"Valuation Bubbles and Sequential Bubbles"[1]

by

Kevin X.D. Huang and Jan Werner

University of Minnesota
Department of Economics
271 l9th Avenue South
Minneapolis, MN 55455

June 1997
revised December 1997

[1]We are indebted to Ed Green, Narayana Kocherlakota, Steve LeRoy, Ket Richter, and Manuel Santos for very helpful comments. Jan Werner greatfully acknowledges the financial support of the Deutsche Forschungsgemeinschaft, SBF 303.

Valuation Bubbles and Sequential Bubbles

by

Kevin X. Huang and Jan Werner

Abstract

Price bubbles in an Arrow-Debreu valuation equilibrium in infinite-time economy are a manifestation of lack of countable additivity of valuation of assets. In contrast, known examples of price bubbles in sequential equilibrium in infinite time cannot be attributed to the lack of countable additivity of valuation. In this paper we develop a theory of valuation of assets in sequential markets (with no uncertainty) and study the nature of price bubbles in light of this theory. We consider an operator, called payoff pricing functional, that maps a sequence of payoffs to the minimum cost of an asset holding strategy that generates it. We show that the payoff pricing functional is linear and countably additive on the set of positive payoffs if and only if there is no Ponzi scheme, and provided that there is no restriction on long positions in the assets. In the known examples of equilibrium price bubbles in sequential markets valuation is linear and countably additive. The presence of a price bubble indicates that the asset's dividends can be purchased in sequential markets at a cost lower than the asset's price. We also present examples of equilibrium price bubbles in which valuation is nonlinear or linear but not countably additive.

Field designation: General Equilibrium, Financial Economics.

1 Introduction

Gilles and LeRoy (1992a, 1992b) argued that asset price bubbles in infinite-time economies are a manifestation of lack of countable additivity of asset valuation. If the value of an asset differs from the infinite sum of values of dividends at every date, then the difference between the two is a price bubble. In an Arrow-Debreu equilibrium in an infinite-time economy with complete forward markets asset valuation may lack countable additivity for a certain class of consumers' preferences (see Bewley (1972) and Epstein and Wang (1985)).

Asset price bubbles in sequential markets are not usually attributed to the lack of countable additivity of valuation. In sequential markets, there is typically a natural notion of the "present value" of the asset's dividend at each date, and a price bubble is the difference between the asset's price and the infinite sum of present values of the dividends. One of the important examples of price bubbles in equilibrium in sequential markets (see Santos and Woodford (1997)) is fiat money with positive price. Since fiat money is an asset with zero dividend, its positive price cannot be attributed to the lack of countable additivity of valuation. The value of an asset with zero dividend is zero under any linear valuation.

The objective of this paper is to develop a theory of valuation of assets in sequential markets with infinite time, and to investigate the nature of asset price bubbles in light of this valuation theory. Of particular interest is the question whether valuation is linear and/or countably additive in known and new examples of price bubbles in sequential markets.

In an economy with sequential markets and infinite time a constraint on asset holdings has to be imposed, since otherwise Ponzi schemes would be feasible. A Ponzi scheme is a strategy of rolling over a debt forever and thereby never paying it back. If Ponzi schemes of arbitrary scale are feasible, then there cannot exist an optimal asset holding strategy for an agent who values consumption at every date. There are various constraints that one can consider. The form of a constraint is crucial for the existence and the nature of price bubbles in sequential equilibrium (see Kocherlakota (1992)). For instance, a constraint that precludes Ponzi schemes if there is no price bubble but permits Ponzi schemes of arbitrary scale when there is a price bubble cannot give rise to an equilibrium with price bubble. An example of such a constraint is when unbounded short selling is prohibited.

Constraints that preclude Ponzi schemes regardless of whether there is a price bubble or not have a potential of giving rise to price bubbles in equilibrium. Santos and Woodford (1997) provide sufficient conditions for there being no equilibrium price bubbles in sequential markets with a constraint that borrowing be limited by a pre-specified bound which may be price- and time-dependent. If the asset is in positive supply and the present value of the aggregate endowment is finite then there is no price bubble. These conditions are fairly restrictive and there is a number of well-known examples of sequential equilibria with price bubbles. In some of these examples the condition of the finite present value of the aggregate endowment is violated; in others the present value is finite but the asset is in zero net-supply. We show that the former examples are associated with Pareto inefficiency of the equilibrium allocation. Equilibria with positive price of fiat money and zero bound on borrowing belong to that class.

In order to study the nature of price bubbles in sequential markets we develop a theory of valuation of assets in such markets (with no uncertainty). We define the payoff pricing functional as an operator that maps a sequence of payoffs to the minimum cost of an asset holding strategy that generates it. Our main result is that the payoff pricing functional is linear and countably additive on the set of positive payoffs if and only if, there is no arbitrage opportunity (in particular, no Ponzi scheme), provided that the constraint on asset holdings does not restrict long positions in the asset. This result applies to any sequential equilibrium with a constraint such as zero bound on borrowing or bounded short-selling. Yet, there are examples of price bubbles in sequential equilibrium with zero bound on borrowing. In these examples the price of an asset exceeds the value of the asset's dividends measured by the payoff pricing functional. This implies that the dividends can be purchased in sequential markets at a cost lower than the price. The constraint on asset holdings prevents agents from earning the arbitrage profit resulting from the difference between the price in sequential markets and the value of the asset.

For some constraints on asset holdings such as a short-sales constraint with a nonzero bound, there may be a Ponzi scheme of limited scale in a sequential equilibrium. When this happens, the payoff pricing functional is nonlinear and there is a price bubble.

In these two cases price bubbles cannot be attributed to the lack of countable additivity of valuation, and hence are different from the Gilles and LeRoy (1992b) notion of price bubbles. However, a price bubble in sequential markets may also occur when the payoff pricing functional is linear but not countably additive. We provide an example of an equilibrium with such price bubble when asset holdings are restricted to simple asset holdings, that is, when an agent can trade the asset at an arbitrary but finite number of dates after which the asset is held without further retrading. Simple asset holdings restrict long positions in the asset, so that our fundamental result about countable additivity of valuation does not apply. In this example an Arrow-Debreu equilibrium that lacks countable additivity of pricing is implemented as a sequential equilibrium.

The paper is organized as follows: In section 2 we define a price bubble in an Arrow-Debreu valuation equilibrium in an infinite-time economy. Conditions under which a valuation equilibrium is not countably additive and there is a price bubble (valuation bubble) can be inferred from the work of Bewley (1972), and pertain to the continuity of agents' preferences. We present an example of a valuation equilibrium with price bubble. In sections 3 and 4 we introduce basic concepts of sequential markets such as arbitrage opportunities, Ponzi schemes, and asset holding constraints, and we define a sequential equilibrium and a price bubble (sequential bubble). In section 5 we prove our main result about linearity and countable additivity of valuation in sequential markets. In section 6 we investigate the possibility of sequential bubbles. We present a generalization (in the setting of no uncertainty) of the result of Santos and Woodford (1997) establishing sufficient conditions for there being no sequential bubbles in equilibrium, and discuss several examples of sequential bubbles. In particular, we present an example with linear but not countably additive valuation, and an example with nonlinear valuation. Section 7 contains remarks about an application of our results to economies with infinitely many agents such as overlapping generations economies. Most of the proofs can be found in the Appendix which also includes three theorems about equivalence of sequential equilibria and valuation equilibria in different settings.

2 Valuation Equilibrium with Bubbles

When consumption takes place over an infinite sequence of dates $t = 0, 1, \dots,$ and there is a single consumption good, consumption plans are infinite sequences of numbers. Let C be a linear space of such sequences. An example of space C is the space of bounded sequences ℓ_∞. The cone of positive sequences in C is the consumption set denoted by C_+.

Agents ($i = 1, \dots, I$) have initial endowments $\omega^i \in C_+$, and utility functions $u^i : C_+ \to \mathcal{R}$, which are assumed to be quasi-concave, increasing, and nonsatiated. We shall assume that there are finitely many agents ($I < \infty$). Most of our results extend to the case of infinitely many agents, see Section 6. Prices are described in an abstract way by a linear functional $P : C \to \mathcal{R}$ which assigns price $P(c)$ to each consumption plan $c \in C$.

Valuation equilibrium is a linear pricing functional P and a consumption allocation $\{c^i\}$ such that markets clear ($\sum_1^I c^i = \sum_1^I \omega^i$), and c^i maximizes agent's i utility $u^i(c)$ subject to the budget constraint $P(c) \leq P(\omega^i)$ and $c \in C_+$.

If the space C were finite-dimensional (say, $C = \mathcal{R}^T$) then pricing functional would necessarily have a scalar product representation as $P(c) = \sum_{t=0}^T p_t c_t$ with price p_t of consumption at date t. Valuation equilibrium would then be the standard Arrow-Debreu equilibrium.

In the present infinite-dimensional setting we can identify the price of consumption at date t as $p_t = P(e^t)$, where $e^t \in C$ denotes a consumption plan of one unit at date t and zero units at all other dates. Linearity of P implies that

$$P(z) = \sum_{t=0}^{\infty} p_t z_t \tag{1}$$

holds for every commodity bundle z which is nonzero at finitely many dates. It may or may not hold for commodity bundles which are nonzero at infinitely many dates. Gilles and LeRoy (1992a, 1992b) argued that if

$$P(z^*) \neq \sum_{t=0}^{\infty} p_t z_t^* \tag{2}$$

for some commodity bundle $z^* \in C$, then there is a *price bubble* on z^*. In this terminology price bubbles are cases of lack of countable additivity of pricing.

Pricing functional P is positive in a valuation equilibrium with monotone preferences. Therefore, $P(z) \geq P(z^T)$ for every positive commodity bundle $z \in C_+$ and every T, where z^T is a commodity bundle equal to z in the first T dates, and equal to zero after date T. Since $P(z^T) = \sum_{t=0}^{T} p_t z_t$, it follows that an equilibrium price bubble cannot be negative on a positive commodity bundle.

Countable additivity is a question of continuity of the pricing functional in a suitable topology. Typically, equilibrium pricing functional is continuous in a topology of C with respect to which agents' utility functions are continuous. Under continuous equilibrium pricing, consumption plans which agents consider as near-substitutes have almost equal prices in equilibrium. Countably additivity of pricing functional P is equivalent to

$$P(z) = \lim_{T} P(z^T) \tag{3}$$

for every $z \in C$. Hence, P is countably additive if it is continuous in a topology in which z^T converges to z.

Equilibrium analysis of economies with consumption space $C = \ell_\infty$ due to Bewley (1972) reveals a possibility of a valuation equilibrium with linear but not countably additive pricing. Conditions under which a valuation equilibrium exists are significantly weaker than those under which a valuation equilibrium with countably additive pricing exists. These conditions pertain mainly to agents' utility functions. If the utility functions are continuous in the norm topology of ℓ_∞, then a valuation equilibrium exists. If they are continuous in a stronger topology - the Mackey $\tau(\ell_\infty, \ell_1)$ topology - then a valuation equilibrium with countably additive pricing exists. Function $u(c) = \inf_t c_t$ is the simplest example of a utility function continuous in the norm topology of ℓ_∞ but not continuous in the Mackey topology. Epstein and Wang (1985) provide a discussion of a general class of such utility functions and a relationship with "uncertainty aversion."

We present an example of a valuation equilibrium with linear but not countably additive pricing, that is, an example of a valuation bubble.

Example 2.1 The consumption space is $C = \ell_\infty$ and there are two agents with utility functions on C_+ given by

$$u^1(c) = \inf_{t\in E} c_t + 2 \inf_{t\in O} c_t, \qquad u^2(c) = \sum_{t=0}^{\infty} \frac{1}{2^t} c_t, \tag{4}$$

where E (O) denotes the set of even (odd, respectively) dates. Their initial endowments are $\omega^1 = (1, 1, \ldots)$ and $\omega^2 = (1, 0, \ldots)$. Both utility functions are continuous in the norm topology of ℓ_∞, but only u^2 and not u^1 is continuous in the Mackey topology. Consider a linear functional P on the consumption space ℓ_∞ given by

$$P(z) = \sum_{t=0}^{\infty} \frac{1}{2^t} z_t + 4 \int_O z_t \mu(dt), \tag{5}$$

where μ is a density charge – a finitely additive measure on the subsets of natural numbers with the property that $\int x_t \mu(dt) = \lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T} x_t$ for every sequence x for which the limit exists (see Rao and Rao (1983)). Functional P is linear and norm continuous on ℓ_∞ but not countably additive. Under the pricing functional P, we have $p_t = 1/2^t$, $P(e^E) = 4/3$ and $P(e^O) = 8/3$, where $e^E = (1, 0, 1, 0, \ldots)$ and $e^O = (0, 1, 0, 1, \ldots)$.

This pricing functional leads to an equilibrium with no trade. Agent 1 always chooses constant consumption levels in odd dates and in even dates. If the ratio of prices of consumption plans e^O and e^E is 2, as it is under P, she is indifferent between any two consumption plans of the same value and with constant consumption levels in odd dates and in even dates. Therefore ω^1 is utility maximizing for agent 1. Agent's 2 utility function coincides with the pricing functional for all consumption plans that involve nonzero consumption in finitely many odd dates. Consequently, she is indifferent between any two such consumption plans of the same value. Consumption plans that involve nonzero consumption in infinitely many odd dates are "overpriced" for agent 2 and therefore she will not choose any of them. Therefore ω^2 is utility maximizing for agent 2. $\square$

The above example is not the simplest possible of a valuation equilibrium with linear but not countably additive pricing, but it has a number of important additional features. Most important is that the price bubble is "essential" in the sense that the same allocation is not an equilibrium under the countably additive "part" $\sum_{t=0}^{\infty} \frac{1}{2^t} z_t$ of the pricing functional P. Furthermore, the aggregate consumption at each date is bounded away from zero.

3 Sequential Markets

In a valuation equilibrium agents trade consumption goods for future dates in simultaneous forward markets. An alternative trading system are sequential spot markets. At any date t a single infinitely-lived asset is traded in exchange for consumption at that date. The asset pays dividend d_t at date t, starting with date 1. We assume that $d_t \geq 0$ for every t. If $d_t = 0$ for every t, the asset is called *fiat money*.

Let q_t be the price of the asset at date t. The sequence of budget constraints of agent i with endowment y_t^i of the consumption good at date t and the initial asset holding $\bar{a}_0^i$ is

$$c_0 + q_0 a_0 = y_0^i + q_0 \bar{a}_0^i, \tag{6}$$

and

$$c_t + q_t a_t = y_t^i + (q_t + d_t) a_{t-1} \tag{7}$$

for $t = 1, 2, \ldots$, where a_t is asset holding at t. In addition we require that $c \in C_+$.

The (net) payoff of an asset holding strategy $a = \{a_t\}$ at date $t \geq 1$ is

$$z_t(a) = (q_t + d_t)a_{t-1} - q_t a_t. \tag{8}$$

Its date-0 cost is

$$q_0 a_0. \tag{9}$$

The *present value*, as of date 0, of one unit of consumption at date t is the cost of an asset holding strategy that has payoff equal to one at date t and zero at every date prior to t, and involves no asset holding after t.

Let us assume that $q_t > 0$. It is a simple exercise to find that p_t is given by

$$p_t = \frac{1}{(q_t + d_t)} \frac{q_{t-1}}{(q_{t-1} + d_{t-1})} \cdot \ldots \cdot \frac{q_1}{(q_1 + d_1)} \cdot q_0, \tag{10}$$

for $t \geq 1$. In recursive form, we have

$$p_{t+1} = \frac{q_t}{q_{t+1} + d_{t+1}} p_t \tag{11}$$

for $t \geq 0$, with $p_0 = 1$.

The present value of the asset's dividends $\{d_t\}$,

$$\sum_{t=1}^{\infty} p_t d_t, \tag{12}$$

is usually termed the *fundamental value* of the asset. If the price of the asset at date 0 equals the fundamental value,

$$q_0 = \sum_{t=1}^{\infty} p_t d_t, \tag{13}$$

then we say that there is no *price bubble* on $\{d_t\}$. Otherwise, there is a price bubble equal to the difference between the asset's price and its fundamental value.

Two properties of price bubbles in sequential markets are worth mentioning: First, a necessary and sufficient condition for there being no price bubble is that

$$\lim_{t \to \infty} p_t q_t = 0. \tag{14}$$

Second, the price bubble cannot be negative. These properties are proved as follows: Equation (11) implies that

$$(q_t + d_t)p_t = q_{t-1}p_{t-1}, \tag{15}$$

for every $t \geq 1$. If we add these equations over all t from 1 to some τ, we obtain

$$q_0 = \sum_{t=1}^{\tau} p_t d_t + p_\tau q_\tau \tag{16}$$

Since $p_t q_t > 0$, we have $q_0 > \sum_{t=1}^{\tau} p_t d_t$ for every τ, and therefore

$$q_0 \geq \sum_{t=1}^{\infty} p_t d_t. \tag{17}$$

Thus, price bubble cannot be negative. Moreover, $q_0 = \sum_{t=1}^{\infty} p_t d_t$ if and only if (14) holds. The sequence $\{p_t q_t\}$ is nonincreasing in t. If the asset is fiat money, then it is a constant sequence.

4 Arbitrage and Asset Holding Constraints

An *arbitrage opportunity* at prices $\{q_t\}$ is an asset holding strategy a such that $z_t(a) \geq 0$ for every $t \geq 1$, and $q_0 a_0 \leq 0$, with at least one strict inequality. One can distinguish two types of arbitrage opportunities: finite arbitrage and infinite arbitrage. A *finite arbitrage* opportunity is an arbitrage opportunity $\{a_t\}$ that involves nonzero holding of the asset only at finitely many dates, i.e., such that $a_t = 0$ for $t \geq \tau$, for some τ. It should be clear that there is no finite arbitrage if asset price q_t is strictly positive at every date. [2]

Any arbitrage opportunity that is not finite is an *infinite arbitrage* opportunity. An important example is a *Ponzi scheme*. Ponzi scheme is a strategy of short-selling the asset at date 0 and rolling-over the debt at every trading date t, $t \geq 1$.

[2] Negative prices can be arbitrage-free, too. For instance, if $q_t + d_t < 0$ for every t, then $\{q_t\}$ is arbitrage-free. We shall restrict our attention to strictly positive prices.

It is given by

$$a_0 = -\frac{\lambda}{q_0} \tag{18}$$

for some scale $\lambda > 0$, and

$$a_t = \frac{q_t + d_t}{q_t} a_{t-1} \tag{19}$$

for $t \geq 1$, so that $z_t(a) = 0$ and $q_0 a_0 = -\lambda$. Using present value factors p_t, such date-0 Ponzi scheme of scale λ is found to be given by

$$a_t = \frac{-\lambda}{p_t q_t}, \tag{20}$$

for each $t \geq 0$, where $\lambda > 0$.

More generally, one can consider a Ponzi scheme of short-selling the asset at an arbitrary date τ and rolling-over the debt. It is given by $a_s = 0$ for $s < \tau$ and

$$a_t = \frac{-\lambda p_\tau}{p_t q_t} \tag{21}$$

for each $t \geq \tau$.

If there is no price bubble so that (14) holds, then Ponzi schemes are unbounded. Otherwise, if there is price bubble, Ponzi schemes are bounded.

In contrast to finite arbitrage opportunities, Ponzi schemes cannot be precluded by asset prices. For arbitrary strictly positive asset prices there exist Ponzi schemes of arbitrary scale. The presence of Ponzi schemes of arbitrary scale is incompatible with the existence of an optimal consumption plan in sequential markets for an agent with monotone preferences. Therefore any consistent model of competitive sequential markets has to involve a constraint on asset holdings. There are various possible constraints. We name only few examples:

- the *zero short-sales* constraint, $a_t \geq 0$;
- the *bounded short-sales* constraint, $\inf_t a_t > -\infty$;
- the (uniform) *nonzero short-sales* constraint, $a_t \geq -b$ for some $b > 0$;
- the (uniform) *nonzero borrowing* constraint, $q_t a_t \geq -b$ for some $b > 0$;

- the *wealth* constraint, $q_t a_t \geq -(1/p_t) \sum_{s=t+1}^{\infty} p_s y_s^i$;
- the constraint of *simple asset holdings.*

The wealth constraint says that current borrowing cannot exceed the present value of future endowments. An asset holding strategy a is *simple* if $a_t = a_\tau$ for $t \geq \tau$, for some τ (i.e, if a is eventually constant).

These constraints differ significantly in the way they preclude Ponzi schemes. Using the form (21) of a Ponzi scheme, one can easily see that the zero short-sales constraint and the wealth constraint preclude Ponzi schemes for arbitrary asset prices. The same holds for simple asset holdings as long as the asset is not fiat money. The bounded short-sales constraint precludes Ponzi schemes if there is no price bubble but permits Ponzi schemes of arbitrary scale if there is a price bubble. The nonzero short-sales constraint precludes Ponzi schemes if there is no price bubble, and permits Ponzi schemes of a bounded scale if there is a price bubble. The nonzero borrowing constraint precludes Ponzi schemes if the present value factor p_t converges to zero, and permits Ponzi schemes of a bounded scale if p_t is bounded away from zero. These differences are very important for the possibility and the form of price bubbles in an equilibrium.

We shall denote by A_i a constrained set of asset holdings from which agent i can choose. A *sequential equilibrium* consists of asset prices $\{q_t\}$, consumption plans $\{c_t^i\}$, and asset holdings $\{a_t^i\}$ such that $\{c_t^i\}$ and $\{a_t^i\}$ maximize agent's i utility $u^i(c)$ subject to the sequence of budget constraints (1) and (2), with $a \in A_i$, and the markets clear. Market clearing in sequential markets means that

$$\sum_{i=1}^{I} c_t^i = \sum_{i=1}^{I} y_t^i + \sum_{i=1}^{I} \bar{a}_0^i d_t \tag{22}$$

and

$$\sum_{i=1}^{I} a_t^i = \sum_{i=1}^{I} \bar{a}_0^i \tag{23}$$

holds for every date t.

5 Valuation in Sequential Markets

We define the *asset span* of sequential markets as

$$M = \{z \in \mathcal{R}^\infty : z = z(a) \text{ for some } a \in A\}. \tag{24}$$

where A is a set of asset holdings.[3] *Payoff pricing functional* $V : M \to \mathcal{R}$ is given by

$$V(z) = \inf\{q_0 a_0 : a \in A, \text{ with } z(a) = z\} \tag{25}$$

for $z \in M$. Both the asset span and the payoff pricing functional depend on asset prices q which are assumed strictly positive so that there is no finite arbitrage.

For a payoff stream z, $V(z)$ is the lowest cost of an asset holding strategy that generates it. In the presence of asset holding constraints, two asset holding strategies with the same payoff may have different initial costs. This form of payoff pricing functional is standard in asset pricing models with trading constraints or transaction costs (see Luttmer (1996)).

Our objective is to see whether the payoff pricing functional is linear and countably additive for different asset holding constraints. We derive general conditions on asset holding constraints which are sufficient for linearity and countable additivity of functional V. The two relevant properties of a set A of asset holdings are:

(D) if $a \in A$, then $a^\tau \in A$ for every date τ, where a^τ is such that $a^\tau_t = 0$ for $t \leq \tau$, and $a^\tau_t = a_t$ for $t > \tau$,

(UL) if $a \geq 0$, then $a \in A$.

Property (D) says that each asset holding strategy can be initiated with an arbitrary delay. That is, modifying each strategy by no-trade in an initial time period of arbitrary length leads to an admissible strategy. Property (UL) means that long positions in the asset are unrestricted. All asset holding constraints introduced in

[3]In order to simplify notation, we disregard the fact that A may be different for different agents.

Section 4 satisfy property (D). Property (UL) is satisfied by all constraints with the exception of simple asset holdings.

Our main result is the following

Theorem 5.1 *Suppose that the set A of asset holding strategies satisfies (D) and (UL). Then*

$$V(z) = \sum_{t=1}^{\infty} p_t z_t \tag{26}$$

for every $z \in M$, $z \geq 0$, if and only if there is no arbitrage opportunity in A. [4]

Proof: We show first that if functional V satisfies (26), then there is no arbitrage opportunity. Suppose, by contradiction, that there is an arbitrage opportunity $a \in A$. Then

$$\begin{aligned} q_0 a_0 &\leq 0 \\ z_t \equiv (q_t + d_t)a_{t-1} - q_t a_t &\geq 0, \ \ \forall t > 0 \end{aligned}$$

with at least one strict inequality. Since $p_t > 0$ for any $t \geq 0$, we have

$$V(z) \leq q_0 a_0 \leq 0 \leq \sum_{t=1}^{\infty} p_t z_t$$

where at least one inequality is strict. This is a contradiction to (26).

In order to prove that (26) holds if there is no arbitrage, we show first that

$$V(z) \geq \sum_{t=1}^{\infty} p_t z_t$$

for any $z \in M, z \geq 0$. Let $a \in A$ be an asset holding strategy that finances $z \in M, z \geq 0$. Then

$$(q_t + d_t)a_{t-1} - q_t a_t = z_t \geq 0, \ \ t > 0.$$

We shall prove that

$$q_t a_t \geq 0$$

[4]Since Ponzi schemes are arbitrage opportunities (see Section 3), no-arbitrage implies that there is no Ponzi scheme.

for any $t \geq 0$. Clearly it holds for $t = 0$ since otherwise a would be an arbitrage. Suppose there is some $T > 0$ such that $q_T a_T < 0$. Define an asset holding strategy $\hat{a}$ by

$$\begin{aligned} \hat{a}_t &= a_t, \ \ if \ \ t \geq T \\ &= 0, \ \ if \ \ t < T, \end{aligned}$$

then $\hat{a} \in A$ for A satisfies (D). Notice that $\hat{a}$ has zero date-0 cost and finances the payoff that is equal to z at any date $t > T$, is strictly positive at date T, and is zero prior to T. Hence $\hat{a}$ is an arbitrage opportunity in A, a contradiction to the no-arbitrage hypothesis.

It follows from (8) and (11) that

$$q_0 a_0 = \sum_{t=1}^{T} p_t z_t + p_T q_T a_T, \ \ \forall \ \ T \geq 1.$$

Since $p_t z_t \geq 0$ and $p_t q_t a_t \geq 0$ for any $t \geq 1$, it follows that

$$q_0 a_0 \geq \sum_{t=1}^{\infty} p_t z_t. \tag{27}$$

Since (33) holds for any asset holding strategy a that finances z, we conclude, using the definition of V, that

$$V(z) \geq \sum_{t=1}^{\infty} p_t z_t. \tag{28}$$

We next show that $V(z) \leq \sum_{t=1}^{\infty} p_t z_t$ for every $z \in M, \ \ z \geq 0$. This is clear if $\sum_{t=1}^{\infty} p_t z_t = \infty$. If $\sum_{t=1}^{\infty} p_t z_t < \infty$, then we define a trading strategy a^* by

$$a_t^* = \frac{1}{p_t q_t} \sum_{s=t+1}^{\infty} p_s z_s, \tag{29}$$

for all $t \geq 0$.Since $a^* \geq 0$ and A satisfies (UL), it follows that $a^* \in A$. It is straightforward to verify that $z_t(a^*) = z_t$ for all $t \geq 1$, hence a^* finances z at the date-0 cost $\sum_{t=0}^{\infty} p_t z_t$. It follows from the definition of V that

$$V(z) \leq \sum_{t=1}^{\infty} p_t z_t. \tag{30}$$

Inequalities (36) and (38) imply that (26) holds. □

Payoff pricing functional (26) displays valuation according to the present value rule. It is linear and countably additive. Thus, linear and countably additive valuation holds in sequential markets whenever there is no arbitrage (in particular, no Ponzi scheme) and the constraint on asset holdings satisfies properties (UL) and (D).

The zero short-sales constraint, the wealth constraint, and the bounded short-sales constraint satisfy (UL) and (D). Furthermore, there cannot be an arbitrage opportunity in equilibrium with these constraints. Consequently, linear and countably additive valuation holds in any equilibrium with these constraints.

The nonzero short-sales constraint and the nonzero borrowing constraint satisfy (UL) and (D), too. However, there can be a Ponzi scheme in an equilibrium with these constraints. Linear and countably additive valuation holds in an equilibrium if and only if there is no arbitrage.

The simple asset holding constraint does not satisfy property (UL) and so Theorem 5.1 does not apply. Since every payoff in the asset span of simple asset holdings can be financed by a unique simple asset holding strategy, the payoff pricing functional is clearly linear. It may or may not be countably additive. Linearity implies only that (26) holds for payoffs that are nonzero in finitely many dates.

If the payoff pricing functional satisfies (26) and there is a price bubble, then the asset's price exceeds the value of the dividends under the payoff pricing functional,

$$q_0 > V(d), \tag{31}$$

implying that the dividends can be purchased at a cost lower than the asset's price. Equilibrium price bubbles can also occur when the payoff pricing functional is nonlinear and when it is linear but not countably additive. Examples of sequential equilibrium price bubbles of all these three types will be given in Section 6.

6 Sequential Equilibrium with Bubbles

Whether price bubble can occur in a sequential equilibrium depends crucially on the form of constraints on asset holdings. For a constraint that permits Ponzi schemes of arbitrary scale when there is a price bubble, there cannot be an equilibrium with price bubble. This is the case for the bounded short-sales constraint. We show in Appendix A, Theorem 9.1, that sequential equilibria with bounded short sales coincide with valuation equilibria with countably additive pricing $\sum_{t=0}^{\infty} p_t c_t$. The payoff pricing functional is linear and countably additive in a sequential equilibrium with the bounded short-sales constraint.

For constraints that preclude Ponzi schemes even if there is a price bubble, an equilibrium with price bubble is potentially possible. In order to study whether a price bubble can actually occur in an equilibrium with any such constraint we shall first consider a general class of constraints of the form

$$a_t \geq -b_t^i, \text{ for every } t, \tag{32}$$

where the bound b_t^i for agent i is nonnegative and may depend on asset prices and the agent's endowments. The zero and nonzero short-sales constraints, the nonzero borrowing constraint, and the wealth constraint belong to this class of constraints.

We present sufficient conditions for there being no price bubble in equilibrium with constraint (32) and examples of price bubbles when these conditions do not hold. To this end, let

$$s^i \equiv -\liminf_{t\to\infty} b_t^i, \tag{33}$$

be the limiting value of the bound on short-selling. Note that $s^i \leq 0$. Let $y_t = \sum_i y_t^i$ denote the aggregate endowment of the consumption good at t, and $\bar{a}_0 = \sum_i \bar{a}_0^i$, the total supply of the asset. We assume that the supply of the asset is positive or zero, $\bar{a}_0 \geq 0$, and that $\bar{a}_0 > -\sum_i b_t^i$ for all t, i.e., that the constraints permit asset trading. The following result is closely related to Theorem 3.1 in Santos and Woodford (1997).

Theorem 6.1 *In a sequential equilibrium with asset holding constraint (32), if the present value of the aggregate endowment $\sum_0^\infty p_t y_t$ is finite, and $\sum_i s^i < \bar{a}_0$, then there is no price bubble.*

The proof can be found in Appendix B.

If the supply of the asset is strictly positive, then there cannot be a price bubble in an equilibrium with finite value of the aggregate endowment (see Santos and Woodford (1997)). For a constraint on asset holdings such that $s^i < 0$ (e.g., nonzero short-sales constraint), there cannot be a price bubble in an equilibrium with finite value of the aggregate endowment even if the supply of the asset is zero.

A sufficient condition for the finite value of the aggregate endowment is that the equilibrium allocation is Pareto optimal.

Theorem 6.2 *Suppose that each utility function is Gateaux differentiable. In a sequential equilibrium with asset holding constraint (32), if the equilibrium allocation $\{c^i\}$ is Pareto optimal and eventually interior (i.e., $c_t^i > 0$ for every i and every $t \geq \tau$, for some τ), then the present value of the aggregate endowment is finite.*

A definition of Gateaux differentiability and the proof of Theorem 6.2 can be found in Appendix B.

Theorem 6.1 provides sufficient conditions for there being no price bubble in equilibrium. It is well-known that if these conditions are not satisfied, a price bubble may occur. One class of examples is when the present value of the aggregate endowment is infinite.

6.1 Infinite value of the aggregate endowment.

Examples of monetary equilibria with positive price of fiat money and the zero short-sales constraint (see Bewley (1980) and Kocherlakota (1992), see also Santos and Woodford (1997)) belong to this class. Since fiat money is in positive supply and $s^i = 0$ for the zero short-sales constraint, the condition $\sum_i s^i < \bar{a}_0$ is satisfied.

By Theorem 6.2, consumption allocations in these monetary equilibria are not Pareto optimal.

The payoff pricing functional is linear and countably additive in equilibrium with the zero short-sales constraint. A price bubble occurs when the asset's price exceeds the value of the asset's dividends under the payoff pricing functional.

Sequential equilibria with a price bubble and an infinite value of aggregate endowment can also occur with nonzero short-sales constraint and nonzero borrowing constraint. We present an example of such equilibrium when the asset is fiat money. An interesting feature of this example is that Ponzi schemes of bounded scale are feasible at equilibrium prices. The presence of a Ponzi scheme implies that the payoff pricing functional is nonlinear since the value of the zero payoff is negative and equal to the negative of date-0 cost of the Ponzi scheme of the largest scale. Thus, linear valuation fails. This example is closely related to Example 1 in Kocherlakota (1992).

Example 6.1 Two infinitely lived agents have the same utility function

$$u(c) = \sum_{t=0}^{\infty} \beta^t ln(c_t), \tag{34}$$

where $0 < \beta < 1$. Their consumption endowments are

$$\begin{aligned} y_0^1 &= B + \eta, & y_0^2 &= A - \eta, & & \\ y_t^1 &= B\rho^t, & y_t^2 &= A\rho^t, & & for\ \ t\ \ even, \\ y_t^1 &= A\rho^t, & y_t^2 &= B\rho^t, & & for\ \ t\ \ odd, \end{aligned}$$

where $\rho \geq 1$ is the factor of growth of the economy (no growth if $\rho = 1$), and $\beta A > B > 0$. The only asset traded is fiat money with zero dividend and the supply equal to 1. The agents' initial holdings of fiat money are $\bar{a}_0^1 = 1$ and $\bar{a}_0^2 = 0$. Each agent can short sell at most one unit of the asset.

Suppose that

$$\eta = \frac{\beta A - B}{(1+\beta)}. \tag{35}$$

Then there is an equilibrium in which fiat money has a positive price

$$q_t = \eta\rho^t, \qquad \forall\ t \geq 0, \tag{36}$$

and the consumption and asset holding allocation is

$$\begin{aligned}
c_t^1 &= (B+3\eta)\rho^t, \qquad for\ \ t\ \ even \\
&= (A-3\eta)\rho^t, \qquad for\ \ t\ \ odd \\
c_t^2 &= (A-3\eta)\rho^t, \qquad for\ \ t\ \ even \\
&= (B+3\eta)\rho^t, \qquad for\ \ t\ \ odd \\
a_t^1 &= -1, \qquad for\ \ t\ \ even \\
&= 2, \qquad for\ \ t\ \ odd \\
a_t^2 &= 2, \qquad for\ \ t\ \ even \\
&= -1, \qquad for\ \ t\ \ odd
\end{aligned}$$

To verify that this is indeed an equilibrium, note that markets clear at every date, and the budget and the short-sales constraints are all satisfied. It remains to verify that the consumption and asset holding plans are optimal for each agent. At each odd date t, agent's 1 first-order condition for the choice of optimal consumption and asset holding is

$$u_t(c^1)q_t = u_{t+1}(c^1)q_{t+1}, \tag{37}$$

where $u_t(c^1) = \beta^t/c_t^1$ is the partial derivative of u with respect to date-t consumption. Using (35), one can easily verify that this condition is satisfied. At each even date t, the first-order condition is

$$u_t(c^1)q_t \geq u_{t+1}(c^1)q_{t+1}, \tag{38}$$

since agent's 1 short-sales constraint is binding. It is satisfied, as well. A suitable transversality condition (see Kocherlakota (1992), Proposition 2) can also be verified to hold. We can similarly verify the optimality for agent 2.

The present value of the aggregate endowment is infinite in this equilibrium since $p_t = \rho^{-t}$ and $y_t = (A+B)\rho^t$ for $t \geq 0$, and so $\sum_{t=0}^{\infty} p_t y_t = \infty$. Theorem 6.2

implies that the equilibrium allocation is not Pareto efficient. This is indeed the case: the two agents' intertemporal marginal rates of substitutions are different at every date.

Note that $a_t = -1$ for every t is a Ponzi scheme which satisfies the imposed short-sales constraint. When the growth factor ρ equals one, the imposed short-sales constraint is equivalent to a borrowing constraint with the uniform bound of η. □

6.2 Zero asset supply.

A different class of examples of price bubbles is when the present value of the aggregate endowment is finite but the supply of the asset equals $\sum_i s^i$. Examples of price bubbles in equilibrium with the wealth constraint and zero supply of the asset belong to that class. If there is a price bubble, then s^i equals zero for the wealth constraint.

The payoff pricing functional is linear and countably additive in an equilibrium with the wealth constraint. There is a price bubble if the asset's price exceeds the value of the asset's dividends under the payoff pricing functional.

We show in Theorem 9.2, Appendix A, (see also Wright ((1987)) that equilibrium allocations in sequential markets with the wealth constraint coincide with valuation equilibrium allocations with countably additive pricing functional

$$P(c) = \sum_0^\infty p_t c_t, \tag{39}$$

and the ith agent's budget constraint

$$P(c) = P(\omega^i) + \eta \bar{a}_0^i, \tag{40}$$

where $\omega^i = y^i + \bar{a}_0^i d$ denotes the consumption endowment including the dividend payment on the initial asset holding, and $\eta = q_0 - \sum_{t=1}^\infty p_t d_t$ denotes the sequential price bubble. The agent's income in this valuation equilibrium consists of the value of the consumption endowment ω^i plus the bubble payment on the initial asset holding. Clearly, Walras law in the valuation equilibrium

$$\sum_i P(c^i) = \sum_i P(\omega^i) \tag{41}$$

implies that the price bubble has to be zero if the total supply of the asset is strictly positive.

If the asset supply is zero, then there are sequential equilibria with price bubbles. A particularly simple case is when initial asset holding $\bar{a}_0^i$ equals zero for each agent (see Kocherlakota (1992), and Magill and Quinzii (1997)). Then a valuation equilibrium with ith agent's income $P(\omega^i)$ gives rise to a continuum of sequential equilibria. All these sequential equilibria have the same consumption allocation which is Pareto optimal, but different equilibrium asset prices. Asset prices are obtained from valuation equilibrium prices $\{p_t\}$ by equation (15) with the initial price q_0 given by $q_0 = \eta + \sum_t p_t d_t$ for arbitrary nonnegative value of the bubble η. Thus, the price bubble represents indeterminacy of asset prices in sequential equilibrium.

A different case is when $\bar{a}_0^i$ is nonzero for some agent (but the total supply is zero). A valuation equilibrium with income $P(\omega^i)$ gives rise to a sequential equilibrium with no price bubble, $\eta = 0$. A valuation equilibrium with income $P(\omega^i) + \eta\bar{a}_0^i$, for $\eta > 0$, gives rise to a sequential equilibrium with price bubble η. This latter valuation equilibrium typically exists for small values of η since the part $\eta\bar{a}_0^i$ of the income of agent i can be regarded as a transfer of date-0 consumption. [5] In general, equilibrium consumption allocations are different for different values of η. Thus price bubbles have a real effect in this case. All equilibrium allocations are Pareto optimal.

6.3 Other asset holding constraints.

Theorem 6.1 does not apply to constraints on asset holdings that are not of the form (32). We have already seen that there cannot be a price bubble in equilibrium

[5] Note that prices are normalized so that $p_0 = 1$.

with the bounded short-sales constraint. The situation is different for simple asset holdings. There can be a price bubble in equilibrium with simple asset holdings even with positive supply of the asset and finite value of the aggregate endowment.

In an equilibrium with simple asset holdings the payoff pricing functional is linear. Since the buy-and-hold strategy is the only simple asset holding strategy that finances the asset's dividends, the asset's price equals the value of the dividends under the payoff pricing functional. A price bubble occurs when the payoff pricing functional is not countably additive.

We present an example of a price bubble in a sequential equilibrium with simple asset holdings. In this example the valuation equilibrium of Example 2.1 is implemented as a sequential equilibrium.

Example 6.2 There are two agents with utility functions u^1 and u^2 specified in Example 2.1. Their endowments of consumption are $y^1 = y^2 = (1, 0, \ldots)$, the initial asset holdings are $\bar{a}_0^1 = 1$ and $\bar{a}_0^2 = 0$ so that $\bar{a}_0 = 1$. Note that $y^1 + \bar{a}_0^1 d = \omega^1$ and $y^2 + \bar{a}_0^2 d = \omega^2$ holds for ω^1 and ω^2 of Example 2.1. There is a single asset with dividend $d_t = 1$ for every date $t \geq 1$.

Let the consumption space in this two-agents economy be the space of eventually constant sequences. It follows from Theorem 9.3 in Appendix A that sequential equilibrium allocations with simple asset holdings coincide with valuation equilibrium allocations in the consumption space of eventually constant sequences with pricing functional of the form

$$P(c) = \sum_{0}^{\infty} p_t c_t + \eta \lim_{t \to \infty} c_t, \tag{42}$$

where η is the price bubble in sequential equilibrium. The budget constraint in the valuation equilibrium is

$$P(c) = P(\omega^i). \tag{43}$$

Set $p_t = \frac{1}{2^t}$ and $\eta = 2$. Using $\bar{c}$ to denote the constant consumption level that the consumption plan $c = \{c_t\}$ eventually reaches the functional P can be expressed as

$$P(c)=\sum_{t=0}^{\infty}\frac{1}{2^t}\,c_t+2\bar{c}. \tag{44}$$

This functional P coincides with the equilibrium pricing functional of Example 2.1 restricted to the subspace of eventually constant sequences. Valuation equilibrium of Example 2.1 in the consumption space ℓ_∞ is also a valuation equilibrium in the subspace of eventually constant sequences. The equilibrium allocation $c^1 = (1,1,\ldots)$ and $c^2=(1,0,\ldots)$ is therefore a sequential equilibrium with simple asset holdings. Asset prices in the sequential equilibrium are $q_t = 2q_{t-1}-1$ for $t\geq 1$ with $q_0=3$ (see Appendix A, (45) and (46)). Since $\sum_{t=1}^{\infty} p_t = 1 < q_0$, there is a price bubble on the asset. Equilibrium asset holdings are $a_t^1 = 1$ and $a_t^2=0$ for every t.

The payoff pricing functional is linear but it is not countably additive since $\sum_{t=1}^{\infty}\frac{1}{2^t}d_t = 1$ while $V(d)=2$. $\square$

7 Infinitely Many Agents

We have assumed thus far that the number of agents is finite. This excludes an important example of an overlapping generations model in which there is an infinite number of finitely lived agents. One can easily see that Theorem 6.1 extends to the case of infinitely many agents (see also Santos and Woodford (1997)). However, Theorem 6.2 depends crucially on the assumption that there are finitely many agents.

There are well-known examples of price bubbles in sequential equilibrium in an overlapping generations economy (see Santos and Woodford (1997), and LeRoy (1997) for most recent discussions). A common feature of these examples is that the present value of aggregate endowment is infinite.

Our theory of valuation of assets in sequential markets of Section 5 and the main result, Theorem 5.1, do not depend on the number of agents and can be applied to overlapping generations economies. The most natural constraint on asset holdings in an overlapping generations economy is that an agent's asset holding is nonzero only during his finite lifetime. This set of asset holdings leads to the asset span of payoffs that are nonzero only in the finite time period of the agent's life. In order to investigate the valuation of infinitely-lived assets, it is necessary to consider alternative specifications of the feasible set of asset holdings that include infinite buy-and-hold strategies (LeRoy (1997)). In light of the results of this paper it should be no surprise that a specification of the set of asset holdings determines to a large extent the nature of asset valuation and price bubbles. One possibility is to expand the set of feasible asset holdings to include all simple asset holdings (see LeRoy (1997)). If the asset is not fiat money (for otherwise there could be a Ponzi scheme among simple asset holdings), agents' asset holdings in equilibrium would remain unchanged. Simple asset holdings give rise to payoff pricing functional that is linear but may be not countably additive. It is not countably additive, if and only if there is a price bubble. Another possibility is to expand agents' asset holdings to those that satisfy zero short-sales constraint after the agent's lifetime. Again, agents' equilibrium asset holdings would remain unchanged. The results of Section 5 indicate that the payoff pricing functional is linear and countably additive and that a price bubble occurs when the asset's price differs from the value of dividends under the payoff pricing functional. Needless to say, these two specifications of feasible asset holdings are associated with different asset spans.

8 REFERENCES

Bewley, T. (1980): "The Optimum Quantity of Money," in *Models of Monetary Economies*, ed. by J. Kareken and N. Wallace, Minneapolis: Federal Reserve Bank.

Bewley, T. (1972): "Existence of Equilibrium in Economies with Infinite Many Commodities," *Journal of Economic Theory*, 4, 514-540.

Epstein L. and T. Wang (1985): "Uncertainty, Risk-Neutral Measures and Security Price Booms and Crashes," *Journal of Economic Theory*, 67, 40-83.

Gilles, C., and S. LeRoy (1992): "Asset Price Bubbles," in *The New Palgrave Dictionary of Money and Finance.*

Gilles, C., and S. LeRoy (1992): "Bubbles and Charges," *International Economic Review*, 33, 323-339.

Kocherlakota, N. (1992): "Bubbles and Constraints on Debt Accumulation," *Journal of Economic Theory*, 57, 245-256.

LeRoy, S. (1997): "Tiny Bubbles," mimeo, University of California at Santa Barbara.

Kandori, M. (1988): "Equivalent Equilibria," *International Economic Review*, 29, No. 3, 401-417.

Luenberger, D.G. (1969): "Optimization by Vector Space Methods,", New York: John Wiley and Sons, Inc.

Luenberger, D.G. (1973): "Introduction to Linear and Nonlinear Programming," Reading, Massachusetts: Addison-Wesley Publishing Company, Inc.

Luttmer, E. G. J. (1996): "Asset Pricing in Economies with Frictions," *Econometrica*, 64, No. 6, 1439-1467.

Magill, M. J. P., and Quinzii, M. (1997): "Incomplete Markets Over an Infinite Horizon: Long Lived Securities and Speculative Bubbles," *Journal of Mathematical Economics*, 26, No. 1, 133-170.

Tirole, J. (1992): "On the Possibility of Speculation Under Rational Expectations," *Econometrica*, 1163-1182.

Rao, K. P. S., and M. Rao (1983): "Theory of Charges," New York: Academic Press.

Santos, M., and M. Woodford (1997): "Rational Asset Pricing Bubbles," *Econometrica*, 65, No. 1, 19-57.

Werner, J. (1997): "Arbitrage, Bubbles, and Valuation," *International Economic Review.*

Wright, R. (1987): "Market Structure and Competitive Equilibrium in Dynamic Equilibrium Models," *Journal of Economic Theory*, 41, 189-201.

9 Appendix A: Equivalent Equilibria

In this section we establish equivalence between sequential equilibria and valuation equilibria in three different settings. Theorem 9.1 extends a related result by Kandori (1988), while Theorem 9.2 extends a result by Wright (1987).

In the following theorems asset prices q are associated with consumption prices p (with $p_0 = 1$) and a price bubble $\eta \geq 0$ in a one-to-one fashion via equations

$$p_{t+1} = \frac{q_t}{q_{t+1} + d_{t+1}} p_t \tag{45}$$

for ≥ 1, and

$$\eta = q_0 - \sum_{t=1}^{\infty} p_t d_t. \tag{46}$$

An agent's endowments in a valuation equilibrium and in a sequential equilibrium are related via

$$\omega_t = y_t + \bar{a}_0 d_t. \tag{47}$$

An agent's income in a valuation equilibrium is the value $P(\omega^i)$ of his endowment, unless otherwise specified. In the first two theorems 9.1 and 9.2 the consumption space is arbitrary.

Our first equivalence result applies to sequential equilibria with bounded short sales. Since there cannot be price bubbles in such equilibria (see Section 4), η is set equal to zero in (46).

Theorem 9.1 *Suppose that y^i is eventually bounded relative to d for every i, i.e., $y^i_t \leq \gamma d_t \; \forall t \geq \tau$, for some $\gamma > 0$, $\tau > 0$. Asset prices $\{q_t\}$ and allocation $\{c^i, a^i\}$ of consumption and asset holdings are a sequential equilibrium with bounded short sales if and only if the consumption allocation $\{c^i\}$ and the pricing functional $P(c) = \sum_0^\infty p_t c_t$ are a valuation equilibrium.*

Proof: We first show that each agent faces identical budget sets of consumptions in sequential markets under prices q and in complete Arrow-Debreu markets under pricing P. Note that $\sum_{t=0}^{\infty} p_t y^i_t < \infty$ holds in both a sequential equilibrium and a valuation equilibrium. It does hold in a sequential equilibrium since y^i is bounded relative to d and $\sum_{t=0}^{\infty} p_t d_t < \infty$ by (17). To simplify notation we shall drop the index i for an agent.

If c is budget feasible in sequential markets with bounded short sales, then there exists $a = \{a_t\}_{t=0}^{\infty}$ such that for

$$\begin{aligned} c_0 + q_0 a_0 &\leq y_0 + q_0 \bar{a}_0 \\ c_t + q_t a_t &\leq y_t + (q_t + d_t) a_{t-1}, \ \ t \geq 1 \\ \inf_t a_t &> -\infty. \end{aligned}$$

It follows from the budget constraints that

$$\sum_{t=0}^{T} p_t c_t + p_T q_T a_T \leq \sum_{t=0}^{T} p_t y_t + q_0 \bar{a}_0, \tag{48}$$

for any $T \geq 0$. Since there cannot be a price bubble in the sequential equilibrium with bounded short sales, we have $\lim p_T q_T = 0$ which implies that $\lim p_T q_T a_T \geq 0$. Passing to the limit as T goes to infinity, we obtain

$$\sum_{t=0}^{\infty} p_t c_t \leq \sum_{t=0}^{\infty} p_t y_t + q_0 \bar{a}_0. \tag{49}$$

Using (47) and (46) (with $\eta = 0$), inequality (49) simplifies to $P(c) \leq P(\omega)$ so that c satisfies the budget constraint in the Arrow-Debreu markets.

Conversely, suppose that c is budget feasible for the agent in the Arrow-Debreu markets under pricing P. Let us define an asset holding strategy a by

$$a_t = \frac{P(c^t) - P(y^t)}{p_t q_t}, \qquad t \geq 0, \tag{50}$$

where c^t is a consumption plan equal to c from date $t+1$ on, and equal to zero prior to t and at t. More explicitly, (50) can be written as

$$a_t = -\frac{\sum_{s=t+1}^{\infty} p_s (y_s - c_s)}{p_t q_t}. \tag{51}$$

Notice that a is well defined since both $\sum_{t=0}^{\infty} p_t c_t$ and $\sum_{t=0}^{\infty} p_t y_t$ are finite in valuation equilibrium. Since

$$-a_t \leq \frac{\sum_{s=t+1}^{\infty} p_s y_s}{p_t q_t} \tag{52}$$

and y is bounded relative to d, we have

$$-a_t \leq \frac{\gamma \sum_{s=t+1}^{\infty} p_s d_s}{p_t q_t} \tag{53}$$

for large t. Using 16, we obtain

$$-a_t \leq \gamma \tag{54}$$

for large t. Therefore a satisfies the bounded short-sales constraint.

It follows from (51) that

$$p_t q_t a_t + \sum_{s=t+1}^{\infty} p_s y_s = \sum_{s=t+1}^{\infty} p_s c_s, \quad t \geq 0. \tag{55}$$

For $t \geq 1$, (55) and its one-period lagged version imply that

$$c_t + q_t a_t = y_t + (q_t + d_t) a_{t-1}.$$

For $t = 0$, (55) and the budget constraint $P(c) \leq P(\omega) = P(y) + q_0 \bar{a}_0$ imply that

$$c_0 + q_0 a_0 \leq y_0 + q_0 \bar{a}_0. \tag{56}$$

Thus a so constructed and c satisfy the sequential budget constraints with asset prices q.

The budget constraint in valuation equilibrium must hold with equality due to the nonsatiation of preferences. Therefore the constraint (56) must also hold with equality. This combined with the market clearing in goods implies that asset holdings defined in (50) clear the asset markets. □

Theorem 9.2 *Asset prices $\{q_t\}$ and allocation $\{c^i, a^i\}$ of consumption and asset holdings are a sequential equilibrium with the wealth constraint if and only if the consumption allocation $\{c^i\}$ and the pricing functional $P(c) = \sum_0^{\infty} p_t c_t$ are a valuation equilibrium with agent ith income given by $P(\omega^i) + \eta \bar{a}_0^i$.*

Note that the set of equilibrium allocations is nonempty only if $\eta \sum_i \bar{a}_0^i = 0$.

Proof: The idea of the proof is the same as in Theorem 9.1. We show that the budget sets of consumption plans are the same in sequential and in Arrow-Debreu markets. Note that $\sum_{t=0}^{\infty} p_t y_t^i < \infty$ necessarily holds for every agent in a sequential equilibrium and in a valuation equilibrium.

If c is budget feasible in sequential markets with the wealth constraint, then there exists $a = \{a_t\}_{t=0}^{\infty}$ such that for

$$\begin{aligned} c_0 + q_0 a_0 &\leq y_0 + q_0 \bar{a}_0 \\ c_t + q_t a_t &\leq y_t + (q_t + d_t) a_{t-1}, \;\; t \geq 1 \\ p_t q_t a_t + \sum_{s=t+1}^{\infty} p_s y_s &\geq 0, \;\; t \geq 0 \end{aligned}$$

These budget constraints imply (48). Adding $\sum_{t=T+1}^{\infty} p_t y_t$ to both sides of (48) and using the wealth constraint, we obtain

$$\sum_{t=0}^{\infty} p_t y_t + q_0 \bar{a}_0 \geq \sum_{t=0}^{T} p_t c_t + (p_T q_T a_T + \sum_{t=T+1}^{\infty} p_t y_t) \geq \sum_{t=0}^{T} p_t c_t. \tag{57}$$

Since (57) holds for arbitrary T, we conclude that (49) holds. Using (47) and (46), inequality (49) simplifies to $P(c) \leq P(\omega) + \eta \bar{a}_0$, so that c satisfies the budget constraint in the Arrow-Debreu markets.

Conversely, suppose that c is budget feasible for the agent in the simultaneous Arrow-Debreu markets under pricing P. We define an asset holding strategy a by (50) and observe that (51) holds. Strategy a is well defined since both $\sum_{t=0}^{\infty} p_t c_t$ and $\sum_{t=0}^{\infty} p_t y_t$ are finite. Inequality (55) implies now that a satisfies the wealth constraint. The rest of the proof is the same as in Theorem 9.1.

□

Let $C^c = \{c \in R^{\infty} : \exists\, \gamma > 0, \exists\, \tau \text{ such that } c_t = \gamma d_t \; \forall t \geq \tau\}$ be the consumption space of sequences that are eventually proportional to d.

Theorem 9.3 *Let the consumption space be C^c. Asset prices $\{q_t\}$ and allocation $\{c^i, a^i\}$ of consumption and asset holdings are a sequential equilibrium with simple asset holdings if and only if the consumption allocation $\{c^i\}$ and pricing functional $P(c) = \sum_0^{\infty} p_t c_t + \eta \lim_{t\to\infty} c_t/d_t$ are a valuation equilibrium.*

Proof: Again. the proof follows the line of Theorem 9.1. We show that the sets of budget feasible consumption plans are the same in sequential and in Arrow-Debreu markets.

If c is budget feasible in sequential markets with simple asset holdings, then there exists $a = \{a_t\}_{t=0}^{\infty}$ such that for

$$\begin{aligned} c_0 + q_0 a_0 &\leq y_0 + q_0 \bar{a}_0 \\ c_t + q_t a_t &\leq y_t + (q_t + d_t) a_{t-1}, \;\; t \geq 1, \end{aligned}$$

and $a_t = a^*$ for every $t \geq \tau$, for some τ. Summing up budget constraints from date 0 to some $T \geq \tau$ it follows that

$$\sum_{t=0}^{T} p_t c_t + p_T q_T a^* \leq \sum_{t=0}^{T} p_t y_t + q_0 \bar{a}_0. \tag{58}$$

Budget constraint for $t \geq \tau$ implies

$$a^* \geq \frac{c_t - y_t}{d_t}, \tag{59}$$

or

$$a^* \geq \lim \frac{c_t}{d_t} - \lim \frac{y_t}{d_t}, \tag{60}$$

since $c, y \in C^c$. Using (60) and taking limits in (58) as T goes to infinity, we obtain

$$\sum_{t=0}^{\infty} p_t c_t + \eta \lim \frac{c_t}{d_t} \leq \sum_{t=0}^{\infty} p_t y_t + \eta \lim \frac{y_t}{d_t} + q_0 \bar{a}_0 \tag{61}$$

where we used the fact that $\lim_{T \to \infty} p_T q_T = \eta$. Since $\eta = q_0 - \sum_{t=1}^{\infty} p_t d_t$ and $\omega_t = y_t + \bar{a}_0 d_t$, (61) implies

$$P(c) \leq P(\omega) \tag{62}$$

for $P(c) = \sum_{t=0}^{\infty} p_t c_t + \eta \lim \frac{c_t}{d_t}$. Thus c satisfy the budget constraint in simultaneous markets with pricing functional P.

Conversely, suppose that a consumption plan $c \in C^c$ satisfies budget constraint (62). We define an asset holding strategy a_t by (50). Using (46), we can rewrite (50) as

$$a_t = \frac{\sum_{s=t+1}^{\infty} p_s d_s \left(\frac{c_s - y_s}{d_s}\right) + \left(p_t q_t - \sum_{s=t+1}^{\infty} p_s d_s\right) \lim \frac{c_t - y_t}{d_t}}{p_t q_t}. \tag{63}$$

Since $c, y \in C^c$, (63) implies that a is simple. The same argument as in the proof of Theorem 8.1 implies that a finances the consumption plan c in sequential markets. $\square$

10 Appendix B

Proof of theorem 4.1: Suppose that there is a sequential equilibrium with price bubble such that $\sum_{t=0}^{\infty} p_t y_t$ is finite and $\sum_i s^i < \bar{a}_0$. Let a^i be an equilibrium asset holding and c^i an equilibrium consumption of agent i. Market clearing implies that $\sum_{t=0}^{\infty} p_t c_t^i < \infty$. It follows from budget constraints (6) and (7) that $\lim_{t\to\infty} p_t q_t a_t^i$ exists and is finite. Since there is a price bubble, we have $\lim_{t\to\infty} p_t q_t = \eta > 0$.

Clearly $\lim_{t\to\infty} a_t^i \geq s^i$. We shall prove that

$$\lim_{t\to\infty} a_t^i = s^i, \quad \forall\ i. \tag{64}$$

Suppose by contradiction that $\lim_{t\to\infty} a_t^i > s^i$. Then we can choose $\epsilon > 0$ and τ_1 such that

$$a_t^i > s^i + 2\epsilon$$

for all $t \geq \tau_1$. Let τ_2 be such that

$$s^i + \epsilon \geq -b_t^i$$

for all $t \geq \tau_2$. Let $\tau \equiv max\{\tau_1, \tau_2\}$, then for $t \geq \tau$, we have

$$a_t^i - \epsilon > -b_t^i.$$

Consider a date-τ Ponzi scheme of scale $\lambda = \epsilon\eta/p_\tau$. It is given by

$$\hat{a}_t = -\frac{\lambda p_\tau}{p_t q_t} = -\epsilon\frac{\eta}{p_t q_t}$$

for $t \geq \tau$, and $\hat{a}_t = 0$ for $t < \tau$. Then

$$a_t^i + \hat{a}_t = a_t^i - \epsilon\frac{\eta}{p_t q_t} \geq a_t^i - \epsilon > -b_t^i,$$

for $t \geq \tau$, where the weak inequality follows from the fact that $p_t q_t$ is a nonincreasing sequence. Therefore $a_t^i + \hat{a}_t$ is feasible which contradicts the assumption that (c^i, a^i) is optimal. Consequently, equation (64) must hold.

Using (64) we obtain

$$\bar{a}_0 = \sum_i a_t^i = \sum_i \lim_{t \to \infty} a_t^i = \sum_i s^i,$$

which contradicts the assumption that $\sum_i s^i < \bar{a}_0$. This concludes our proof of theorem 4.1. □

Proof of theorem 6.2: Utility function u^i is *Gateaux differentiable*, if the limit

$$\partial u^i(c^i; h) \equiv \lim_{\alpha \to 0} \left[\frac{u^i(c^i + \alpha h) - u^i(c^i)}{\alpha} \right] \tag{65}$$

exists for every $c^i \in C_+$ and every $h \in C$ such that $c^i + \alpha h \in C_+$, and if $h \mapsto \partial u^i(c^i; h)$ defines a linear functional on C. [6]

Let $\{c^i\}$ be a sequential equilibrium allocation which is Pareto optimal and eventually interior (say, after date τ). The first-order conditions for utility maximization and for Pareto optimality imply that

$$\frac{u_t^i(c^i)}{u_\tau^i(c^i)} = \frac{p_t}{p_\tau}, \;\; \forall\, t \geq \tau, \;\; \forall\, i, \tag{66}$$

where $u_t^i(c^i)$ denotes the partial derivative with respect to date-t consumption.

Since u^i is concave and Gateaux differentiable, we have

$$\partial u^i(c^i; c^i) \leq u^i(c^i) - u^i(0), \tag{67}$$

see Luenberger (1973). Monotonicity of u^i implies that $\partial u^i(c^i; \cdot)$ is a positive functional. It is also linear, and therefore we obtain

$$\sum_{t=1}^{T} u_t^i(c^i) c_t^i \leq \partial u^i(c^i; c^i), \tag{68}$$

[6] Frequently Gateaux differentiability is taken to mean merely that the limit in (65) exists (see Luenberger (1969)). A sufficient condition for the linearity is that the partial derivatives of u^i with respect to date-t consumption are continuous, for every t.

for arbitrary T. Combining (67) and (68), we have

$$\sum_{t=1}^{T} u_t^i(c^i) c_t^i \leq u^i(c^i) - u^i(0), \tag{69}$$

for every T. Consequently,

$$\sum_{t=1}^{\infty} u_t^i(c^i) c_t^i \leq u^i(c^i) - u^i(0) < \infty. \tag{70}$$

The first-order condition (66) implies that

$$\sum_{t=\tau}^{\infty} u_t^i(c^i) c_t^i = \frac{u_\tau^i(c^i)}{p_\tau} \sum_{t=\tau}^{\infty} p_t c_t^i. \tag{71}$$

It follows now from (71) and (70) that the infinite sum $\sum_{t=0}^{\infty} p_t c_t^i$ is finite for every i. Market clearing implies that the present value of the aggregate endowment $\sum_{t=0}^{\infty} p_t y_t$ is finite. $\square$