic case is to consider that Z_1 can not be simulated exactly and that we have to use an Euler approximation of it. The following Lemma will give a weak approximation result for of $\Phi_t(Z_1)$.

LEMMA 3.1 *Let $f \in C_p^4(\mathbb{R})$. Assume that A, B, b and σ are smooth with bounded derivatives, then there exists a positive constant C that may depend on f and t such that*

$$|Ef(\overline{\Phi}_t(Z_1)) - Ef(\overline{\Phi}_t(\overline{Z}_1))| \leq C\delta'.$$

Proof: We start like in the proof of Theorem 3.1. Denote by $\phi_t(x)$ the stochastic flow defined by (14) and assume without loss of generality that $s_M = 1$.

$$
\begin{aligned}
Ef(\overline{\Phi}_t(\overline{Z}_1)) - Ef(\overline{\Phi}_t(Z_1)) &= \sum_{i=1}^M E[f(\overline{\Phi}_t(\phi_1(s_i, \overline{Z}_{s_i}))) - f(\overline{\Phi}_t(\phi_1(s_{i-1}, \overline{Z}_{s_{i-1}})))] \\
&= \sum_{i=1}^M E\Big[f(\overline{\Phi}_t(\phi_1(s_i, \overline{Z}_{s_i}))) - f(\overline{\Phi}_t(\phi_1(s_i, \overline{Z}_{s_{i-1}}))) \\
&\qquad + f(\overline{\Phi}_t(\phi_1(s_i, \overline{Z}_{s_{i-1}}))) - f(\overline{\Phi}_t(\phi_1(s_i, \phi_{s_i}(s_{i-1}, \overline{Z}_{s_{i-1}})))) \Big] \\
&= \sum_{i=1}^M E\Big[\frac{\partial}{\partial x} f(\overline{\Phi}_t(\phi_1(s_i, x)))\Big|_{x=\overline{Z}_{s_{i-1}}} (\overline{Z}_{s_i} - \overline{Z}_{s_{i-1}} \\
&\qquad -(\phi_{s_i}(s_{i-1}, \overline{Z}_{s_{i-1}}) - \overline{Z}_{s_{i-1}})) + R_i(\overline{Z}_{s_i}) - R_i(\phi_{s_i}(s_{i-1}, \overline{Z}_{s_{i-1}})) \Big]
\end{aligned}
$$

where

$$R_i(U) = \int_0^1 \int_0^{\theta_2} \frac{\partial^2}{\partial x^2} f(\overline{\Phi}_t(\phi_1(s_i, x)))(U + \theta_1(U - \overline{Z}_{s_{i-1}}))(U - \overline{Z}_{s_{i-1}})^2 d\theta_1 d\theta_2.$$

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From here we can continue with the argument as in the proof of Theorem 3.1 (in particular the argument after (8)). Therefore the result follows. $\qquad\square$

Therefore one obtains that if one approximates the initial condition with a degree of accuracy δ, the new approximation $\overline{\Phi}_t(\overline{Z}_1)$ is also of order at least δ.

Note that in the proof of this theorem was essential that Z is a diffusion in order to be able to use the Itô-Taylor formula as in Chapter 14 of [16]. This will not be the case in the next section.

Resuming the previous calculations we have the following result.

THEOREM 3.2 *Assume that* $f \in C_p^4(\mathbb{R}^d)$, A, B, b *and* σ *are smooth with bounded derivatives. Then there exists a positive constant* C *such that it depends on* t *and* f *but it is independent of the partitions* π *and* π' *as well as* δ *and* δ' *and it satisfies*

$$|E[f(\Phi_t(Z_1)) - f(\overline{\Phi}_t(\overline{Z}_1))]| \leq C(\delta + \delta').$$

One can also extend this result to high order weak approximations. A previous result of this type was quoted in [2], Theorem 4.2. In that result the hypothesis on X_0 were much more restrictive than the ones presented here. In particular one required that some type of trace for $D_t D_t X_0$ existed. The argument was to use approximations for X_0 that belonged to $\mathcal{S}$.

4   An Euler type scheme for solutions of stochastic differential equations with boundary conditions

Stochastic differential equations with boundary conditions arise naturally in the study of perturbations of ordinary differential equations with boundary conditions as well as stochastic models for many physical phenomena. The behaviour and numerical approximation in the ode case is well understood (see e.g. [7] or [15]).

Stochastic differential equations with boundary conditions have a history that may have started with Kwakerwnaak [21]. Later they have been studied by Ocone-Pardoux [27], Nualart-Pardoux [25], Donati-Martin [8], Garnier [13], among others. These are equations of the type

$$dX_t \;=\; \sigma(X_t) \;\circ\; dW_t \;+\; b(X_t)\,dt, \quad 0 \leq t \leq 1, \tag{15}$$

with a boundary condition of the form

$$h(X_0, X_1) \;=\; \overline{h}. \tag{16}$$

Most of the studies on these equations concern existence and uniqueness of solutions and the Markov field property.

These equations are anticipative in nature due to the boundary condition. Their solutions are evaluations of flows at random variables. That is, the solution to (15) and (16), when it exists, can be written as $X_t \;=\; \Phi_t(X_0)$, where $\{\Phi_t(x)\ t \in [0,1]\}$ is the stochastic flow associated with (15) and X_0 is the unique solution to (16).

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A general type of sde with a given anticipating initial condition was first studied by Ocone-Pardoux [28]. They proved existence and uniqueness for solutions by means of an Itô-Ventzell type formula.

Here we are interested in the rate of convergence for a weak approximation to the solution of (15)-(16). That is, we will define an approximation for the solution of (15)-(16) and prove that the approximation converges at some rate to the solution.

We will consider two cases. First, when the boundary condition is linear and the equation is one dimensional. Existence and uniqueness for this type of equations was considered by Donati-Martin [8].

As in the previous section we want to find an approximation for $E(f(\Phi_t(X_0)))$. Theorem 3.1 is applicable in this case because $X_0 \in \mathbb{D}^\infty$ as will be stated later. The added complication here is the fact that X_0 also needs to be approximated. This approximation is not as smooth as in the case studied in Lemma 3.1. In fact, it is only locally smooth as it will be proved in Lemma 4.1. This introduces a new ingredient to this problem that will be solved through an appropiate localization procedure. We will carry this argument to analyze approximations of densities for $\Phi_t(X_0)$ when they exist.

In the second case we consider a multidimensional equation with $\sigma \equiv B$, a constant matrix and a boundary condition of a general type. In this case one can consider as examples some periodic boundary conditions. Existence and uniqueness was obtained by Nualart and Pardoux [25].

In all the cases considered here it is known (see, [25]) that solutions of (15)-(16) are seldom Markov processes. Therefore the idea of using the classical method of analysis (see e.g. Chapter 14 in [16]) through PDE problems can not be applied here.

In other articles (see [19], [10]), we have considered the necessary preliminaries to study this problem. That is, the existence and smoothness of the density and the strong approximation for the solution of (15)-(16) in the two cases mentioned above. These results are used throughout the text so we will recall them when necessary.

Although here we only consider approximations for the densities one could also have considered approximations for the distribution functions. With these results in hand one could approximate these type of processes using the appropiate Monte Carlo methods.

4.1 The one dimensional case

In this section we will study approximations for densities of solutions to (15) and (16) in the one dimensional case with linear boundary conditions. Existence and uniqueness of solutions for this case was studied in [8]. That is, let σ and b be real functions, F_0, F_1, $h_0 \in \mathbb{R}$. Consider the equation

$$dX_t = \sigma(X_t) \circ dW_t + b(X_t)\,dt, \quad 0 \le t \le 1, \tag{17}$$

$$F_0 X_0 + F_1 X_1 = h_0. \tag{18}$$

Assume from now on that σ and b are smooth functions with bounded derivatives and $F_0\,F_1 > 0$. Without loss of generality we will always take $F_0 > 0$ and $F_1 > 0$. In [8] it is proved that under these assumptions, there exists a unique solution to (17) belonging to the space $\mathbb{L}^{1,\infty}_{C,loc}$. Furthermore, this solution can be written as $\Phi_t(X_0)$, where X_0 is the unique solution to (18).

Next we consider a theorem that states when a smooth density of (17)-(18) exists.

THEOREM 4.1 *(Kohatsu-Higa and Sanz-Solé [19]) Define*

$$\Sigma_0 = \{\sigma\}$$
$$\Sigma_j = \left\{ [\sigma, V], \ [b, V] + \frac{1}{2}[\sigma, [\sigma, V]] \, ; \ V \in \Sigma_{j-1} \right\}, \ j \geq 1,$$

where $[\cdot, \cdot]$ denotes the Lie bracket. Suppose that there exist $j_0 \geq 0$ and $V \in \Sigma_{j_0}$ with $(V(X_0))^{-1} \in \cap_{p \geq 1} L^p(\Omega)$. Then, for any $t \in [0, 1]$ the law of X_t has a C^∞ density.

Furthermore there exists a finite positive constant $C_p(t)$ such that $\|(\Delta_{\Phi_t(X_0)})^{-1}\|_p \leq C_p(t)$, where Δ_F denotes the Malliavin covariance matrix associated with the random variable F.

We will denote the density of X_t at the point y by $p(t, y)$.

Let $C := \|\sigma'\|_\infty + \|b'\|_\infty$. Choose $\delta, M > 0$ satisfying $\delta \vee M < \dfrac{1}{4C}$ and let $W(\Delta_k)$ denote $W(t_{k+1}) - W(t_k)$. With this notation one obtains using a recursive argument that $\overline{\Phi}_t(x)$ is differentiable in x and that $\overline{\Phi}'_0(x) = 1$,

$$\overline{\Phi}'_{t_{k+1}}(x) = \prod_{i=0}^{k} \left(1 + \sigma'\left(\overline{\Phi}_{t_i}(x)\right) W(\Delta_k) + b'\left(\overline{\Phi}_{t_k}(x)\right) \delta \right).$$

Then, on the set $L_M = \left\{ \sup\limits_{0 \leq k \leq N-1} |W(\Delta_k)| < M \right\}$, one has that $|\sigma'\left(\overline{\Phi}_{t_i}(x)\right) W(\Delta_k) + b'\left(\overline{\Phi}_{t_k}(x)\right) \delta| \leq 1/2$ and therefore $\overline{\Phi}'_1(x)$ is strictly positive. Consequently, if $\omega \in L_M$, the function $\overline{G}(x, \omega)$ given by

$$\overline{G}(x, \omega) = F_0 x + F_1 \overline{\Phi}_1(x)(\omega)$$

is monotone. We denote by $\widetilde{X}_0(\omega)$ the unique solution to $\overline{G}(x, \omega) = h_0$, $\omega \in L_M$. Obviously one has that the probabilty of the set $\overline{L}_M$ is small (the bar denotes the complement of the set L_M). That is for $q \geq 2$

$$P(\overline{L}_M) \leq \frac{C_q^N \delta^{\frac{q}{2}}}{M^q} \tag{19}$$

$$C_q^N = E\left(\sup_{0 \leq k \leq N-1} \frac{|W(\Delta_k)|}{\sqrt{\delta}} \right)^{\frac{q}{2}}$$
$$= \frac{2}{\pi} \int_0^\infty v^q N \left(\int_{-v}^{v} \exp(-\frac{u^2}{2}) du \right)^{N-1} \exp(-\frac{v^2}{2}) dv.$$

Furthermore $\sup_{\delta \in (0,1]} \delta C_q^N < \infty$.

Remark 4.1 *The following procedure provides approximations for $\widetilde{X}_0(\omega)$. Assume $F_0 F_1 > 0$. To simplify the notation we skip the dependence on ω. Fix $\widetilde{X}_0^{2,0} \leq \widetilde{X}_0^{1,0}$ such that $\overline{G}(\widetilde{X}_0^{2,0}) \leq h_0$, $\overline{G}(\widetilde{X}_0^{1,0}) \geq h_0$. We proceed inductively as follows. Let $\widetilde{X}_0^{2,i} \leq \widetilde{X}_0^{1,i}$ be such that $\overline{G}(\widetilde{X}_0^{2,i}) \leq h_0$,*

$\overline{G}(\widetilde{X}_0^{1,i}) \geq h_0$, $i \geq 0$. Consider $\dfrac{\widetilde{X}_0^{2,i} + \widetilde{X}_0^{1,i}}{2}$. Then,

$$if \quad \overline{G}\left(\frac{\widetilde{X}_0^{2,i} + \widetilde{X}_0^{1,i}}{2}\right) \; < \; h_0 \;\; , \;\; set \;\; \widetilde{X}_0^{2,i+1} = \frac{\widetilde{X}_0^{2,i} + \widetilde{X}_0^{1,i}}{2} \;\; , \;\; \widetilde{X}_0^{1,i+1} = \widetilde{X}_0^{1,i},$$

$$if \quad \overline{G}\left(\frac{\widetilde{X}_0^{2,i} + \widetilde{X}_0^{1,i}}{2}\right) \; > \; h_0 \;\; , \;\; set \;\; \widetilde{X}_0^{2,i+1} = \widetilde{X}_0^{2,i} \;\; , \;\; \widetilde{X}_0^{1,i+1} = \frac{\widetilde{X}_0^{2,i} + \widetilde{X}_0^{1,i}}{2},$$

$$if \quad \overline{G}\left(\frac{\widetilde{X}_0^{2,i} + \widetilde{X}_0^{1,i}}{2}\right) \; = \; h_0 \;\; , \;\; set \;\; \widetilde{X}_0 = \frac{\widetilde{X}_0^{2,i} + \widetilde{X}_0^{1,i}}{2}.$$

Notice that $\left|\widetilde{X}_0^{2,i} - \widetilde{X}_0^{1,i}\right| \leq \dfrac{\left|\widetilde{X}_0^{2,0} - \widetilde{X}_0^{1,0}\right|}{2^{i-1}}$, $i \geq 1$.

Let $\overline{X}_0 = \widetilde{X}_0 \, \mathbf{1}_{L_M}$. The integer M plays the role of a stability index. When the increments of the Brownian motion are too big, then a solution to $\overline{G}(x, \omega) = h_0$ may not exist.

Therefore $\overline{X}_0$ is an approximation of X_0, defined through (15)-(16) using an Euler approximation (with step size δ) instead the flow Φ in $X = \Phi(X_0)$. In deterministic settings this is known as the shooting method to approximate ordinary differential equations with boundary conditions.

Now we study some stochastic differentiability properties of X_0 and $\overline{X}_0$. For this, let ξ be a random variable such that $\overline{\Phi}_1(X_0) - \overline{\Phi}_1(\overline{X}_0) = \overline{\Phi}_1'(\xi)(X_0 - \overline{X}_0)$. Then, define for $\omega \in L_M$, $\overline{Z}(\xi) = F_0 + F_1 \overline{\Phi}_1'(\xi) = \int_0^1 (F_0 + F_1 \overline{\Phi}_1'(\overline{X}_0 + u(X_0 - \overline{X}_0))) du \neq 0$. This gives

$$X_0 - \overline{X}_0 = -(\overline{Z}(\xi))^{-1} F_1 (\Phi_1(X_0) - \overline{\Phi}_1(X_0)), \tag{20}$$

if $\omega \in L_M$. For $\omega \notin L_M$ we define $\overline{Z}(\xi) = 1$. We will need the following stochastic differentiability properties of X_0 and $\overline{X}_0$.

LEMMA 4.1 *We have that* $X_0 \in \mathbb{D}^\infty$, $\overline{X}_0 \in \mathbb{D}_{loc}^\infty$, $(\overline{Z}(\xi))^{-1} \in \mathbb{D}_{loc}^\infty$. *Also we have for all* $p > 1$, $k = 0, 1, ...$

$$\sup_\delta \; \sup_{(t_1,...,t_k) \in [0,1]^k} \|D_{t_1}...D_{t_k} X_0\|_p + \sup_\delta \; \sup_{(t_1,...,t_k) \in [0,1]^k} \|D_{t_1}...D_{t_k} \overline{X}_0\|_p$$
$$+ \sup_\delta \; \sup_{(t_1,...,t_k) \in [0,1]^k} \|D_{t_1}...D_{t_k} (\overline{Z}(\xi))^{-1}\|_p < \infty.$$

Proof: In Kohatsu-Higa and Sanz-Solé [19] it is proven that $X_0 \in \mathbb{D}^\infty$ and that

$$D_t X_0 = -\frac{F_1 D_t \Phi_1(X_0)}{F_0 + F_1 \Phi_1'(X_0)}. \tag{21}$$

Given that $F_0 F_1 > 0$ we have that $|D_t X_0| \leq C |D_t \Phi_1(X_0)|$. Therefore one obtains that $\sup_t \|D_t X_0\|_p < \infty$ applying Lemma 7.3. The proof of $\sup_{u,t} \|D_u D_t X_0\|_p < \infty$ is similar.

Now we will prove the properties about $\overline{X}_0$ and $(\overline{Z}(\xi))^{-1}$ for $k = 1$.

We use the following localizing sequence: The sets

$$L_M^\epsilon := \{ \max_{0 \leq k \leq N-1} |\omega(\Delta_k)| < M - 2\epsilon \text{ or } \max_{0 \leq k \leq N-1} |\omega(\Delta_k)| > M + \epsilon \} \uparrow \Omega \text{ as } \epsilon \to 0 \tag{22}$$

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and let $h_{M,\epsilon} : \mathbb{R}^N \to [0,1]$ be a bounded smooth function with bounded derivatives such that

$$h_{M,\epsilon}(x_1, ..., x_N) = \begin{cases} 1 & ; \text{ if } \max\{|x_1|, ..., |x_N|\} \leq M - 2\epsilon \\ 0 & ; \text{ if } \max\{|x_1|, ..., |x_N|\} \geq M - \epsilon. \end{cases} \tag{23}$$

Then the localizing random variable is defined by $\overline{X_0}^\epsilon = \overline{X_0} h_{M,\epsilon}(\omega(\Delta_0), ..., \omega(\Delta_{N-1}))$ and $(\overline{Z}(\xi)^{-1})^\epsilon = (\overline{Z}(\xi))^{-1} h_{M,\epsilon}(\omega(\Delta_0), ..., \omega(\Delta_{N-1})) + (1 - h_{M,\epsilon}(\omega(\Delta_0), ..., \omega(\Delta_{N-1})))$. Now in order to prove the differentiability of $\overline{X_0}^\epsilon$ and $(\overline{Z}(\xi)^{-1})^\epsilon$ consider $\omega^n = \omega + n^{-1} \int_0^\cdot g_s ds$ for $g \in L^2([0,1])$, $n \in \mathbb{N}$. To simplify notation we will write $h_{M,\epsilon}(\omega)$ for $h_{M,\epsilon}(\omega(\Delta_0), ..., \omega(\Delta_{N-1}))$. Then consider

$$n(\overline{X_0}^\epsilon(\omega^n) - \overline{X_0}^\epsilon(\omega)) - D_g \overline{X_0}(\omega) h_{M,\epsilon}(\omega) - \overline{X_0}(\omega) D_g h_{M,\epsilon}(\omega) =$$
$$n(\overline{X_0}(\omega^n) - \overline{X_0}(\omega) - n^{-1} D_g \overline{X_0}(\omega)) h_{M,\epsilon}(\omega^n) + D_g \overline{X_0}(\omega)(h_{M,\epsilon}(\omega^n) - h_{M,\epsilon}(\omega)) +$$
$$n(h_{M,\epsilon}(\omega^n) - h_{M,\epsilon}(\omega) - n^{-1} D_g h_{M,\epsilon}(\omega)) \overline{X_0}(\omega). \tag{24}$$

Here,

$$D_g \overline{X_0}(\omega) = -\frac{F_1(D_g \overline{\Phi}_1)(\overline{X_0})(\omega)}{F_0 + F_1 \overline{\Phi}_1'(\overline{X_0})(\omega)} \, \mathbb{1}_{L_M}(\omega) \tag{25}$$

$$D_g h_{M,\epsilon}(\omega) = \sum_{j=1}^N \frac{\partial h_{M,\epsilon}}{\partial x_j}(\omega(\Delta_0), ..., \omega(\Delta_{N-1})) \int_{t_{j-1}}^{t_j} g(s) ds.$$

If we take n big enough so that $\dfrac{\sqrt{\delta}\|g\|_2}{n} < \dfrac{\epsilon}{4}$ and suppose that $h_{M,\epsilon}(\omega^n) \neq 0$, then using the definition of $h_{M,\epsilon}$ we have that

$$\max_{0 \leq k \leq N-1} |\omega(\Delta_k)| \leq \max_{0 \leq k \leq N-1} |\omega^n(\Delta_k)| + \frac{\sqrt{\delta}\|g\|_2}{n}$$
$$\leq M - \frac{3\epsilon}{4}.$$

Therefore we only consider the case $\omega, \quad \omega^n \in L_M$. In such a case, both $\overline{X_0}(\omega)$ and $\overline{X_0}(\omega^n)$ satisfy the boundary condition. Subtracting these boundary conditions we obtain

$$n(\overline{X_0}(\omega^n) - \overline{X_0}(\omega)) = -F_1 n \left(\frac{\overline{\Phi}_1(\omega^n, \overline{X_0}(\omega)) - \overline{\Phi}_1(\omega, \overline{X_0}(\omega))}{F_0 + F_1 \overline{\Phi}_1'(\omega^n, \xi^n(\omega))} \right),$$

where $\xi^n(\omega)$ is a random point between $\overline{X_0}(\omega)$ and $\overline{X_0}(\omega^n)$. Therefore due to the stochastic differentiability of $\overline{\Phi}_1$ and as $\xi^n(\omega)$ converges to $\overline{X_0}(\omega)$ in $L^p(\Omega)$ for any $p > 1$ as $n \to \infty$ and using Lemma 7.3, it follows that

$$E \left| \left(F_1 n \left(\frac{\overline{\Phi}_1(\omega^n, \overline{X_0}(\omega)) - \overline{\Phi}_1(\omega, \overline{X_0}(\omega))}{F_0 + F_1 \overline{\Phi}_1'(\omega^n, \xi^n(\omega))} \right) - \frac{F_1(D_g \overline{\Phi}_1)(\overline{X_0})}{F_0 + F_1 \overline{\Phi}_1'(\overline{X_0})} \right) h_{M,\epsilon}(\omega^n) \right|^p \to 0.$$

The other terms in (24) are dealt in a similar fashion. The property $\sup_{t,\delta} \|D_t \overline{X_0}\|_p < \infty$ is obtained through (25) using Lemma 7.3 and properties of the flow defined by $D_t \overline{\Phi}_s(x)$.

The proof of the stochastic differentiability of $(\overline{Z}(\xi)^{-1})^\epsilon$ uses the same techniques. In fact, consider

$$n((\overline{Z}(\xi)^{-1})(\omega^n) - (\overline{Z}(\xi)^{-1})(\omega)) h_{M,\epsilon}(\omega^n) = h_{M,\epsilon}(\omega^n) F_1$$
$$\frac{\int_0^1 (\overline{\Phi}_1'(\omega^n, [\overline{X_0} + u(X_0 - \overline{X_0})](\omega^n)) - \overline{\Phi}_1'(\omega, [\overline{X_0} + u(X_0 - \overline{X_0})](\omega)) du}{(F_0 + F_1 \int_0^1 (\overline{\Phi}_1'(\omega^n, [\overline{X_0} + u(X_0 - \overline{X_0})](\omega^n)) du)(F_0 + F_1 \int_0^1 (\overline{\Phi}_1'(\omega, [\overline{X_0} + u(X_0 - \overline{X_0})](\omega)) du)}.$$

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Here, we again have that $\overline{X_0}(\omega^n)$ and $\overline{X_0}(\omega)$ satisfy the boundary condition if $h_{M,\epsilon}(\omega^n) \neq 0$. Therefore the differentiability of $(\overline{Z}(\xi)^{-1})^\epsilon$ follows from the differentiability properties of $\overline{X_0}^\epsilon$ and $\overline{\Phi}_1'$. $\quad\square$

LEMMA 4.2 *For $t \in [0,1]$ fixed, k, $p \in \mathbb{N}$ and for every $\epsilon \in (0,1)$ there exists a positive constant $C(\epsilon,k,p)$ such that*

$$\|X_0 - \overline{X_0}^\epsilon\|_{k,p} \leq C(\epsilon,k,p)\delta^{1/2}.$$

Proof:

$$
\begin{aligned}
X_0 - \overline{X_0}^\epsilon &= (X_0 - \overline{X_0})h_{M,\epsilon} + X_0(1 - h_{M,\epsilon}) \\
&= (-\overline{Z}(\xi))^{-1}F_1(\Phi_1(X_0) - \overline{\Phi}_1(X_0))h_{M,\epsilon} + X_0(1 - h_{M,\epsilon}).
\end{aligned}
$$

Lemma 4.1 and Proposition 3.1 give

$$\|(-\overline{Z}(\xi))^{-1}F_1(\Phi_1(X_0) - \overline{\Phi}_1(X_0)))h_{M,\epsilon}\|_{k,p} \leq C(\epsilon,k,p)\delta^{1/2}.$$

The second estimate $\|X_0(1 - h_{M,\epsilon})\|_{k,p} \leq C(\epsilon,k,p)\delta^{1/2}$ follows from (19). $\quad\square$

Now we give the result that shows that the weak rate of convergence for our approximation method is δ. Its proof will give us some important steps to consider later approximations for the density of the solution process.

THEOREM 4.2 *Let $f \in C_p^4(\mathbb{R})$. Then we have that for any $t \in [0,1]$ there is a positive constant C depending on f but independent of π, t and δ such that*

$$|Ef(\Phi_t(X_0)) - Ef(\overline{\Phi}_t(\overline{X_0}))| \leq C\delta.$$

The method of proof shown here has the advantage that it allows the extension to consideration of non-smooth functions in cases where one has some estimations of the Malliavin covariance matrix of the process involved. In order to avoid a long proof with long expressions we will sketch the proof of the above Theorem using analogies with some of the steps taken in the proof of Theorem 3.1.

Sketch of the proof of Theorem 4.2: First, we have that Theorem 3.1 is applicable due to Lemma 4.1 and therefore

$$|E(f(\Phi_t(X_0)) - f(\overline{\Phi}_t(X_0)))| \leq C\delta$$

Now we consider the term

$$
\begin{aligned}
|E(f(\overline{\Phi}_t(X_0)) - f(\overline{\Phi}_t(\overline{X_0})))| &\leq |E(f(\overline{\Phi}_t(X_0)) - f(\overline{\Phi}_t(\overline{X_0})); L_M)| \\
&\quad + |E(f(\overline{\Phi}_t(X_0)) - f(\overline{\Phi}_t(\overline{X_0})); \overline{L}_M)|.
\end{aligned}
$$

The second term on the right is smaller than $C\delta$ due to (19), Cauchy-Schwarz inequality and the fact that $\sup_{\delta \in (0,1]} E|f(\overline{\Phi}_t(X_0)) - f(\overline{\Phi}_t(\overline{X_0}))|^p < \infty$ for any p.

To deal with the first term note that for $\omega \in L_M$, we have using the mean value theorem and (20) that

$$
\begin{aligned}
f(\overline{\Phi}_t(X_0) - y) - f(\overline{\Phi}_t(\overline{X}_0) - y) &= \int_0^1 \frac{d}{dx} f(\overline{\Phi}_t(\cdot - y))(\overline{X}_0 + u(X_0 - \overline{X}_0)) du(X_0 - \overline{X}_0) \\
&= \int_0^1 \frac{d}{dx} f(\overline{\Phi}_t(\cdot - y))(\overline{X}_0 + u(X_0 - \overline{X}_0)) du(-\overline{Z}(\xi))^{-1} F_1(\Phi_1(X_0) - \overline{\Phi}_1(X_0))) .
\end{aligned} \tag{26}
$$

Therefore,

$$
\begin{aligned}
E(f(\overline{\Phi}_t(\overline{X}_0)) - f(\overline{\Phi}_t(X_0)); L_M) &= \sum_{i=1}^N E \Bigg[\int_0^1 \frac{d}{dx} f(\overline{\Phi}_t(\overline{X}_0 + u(X_0 - \overline{X}_0)) - y) du(-\overline{Z}(\xi))^{-1} F_1 \\
&\qquad \left(\frac{\partial}{\partial x} \Phi_t(t_i, x) \bigg|_{x = \overline{\Phi}_{t_{i-1}}(X_0)} \left\{ \overline{\Phi}_{t_i}(X_0) - \overline{\Phi}_{t_{i-1}}(X_0) \right. \right. \\
&\qquad \left. - (\Phi_{t_i}(t_{i-1}, \overline{\Phi}_{t_{i-1}}(X_0)) - \overline{\Phi}_{t_{i-1}}(X_0)) \right\} + R_i(\overline{\Phi}_{t_i}(X_0)) \\
&\qquad \left. - R_i(\Phi_{t_i}(t_{i-1}, \overline{\Phi}_{t_{i-1}}(X_0))) \right) ; L_M \Bigg]
\end{aligned}
$$

This is the analogue to formula (8). The residues R_i are defined considering $f(x) = x$ in (9). As in Theorem 3.1 we arrive at the consideration of terms that are similar. For example, the analogous to (11) is

$$
\begin{aligned}
\int_{t_{i-1}}^{t_i} \int_0^1 & E \left(\frac{d}{dx} f(\overline{\Phi}_t(\overline{X}_0 + v(X_0 - \overline{X}_0)) - y))(-\overline{Z}(\xi))^{-1} F_1 \frac{\partial}{\partial x} \Phi_t(t_i, x) \bigg|_{x = \overline{\Phi}_{t_{i-1}}(X_0)} \right. \\
& \left. \int_{t_{i-1}}^s G(\Phi_u(t_{i-1}, \overline{\Phi}_{t_{i-1}}(x))) dW_u \bigg|_{x = X_0} ; L_M \right) dv \, ds
\end{aligned}
$$

Denote the expression inside the above expectation by Ξ. Then, we have that for a fixed $\epsilon > 0$,

$$
\begin{aligned}
E(\Xi; L_M) &= E(\Xi h_{M, 2\epsilon}) + E(\Xi (\mathbb{1}_{L_M} - h_{M, 2\epsilon})) \\
&\leq E(\Xi^\epsilon h_{M, 2\epsilon}) + C(\epsilon)(E(\Xi^2))^{1/2} P(M - 4\epsilon \leq \sup_{0 \leq k \leq N-1} |W(\Delta_k)| < M)^{1/2}
\end{aligned}
$$

where $C(\epsilon)$ is a positive constant that depends only on ϵ. Ξ^ϵ is the localization of Ξ to the set L_M^ϵ. That is, we put $\overline{X}_0 = \overline{X}_0^\epsilon$ and $\overline{Z}(\xi)^{-1} = (\overline{Z}(\xi)^{-1})^\epsilon$ in the formula for Ξ.

Due to (19), we have that $P(M - 4\epsilon \leq \sup_{0 \leq k \leq N-1} |W(\Delta_k)| < M) \leq C(q) \delta^{\frac{q}{2} - 1}$.

For $E(\Xi^\epsilon h_{M, 2\epsilon})$ one can apply integration by parts. That is,

$$
\begin{aligned}
E(\Xi^\epsilon h_{M, 2\epsilon}) &= E \left(\frac{d}{dx} f(\overline{\Phi}_t(\overline{X}_0^\epsilon + v(X_0 - \overline{X}_0^\epsilon)) - y))((-\overline{Z}(\xi))^{-1})^\epsilon F_1 \frac{\partial}{\partial x} \Phi_t(t_i, x) \bigg|_{x = \overline{\Phi}_{t_{i-1}}(X_0)} \right. \\
&\qquad \left. \int_{t_{i-1}}^s G(\Phi_u(t_{i-1}, \overline{\Phi}_{t_{i-1}}(x))) dW_u \bigg|_{x = X_0} h_{M, 2\epsilon} \right) .
\end{aligned}
$$

Now we can procceed applying Lemma 7.5 to obtain that is enough to find a bound for a series of terms one of which is

$$
\sup_{u \in [t_{i-1}, t_i]} \left| \int_0^1 E(D_u X_0 f(\overline{\Phi}_t(\overline{X}_0 + v(X_0 - \overline{X}_0)) - y))(-\overline{Z}(\xi))^{-1} F_1 \frac{\partial}{\partial x} \Phi_t(t_i, x) \right|_{x = \overline{\Phi}_{t_{i-1}}(X_0)}
$$

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$$\frac{\partial G(\Phi_u(t_{i-1}, \overline{\Phi}_{t_{i-1}}(\cdot)))}{\partial x}(X_0)h_{M,2\epsilon}). \tag{27}$$

Given the bounds for all the processes involved one has that the above expression is bounded by a constant that depends on ϵ. For example, $|((-\overline{Z}(\xi))^{-1})^\epsilon| \leq F_0^{-1}$ and $f(\overline{\Phi}_t(\overline{X}_0^\epsilon + u(X_0 - \overline{X}_0^\epsilon)) - y)) \in L^p(\Omega)$ for any $p > 1$, $\epsilon > 0$, uniformly in $u \in [0,1]$, $\delta \in (0,1]$ and in the partition π.

Therefore the result follows. $\qquad\square$

Now we give the main result of this subsection:

THEOREM 4.3 *Suppose that σ and b are smooth functions with bounded derivatives, $F_0 F_1 > 0$ and that there exist $j_0 \geq 0$ and $V \in \Sigma_{j_0}$ with $(V(X_0))^{-1} \in \cap_{p \geq 1} L^p(\Omega)$. Then*

$$\sup_y |p(t,y) - E\phi_{\delta^{\frac{1}{2}}}(\overline{\Phi}_t(\overline{X}_0) - y)| \leq C\delta.$$

Here ϕ_r denotes the density of a Gaussian random variable with mean 0 and standard deviation r and C is a positive constant independent of δ and the partition π.

Sketch of the proof of Theorem 4.3

First, one considers

$$\begin{aligned}
p(t,y) - E\phi_{\delta^{\frac{1}{2}}}(\overline{\Phi}_t(\overline{X}_0) - y) &= p(t,y) - E\phi_{\delta^{\frac{1}{2}}}(\Phi_t(X_0) - y) \\
&\quad + E\phi_{\delta^{\frac{1}{2}}}(\Phi_t(X_0) - y) - E\phi_{\delta^{\frac{1}{2}}}(\overline{\Phi}_t(\overline{X}_0) - y) \\
&:= A + B.
\end{aligned}$$

Now we prove that

$$\sup_y |A| \leq C\delta. \tag{28}$$

Note that if Z is a random variable with continuous density $q(z)$ one has that

$$q(z) = \lim_{n \to \infty} E[\phi_{n^{-a}}(Z - z)].$$

Therefore,

$$\begin{aligned}
A &= p(t,y) - E\phi_{\delta^{\frac{1}{2}}}(\Phi_t(X_0) - y) \\
&= \lim_{n \to \infty} E[\phi_{n^{-a}}(\Phi_t(X_0) - y) - \phi_{n^{-a}}(\Phi_t(X_0) + \delta^{\frac{1}{2}}\overline{W}_1 - y)],
\end{aligned}$$

where $n \in \mathbb{N}$ and $a > 0$ is fixed. $\overline{W}$ is a Wiener process independent of W and E still denotes the expectation on the extended Wiener space supporting $(W, \overline{W})$. Here we apply a Taylor expansion argument

$$\begin{aligned}
E[\phi_{n^{-a}}(\Phi_t(X_0) - y) - \phi_{n^{-a}}(\Phi_t(X_0) + \delta^{\frac{1}{2}}\overline{W}_1 - y)] &= E[\phi''_{n^{-a}}(\Phi_t(X_0) - y)]\frac{\delta}{2} \\
&\quad + \int_0^1 E[\phi'''_{n^{-a}}(\Phi_t(X_0) - y + u\delta^{\frac{1}{2}}\overline{W}_1)\overline{W}_1^3]du\frac{\delta^{\frac{3}{2}}}{3!}. \tag{29}
\end{aligned}$$

The proof of (28) finishes if we prove that each of the expectations on the right hand side of (29) is bounded. Using Lemma 7.1 we have that for some universal constants a, b, e, k and w

$$|E[\phi_{n-a}''(\Phi_t(X_0) - y)]| = |E[\Psi_{n-a}(\Phi_t(X_0) - y)H^3(\Phi_t(X_0), 1)]|$$
$$\leq \|H^3(\Phi_t(X_0), 1)\|_1 \leq C\|\Delta_{\Phi_t(X_0)}^{-1}\|_a^k \|\Phi_t(X_0)\|_{e,b}^w.$$

Ψ_r denotes the distribution function of a $N(0, r)$ random variable. The right hand of the above equation is bounded due to Theorem 4.1 and Proposition 3.1. Analogously we obtain

$$\sup_n \sup_{u \in [0,1]} \sup_{\delta \in (0,1]} E[\phi_{n-a}'''(\Phi_t(X_0) - y + u\delta^{\frac{1}{2}}\overline{W}_1)\overline{W}_1^3] < \infty$$

This finishes the proof of (28).

Now we procceed to prove that $\sup_y |B| \leq C\delta$. Assume without loss of generality that $t \in \pi$ and let $n(t)$ denote the integer j such that $t_j = t$. Consider

$$
\begin{aligned}
B &:= E[\phi_{\delta^{\frac{1}{2}}}(\Phi_t(X_0) - y) - \phi_{\delta^{\frac{1}{2}}}(\overline{\Phi}_t(\overline{X}_0) - y)] \\
&= E[\phi_{\delta^{\frac{1}{2}}}(\Phi_t(X_0) - y) - \phi_{\delta^{\frac{1}{2}}}(\overline{\Phi}_t(X_0) - y)] \\
&\quad + E[\phi_{\delta^{\frac{1}{2}}}(\overline{\Phi}_t(X_0) - y) - \phi_{\delta^{\frac{1}{2}}}(\overline{\Phi}_t(\overline{X}_0) - y)] \\
&=: B_1 + B_2
\end{aligned}
$$

Lets start with B_1. Define $\varphi_\delta(x) = \phi_{\delta^{\frac{1}{2}}/\sqrt{2}}(x)$ and $Z = \frac{\delta^{\frac{1}{2}}}{\sqrt{2}}\overline{W}_1$ First, apply (8) for $f = \varphi_\delta$ to obtain

$$
\begin{aligned}
E\phi_{\delta^{\frac{1}{2}}}(\Phi_t(X_0) - y) - E\phi_{\delta^{\frac{1}{2}}}(\overline{\Phi}_t(X_0) - y) &= E\left(\varphi_\delta(\Phi_t(X_0) + Z - y) - \varphi_\delta(\overline{\Phi}_t(X_0) + Z - y)\right) \\
&= -\sum_{i=1}^{n(t)} E\left[\frac{d}{dy}\varphi_\delta(\Phi_t(t_i, x) + Z - y)\frac{d}{dx}\Phi_t(t_i, x)\Big|_{x=\overline{\Phi}_{t_{i-1}}(X_0)}\left\{\overline{\Phi}_{t_i}(X_0)\right.\right. \\
&\quad \left. -\overline{\Phi}_{t_{i-1}}(X_0) - (\Phi_{t_i}(t_{i-1}, \overline{\Phi}_{t_{i-1}}(X_0)) - \overline{\Phi}_{t_{i-1}}(X_0))\right\} + R_i(\overline{\Phi}_{t_i}(X_0)) \\
&\quad \left. -R_i(\Phi_{t_i}(t_{i-1}, \overline{\Phi}_{t_{i-1}}(X_0)))\right]
\end{aligned}
$$

where

$$R_i(U) = \int_0^1 \int_0^{s_2} \frac{d^2}{dx^2}\varphi_\delta(\Phi_t(t_i, \cdot) + Z - y)(U + s_1(U - \overline{\Phi}_{t_{i-1}}(X_0)))(U - \overline{\Phi}_{t_{i-1}}(X_0))^2 ds_1 ds_2.$$

Here the analysis goes as in the proof of Theorem 3.1. That is, one finds the expansion of $\overline{\Phi}_{t_i}(X_0) - \overline{\Phi}_{t_{i-1}}(X_0) - (\Phi_{t_i}(t_{i-1}, \overline{\Phi}_{t_{i-1}}(X_0)) - \overline{\Phi}_{t_{i-1}}(X_0))$ using the Itô-Taylor formula.

Now we procceed to find uniform bounds for the expectations of the integrands as in (13). The additional problem that appears in this case is that the derivatives of φ_δ start to appear. Here we apply the integration by parts formula, enough number of times so that we recover the function Ψ which is bounded by 1.

That is, consider for example the term analogous to the one obtained in (13). In such a case we have to prove that there exists a positive constant C independent of the partition π and δ such that

$$, = \sup_{u \in [t_{i-1}, t_i]}\left|E\left(D_u X_0 \frac{\partial G(\Phi_u(t_{i-1}, \overline{\Phi}_{t_{i-1}}(\cdot)))}{\partial x}(X_0)\frac{d}{dy}\varphi_\delta(\Phi_t(t_i, \overline{\Phi}_{t_{i-1}}(X_0))\right.\right.$$

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$$+Z - y)\frac{d}{dx}\Phi_t(t_i, x)\bigg|_{x=\overline{\Phi}_{t_{i-1}}(X_0)}\Bigg)\Bigg| \leq C. \tag{30}$$

To prove the above inequality we will use Proposition 7.1 which applied to , gives that there exists appropiate constants such that

$$
, \quad = \quad \sup_{u\in[t_{i-1},t_i]}\Bigg|E\Bigg(\Psi_{\delta^{1/2}/\sqrt{2}}(\Phi_t(t_i,\overline{\Phi}_{t_{i-1}}(X_0)) + Z - y)H^2\Bigg(\Phi_t(t_i,\overline{\Phi}_{t_{i-1}}(X_0)) + Z, D_u X_0
$$

$$
\frac{\partial G(\Phi_u(t_{i-1},\overline{\Phi}_{t_{i-1}}(\cdot)))}{\partial x}(X_0)\frac{d}{dx}\Phi_t(t_i,x)\bigg|_{x=\overline{\Phi}_{t_{i-1}}(X_0)}\Bigg)\Bigg)\Bigg|
$$

$$
\leq \quad \sup_{u\in[t_{i-1},t_i]}\Bigg\|H^2\Bigg(\Phi_t(t_i,\overline{\Phi}_{t_{i-1}}(X_0)) + Z, D_u X_0\frac{\partial G(\Phi_u(t_{i-1},\overline{\Phi}_{t_{i-1}}(\cdot)))}{\partial x}(X_0)\frac{d}{dx}\Phi_t(t_i,x)\bigg|_{x=\overline{\Phi}_{t_{i-1}}(X_0)}\Bigg)\Bigg\|_1
$$

$$
\leq \quad C\sup_{u\in[t_{i-1},t_i]}\Bigg\{\|\Delta^{-1}_{\Phi_t(t_i,\overline{\Phi}_{t_{i-1}}(X_0))+Z}\|^a_k\|\Phi_t(t_i,\overline{\Phi}_{t_{i-1}}(X_0)) + Z\|^w_{e,b}
$$

$$
\Bigg\|D_u X_0\frac{\partial G(\Phi_u(t_{i-1},\overline{\Phi}_{t_{i-1}}(\cdot)))}{\partial x}(X_0)\frac{d}{dx}\Phi_t(t_i,x)\bigg|_{x=\overline{\Phi}_{t_{i-1}}(X_0)}\Bigg\|_{e',b'}\Bigg\}. \tag{31}
$$

The proof of (30) is finished if we prove the following assertions:

(i) $\displaystyle\sup_{t_{i-1}<t_i\leq t}\|\Phi_t(t_i,\overline{\Phi}_{t_{i-1}}(X_0)) + Z\|_{d,b} < \infty$, for any $d, b \in \mathbb{N}$.

(ii) $\displaystyle\sup_{t_{i-1}<t_i\leq t}\|(\Delta_{\Phi_t(t_i,\overline{\Phi}_{t_{i-1}}(X_0))+Z})^{-1}\|_p < \infty$, for all $p > 1$.

(iii) For all $d, b \in \mathbb{N}$,

$$
\sup_{u\in[0,t];t_{i-1}<u<t_i\leq t}\Bigg\|\Bigg(D_u X_0\frac{\partial}{\partial x}G(\Phi_u(t_{i-1},\overline{\Phi}_{t_{i-1}}(X_0)))\frac{d}{dx}\Phi_t(t_i,x)\bigg|_{x=\overline{\Phi}_{t_{i-1}}(X_0)}\Bigg)\Bigg\|_{d,b} < \infty.
$$

(i) and (iii) follow from flow properties, (21), Lemma 7.3 and Lemma 4.1. (ii) follows from Lemma 7.2 in the Appendix.

Now consider the other term in B, B_2. We have

$$B_2 = E[(\varphi_\delta(\overline{\Phi}_t(X_0) + Z - y) - \varphi_\delta(\overline{\Phi}_t(\overline{X}_0) + Z - y))(1_{L_M} + 1_{\overline{L}_M})] = B_{21} + B_{22}$$

In the case of B_{22}, using (19), one has for fixed $q \geq 5$

$$
\begin{aligned}
B_{22} &= E[\varphi_\delta(\overline{\Phi}_t(X_0) + Z - y) - \varphi_\delta(\overline{\Phi}_t(\overline{X}_0) + Z - y); \overline{L}_M] \\
&\leq C\delta^{-1/2}P(\overline{L}_M) \\
&\leq C\delta^{-1/2}\frac{\delta^{\frac{q}{2}-1}}{M^q} \\
&\leq C\delta, \tag{32}
\end{aligned}
$$

where C is a positive constant that only depends on q and M. Now consider B_{21}. For $\omega \in L_M$, we have using (26)

$$\varphi_\delta(\overline{\Phi}_t(X_0) + Z - y) - \varphi_\delta(\overline{\Phi}_t(\overline{X}_0) + Z - y) = \int_0^1 \frac{d}{dx}\varphi_\delta(\overline{\Phi}_t(\overline{X}_0 + u(X_0 - \overline{X}_0)) + Z - y)\,du \quad (33)$$
$$(-\overline{Z}(\xi))^{-1}F_1(\Phi_1(X_0) - \overline{\Phi}_1(X_0))).$$

Therefore the problem of considering the rate of convergence of B_{21} goes through applying (8) localized on the set L_M for $\Phi_1(X_0) - \overline{\Phi}_1(X_0)$. That is, $f(x) = x$ and $t = 1$ in (8). Then one continues with like in the proof of Theorem 4.2. For example instead of (27) we will have for $\epsilon \in (0,1)$ fixed

$$\sup_{u \in [t_{i-1}, t_i]} \left| \int_0^1 E\left(D_u X_0 \frac{d}{dx}\varphi_\delta(\overline{\Phi}_t(\overline{X}_0^\epsilon + v(X_0 - \overline{X}_0^\epsilon)) + Z - y))((-\overline{Z}(\xi))^{-1})^\epsilon F_1 \right.\right.$$
$$\left.\left. \frac{\partial}{\partial x}\Phi_t(t_i, x)\right|_{x=\overline{\Phi}_{t_{i-1}}(X_0)} \frac{\partial G(\Phi_u(t_{i-1}, \overline{\Phi}_{t_{i-1}}(\cdot)))}{\partial x}(X_0)h_{M,2\epsilon})dv \right| < C(\epsilon).$$

Here one can apply integration by parts and obtain the necessary properties as we have done in (31) (in particular, Lemma 7.2). Therefore the result follows. $\qquad\square$

As in the proof of Theorem 3.1 one has to deal with the residues $R_i(U)$ using the integration by parts formula (5).

With the same techniques as in this proof one can obtain results for approximations of distribution functions.

4.2 Weak approximation for general boundary conditions in the multidimensional case

We will now briefly indicate how to obtain a weak approximation result for the multidimensional sde with boundary condition considered in [25]. In this section we will consider the stochastic differential equation

$$dX_t + F(X_t)\,dt = B\,dW_t, \quad t \in [0,1], \tag{34}$$

with boundary condition

$$h(X_0, X_1) = \overline{h},. \tag{35}$$

for $h : \mathbb{R}^{2d} \to \mathbb{R}^d$. Here $\{X_t, t \in [0,1]\}$ is an $\mathbb{R}^d$- valued continuous stochastic process and $\{W_t, t \in [0,1]\}$ is a $\mathbb{R}^k$- valued Brownian motion with $k \geq d$, $F : \mathbb{R}^d \to \mathbb{R}^d$ takes the form

$$F(x) = Ax + B\widetilde{f}(x),$$

where A is a $d \times d$ matrix, $\widetilde{f} : \mathbb{R}^d \to \mathbb{R}^k$ is measurable and locally bounded and B is a $d \times k$ matrix. In [25] a theorem on existence and uniqueness of solution for this kind of equations has been established. More explicitely, let $\mathcal{C}_0\left([0,1];\mathbb{R}^k\right)$ be the set of continuous, $\mathbb{R}^d$- valued functions vanishing at 0; set $\Lambda = \left\{\int_0^1 e^{At}B\,d\varphi(t) \; ; \; \varphi \in \mathcal{C}\left([0,1];\mathbb{R}^k\right)\right\}$, where the integrals are defined using integration by parts. Assume

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(H1) For any $z \in \Lambda$ the equation $h\left(y, e^{-A}(y+z)\right) = \overline{h}$ has a unique solution $y = g(z)$.

In order to find the solution of (34), (35) we consider the linear equation

$$dY_t + AY_t \, dt = B \, dW_t, \quad t \in [0,1],$$

with boundary condition (35). This equation has a unique solution given by

$$Y_t = e^{-At} \left[g\left(\int_0^1 e^{As} B \, dW_s \right) + \int_0^t e^{As} B \, dW_s \right].$$

Let $\Sigma = \left\{ \xi \in \mathcal{C}\left([0,1]; \mathbb{R}^k\right) \ , \ \xi_t - \xi_0 + \int_0^t A\xi_s \, ds \in \mathrm{Im}B, \ 0 \le t \le 1, h(\xi_0, \xi_1) = \overline{h} \right\}$. Then there exists a bijection $\psi : \mathcal{C}_0\left([0,1]; \mathbb{R}^k\right) \to \Sigma$ such that $Y_t = (\psi(W))_t$. Finally we define the mapping $T : \mathcal{C}_0\left([0,1]; \mathbb{R}^k\right) \to \mathcal{C}_0\left([0,1]; \mathbb{R}^k\right)$ by

$$T(\theta) = \theta + \int_0^\cdot \widetilde{f}((\psi(\theta))_s) \, ds.$$

THEOREM 4.4 *(Nualart and Pardoux, [25])   Assume T is a bijection and* (H1). *Then equation (34) with boundary condition (35) possess a unique solution in $C([0,T]; \mathbb{R}^d)$ given by*

$$X = \psi\left(T^{-1}(W)\right).$$

Momentarily assume that T is a bijection, then one can give a different way of expressing the solution to (34), (35). For this let $\theta = T^{-1}(W)$, $\xi_t = \int_0^t e^{Au} B \, dW_u$ and $\varphi_t = \int_0^t e^{Au} B \, d\theta_u$. Then,

$$X_t = e^{-At} \left[g\left(\int_0^1 e^{Au} B \, d\theta_u \right) + \int_0^t e^{Au} B \, d\theta_u \right]$$

$$= e^{-At} \left[g(\varphi_1) + \varphi_t \right].$$

In [10], it was proved that $\varphi_t = \xi_t + u_t$, where

$$u_t = - \int_0^t e^{As} B\widetilde{f}\left(e^{-As} \left[g(\xi_1 + u_1) + (\xi_s + u_s) \right] \right) \, ds.$$

Fix $y \in \mathbb{R}$ and assume

$$u_t(y) = - \int_0^t e^{As} B\widetilde{f}\left(e^{-As} \left[g(\xi_1 + y) + \xi_s + u_s(y) \right] \right) \, ds \tag{36}$$

has a unique solution. Moreover, suppose that the mapping $y \mapsto u_1(y)$ has a unique fixed point Y. Then, clearly

$$\varphi_t = \xi_t + u_t(Y).$$

Let $|M|$ denote the norm of a matrix M, that is, $|M| = \sup_{|x|=1} |Mx|$. We need the following assumption:

(H2) $g, \widetilde{f} \in C^1(\mathbb{R}^d, \mathbb{R}^d)$ and

$$K\left(L_{\widetilde{f}}, L_g\right) = L_g \left[\exp\left[\left(e^{2|A|} - 1 \right) \frac{|B|}{2|A|} L_{\widetilde{f}} \right] - 1 \right] < 1, \tag{37}$$

where $\widetilde{f}$ and g are Lipschitz functions with Lipschitz constant $L_{\widetilde{f}}$ and L_g, respectively. Under (H2) T is a bijection and therefore there is a unique solution to (34) and (35).

Here we will work with the same approximation scheme as introduced in [10]. That is, define $\overline{\xi}_t$, $\overline{u}_t(y)$ by

$$\overline{\xi}_t = \int_0^t e^{A\eta(s)} B \, dW_s,$$

$$\overline{u}_t(y) = -\int_0^t e^{A\eta(s)} B\widetilde{f}\left(e^{-A\eta(s)}\left[g\left(\overline{\xi}_1 + y\right) + \overline{\xi}_{\eta(s)} + \overline{u}_{\eta(s)}(y)\right]\right) \, ds.$$

In Section 3 of [10] it is proven that under (H2) $y \mapsto \overline{u}_1(y)$ has a unique fixed point, say $\overline{Y}$. Let $\overline{\varphi}_t = \overline{\xi}_t + \overline{u}_t(\overline{Y})$ and

$$\overline{X}_t = e^{-At}\left[g\left(\overline{\varphi}_1\right) + \overline{\varphi}_t\right]. \tag{38}$$

Now we introduce some preliminary lemmas.

LEMMA 4.3 *Assume* (H2). *Then* $I - u_1'(y)$ *and* $I - \overline{u}_1'(y)$ *are invertible matrices for all* $y \in \mathbb{R}^d$ *a.s.*

Proof: We will sketch the proofs for $I - \overline{u}_1'(y)$.

It is enough to prove that the maximum eigenvalue of $\overline{u}_1'(y)$ is strictly smaller than 1.

For this is enough to prove that $|\overline{u}_1(y_1) - \overline{u}_1(y_2)| \le K\left(L_{\widetilde{f}}, L_g\right)|y_1 - y_2| < |y_1 - y_2|$. This is exactly (3.16) in [10]. □

The following result is an extension of Lemma 3.2 in [19]. The proof is obtained through standard methods of calculation for stochastic derivatives.

LEMMA 4.4 *Assume* (H2). *Then*

(i) $u_t(y), \overline{u}_t(y) \in \mathbb{D}^{1,\infty}$, *for all* $y \in \mathbb{R}^d$ *and* $t \in [0,1]$.

(ii) $Y, \overline{Y} \in \mathbb{D}^{1,\infty}$ *with* $D_s Y = (I - u_1'(Y))^{-1}(D_s u_t)(Y)$ *and* $D_s\overline{Y} = (I - \overline{u}_1'(\overline{Y}))^{-1}(D_s\overline{u}_t)(\overline{Y})$.

(iii) $u_t(Y), \overline{u}_t(\overline{Y}) \in \mathbb{D}^{1,\infty}$ *and the chain rule is satisfied for both processes for all* $t \in [0,1]$.

(iv) $\sup_s E[\sup_t |D_s(u_t(Y))|^p] < \infty$ $\sup_s E[\sup_t |D_s(\overline{u}_t(\overline{Y}))|^p] < \infty$ *for all* $p > 1$.

The following result gives the rate of convergence for the weak approximation. Strong approximations where studied in [10].

THEOREM 4.5 *Assume* (H1), (H2) *and that* f, $\widetilde{f}$ *and* g *are in* C_p^2. *Then there exists a positive constant* C *independent of* δ *and* π

$$|E(f(X_t) - f(\overline{X}_t))| \le C\delta.$$

Proof: First, note that

$$f(X_t) - f(\overline{X}_t) = \Upsilon_t^1[\Upsilon_t^2(\varphi_1 - \overline{\varphi}_1) + \varphi_t - \overline{\varphi}_t].$$

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where

$$\Upsilon_t^1 = \int_0^1 f'(\overline{X}_t + \alpha(X_t - \overline{X}_t))d\alpha e^{-At}$$

$$\Upsilon_t^2 = \int_0^1 g'(\overline{\varphi}_1 + \alpha(\varphi_1 - \overline{\varphi}_1))d\alpha.$$

Due to Lemma 4.4, we have that Υ^1 and $\Upsilon^2 \in \mathbb{D}^{1,\infty}$.

Therefore we consider $\varphi_t - \overline{\varphi}_t = \xi_t - \overline{\xi}_t + (u_t(Y) - \overline{u}_t(\overline{Y}))$. The difference $\xi_t - \overline{\xi}_t$ can obviously be written as an expression of order δ. That is,

$$\xi_t - \overline{\xi}_t = \int_0^t \left(e^{As} - e^{A\eta(s)} \right) B \, dW_s,$$

where $| e^{As} - e^{A\eta(s)} | \leq e^{2|A|}\delta$ and therefore $\sup_{t \in [0,1]} \|\xi_t - \overline{\xi}_t\|_p \leq C(p)\delta$ for a positive constant $C(p)$ independent of δ and π but dependent on $p > 1$.

To simplify the notation define

$$k(x_1, ..., x_5) = -e^{Ax_1} B\widetilde{f}\left(e^{-Ax_1} \left[g\left(x_2 + x_3 \right) + x_4 + x_5 \right] \right). \tag{39}$$

Now consider

$$
\begin{aligned}
u_t(Y) - \overline{u}_t(\overline{Y}) &= \int_0^t k\left(s, \xi_1, Y, \xi_s, u_s(Y)\right) - k\left(\eta(s), \overline{\xi}_1, Y, \overline{\xi}_{\eta(s)}, u_s(Y)\right) \, ds \\
&+ \int_0^t k\left(\eta(s), \overline{\xi}_1, Y, \overline{\xi}_{\eta(s)}, u_s(Y)\right) - k\left(\eta(s), \overline{\xi}_1, \overline{Y}, \overline{\xi}_{\eta(s)}, u_s(Y)\right) ds \\
&+ \int_0^t k\left(\eta(s), \overline{\xi}_1, \overline{Y}, \overline{\xi}_{\eta(s)}, u_s(Y)\right) - k\left(\eta(s), \overline{\xi}_1, \overline{Y}, \overline{\xi}_{\eta(s)}, \overline{u}_s(\overline{Y})\right) ds \\
&+ \int_0^t k\left(\eta(s), \overline{\xi}_1, \overline{Y}, \overline{\xi}_{\eta(s)}, \overline{u}_s(\overline{Y})\right) - k\left(\eta(s), \overline{\xi}_1, \overline{Y}, \overline{\xi}_{\eta(s)}, \overline{u}_{\eta(s)}(\overline{Y})\right) ds \\
&= I_1(t) + I_2(t) + I_3(t) + I_4(t). \tag{40}
\end{aligned}
$$

Lets start with I_4. Consider

$$
\begin{aligned}
|I_4(t)| &\leq \int_0^t e^{2|A|\eta(s)} BL_{\widetilde{f}} |\overline{u}_s(\overline{Y}) - \overline{u}_{\eta(s)}(\overline{Y})| ds \\
&\leq \delta BL_{\widetilde{f}} \int_0^t e^{2|A|\eta(s)} |k\left(\eta(s), \overline{\xi}_1, \overline{Y}, \overline{\xi}_{\eta(s)}, \overline{u}_{\eta(s)}(\overline{Y})\right)| ds
\end{aligned}
$$

Therefore it follows from Lemma 7.6 in the Appendix that $\| \sup_{s \in [0,t]} |I_4(s)| \|_p \leq C(p)\delta$ for a positive constant $C(p)$ independent of δ and π but depend on $p > 1$. Now consider I_1. We will divide it into 4 terms

$$
\begin{aligned}
I_1(t) &= \int_0^t k\left(s, \xi_1, Y, \xi_s, u_s(Y)\right) - k\left(\eta(s), \xi_1, Y, \xi_s, u_s(Y)\right) ds \\
&+ \int_0^t k\left(\eta(s), \xi_1, Y, \xi_s, u_s(Y)\right) - k\left(\eta(s), \overline{\xi}_1, Y, \xi_s, u_s(Y)\right) ds \\
&+ \int_0^t k\left(\eta(s), \overline{\xi}_1, Y, \xi_s, u_s(Y)\right) - k\left(\eta(s), \overline{\xi}_1, Y, \overline{\xi}_s, u_s(Y)\right) ds \\
&+ \int_0^t k\left(\eta(s), \overline{\xi}_1, Y, \overline{\xi}_s, u_s(Y)\right) - k\left(\eta(s), \overline{\xi}_1, Y, \overline{\xi}_{\eta(s)}, u_s(Y)\right) ds \\
&= I_{11}(t) + I_{12}(t) + I_{13}(t) + I_{14}(t).
\end{aligned}
$$

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As in the case of I_4 and using the hypotheses on $\widetilde{f}$ and g one obtains that $\|\sup_{s\in[0,t]}|I_{1i}(s)|\|_p \leq C(p)\delta$ is satisfied for $i=1,2,3$. I_{14} can be written as (using the mean value theorem)

$$I_{14}(t) = \Upsilon_s^3 (W_s - W_{\eta(s)}). \tag{41}$$

where

$$\Upsilon_s^3 = e^{A\eta(s)}\, B \int_0^1 \widetilde{f}'(e^{-A\eta(s)}\left[g\left(\overline{\xi}_1 + Y\right) + \overline{\xi}_{\eta(s)} + \alpha(\overline{\xi}_s - \overline{\xi}_{\eta(s)}) + u_s(Y)\right])\, d\alpha\, e^{A\eta(s)}$$

Note that due to Lemma 4.4, one has that $\Upsilon^3 \in \mathbb{D}^{1,\infty}$

Now we analyze the term I_3. We have by the mean value theorem

$$I_3(t) \quad = \quad -\int_0^t \gamma_s \left(u_s(Y) - \overline{u}_s(\overline{Y})\right)ds$$

where

$$\gamma_s = e^{A\eta(s)} B \int_0^1 \widetilde{f}'(e^{-A\eta(s)}\left[g\left(\overline{\xi}_1 + \overline{Y}\right) + \overline{\xi}_{\eta(s)} + u_s(Y) + \theta(\overline{u}_s(\overline{Y}) - u_s(Y))\right])e^{-A\eta(s)}\, d\theta.$$

Similar calculations are applied to I_2 to obtain

$$I_2(t) \quad = \quad -\int_0^t \beta_s (Y - \overline{Y})ds$$

where

$$\beta_s = e^{A\eta(s)} B \int_0^1 \widetilde{f}'(e^{-A\eta(s)}\left[g\left(\overline{\xi}_1 + Y + \theta(\overline{Y} - Y)\right) + \overline{\xi}_{\eta(s)} + u_s(Y)\right])e^{-A\eta(s)}g'(\overline{\xi}_1 + Y + \theta(\overline{Y} - Y))d\theta.$$

Resuming these calculations we have that we can write (40) as:

$$u_t(Y) - \overline{u}_t(\overline{Y}) = \int_0^t \alpha_s ds + \int_0^t \beta_s (Y - \overline{Y})ds + \int_0^t \gamma_s \left(u_s(Y) - \overline{u}_s(\overline{Y})\right)ds. \tag{42}$$

This is a linear equation in $u_t(Y) - \overline{u}_t(\overline{Y})$ where α is defined so that

$$\int_0^t \alpha_s ds = (I_1 + I_4)(t).$$

Using classical results (see Dunford and Schwartz [9], vol II, page 1282) we solve (42):

$$u_t(Y) - \overline{u}_t(\overline{Y}) = \sum_{j=0}^\infty \Psi^j G(t), \tag{43}$$

where $G(t) = \int_0^t \alpha_s ds + \int_0^t \beta_s ds(Y - \overline{Y})$ and $(\Psi Z)(t) = \int_0^t \gamma_s Z(s)ds$. Here one has that $\sup_{s\in[0,1]} |\gamma_s| < \infty$ a.e. (see the proof of Lemma 7.7 in the Appendix) and therefore the series above converge uniformly for t in compact sets.

In order to solve for $Y - \overline{Y}$ in (43) for $t = 1$ one proves by induction (see Lemma 7.7 in the Appendix) that for $j = 1, 2, \ldots$

$$|\Psi^{j-1} B(1)| \leq \frac{(e^{2|A|} - 1)^j}{j! 2^j |A|^j} |B|^j L_{\widetilde{f}}^j L_g, \tag{44}$$

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therefore by (37) one has that $|\sum_{j=0}^{\infty} \Psi^j B(1)| < 1$ and therefore the inverse of $I - \sum_{j=0}^{\infty} \Psi^j B(1)$ (45) exists.

Replacing $t = 1$ in (43), we have that using (37),

$$Y - \overline{Y} = (I - \sum_{j=0}^{\infty} \Psi^j B(1))^{-1} \sum_{j=0}^{\infty} \Psi^j A(1), \tag{45}$$

where $A(t) = \int_0^t \alpha_s ds + I_1(t) + I_4(t)$ and $B(t) = \int_0^t \beta_s ds$.

Using (45) in (43) we also have

$$u_t(Y) - \overline{u}_t(\overline{Y}) = \sum_{j=0}^{\infty} \Psi^j A(t) + \sum_{j=0}^{\infty} \Psi^j B(t)(I - \sum_{j=0}^{\infty} \Psi^j B(1))^{-1} \sum_{j=0}^{\infty} \Psi^j A(1). \tag{46}$$

Note that $\|\sup_{t \in [0,1]} |A(t) - I_{14}(t)|\|_p \leq C\delta$. Now if we put all these estimates together, we have:

$$\begin{aligned} E[f(X_t) - f(\overline{X}_t)] &= E[\Upsilon_t^1[\Upsilon_t^2(\xi_1 - \overline{\xi}_1) + \xi_t - \overline{\xi}_t]] \\ &\quad + E[\Upsilon_t^1 \Upsilon_t^2 (Y - \overline{Y})] + E[\Upsilon_t^1(u_t(Y) - \overline{u}_t(\overline{Y}))] = A_1 + A_2 + A_3 \end{aligned}$$

First it is easy to see that $|A_1| \leq C\delta$ given that $\sup_{t \in [0,1]} \|\xi_t - \overline{\xi}_t\|_p \leq C\delta$ and Υ^1 and $\Upsilon^2 \in \mathbb{D}^{1,\infty}$.

Next, as A_2 and A_3 are of the same nature we only consider A_3 leaving A_2 to the reader. Using (46) one has

$$\begin{aligned} A_3 &= E[\Upsilon_t^1\{\sum_{j=0}^{\infty} \Psi^j (A - I_{14})(t) + \sum_{j=0}^{\infty} \Psi^j B(t)(I - \sum_{j=0}^{\infty} \Psi^j B(1))^{-1} \sum_{j=0}^{\infty} \Psi^j (A - I_{14})(1)\}] \\ &\quad + E[\Upsilon_t^1\{\sum_{j=0}^{\infty} \Psi^j I_{14}(t) + \sum_{j=0}^{\infty} \Psi^j B(t)(I - \sum_{j=0}^{\infty} \Psi^j B(1))^{-1} \sum_{j=0}^{\infty} \Psi^j I_{14}(1)\}] = A_{31} + A_{32} \end{aligned}$$

In Lemma 7.8 in the Appendix it is proven that $|A_{31}| \leq C\delta$. To prove $|A_{32}| \leq C\delta$ one needs to apply the integration by parts formula (5) because the rate is being carried by $(W_s - W_{\eta(s)})$ in the definition of I_{14} in (41). This is done in Lemma 7.9 in the Appendix. This finishes the proof. $\square$

Further refining this proof one could consider the approximations for densities or distribution functions of solutions to (34). The hypothesis will require further smoothness of $\widetilde{f}$ and g. The study of existence and smoothness of densities for multidimensional stochastic differential equations with boundary conditions was carried out in [19]. The statement in the case of density approximations is

THEOREM 4.6 *Assume (H1) and (H2). Also assume that g and $\widetilde{f}$ are elements of C_p^{∞} and that $g'(x)$ (respectively $I + g'(x)$) has an inverse for all $x \in \mathbb{R}^d$ and that its inverse has at most polynomial growth at infinity. Then, if $\det BB^T \neq 0$ one has*

$$\sup_x |p(t, x) - E\phi_{\delta^{1/2}}(\overline{X}_t - x)| \leq C\delta,$$

for a positive constant C independent of δ and the partition π and for $t \in [0, 1)$ (respectively $t \in (0, 1]$).

5 An Example

In this section we will discuss informally an example of stochastic equation with boundary conditions. There are various examples were the methods introduced here can be applied. These include the smoothing problem (see [30]), the problem of estimation of maximum a posteriori for trajectories of diffusions processes (see [32] and [33]) and the study of some classes of reciprocal processes (see [20]). These equations also appear in the asymptotical study of waves in random media (see [11] and [29]) and in the study of second order stochastic differential equations (see [26]). In general, most of the control systems where differential equations with boundary conditions appear will have an associated stochastic differential equation with boundary conditions when noise is introduced into it. Usually the boundary conditions appear from using a space variable instead of a time variable. We will briefly discuss one of the possible applications where this is exactly the case. A more detailed account will be given in a future publication.

As a simple example we will give a description of the equation considered in [29]. In this article they considered a transmission-reflection problem for a one dimensional equation in a random slab. The coefficients are assumed to randomly fluctuate in a small scale therefore producing a limit equation which will be a linear stochastic differential equation with linear boundary conditions. This limit equation is obtained using techniques of diffusion approximation. To describe the situation, let $L > 0$ be fixed. The one dimensional acoustic wave equation in the interval $[0, L]$ is

$$
\begin{aligned}
\rho(x)\frac{\partial u}{\partial t} + \frac{\partial p}{\partial x} &= 0 \\
\frac{1}{K(x)}\frac{\partial p}{\partial t} + \frac{\partial u}{\partial x} &= 0
\end{aligned}
$$

with some boundary conditions which will be described later. Here $u(x,t)$ is the velocity, $p(x,t)$ is the pressure, $\rho(x)$ is the density and $K(x)$ is the bulk modulus. Then one considers that $\rho(x) = 1 + \beta(Z_x^\epsilon)$ for some smooth function taking values in $[-c, c]$ for $c < 1$ and Z_x^ϵ is a Markov processes with certain properties. One also assumes that $K \equiv 1$ which corresponds to the homogeneous case. Then one defines $A = u + p$ and $B = u - p$ called the right and left going wave respectively. The boundary conditions are then set for $A(0,t)$ and $B(L,t)$. These conditions correspond to entering pulses at time $t = 0$ at $x = 0$ and at $x = L$. A combination of conditions in $x = 0$ and $x = L$ can also be considered. As described in [29] the general problem is given these boundary conditions what can we say about the medium if we have small-scale inhomogeneities present? (these being represented by Z_x^ϵ).

In order to transform the above problem into a stochastic differential equation with boundary conditions one considers the Fourier transform of A and B which are random and will depend in general of ϵ. Under certain conditions the limit equation for the Fourier transforms is given by

$$
\frac{d}{dx}\begin{pmatrix} \tilde{A}^\epsilon \\ \tilde{B}^\epsilon \end{pmatrix} = \frac{iw}{2\epsilon}\beta(Z_x^\epsilon)\begin{pmatrix} 1 & e^{-2iwx/\epsilon} \\ -e^{-2iwx/\epsilon} & -1 \end{pmatrix}\begin{pmatrix} \tilde{A}^\epsilon \\ \tilde{B}^\epsilon \end{pmatrix},
$$

where $\tilde{A}^\epsilon(x, w) = A^\epsilon(x, w)e^{iwx/\epsilon}$ and $\tilde{B}^\epsilon(x, w) = B^\epsilon(x, w)e^{iwx/\epsilon}$. A^ϵ and B^ϵ being the Fourier transforms of A and B. Some of the physical quantities of interest are $R = \tilde{B}^\epsilon/\tilde{A}^\epsilon$, the reflection coefficient

and $T = 1/\bar{\bar{A}}^\epsilon$ the transmission coefficient.

These problems fall into a general category of diffusion approximation theorem from [13] which we quote here. In our case we will have $f = 0$, $L = 1$ and $\eta^k(t/\epsilon^2) = \beta(Z_t^\epsilon)$.

THEOREM 5.1 *Let X^ϵ be the smallest solution of the system*

$$\frac{dX_t^\epsilon}{dt} = f(t, X_t^\epsilon) + \sigma^k(t, X_t^\epsilon)\frac{1}{\epsilon}\eta^k(\frac{t}{\epsilon^2}),$$
$$H_0 X_0^\epsilon + H_1 X_1^\epsilon = V_0$$

Here $\eta^k(t)$ are independent Markov processes with a unique invariant probability measure under which they are ergodic and fulfill Doeblin's condition. Denote $\alpha^k = \int_0^\infty E[\eta^k(0)\eta^k(t)]dt \in (0, \infty)$. Let X be the solution of

$$dX_t = f(t, X_t)dt + \alpha^k \sigma^k(t, X_t) \circ dW_t^k \tag{47}$$

$$H_0 X_0 + H_1 X_1 = V_0. \tag{48}$$

Then under conditions ($\mathcal{H}$1 and ($\mathcal{H}$2 (see [13]) there exists a unique solution X in $\mathbb{L}_{C,loc}^{1,8}$ and furthermore X^ϵ converges weakly to X in the uniform topology.

Therefore according to the properties of the process η one will find a whole array of processes X which satisfy Equations (47) and (48) which according to our results can be approximated. As in the example considered, it is common that stochastic differential equations with boundary conditions appear when the variable t represents a space variable rather than a time variable.

The conditions we have required in Theorem 4.3 are sufficient to obtain that ($\mathcal{H}$1) and ($\mathcal{H}$2) are satisfied. In the multidimensional case there are cases that we have considered here were ($\mathcal{H}$2) is not satisfied.

6   Conclusions

We have considered an alternative method of proof for weak approximations of solutions of stochastic differential equations. This method should be useful in many cases, in particular, when non-adapted processes are considered or when we are interested in approximating the distribution or the density function of a locally smooth process.

Many variations of this argument can be implemented. For example, one can obtain with some further work an expansion of the error in terms of powers of the step size. One possible disadvantage of this method is that the calculation of the constants in this expansion is quite cumbersome. It does not seem to have a nice expression like in the case of diffusions (see e.g. [31]). Another problem of interest is to consider the generalization of Theorem 4.3 to many dimensions using the technique of stochastic invariant imbedding introduced in [13].

Another argument to approach the analysis of numerical schemes in non-linear problems has been used to study the McKean-Vlasov equation (see [18]) and the author is currently considering applying a combination of both techniques to the case of the Burgers equation.

Further extensions of this method can be investigated in order to obtain weak approximation results for higher order schemes and to develop the error in powers of the step size of the approximation which should help the design of interpolation schemes.

Also these ideas should give some light about the behaviour of weak approximations to higher order sde's with boundary conditions as well as stochastic partial differential equations with boundary conditions. These problems need to be studied further.

7 Appendix

In this section we will prove accesory results used in other sections.

The following Proposition is the basic result to obtain properties about the densities of smooth random variables.

PROPOSITION 7.1 *Let* $F = (F^1, ..., F^d) \in (\mathbb{D}^\infty)^d$ *such that* $(\det \Delta_F)^{-1} \in \cap_{p \geq 2} L^p(\Omega)$. *Let* $G \in \mathbb{D}^\infty$ *and let* $g \in C_p^\infty(\mathbb{R}^d)$. *Then* $(\det \Delta_F)^{-1} \in \mathbb{D}^\infty$ *and for any multi-index* $\alpha \in \{1, ..., d\}^k$, $k \geq 1$, *there exists an element* $H_\alpha(F, G) \in \mathbb{D}^\infty$ *such that:*

$$E[(\partial_\alpha g)(F)G] = E[g(F)H_\alpha(F, G)].$$

Furthermore for any multi-index α, *integers* p *and* q *there exists constants* $C(p, q, \alpha)$, e, b, e', b', a, k *and* w *such that*

$$\|H_\alpha(F, G)\|_{p,q} \leq C(p, q, \alpha)\|\Delta_F^{-1}\|_k^a \|F\|_{e,b}^w \|G\|_{e',b'}.$$

Proof: The proof of this proposition can be obtained by performing some aditional calculations to the usual proof (see for example, Proposition 3.2.2 in Nualart [24]).

For example, let $\alpha = (1)$. Then it follows that

$$H_{(1)}(F, G) = \sum_{j=1}^d \delta(G(\Delta_F^{-1})^{1j} DF^j).$$

Now using the continuity of the adjoint operator δ and some standard properties of the norms $\|\cdot\|_{p,q}$ we have that

$$
\begin{aligned}
\|\sum_{j=1}^d \delta(G(\Delta_F^{-1})^{1j} DF^j)\|_{p,q} &\leq \|\sum_{j=1}^d G(\Delta_F^{-1})^{1j} DF^j\|_{p+1,q} \\
&\leq C\|\Delta_F^{-1}\|_{p+1,b}, \|F\|_{p+2,b'}\|G\|_{p+1,b''}
\end{aligned}
$$

for some integers b, b', b''. To finish the proof is enough to note that

$$D[(\Delta_F^{-1})^{ij}] = -\sum_{k,l=1}^d (\Delta_F^{-1})^{ik}(\Delta_F^{-1})^{jl} D[\Delta_F^{kl}].$$

$\square$

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In the one dimensional case (d=1) we write $H(F, G) = H_{(1)}(F, G)$. By induction also define $H^j(F, G) = H^{j-1}(F, H(F, G))$.

Now we give some results on the estimation of Malliavin covariance matrices. In the following F_δ^u and F^u denote measurable random fields.

LEMMA 7.1 Let F_δ^u, $F^u \in (\mathbb{D}^{1,\infty})^m$ and suppose that

(i) there exists $\gamma > 0$ such that $\sup_u \|F_\delta^u - F^u\|_{1,p} = O(\delta^\gamma)$, for all $p > 1$.

(ii) $\sup_u \|(\det \Delta_{F^u})^{-1}\|_p < \infty$, for all $p > 1$.

(iii) For all p, there exists $\nu(p) > 0$ such that $\sup_u \|(\det \Delta_{F_\delta^u})^{-1}\|_p = O(\delta^{-\nu(p)})$.

Then we have $\sup_\delta \sup_u \|(\det \Delta_{F_\delta^u})^{-1}\|_p < \infty$ for all $p > 1$.

Proof: Consider $A = [|\det \Delta_{F_\delta^u} - \det \Delta_{F^u}| \leq \frac{1}{2} |\det \Delta_{F^u}|]$. Then,

$$E(|(\det \Delta_{F_\delta^u})^{-1}|^p; A) \leq 2^p E(|(\det \Delta_{F^u})^{-1}|^p) < \infty.$$

$$E(|(\det \Delta_{F_\delta^u})^{-1}|^p; A^c) \leq E(|(\det \Delta_{F_\delta^u})^{-1}|^{2p})^{1/2} P(A^c)^{1/2}. \tag{49}$$

The proof finishes by noting that

$$P(A^c) \leq 2^k E(|(\det \Delta_{F^u})^{-1}|^k |\det \Delta_{F_\delta^u} - \det \Delta_{F^u}|^k) = O(\delta^{\gamma k}),$$

for any k. Taking k big enough, one obtains that (49) is bounded. $\square$

As an application of this Lemma we consider:

LEMMA 7.2 Let X_0 and $\overline{X}_0^\epsilon$ be as defined in Section 4.1. Also let $Z = \frac{\delta^{\frac{1}{2}}}{\sqrt{2}} \overline{W}_1$ where $\overline{W}$ is a Wiener process independent of W. Then, for fixed $t > 0$,

(I) $\sup_{t_{i-1} < t_i \leq t} \|(\Delta_{\Phi_t(t_i, \overline{\Phi}_{t_{i-1}}(X_0)) + Z})^{-1}\|_p < \infty$, for all $p > 1$.

(II) $\sup_{u \in [0,1]} \|(\Delta_{\overline{\Phi}_t(\overline{X}_0^\epsilon + u(X_0 - \overline{X}_0^\epsilon)) + Z})^{-1}\|_p < \infty$, for all $p > 1$.

Proof: In both cases we only need to check that the conditions in Lemma 7.1 are satisfied. For (i) we have that due to Proposition 3.1 and Lemma 4.2:

$$\|\Phi_t(t_i, \overline{\Phi}_{t_{i-1}}(X_0)) + Z - \Phi_t(X_0)\|_{1,p} \leq C\delta^{1/2}$$

$$\|\sup_{u \in [0,1]} \overline{\Phi}_t(\overline{X}_0^\epsilon + u(X_0 - \overline{X}_0^\epsilon)) + Z - \Phi_t(X_0)\|_{1,p} \leq C\delta^{1/2},$$

where C is a positive constant that does not depend on δ or the partition π. (ii) follows from Theorem 4.1. (iii) is similar in both cases so we will only do the first:

$$\Delta_{\Phi_t(t_i, \overline{\Phi}_{t_{i-1}}(X_0)) + Z} = \int_0^1 |D_s \Phi_t(t_i, \overline{\Phi}_{t_{i-1}}(X_0))|^2 + |\overline{D}_s Z|^2 ds \geq \int_0^1 |\overline{D}_s Z|^2 ds$$

$$\geq \frac{\delta}{2}.$$

From here the proof follows. $\square$

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LEMMA 7.3 *Let* $\{F_t(x), \ t \in [0,1]\}$, $x \in \mathbb{R}^k$, *be a family of stochastic processes such that there exists* $\alpha \geq 1$ *with*

$$\sup_{t \in [0,1]} E\big(\sup_{|x| \leq 1} |F_t(\cdot, nx)|^p\big) \leq K(p) n^{\alpha p}, \tag{50}$$

for any $p \geq 1$, *where* K *is a constant depending on* p *and* α. *Let* $Y : \Omega \to \mathbb{R}^k$ *be a random vector belonging to* $\cap_{p \geq 1} L^p(\Omega)$. *Then*

$$\sup_{t \in [0,1]} E(|F_t(Y)|^p) \leq 2K(pr)^{1/r}(1 + E|Y|^{(\alpha p+1)s})^{\frac{1}{s}}, \ \textit{for any } p \geq 1 \ \textit{and } \frac{1}{r} + \frac{1}{s} = 1 \ r, s > 1.$$

This result is Lemma 2.1 in [19]. The above bound can be obtained from the proof. As an application one has the following result.

LEMMA 7.4 *Suppose that* σ *and* m *have bounded derivatives up to order* k. *Also let* $Y : \Omega \to \mathbb{R}$ *be a random variable belonging to* $\cap_{p \geq 1} L^p(\Omega)$. *Then, for each* $p \geq 2$, *there exist a constant* $C_{k,p}$ *such that*

$$\sup_{s \leq 1} E \sup_{t \in [s,1]} \| \nabla^k \Phi_t(s, Y) \|^p \leq C_{k,p},$$

$$\sup_{s \leq 1} E \sup_{t \in [s,1]} \| \nabla^k \overline{\Phi}_t(s, Y) \|^p \leq C_{k,p},$$

$$\sup_{s \leq 1} E \sup_{t \in [s,1]} \| \nabla^k (\Phi_t(s, Y) - \overline{\Phi}_t(s, Y)) \|^p \leq C_{k,p} \delta^{p/2}.$$

The proofs of the above statements are obtained as in the proof of Proposition 3.1. That is, one applies Lemma 7.3 after checking that the condition (50) is satisfied. This is done for every k using the Faá di Bruno formula (see [12]).

In the next lemma we obtain a formula to evaluate the expectations of products of random variables with stochastic integrals. The main ingredient of the proof is the integration by parts formula of Proposition 7.1.

LEMMA 7.5 *Let* Y *be a random variable such that* $Y \in \cap_{p \geq 2} L^p(\Omega)$ *and* $Y \in \mathbb{D}^{1,4}$.

Also let $\{u_t(x), t \in [0,1]\}$, $x \in \mathbb{R}$, *be a family of measurable adapted random fields such that* $u_t(x)$ *and* $\frac{\partial^j u_t}{\partial x^j}$ *satisfy (50) for* $j = 1, 2$.

Then for $a, b \in [0,1]$ *and* $X_0 \in \mathbb{D}^{1,\infty}$, *the following formula holds:*

$$E\Big(\int_a^b u_t^\alpha(x) dW_t \Big|_{x=X_0} Y\Big) = E\Big(\int_a^b Y D_t X_0 \frac{\partial u_t^\alpha}{\partial x}(X_0) + D_t Y u_t^\alpha(X_0) dt\Big). \tag{51}$$

Sketch of the proof: Let $\phi_r(x)$ denote the density of a normal random variable with standard deviation r. The proof is obtained through the justification of the following steps:

$$E\Big(\int_a^b u_t(x) dW_t \Big|_{x=X_0} Y\Big) \quad = \quad \lim_{r \to 0} E\Big(\int_\mathbb{R} \int_a^b u_t(x) dW_t \phi_r(x - X_0) Y dx\Big) \tag{52}$$

$$= \quad \lim_{r \to 0} \int_\mathbb{R} E\Big(\int_a^b u_t(x) dW_t \phi_r(x - X_0) Y\Big) dx \tag{53}$$

$$= \lim_{r \to 0} \int_{\mathbb{R}} E\Big(\int_a^b \big(-\phi_r'(x - X_0)D_t X_0 Y + \phi_r(x - X_0)D_t Y\big) u_t(x)dt\Big)dx \quad (54)$$

$$= \lim_{r \to 0} E\Big(\int_a^b \int_{\mathbb{R}} \phi_r(x - X_0)\big(D_t X_0 Y u_t'(x) + D_t Y u_t(x)\big)dxdt\Big)$$

$$= E\Big(\int_a^b Y D_t X_0 \frac{\partial u_t}{\partial x}(X_0) + D_t Y u_t(X_0)dt\Big).$$

First the term on the left hand side of (51) is well defined due to the hypothesis and the application of Lemma 3.2.2 in [23] and Lemma 7.3. (52) follows because $\int_{\mathbb{R}} \int_a^b u_t(x)dW_t \phi_r(x - X_0)Y\,dx$ converges pointwise to $\int_a^b u_t(x)dW_t\Big|_{x=X_0} Y$ and this sequence is uniformly integrable. That is, one uses the hypothesis and Lemma 7.3 to prove that

$$\sup_r E\Big(\int_{\mathbb{R}} \int_a^b u_t(x)dW_t \phi_r(x - X_0)Y\,dx\Big)^2 < \infty.$$

To prove the above one follows the following steps

$$E\Big(\int_{\mathbb{R}} \int_a^b u_t(x)dW_t \phi_r(x - X_0)Y\,dx\Big)^2 = \sum_{n=0}^{\infty} E\Big(\mathbb{1}(n \le |X_0| < n+1)Y^2\Big(\int_{\mathbb{R}} \int_a^b u_t(x)dW_t \phi_r(x - X_0)dx\Big)^2\Big)$$

$$\le \sum_{n=0}^{\infty} E\Big(\mathbb{1}(n \le |X_0| < n+1)Y^2\Big(\int_{\{|x - X_0| \le \frac{|x|}{2}\}} \sup_{|x| \le 2(n+1)} \Big|\int_a^b u_t(x)dW_t\Big| \phi_r(x - X_0)dx$$

$$+ \int_{\{|x - X_0| > \frac{|x|}{2}\}} \Big|\int_a^b u_t(x)dW_t\Big| \phi_r(\tfrac{x}{2})dx\Big)^2\Big)$$

$$\le C\big((EY^4)^{1/2}(E|X_0|^k)^{1/4}\big(\sum_{n=0}^{\infty} \frac{1}{n^{k/4}}(E \sup_{|x| \le 2(n+1)} \Big|\int_a^b u_t(x)dW_t\Big|^8)^{1/4}$$

$$+ \sum_{n=0}^{\infty} \frac{1}{n^{k/4}}\big(\int_{\mathbb{R}} E\Big|\int_a^b u_t(x)dW_t\Big|^8 \phi_r(\tfrac{x}{2})dx\big)^{1/4}\big), \quad \text{for any } k > 0.$$

From here using the Sobolev embedding theorem and the hypothesis, one finds bounds for the expectations above. These bounds are polynomials in n. Therefore taking k big enough the uniform integrability follows.

(53) follows by Fubini theorem. (54) is a consequence of the integration by parts formula. The last two steps in the proof are a repetition of the arguments used in (52) and (53). $\qquad\square$

LEMMA 7.6 *Let* $k : [0,1]^2 \times (\mathbb{R}^d)^4 \to \mathbb{R}^{d \times d}$ *be defined by (39). Then the following properties are verified:*

(i) *The function* k *belongs to* $C_p^2([0,1]^2 \times (\mathbb{R}^d)^4)$ *and furthermore the following inequalities are satisfied*

$$|k(x, x_2, ..., x_5) - k(y, x_2, ..., x_5)| \le e^{2|A|}B|x - y|\{|\widetilde{f}\big(e^{-Ax}\,[g(x_2 + x_3) + x_4 + x_5]\big)|$$

$$+ L_{\widetilde{f}}|g(x_2 + x_3) + x_4 + x_5|\}$$

$$|k(x_1, x, x_3, ..., x_5) - k(x_1, y, x_3, ..., x_5)| \le e^{2|A|}B L_{\widetilde{f}} L_g |x - y|$$

$$|k(x_1, ..., x, x_5) - k(x_1, ..., y, x_5)| \le e^{2|A|}B L_{\widetilde{f}}|x - y|$$

(ii)

$$E \sup_{s \in [0,t]} |k(\eta(s), \overline{\xi}_1, \overline{Y}, \overline{\xi}_{\eta(s)}, \overline{u}_{\eta(s)}(\overline{Y}))|^p \leq C(p),$$

$$E \sup_{s \in [0,t]} |k(s, \xi_1, Y, \xi_s, u_s(Y))|^p \leq C(p),$$

for a positive constant $C(p)$ independent of δ and π

LEMMA 7.7 *Under the conditions of Theorem 4.5 we have for $j = 1, 2, ..$*

$$|\Psi^{j-1} B(t)| \leq \frac{(e^{2|A|t} - 1)^j}{j! 2^j |A|^j} |B|^j L_{\widetilde{f}}^j L_g,$$

Proof: Note that

$$|\gamma_s| \leq e^{2|A|\eta(s)} B L_{\widetilde{f}}$$

$$|\beta_s| \leq e^{2|A|\eta(s)} B L_{\widetilde{f}} L_g$$

Therefore for $j = 1$ we have

$$|B(t)| \leq \int_0^t |\beta_s| ds \leq \frac{(e^{2|A|t} - 1) B L_{\widetilde{f}} L_g}{2|A|}.$$

Now assume that the assertion is true for j. Then

$$\begin{aligned}
|\Psi^j B(t)| &\leq \int_0^t |\Psi^{j-1} B(s)| |\gamma_s| ds \\
&\leq |B|^j L_{\widetilde{f}}^j L_g \int_0^t \frac{(e^{2|A|s} - 1)^j}{j! 2^j |A|^j} e^{2|A|\eta(s)} B L_{\widetilde{f}} ds \\
&\leq \frac{(|B| L_{\widetilde{f}})^{j+1} L_g}{j! 2^j |A|^j} \int_0^t (e^{2|A|s} - 1)^j e^{2|A|s} ds \\
&\leq \frac{(|B| L_{\widetilde{f}})^{j+1} L_g}{j+1! 2^{j+1} |A|^{j+1}}.
\end{aligned}$$

$\square$

LEMMA 7.8 *For $A(t) = I_1(t) + I_4(t)$, we have*

$$|E[\Upsilon_t^1 \sum_{j=0}^{\infty} \Psi^j (A - I_{14})(t) + \sum_{j=0}^{\infty} \Psi^j B(t) (I - \sum_{j=0}^{\infty} \Psi^j B(1))^{-1} \sum_{j=0}^{\infty} \Psi^j (A - I_{14})(1)]| \leq C\delta$$

for some positive constant C independent of δ and the partition π.

Proof: Given that $\Upsilon_t^1 \in L^p(\Omega)$ for any $p > 1$, and $|\sum_{j=0}^{\infty} \Psi^j B(t)| < 1$ it is enough to prove that

$$E(\sum_{j=0}^{\infty} |\Psi^j (A - I_{14})(t)|)^p \leq C(p)\delta^p$$

$$E|(I - \sum_{j=0}^{\infty} \Psi^j B(1))^{-1} \sum_{j=0}^{\infty} \Psi^j (A - I_{14})(1)|^p \leq C(p)\delta^p \tag{55}$$

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To prove the first inequality one proves by induction that

$$|\Psi^j(A - I_{14})(t)| \leq \sup_{s \in [0,t]} |(A - I_{14})(s)| \frac{(e^{2|A|} - 1)^j}{j! 2^j |A|^j} |B|^j L_{\widetilde{f}}^j,$$

for $j = 0, 1, \dots$. From here the inequality follows because

$$E(\sup_{s \in [0,t]} |(A - I_{14})(s)|^p) = E(\sup_{s \in [0,t]} |I_{11}(t) + I_{12}(t) + I_{13}(t) + I_4(t)|^p) \leq C(p)\delta^p.$$

The second inequality in (55) is analogous except one also needs $(I - \sum_{j=0}^{\infty} \Psi^j B(1))^{-1} \in L^p(\Omega)$ for any $p > 1$. This follows from (44) as

$$|(I - \sum_{j=0}^{\infty} \Psi^j B(1))^{-1}| \leq \frac{1}{1 - K\left(L_{\widetilde{f}}, L_g\right)}.$$

$\square$

LEMMA 7.9

$$|E[\Upsilon_t^1 \sum_{j=0}^{\infty} \Psi^j I_{14}(t) + \sum_{j=0}^{\infty} \Psi^j B(t)(I - \sum_{j=0}^{\infty} \Psi^j B(1))^{-1} \sum_{j=0}^{\infty} \Psi^j I_{14}(1)]| \leq C\delta,$$

where C is a positive constant that does not depend on δ or the partition π.

Proof: Consider first for $j \geq 1$, the term

$$
\begin{aligned}
E[\Upsilon_t^1 \Psi^j I_{14}(t)] &= \int_0^t \int_0^{s_{j-1}} \cdots \int_0^{s_2} E[\prod_{l=1}^{j-1} \gamma_{s_l} \Upsilon_t^1 \Upsilon_{s_1}^3 (W_{s_1} - W_{\eta(s_1)})] ds_1 \dots ds_{j-1} \\
&= \int_0^t \int_0^{s_{j-1}} \cdots \int_0^{s_2} \int_{\eta(s_1)}^{s_1} E[D_u(\Upsilon_t^1 \Upsilon_{s_1}^3) \prod_{l=1}^{j-1} \gamma_{s_l} \\
&\quad + \sum_{k=1}^{j-1} D_u(\gamma_{s_k}) \prod_{l=1, l \neq k}^{j-1} \gamma_{s_l} \Upsilon_t^1 \Upsilon_{s_1}^3] du ds_1 \dots ds_{j-1} \\
&\leq C(1 + C_1(j-1))\delta \frac{(e^{2|A|} - 1)^{j-1}}{j - 1! 2^{j-1} |A|^{j-1}} |B|^{j-1} L_{\widetilde{f}}^{j-1}.
\end{aligned}
$$
(56)

The above calculation follows from (5) and previous estimates on Υ^1, Υ^3 and γ. Also note that a calculation gives that $\sup_{u \leq s} \|D_u \gamma_s\|_p \leq C(p) e^{2|A|\eta(s)} B$.

A calculation for the other terms is done noting that we have to prove some differentiability properties of $\sum_{j=0}^{\infty} \Psi^j B(t)(I - \sum_{j=0}^{\infty} \Psi^j B(1))^{-1}$. Given that $\beta_s \in \mathbb{D}^{1,\infty}$ uniformly in $s \in [0,1]$ one can obtain the same property for $\sum_{j=0}^{\infty} \Psi^j B(t)$ as we have shown in (56). Similarly for $(I - \sum_{j=0}^{\infty} \Psi^j B(1))^{-1}$ if one uses the inversion formula $(I - M)^{-1} = \sum_{j=0}^{\infty} M^j$ for a matrix M with norm smaller than 1. $\square$

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