On the throughput-WIP trade-off in queueing systems, diminishing returns and the Threshold Property: A linear programming approach

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Abstract

We present a new unifying framework for investigating throughput-WIP (Work-in-Process) optimal control problems in queueing systems, based on reformulating them as linear programming (LP) problems with special structure: We show that if a throughput-WIP performance pair in a stochastic system satisfies the Threshold Property we introduce in this paper, then we can reformulate the problem of optimizing a linear objective of throughput-WIP performance as a (semi-infinite) LP problem over a polygon with special structure (a threshold polygon). The strong structural properties of such polygones explain the optimality of threshold policies for optimizing linear performance objectives: their vertices correspond to the performance pairs of threshold policies. We analyze in this framework the versatile input-output queueing intensity control model introduced by Chen and Yao (1990), obtaining a variety of new results, including (a) an exact reformulation of the control problem as an LP problem over a threshold polygon; (b) an analytical characterization of the Min WIP function (giving the minimum WIP level required to attain a target throughput level); (c) an LP Value Decomposition Theorem that relates the objective value under an arbitrary policy with that of a given threshold policy (thus revealing the LP interpretation of Chen and Yao's optimality conditions); (d) diminishing returns and invariance properties of throughput-WIP performance, which underlie threshold optimality; (e) a unified treatment of the time-discounted and time-average cases.

Key words. Throughput-WIP (Work-in-Process) optimal queueing control, threshold optimality, achievable performance region.

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1 Introduction

What is the minimum WIP (Work-in-Process) performance level required to attain a target throughput level in a queueing system? When are threshold policies (which control the system by setting a WIP cap) optimal for achieving a linear throughput-WIP performance objective? To answer these questions, which are important for the efficient operation of modern manufacturing and communications systems (see, e.g., Hopp and Spearman (1996)), we need to understand the trade-off relation between throughput and WIP and the structure of the region of achievable throughput-WIP performance pairs.

A wide variety of optimal intensity control problems in queueing systems have been shown to be solved optimally by threshold policies. The way to establish threshold optimality has been typically ad hoc. Although these problems can usually be formulated in the framework of dynamic programming, it is often not clear how to derive this structural property from Bellman's equations. This has led to a typical line of attack based on studying the properties of the optimal value function in a particular model, and deriving from them the threshold optimality result (see, e.g., Chen and Yao (1990)).

The dynamic programming approach based on studying the problem's optimal value function has been developed into a general framework for investigating monotonicity properties of optimal policies, of which threshold optimality is the simplest case (see, e.g., Glasserman and Yao (1994)). This approach, however, as it focuses on the value function, does not answer important performance questions such as the throughput-WIP trade-off issues discussed above.

In this paper we present a new unifying framework for investigating the relation between throughput and WIP, and the optimality of threshold policies, in queueing systems. Our approach is based, not on a study of value functions (as in the traditional dynamic programming approach), but on a study of the system's region of achievable throughput-WIP performance.

This paper thus extends the *performance region approach* to stochastic optimization, which has up to now been developed only for stochastic scheduling problems (see, e.g., Niño-Mora (1995), and Bertsimas and Niño-Mora (1996)), to a new domain: that of optimal *intensity control* problems in queueing systems.

Our framework is based on investigating the throughput-WIP performance region of a general stochastic system under an assumption which we call the *Threshold Property*, strongly related to the satisfaction of certain diminishing returns properties of throughput with respect to WIP. We show that, under the Threshold Property, the throughput-WIP performance region has a polygonal structure, which allows to reformulate an optimal control problem with a linear performance objective as a (semi-infinite) LP problem over a special polygon (a threshold polygon). The strong structural properties of such polygones explain threshold optimality, since their vertices are achieved by threshold policies. Furthermore, this knowledge of the system's performance region allows us to give a general analytical characterization of the min WIP function, giving the minimum WIP performance level required to attain a target throughput level.

The structure of the paper is as follows: Section 2 presents our LP framework for studying threshold optimality. Section 3 describes a specific model we analyze within this framework: the versatile optimal queueing intensity control problem introduced by Chen and Yao (1990). Section 4 presents the new results obtained from such analysis, including an analytical characterization of the min WIP function. Section 5 presents several key diminishing returns properties of threshold policies, which are essential to the proof of the Threshold Property for this model. Section 6 presents a new LP Value Decomposition property which relates the value of an arbitrary policy with that of a given threshold policy, thus giving directly LP optimality conditions (including Chen and Yao's). Section 7 presents the results of the LP analysis under the time-average criterion. Appendix A contains technical lemmas used for our model's analysis.

2 An LP framework for threshold optimality in stochastic throughput-WIP control

Consider a general dynamic and stochastic system, controlled by a *policy* in a class $\mathcal{U}$ of *admissible* policies. Its performance under policy $u \in \mathcal{U}$ is given by a pair $(\tilde{\mu}^u, \tilde{L}^u) \geq \mathbf{0}$, which must be an expectation. We call $(\tilde{\mu}^u, \tilde{L}^u)$ a *throughput-WIP (Work-in-Process)* performance pair, in accordance with its typical interpretation in applications of this framework.

We are interested in solving the optimal control problem

$$\tilde{V}^*(c) = \max \left\{ \tilde{\mu}^u - c\,\tilde{L}^u : u \in \mathcal{U} \right\}, \tag{1}$$

for each cost $c > 0$. A wide variety of optimal control models can be cast as special cases of problem (1), which seeks to strike an optimal trade-off between the benefits of high system throughput and the costs of high WIP levels.

We are given a family of threshold-type policies, indexed by their integer *critical threshold values* $b \geq 0$. We denote by $(\tilde{\mu}^b, \tilde{L}^b)$ the performance pair for the *b-threshold policy*, and write its objective value as

$$\tilde{V}^b(c) = \tilde{\mu}^b - c\,\tilde{L}^b.$$

We present in this section a set of conditions on performance pair $(\tilde{\mu}^u, \tilde{L}^u)$, summarized in the *Threshold Property* in Definition 1, which guarantee that problem (1) is solved by a threshold policy, for each $c > 0$. The first part of the Threshold Property requires the performance pairs of threshold policies to satisfy the set of conditions we state next.

Assumption 1 (Diminishing Vanishing Returns) *The performance pairs $\{(\tilde{\mu}^b, \tilde{L}^b)\}_{b \geq 0}$ of threshold policies satisfy the following conditions:*
(i)[Strict Monotonicity (Increasing) of $\tilde{\mu}^b$]

$$0 \leq \tilde{\mu}^b < \tilde{\mu}^{b+1}, \qquad \text{for } b \geq 0;$$

(ii)[Strict Monotonicity (Increasing) of $\tilde{L}^b$]

$$0 \leq \tilde{L}^b < \tilde{L}^{b+1}, \quad \text{for } b \geq 0;$$

(iii)[Diminishing Marginal Returns of $\tilde{\mu}^b$ with respect to $\tilde{L}^b$]

$$\frac{\tilde{\mu}^{b+1} - \tilde{\mu}^b}{\tilde{L}^{b+1} - \tilde{L}^b} < \frac{\tilde{\mu}^b - \tilde{\mu}^{b-1}}{\tilde{L}^b - \tilde{L}^{b-1}}, \qquad \text{for } b \geq 1;$$

(iv)[Vanishing Marginal Returns of $\tilde{\mu}^b$ with respect to $\tilde{L}^b$]

$$\lim_{b \to \infty} \frac{\tilde{\mu}^b - \tilde{\mu}^{b-1}}{\tilde{L}^b - \tilde{L}^{b-1}} = 0;$$

(v) *there is a policy $u_0 \in \mathcal{U}$ such that*

$$\tilde{L}^{u_0} = \min\{\tilde{L}^u : \tilde{\mu}^u = 0, u \in \mathcal{U}\},$$

and

$$\tilde{L}^{u_0} \geq \tilde{L}^0,$$

with

$$\tilde{L}^{u_0} > \tilde{L}^0 \quad \text{if and only if} \quad 0 = \tilde{\mu}^{u_0} < \tilde{\mu}^0.$$

Let us define a sequence of *critical cost parameters* $\mathbf{c} = \{c^b\}_{b \geq 0}$ by

$$c^b = \begin{cases} \frac{\tilde{\mu}^b - \tilde{\mu}^{b-1}}{\tilde{L}^b - \tilde{L}^{b-1}} & \text{if } b \geq 1 \\ \frac{\tilde{\mu}^0 - \tilde{\mu}^{u_0}}{\tilde{L}^0 - \tilde{L}^{u_0}} = \frac{\tilde{\mu}^0}{\tilde{L}^0 - \tilde{L}^{u_0}} & \text{if } b = 0 \text{ and } \tilde{L}^0 < \tilde{L}^{u_0} \\ 0 & \text{otherwise,} \end{cases} \tag{2}$$

and a sequence $\boldsymbol{d} = \{d^b\}_{b \geq 0}$ by

$$d^b = \tilde{V}^b(c^b) = \tilde{\mu}^b - c^b\,\tilde{L}^b, \qquad \text{for } b \geq 0. \tag{3}$$

Definition 1 (Threshold Property) We say that performance pair $(\tilde{\mu}^u, \tilde{L}^u)$ satisfies the *Threshold Property* with respect to the given family of threshold policies if
(i) assumption 1 holds;
(ii) under *any* admissible policy $u \in \mathcal{U}$,
(ii.a) $\tilde{\mu}^u - c^0\,\tilde{L}^u \geq d^0$;
(ii.b) $\tilde{\mu}^u - c^b\,\tilde{L}^u \leq d^b$, $\qquad$ *for* $b \geq 1$.

Remarks:

1. Part (i) of the Threshold Property (Assumption 1) implies that
$$c^1 > c^2 > \cdots > 0 \geq c^0 \qquad \text{and} \qquad \lim_{b \to \infty} c^b = 0.$$
2. Parts (ii.a-b) of the Threshold Property, together with $c^1 > c^2 > \cdots > 0 \geq c^0$, imply that, for $b \geq 0$, $(\tilde{\mu}^b, \tilde{L}^b)$ is the *unique* solution of linear system
$$\begin{aligned} \mu - c^b\, L &= d^b, \\ \mu - c^{b+1}\, L &= d^{b+1}. \end{aligned}$$
3. Part (ii.a) says that the 0-threshold policy is optimal for control problem
$$\min \{\tilde{\mu}^u - c^0\, \tilde{L}^0 : u \in \mathcal{U}\}.$$
4. Part (ii.b) says that, for $c = c^b$ ($b \geq 1$), the b-threshold policy solves problem (1).

To establish the Threshold Property in a model we need, apart from showing that Assumption 1 holds, to prove the optimality of threshold policies *only for a countable set* of critical cost parameters $\{c^b\}_{b \geq 0}$. This turns out to be enough to ensure that threshold policies are also optimal for *all* costs $c > 0$.

We shall establish this *threshold optimality* result through an analysis of the system's throughput-WIP *region of achievable performance*, or *performance region*, defined by
$$\mathcal{X} = \left\{ (\tilde{\mu}^u, \tilde{L}^u) : u \in \mathcal{U} \right\}. \tag{4}$$
In particular, we shall show that, under the Threshold Property, performance region $\mathcal{X}$ is closely related to the polygon (with a countable number of constraints)
$$\mathcal{P}(\boldsymbol{c}, \boldsymbol{d}) = \left\{ (\mu, L) \geq \mathbf{0} : \mu - c^0\, L \geq d^0 \qquad \text{and} \qquad \mu - c^b\, L \leq d^b, \quad \text{for } b \geq 1 \right\},$$
in the sense that, for any $c > 0$, we can reformulate optimal control problem (1) as a semi-infinite LP problem over $\mathcal{P}(\boldsymbol{c}, \boldsymbol{d})$:
$$\tilde{V}^*(c) = \max \{\mu - c\, L : (\mu, L) \in \mathcal{X}\} = \max \{\mu - c\, L : (\mu, L) \in \mathcal{P}(\boldsymbol{c}, \boldsymbol{d})\}.$$
This result, together with the structural properties of such polygones, accounts for threshold optimality, as we will show in Section 2.2.

2.1 Threshold polygones

In this section we study the properties of polygones of the form
$$\mathcal{P}(\boldsymbol{c}, \boldsymbol{d}) = \left\{ (\mu, L) \geq \mathbf{0} : \mu - c^0\, L \geq d^0 \qquad \text{and} \qquad \mu - c^b\, L \leq d^b, \quad \text{for } b \geq 1 \right\},$$
where $\boldsymbol{c} = \{c^b\}_{b=0}^{\infty}$ and $\boldsymbol{d} = \{d^b\}_{b=0}^{\infty}$ are real sequences satisfying
$$c^1 > c^2 > \cdots > 0 \geq c^0 \qquad \text{and} \qquad \lim_{b \to \infty} c^b = 0, \tag{5}$$
and
$$d^0 = 0 \qquad \text{if and only if} \qquad c^0 = 0. \tag{6}$$
We present next the geometric counterpart of the Threshold Property.

Definition 2 (Threshold Polygon) We say that $\mathcal{P}(\boldsymbol{c}, \boldsymbol{d})$ is a *threshold polygon* with parameters $\boldsymbol{c}$ and $\boldsymbol{d}$ if, for any integer $b \geq 0$, the unique solution (μ^b, L^b) of linear system
$$\mu - c^b\, L = d^b,$$
$$\mu - c^{b+1}\, L = d^{b+1}$$
satisfies $(\mu^b, L^b) \in \mathcal{P}(\boldsymbol{c}, \boldsymbol{d})$.

Remark: Notice that it follows from Definition 2 that each (μ^b, L^b) is an extreme point of threshold polygon $\mathcal{P}(\boldsymbol{c}, \boldsymbol{d})$. We have represented in Figure 1 an example of a threshold polygon.

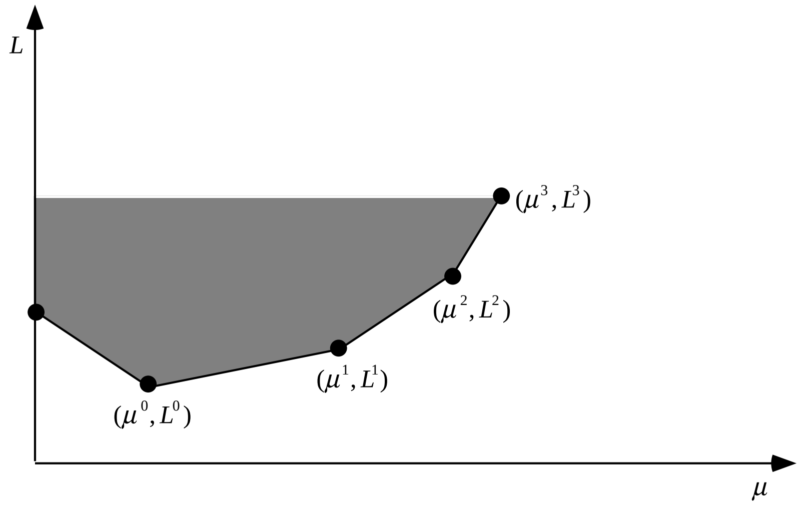

Figure 1: A threshold polygon.

LP over threshold polygones

We show in this section that the optimal solution to an LP problem over a threshold polygon can be characterized by a critical threshold value.

Consider the (semi-infinite) LP problem,

$$\begin{aligned} V^{LP} &= \max \mu - c\,L \\ &\quad \text{subject to} \\ &\quad \mu - c^b\,L \le d^b \quad \text{for } b \ge 1 \\ &\quad \mu - c^0\,L \ge d^0 \\ &\quad \mu, L \ge 0, \end{aligned} \tag{7}$$

and its dual LP problem, having a dual variable y^b corresponding to the bth primal constraint,

$$\begin{aligned} V^D &= \min \sum_{b=0}^{\infty} d^b\,y^b \\ &\quad \text{subject to} \\ &\quad \sum_{b=0}^{\infty} y^b \ge 1 : \mu \\ &\quad \sum_{b=0}^{\infty} c^b\,y^b \le c : L \\ &\quad y^b \ge 0, \quad \text{for } b \ge 1 \\ &\quad y^0 \le 0. \end{aligned} \tag{8}$$

Let us define the *critical threshold* function $b^*(\cdot)$ by

$$b^*(c) = \min \left\{ b \ge 0 : c^{b+1} \le c \right\},$$

which, by (5), is well defined for $c > 0$.

Proposition 1 (LP over Threshold Polygones) *LP problem (7) is solved optimally by pair* $\left(\mu^{b^*(c)}, L^{b^*(c)}\right)$*, for each* $c > 0$.

Proof
Let us write, for ease of notation, $b^* = b^*(c)$. We consider two cases: $b^* = 0$ and $b^* \ge 1$.

Case I: $b^* = 0$, i.e., $c \geq c^1$. Define $\bar{\boldsymbol{y}} = \{\bar{y}^b\}_{b=0}^{\infty}$ by $\bar{y}^0 = (c - c^1)/(c^0 - c^1)$, $\bar{y}^1 = 1 - \bar{y}^0$, and $\bar{y}^b = 0$ for $b \geq 2$. Then, $\bar{\boldsymbol{y}}$ is a feasible solution for dual LP (8) satisfying complementary slackness with primal solution (μ^0, L^0), which proves the optimality of both.

Case II: for some $b^* \geq 1$, $c^{b^*+1} \leq c < c^{b^*}$. Define $\bar{\boldsymbol{y}}$ as follows: $\bar{y}^{b^*}$ and $\bar{y}^{b^*+1}$ are defined as the solution to linear system

$$\begin{aligned} y^{b^*} + y^{b^*+1} &= 1, \\ c^{b^*} y^{b^*} + c^{b^*+1} y^{b^*+1} &= c, \end{aligned}$$

and $\bar{y}^b = 0$ for all other b. Again, we have that $\bar{\boldsymbol{y}}$ is a feasible solution for dual LP (8) satisfying complementary slackness with primal solution $\left(\mu^{b^*}, L^{b^*}\right)$, which proves the optimality of both. □

2.2 The Threshold Property and threshold optimality

We show in this section that the Threshold Property in Definition 1 implies the optimality of threshold policies for solving problem (1).

We first reformulate this optimal control problem as the mathematical program

$$\tilde{V}^*(c) = \max\{\mu - c\,L : (\mu, L) \in \mathcal{X}\}, \tag{9}$$

where $\mathcal{X}$ is the system's throughput-WIP performance region defined by (4).

Consider now polygon $\mathcal{P}(\boldsymbol{c}, \boldsymbol{d})$, where $\boldsymbol{c} = \{c^b\}_{b \geq 0}$ is given by (2), and $\boldsymbol{d} = \{d^b\}_{b \geq 0}$ is given by (3). Now, the Threshold Property says that $\mathcal{X} \subseteq \mathcal{P}(\boldsymbol{c}, \boldsymbol{d})$, hence the semi-infinite LP problem

$$V^{LP}(c) = \max\{\mu - c\,L : (\mu, L) \in \mathcal{P}(\boldsymbol{c}, \boldsymbol{d})\} \tag{10}$$

is an *LP relaxation* of problem (9), and hence of (1), i.e.,

$$V^{LP}(c) \geq \tilde{V}^*(c).$$

Our main result in this section says that this LP is in fact exact.

Theorem 1 (Semi-infinite LP Formulation) *Under the Threshold Property, $\mathcal{P}(\boldsymbol{c}, \boldsymbol{d})$ is a threshold polygon and, for any $c > 0$,*

$$V^{LP}(c) = \tilde{V}^*(c).$$

Proof
The results that $\mathcal{P}(\boldsymbol{c}, \boldsymbol{d})$ is a threshold polygon and $\mathcal{X} \subseteq \mathcal{P}(\boldsymbol{c}, \boldsymbol{d})$, hence $V^{LP}(c) \geq \tilde{V}^*(c)$, follow directly from the Threshold Property.

Furthermore, by Proposition 1 the optimal solution to LP problem (10) is always achievable, as it corresponds to some $(\tilde{\mu}^b, \tilde{L}^b)$. This shows the other inequality, $V^{LP}(c) \leq \tilde{V}^*(c)$, which completes the proof. □

The characterization of the optimal LP solution over a threshold polygon given in Proposition 1 translates now, by Theorem 1, into the optimality of threshold policies. Let us define the *critical threshold* function $b^*(\cdot)$ by

$$b^*(c) = \min\left\{b \geq 0 : c^{b+1} \leq c\right\}, \qquad \text{for } c > 0.$$

Corollary 1 (Threshold Optimality) *Under the Threshold Property, the $b^*(c)$-threshold policy solves optimally control problem (1), for $c > 0$.*

We present next another important consequence of the Threshold Property. Let us define the *Minimum WIP* function $\tilde{L}_{\min}(\cdot)$ by

$$\begin{aligned} \tilde{L}_{\min}(\mu) &= \min\{\tilde{L}^u : \tilde{\mu}^u = \mu, u \in \mathcal{U}\} \\ &= \min\{L : (\mu, L) \in \mathcal{X}\}, \end{aligned}$$

so that $\tilde{L}_{\min}(\mu)$ is the minimum WIP level required to attain a target throughput level μ. Our next result gives an analytical characterization of the Minimum WIP function $\tilde{L}_{\min}(\cdot)$ as a piece-wise linear function of throughput μ.

Corollary 2 (Min WIP Characterization) *Under the Threshold Property,*

$$\tilde{L}_{\min}(\mu) = \tilde{L}^{b-1} + \frac{1}{c^b}\,(\mu - \tilde{\mu}^{b-1}), \qquad \textit{for } \mu \in \left[\tilde{\mu}^{b-1}, \tilde{\mu}^b\right],\ b \geq 1.$$

We also state the following conjecture on the structure of performance region $\mathcal{X}$.

Conjecture 1 (Performance Region Characterization) *Suppose the Threshold Property holds and, in addition,*
(i) *the class $\mathcal{U}$ of admissible policies is closed under randomization (or, equivalently, performance region $\mathcal{X}$ is convex);*
(ii) *if WIP performance level L is achievable, then L' is also achievable, for any $L' > L$.*
Then,

$$\mathcal{X} = \overline{\mathcal{P}(\boldsymbol{c}, \boldsymbol{d})},$$

where $\overline{\mathcal{P}(\boldsymbol{c}, \boldsymbol{d})}$ denotes the closure of threshold polygon $\mathcal{P}(\boldsymbol{c}, \boldsymbol{d})$.

3 Model: Intensity control of a queueing system

In this section we analyze within the LP framework developed in Section 2 the versatile input-output queueing intensity control model introduced by Chen and Yao (1990).

Model description and formulation

The queueing system of interest consists of a service facility which services a single customer class. Let $L(t)$ denote the number of customers in the system (waiting or in service) at time t. We can control the number-in-system process $\{L(t)\}_{t \geq 0}$ through a *policy* that sets the current *stochastic intensities*, $\lambda(t)$ and $\mu(t)$, of the *point processes* (see, e.g., Baccelli and Brémaud (1994)), modelling customer arrivals and departures, respectively. To be *admissible*, such a policy must: (1) be *adapted* to the current system's history $\mathcal{F}_t^L = \sigma\{L(\tau), \tau \in [0, t]\}$; (2) induce a *stable* number-in-system process; and (3) satisfy the sample-path *capacity constraints* $\lambda(t) \leq \bar{\lambda}_{L(t)}$ and $\mu(t) \leq \bar{\mu}_{L(t)}$, where $\{\bar{\lambda}_k\}_{k \geq 0}$ and $\{\bar{\mu}_k\}_{k \geq 0}$ are given sequences of *positive* input and output capacity limits (except for $\bar{\mu}_0 = 0$). We denote by $\mathcal{U}$ the class of admissible policies.

We present next two sets of conditions on capacity limits which, as we shall see, are essential for casting the model in the LP framework of Section 2. They are related to, but not identical, to the conditions assumed by Chen and Yao (see Remarks below). Let us denote the marginal input and output capacities, respectively, by $\Delta\bar{\lambda}_k = \bar{\lambda}_k - \bar{\lambda}_{k-1}$ and $\Delta\bar{\mu}_k = \bar{\mu}_k - \bar{\mu}_{k-1}$, for $k \geq 1$ (notice that $\Delta\bar{\mu}_1 = \bar{\mu}_1$).

Assumption 2 (Vanishing Marginal Output Capacity) *Marginal output capacity satisfies*
(i) $0 \leq \cdots \leq \Delta\bar{\mu}_3 \leq \Delta\bar{\mu}_2 \leq \Delta\bar{\mu}_1 = \bar{\mu}_1 > 0$;
(ii) $\lim_{k \to \infty} \Delta\bar{\mu}_k = 0$.

Assumption 3 (Decreasing Input Capacity) *Input capacity is decreasing on the number of customers in the system:*

$$\bar{\lambda}_0 \geq \bar{\lambda}_1 \geq \bar{\lambda}_2 \geq \cdots > 0.$$

Remarks:

1. Assumption 2(i) is condition (5.5b) in Chen and Yao. We add Assumption 2(ii) so that problem (11) below can be solved within our LP framework for *any* cost rate $c > 0$.
2. Our Assumption 3, combined with Assumption 2(i), is *stronger* than condition (5.5a) of Chen and Yao ($\Delta\bar{\lambda}_k \leq \Delta\bar{\mu}_k$). However, we need our stronger assumption to establish the Threshold Property.

Time-discounted optimal control problem

We consider the following economic structure: A unit (dollar) reward is received at the service completion epoch of a customer. Furthermore, a customer in the system (waiting or in service), incurs holding costs at a rate of $c > 0$ dollars per unit time. Rewards and costs are continuously discounted in time with discount factor $\alpha > 0$.

For a given policy $u \in \mathcal{U}$ and initial state $L(0) = k$, we denote by $\tilde{V}^{k,u}(c)$ the corresponding objective value, representing the total expected discounted value of rewards earned minus costs incurred over an infinite horizon. The optimal control problem of interest consists in finding an admissible policy that maximizes such objective value:

$$\tilde{V}^{k,*}(c) = \max\{\tilde{V}^{k,u}(c) : u \in \mathcal{U}\}. \tag{11}$$

Time-discounted performance measures and problem formulation

We introduce next two natural performance measures for formulating problem (11). For a given initial state $L(0) = k \geq 0$ and policy $u \in \mathcal{U}$, we define $\tilde{\mu}^{k,u}$ as the corresponding total expected discounted number of service completions (scaled by α):

$$\tilde{\mu}^{k,u} = \alpha\, E_u \left[\int_0^\infty \mu(t)\, e^{-\alpha t}\, dt \mid L(0) = k \right].$$

We further define $\tilde{L}^{k,u}$ as the total expected discounted number of customers in the system (scaled by α):

$$\tilde{L}^{k,u} = \alpha\, E_u \left[\int_0^\infty L(t)\, e^{-\alpha t}\, dt \mid L(0) = k \right].$$

The corresponding (scaled) marginal performance measures are denoted by

$$\Delta\tilde{\mu}^{k,u} = \frac{1}{\alpha}\, \tilde{\mu}^{k,u} - \tilde{\mu}^{k-1,u},$$

and

$$\Delta\tilde{L}^{k,u} = \frac{1}{\alpha}\, \tilde{L}^{k,u} - \tilde{L}^{k-1,u}, \qquad \text{for } k \geq 1.$$

We can thus reformulate optimal control problem (11) as

$$\tilde{V}^{k,*}(c) = \max\{\tilde{\mu}^{k,u} - c\,\tilde{L}^{k,u} : u \in \mathcal{U}\}. \tag{12}$$

Threshold policies

We shall consider the following family of threshold policies: For each integer critical threshold value $b \geq 0$, the b-*threshold policy* sets the input intensity at full capacity if $L(t) < b$; otherwise, input intensity is set to 0. Output intensity is always set at full capacity. We denote by $(\tilde{\mu}^{k,b}, \tilde{L}^{k,b})$ the performance pair achieved by such b-threshold policy, with $L(0) = k$, and write its objective value as

$$\tilde{V}^{k,b}(c) = \tilde{\mu}^{k,b} - c\,\tilde{L}^{k,b}.$$

We further define the corresponding (scaled) marginal value function as

$$\tilde{v}^{k,b}(c) = \frac{1}{\alpha}\left(\tilde{V}^{k,b}(c) - \tilde{V}^{k-1,b}(c)\right) = \Delta\tilde{\mu}^{k,b} - c\,\Delta\tilde{L}^{k,b}, \qquad \text{for } k \geq 1, b \geq 0.$$

These functions are characterized by the recursions given in Lemmas 7 and 8 in Appendix A.

In addition, we define u_0 to be the policy that lets no customers in nor out of the system, so that $L(t) = L(0)$ for $t \geq 0$.

4 Model analysis via the Threshold Property

We show in this section how to formulate control problem (12) in the LP framework of Section 2, and the results that follow from such formulation.

Let us define the *critical cost parameters* $c_\alpha^{k,0}$ and $\boldsymbol{c}_\alpha = \{c_\alpha^b\}_{b \geq 1}$ by

$$c_\alpha^{k,0} = \begin{cases} -\alpha & \text{if } k \geq 1 \\ 0 & \text{if } k = 0, \end{cases} \tag{13}$$

and

$$c_\alpha^b = \frac{\Delta\tilde{\mu}^{b,b}}{\Delta\tilde{L}^{b,b}} \qquad \text{for } b \geq 1. \tag{14}$$

As we shall see, sequence $(c^{k,0}, \boldsymbol{c}_\alpha)$ corresponds precisely to the definition of $\boldsymbol{c}$ given in (2). This is so because of certain invariance properties on the system's initial state, proven in Lemma 3. We show in Figure 2 a recursive algorithm for computing the c^b's.

We present next the main result of this section, which says that, under Assumptions 2 and 3, $(\tilde{\mu}^{k,u}, \tilde{L}^{k,u})$ satisfies the Threshold Property.

Theorem 2 (Threshold Property: Time-discounted Case) *Suppose capacity limits satisfy Assumptions 2 and 3. Then, for any $L(0) = k \geq 0$, performance pair $(\tilde{\mu}^{k,u}, \tilde{L}^{k,u})$ satisfies the Threshold Property:*
(a) *the performance pairs of threshold policies, $\{(\tilde{\mu}^{k,b}, \tilde{L}^{k,b})\}_{b \geq 0}$, satisfy Assumption 1;*
(b) *under* any *admissible policy $u \in \mathcal{U}$,*
(b.1) $\tilde{\mu}^{k,u} - c_\alpha^{k,0}\, \tilde{L}^{k,u} \geq \tilde{V}^{k,0}(c_\alpha^{k,0})$;
(b.2) $\tilde{\mu}^{k,u} - c_\alpha^{b}\, \tilde{L}^{k,u} \leq \tilde{V}^{k,b}(c_\alpha^{b})$, *for $b \geq 1$.*

Proof
Part (a) is proven in Proposition 2, in Section 5. Part (b) is precisely Corollary 9 in Section 6. □

We present next some important consequences of Theorem 6. Let us consider the model's throughput-WIP achievable performance region

$$\mathcal{X}_\alpha^k = \{(\tilde{\mu}^{k,u}, \tilde{L}^{k,u}) : u \in \mathcal{U}\},$$

and the polygon

$$\mathcal{P}(c_\alpha^{k,0}, \boldsymbol{c}_\alpha, \boldsymbol{d}_\alpha^k) = \left\{(\mu, L) \geq \boldsymbol{0} : \mu - c_\alpha^{k,0}\, L \geq d_\alpha^{k,0} \quad \text{and} \quad \mu - c_\alpha^{b}\, L \leq d_\alpha^{k,b}, \quad \text{for } b \geq 1\right\},$$

where sequence $\boldsymbol{d}_\alpha^k = \{d_\alpha^{k,b}\}_{b\geq 0}$ is defined, analogously as in (3), by

$$d_\alpha^{k,b} = \begin{cases} \tilde{V}^{k,0}(c_\alpha^{k,0}) & \text{if } b = 0 \\ \tilde{V}^{k,b}(c_\alpha^{b}) & \text{if } b \geq 1. \end{cases}$$

Since the Threshold Property holds, Theorem 1 applies, giving an exact semi-infinite LP formulation of optimal control problem (12).

Corollary 3 (Semi-infinite LP Formulation) *Under Assumptions 2 and 3, $\mathcal{P}(c_\alpha^{k,0}, \boldsymbol{c}_\alpha, \boldsymbol{d}_\alpha^k)$ is a threshold polygon, whose vertex set includes $\{(\tilde{\mu}^{k,b}, \tilde{L}^{k,b}), b \geq 0\}$. Furthermore,*

$$\tilde{V}^{k,*}(c) = \max\{\mu - c\, L : (\mu, L) \in \mathcal{P}(c_\alpha^{k,0}, \boldsymbol{c}_\alpha, \boldsymbol{d}_\alpha^k)\}.$$

Furthermore, Corollary 1 of the LP framework yields the characterization of the optimal threshold policy we present next. Let us define the *critical threshold function* $b_\alpha^*(\cdot)$ by

$$b_\alpha^*(c) = \min\left\{b \geq 0 : c_\alpha^{b+1} \leq c\right\}, \qquad \text{for } c > 0.$$

Corollary 4 (Threshold Optimality) *Under Assumptions 2 and 3 on capacity limits, control problem (12) is solved optimally by the $b_\alpha^*(c)$-threshold policy, for $c > 0$.*

In addition, Corollary 2 gives an analytical characterization of the *Min WIP* function $\tilde{L}_{\min}^k(\cdot)$, defined by

$$\tilde{L}_{\min}^k(\mu) = \min\{\tilde{L}^{k,u} : \tilde{\mu}^{k,u} = \mu, u \in \mathcal{U}\} = \min\{L : (\mu, L) \in \mathcal{X}_\alpha^k\}.$$

Corollary 5 (Min WIP Characterization)

$$\tilde{L}_{\min}^k(\mu) = \tilde{L}^{k,b-1} + \frac{1}{c_\alpha^b}\,(\mu - \tilde{\mu}^{k,b-1}), \qquad \textit{for } \mu \in \left[\tilde{\mu}^{k,b-1}, \tilde{\mu}^{k,b}\right],\ b \geq 1.$$

In the following two sections we develop the results that lead to the Threshold Property in Theorem 6.

5 Diminishing returns under threshold policies

We present in this section several properties of throughput-WIP performance measures under threshold policies, which are essential to our proof of the Threshold Property in Theorem 6. They say, in short, that the marginal throughput performance level is subject to *diminishing returns* with respect to the marginal WIP level, both on the critical threshold value and on the initial state.

First we present a preliminary result which establishes that our definition of the critical cost parameter sequence $(c_\alpha^k, \boldsymbol{c}_\alpha)$ in (13) and (22) is consistent with that of $\boldsymbol{c}$ in the general framework, as defined in (2). Notice that these critical cost parameters represent the successive relative marginal throughput/WIP performance levels, as the threshold value varies.

Lemma 3 (Invariance on Initial State) *Suppose Assumptions 2 and 3 hold. Then, for* $b \geq 1$*:*
(a) *For* $k \geq 0$, $\tilde{\mu}^{k,b+1} > \tilde{\mu}^{k,b}$ *and* $\tilde{L}^{k,b+1} > \tilde{L}^{k,b}$. *Furthermore,*

$$\frac{\Delta\tilde{\mu}^{b+1,b+1}}{\Delta\tilde{L}^{b+1,b+1}} = \frac{\tilde{\mu}^{k,b+1} - \tilde{\mu}^{k,b}}{\tilde{L}^{k,b+1} - \tilde{L}^{k,b}}, \qquad \text{for } k \geq 0;$$

(b) $\frac{1-\Delta\tilde{\mu}^{k,0}}{\Delta\tilde{L}^{k,0}} = \alpha, \qquad$ *for* $k \geq 1$*;*
(c) $\frac{\tilde{\mu}^{k,0}}{k-\tilde{L}^{k,0}} = \alpha, \qquad$ *for* $k \geq 1$.

Proof
Part (a) is precisely Corollary 6(c).
(b) By Lemma 7(b, c) in Appendix A we have

$$\tilde{\mu}^{1,0} = \frac{\alpha\,\bar{\mu}_1}{\bar{\mu}_1 + \alpha} \qquad \text{and} \qquad \tilde{L}^{1,0} = \frac{\alpha}{\bar{\mu}_1 + \alpha}.$$

Since $\Delta\tilde{\mu}^{1,0} = \tilde{\mu}^{1,0}/\alpha$ and $\Delta\tilde{L}^{1,0} = \tilde{L}^{1,0}/\alpha$, this proves the result for $k = 1$. For $k \geq 2$ we have, by Lemma 8(b, c),

$$(\bar{\mu}_k + \alpha)\,\alpha\,\Delta\tilde{L}^{k,0} = \bar{\mu}_{k-1}\,\alpha\,\Delta\tilde{L}^{k-1,0} + \alpha$$

and

$$(\bar{\mu}_k + \alpha)\,\Delta\tilde{\mu}^{k,0} = \bar{\mu}_{k-1}\,\Delta\tilde{\mu}^{k-1,0} + \Delta\bar{\mu}_k.$$

Furthermore, since

$$(\bar{\mu}_k + \alpha) = \bar{\mu}_{k-1} + \Delta\bar{\mu}_k + \alpha,$$

it follows that

$$(\bar{\mu}_k + \alpha)\,(1 - \Delta\tilde{\mu}^{k,0}) = \bar{\mu}_{k-1}\,(1 - \Delta\tilde{\mu}^{k-1,0}) + \alpha.$$

Thus sequences $\{\alpha\,\Delta\tilde{L}^{k,0}\}_{k\geq 1}$ and $\{1 - \Delta\tilde{\mu}^{k,0}\}_{k\geq 1}$ are generated by the same recursion. Since we have shown that $\alpha\,\Delta\tilde{L}^{1,0} = 1 - \Delta\tilde{\mu}^{1,0}$, it follows that the two sequences are identical, proving the result.

(c) From the expressions for $\tilde{\mu}^{1,0}$ and $\tilde{L}^{1,0}$ in the proof of part (b), it follows that the result holds for $k = 1$. For $k \geq 2$ we have, by Lemma 7(b, c),

$$(\bar{\mu}_k + \alpha)\,\tilde{\mu}^{k,0} = \bar{\mu}_k\,\tilde{\mu}^{k-1,0} + \alpha\,\bar{\mu}_k$$

and

$$(\bar{\mu}_k + \alpha)\,\tilde{L}^{k,0} = \bar{\mu}_k\,\tilde{L}^{k-1,0} + \alpha\,k.$$

Since

$$(\bar{\mu}_k + \alpha)\,k = \bar{\mu}_k\,(k-1) + \alpha\,k + \bar{\mu}_k,$$

it follows that

$$(\bar{\mu}_k + \alpha)\,\alpha\,(k - \tilde{L}^{k,0}) = \bar{\mu}_k\,\alpha\,(k - 1 - \tilde{L}^{k-1,0}) + \alpha\,\bar{\mu}_k.$$

Thus sequences $\{\tilde{\mu}^{k,0}\}_{k\geq 1}$ and $\{\alpha\,(k - \tilde{L}^{k,0})\}_{k\geq 1}$ are defined by the same recursion and, since $\tilde{\mu}^{1,0} = \alpha\,(1 - \tilde{L}^{1,0})$, it follows they are identical, which completes the proof. □

We next establish the first part of the Threshold Property in Theorem 6. This includes the result that the relative marginal throughput/WIP performance is subject to diminishing and vanishing returns on the threshold value.

Proposition 2 *Under Assumptions 2 and 3, the performance pairs of threshold policies* $\{(\tilde{\mu}^{k,b}, \tilde{L}^{k,b})\}_{b=0}^{\infty}$ *satisfy Assumption 1, for* $k \geq 0$*:*
(a)[Strict Monotonicity (increasing) of $\tilde{\mu}^{k,b}$]

$$\tilde{\mu}^{k,b} < \tilde{\mu}^{k,b+1}, \qquad \text{for } b \geq 0;$$

(b)[Strict Monotonicity (Increasing) of $\tilde{L}^{k,b}$]

$$\tilde{L}^{k,b} < \tilde{L}^{k,b+1}, \quad \text{for } b \geq 0;$$

(c)[Diminishing Marginal Returns of $\tilde{\mu}^{k,b}$ with respect to $\tilde{L}^{k,b}$]

$$\frac{\tilde{\mu}^{k,b+1} - \tilde{\mu}^{k,b}}{\tilde{L}^{k,b+1} - \tilde{L}^{k,b}} < \frac{\tilde{\mu}^{k,b} - \tilde{\mu}^{k,b-1}}{\tilde{L}^{k,b} - \tilde{L}^{k,b-1}}, \qquad \text{for } b \geq 1;$$

(d)[Vanishing Marginal Returns of $\tilde{\mu}^{k,b}$ with respect to $\tilde{L}^{k,b}$]

$$\lim_{b \to \infty} \frac{\tilde{\mu}^{k,b} - \tilde{\mu}^{k,b-1}}{\tilde{L}^{k,b} - \tilde{L}^{k,b-1}} = 0;$$

(e) *there is a policy* $u_0 \in \mathcal{U}$ *such that*

$$\tilde{L}^{k,u_0} = \min \{\tilde{L}^{k,u} : \tilde{\mu}^{k,u} = 0, u \in \mathcal{U}\},$$

and

$$\tilde{L}^{k,u_0} \geq \tilde{L}^{k,0},$$

with

$$\tilde{L}^{k,u_0} > \tilde{L}^{k,0} \quad \textit{if and only if} \quad 0 = \tilde{\mu}^{k,u_0} < \tilde{\mu}^{k,0}.$$

Proof

(a) The result follows from Corollary 6(c) combined with Lemma 9(b).

(b) Similarly, this follows from Corollary 6(c) combined with Lemma 9(c).

(c) In light of Lemma 3, we need to show that

$$\frac{\Delta\tilde{\mu}^{b+1,b+1}}{\Delta\tilde{L}^{b+1,b+1}} < \frac{\Delta\tilde{\mu}^{b,b}}{\Delta\tilde{L}^{b,b}}, \qquad \text{for } b \geq 1.$$

This follows since

$$\frac{\Delta\tilde{\mu}^{b+1,b+1}}{\Delta\tilde{L}^{b+1,b+1}} = \frac{\Delta\tilde{\mu}^{b,b} + \frac{\Delta\bar{\mu}_{b+1}}{\bar{\mu}_b}}{\Delta\tilde{L}^{b,b} + \frac{1}{\bar{\mu}_b}} \tag{15}$$

$$< \frac{\Delta\tilde{\mu}^{b,b}}{\Delta\tilde{L}^{b,b}}, \tag{16}$$

where identity (15) follows from Lemma 9, and inequality (16) follows from Lemma 13(b) since, by Lemma 14 and Assumption 2(i) we have

$$\Delta\bar{\mu}_{b+1} \leq \Delta\bar{\mu}_b < \frac{\Delta\tilde{\mu}^{b,b}}{\Delta\tilde{L}^{b,b}}.$$

(d) This follows from Lemma 3 and Lemma 11(c). This is proven in Lemma 11(c) below.

(e) Take as u_0 the policy defined above that lets no customers in nor out of the system, so that $L(t) = L(0)$ for $t \geq 0$. It is trivial to check that this policy satisfies the stated conditions. □

Our next result shows that the relative marginal throughput/WIP level is subject to diminishing returns on the system's initial state. This property is essential for threshold optimality, as we will see in the next section.

Lemma 4 (Diminishing Returns on Initial State) *Under Assumptions 2 and 3,*

(a)

$$\frac{\Delta\tilde{\mu}^{b,b}}{\Delta\tilde{L}^{b,b}} < \frac{\Delta\tilde{\mu}^{k,b}}{\Delta\tilde{L}^{k,b}}, \qquad \textit{for } 1 \leq k \leq b-1;$$

(b)

$$\frac{\Delta\tilde{\mu}^{k+1,b}}{\Delta\tilde{L}^{k+1,b}} < \frac{\Delta\tilde{\mu}^{k,b}}{\Delta\tilde{L}^{k,b}}, \qquad \textit{for } k \geq b-1.$$

Proof

(a) We use induction on $b \geq 1$. The case $b = 1$ is trivial. Now, suppose for some $b \geq 2$

$$\frac{\Delta\tilde{\mu}^{b-1,b-1}}{\Delta\tilde{L}^{b-1,b-1}} < \frac{\Delta\tilde{\mu}^{k,b-1}}{\Delta\tilde{L}^{k,b-1}}, \qquad \text{for } 1 \leq b-2.$$

Let $1 \leq k \leq b-2$. We have, by Lemma 3(a), Proposition 2(c) and the induction hypothesis, that

$$\begin{aligned} \frac{\Delta\tilde{\mu}^{k,b} - \Delta\tilde{\mu}^{k,b-1}}{\Delta\tilde{L}^{k,b} - \Delta\tilde{L}^{k,b-1}} &= \frac{\Delta\tilde{\mu}^{b,b}}{\Delta\tilde{L}^{b,b}} \\ &< \frac{\Delta\tilde{\mu}^{b-1,b-1}}{\Delta\tilde{L}^{b-1,b-1}} \\ &< \frac{\Delta\tilde{\mu}^{k,b-1}}{\Delta\tilde{L}^{k,b-1}}. \end{aligned}$$

Therefore, by Lemma 13(a),

$$\frac{\Delta\tilde{\mu}^{b,b}}{\Delta\tilde{L}^{b,b}} < \frac{\Delta\tilde{\mu}^{k,b}}{\Delta\tilde{L}^{k,b}},$$

which completes the induction proof.

(b) We use induction on $k \geq b-1$. The case $k = b-1$ was proven in part (a). Suppose now that, for some $k \geq b-1$,

$$\frac{\Delta\tilde{\mu}^{k+1,b}}{\Delta\tilde{L}^{k+1,b}} < \frac{\Delta\tilde{\mu}^{k,b}}{\Delta\tilde{L}^{k,b}}.$$

Now, for such a k Lemma 8(b, c) yields

$$(\bar{\mu}_{k+1} + \alpha)\,\Delta\tilde{\mu}^{k+1,b} = \bar{\mu}_k\,\Delta\tilde{\mu}^{k,b} + \Delta\bar{\mu}_{k+1},$$

and

$$(\bar{\mu}_{k+1} + \alpha)\,\Delta\tilde{L}^{k+1,b} = \bar{\mu}_k\,\Delta\tilde{L}^{k,b} + 1.$$

We can thus rewrite the induction hypothesis as

$$\frac{\Delta\tilde{\mu}^{k+1,b}}{\Delta\tilde{L}^{k+1,b}} < \frac{(\bar{\mu}_{k+1} + \alpha)\,\Delta\tilde{\mu}^{k+1,b} - \Delta\bar{\mu}_{k+1}}{(\bar{\mu}_{k+1} + \alpha)\,\Delta\tilde{L}^{k+1,b} - 1}.$$

Now, this inequality is equivalent, by Lemma 13(a), to

$$\Delta\bar{\mu}_{k+1} < \frac{\Delta\tilde{\mu}^{k+1,b}}{\Delta\tilde{L}^{k+1,b}}.$$

Since, by Assumption 2(i), $\Delta\bar{\mu}_{k+1} \leq \Delta\bar{\mu}_k$, it follows hat

$$\Delta\bar{\mu}_{k+2} < \frac{\Delta\tilde{\mu}^{k+1,b}}{\Delta\tilde{L}^{k+1,b}},$$

which, by Lemma 13(b), is equivalent to the inequality

$$\frac{\Delta\tilde{\mu}^{k+2,b}}{\Delta\tilde{L}^{k+2,b}} = \frac{\bar{\mu}_{k+1}\,\Delta\tilde{\mu}^{k+1,b} + \Delta\bar{\mu}_{k+2}}{\bar{\mu}_{k+1}\,\Delta\tilde{L}^{k+1,b} + 1} \leq \frac{\Delta\tilde{\mu}^{k+1,b}}{\Delta\tilde{L}^{k+1,b}}.$$

This completes the proof. □

6 LP value decomposition, optimality conditions and threshold optimality

To establish the Threshold Property we still need to prove the optimality of threshold policies for the corresponding sequence of critical cost parameters (Theorem 6(b)). This section is devoted to accomplishing such task. We shall present sufficient optimality conditions for threshold policies, and then show that the critical cost parameters satisfy them.

The first optimality conditions we present are those first given by Chen and Yao (in their Theorem 4.1). Our approach to their derivation is, however, radically different, and yields new insights: While their proof is based on ad hoc probabilistic arguments, ours reveals that those conditions are precisely the standard LP optimality conditions. Consider the (infinite dimensional) LP problem:

$$\begin{aligned} V^{LP,k} \quad = \quad & \max \sum_{l=1}^{\infty} \mu_l - c \sum_{l=1}^{\infty} l\,p_l \\ & \text{subject to} \\ & \lambda_l - \bar{\lambda}_l\,p_l \leq 0, \quad \text{for } l \geq 0 : \pi_l^{\lambda} \\ & \mu_l - \bar{\mu}_l\,p_l \leq 0, \quad \text{for } l \geq 1 : \pi_l^{\mu} \\ & \mu_{l+1} - \lambda_l + \alpha \sum_{m=l+1}^{\infty} p_m = \alpha\,1\{l < k\}, \quad \text{for } l \geq 0 : v^{l+1} \\ & \sum_{l=0}^{\infty} p_l = 1 : V^0 \\ & p_l, \lambda_l, \mu_{l+1} \geq 0, \qquad \text{for } l \geq 0. \end{aligned} \tag{17}$$

The importance of LP problem (17) lies in the fact that, as we will see, it represents an *exact* LP formulation of optimal control problem (12). The simpler result that LP (17) is a *relaxation* of (12) follows by letting the variables of that LP correspond to the *auxiliary performance measures* defined by

- $\tilde{p}_l^{k,u} = \alpha\, E_u \left[\int_0^\infty 1\{L(t) = l\}\, e^{-\alpha\, t}\, dt \mid L(0) = k \right]$;
- $\tilde{\lambda}_l^{k,u} = \alpha\, E_u \left[\int_0^\infty \lambda(t)\, 1\{L(t) = l\}\, e^{-\alpha\, t}\, dt \mid L(0) = k \right]$;
- $\tilde{\mu}_l^{k,u} = \alpha\, E_u \left[\int_0^\infty \mu(t)\, 1\{L(t) = l\}\, e^{-\alpha\, t}\, dt \mid L(0) = k \right]$,

and then applying Lemma 16 in Appendix A. We have indicated beside each constraint of LP problem (17) the name of the corresponding dual variable. The dual LP problem is

$$
\begin{aligned}
V^{D,k} \quad = \quad & \min\, V^0 + \alpha \sum_{0 \le l < k} v^{l+1} \\
& \text{subject to} \\
& V^0 - \bar{\lambda}_0\, \pi_0^\lambda \ge 0 : p_0 \\
& V^0 - \bar{\lambda}_l\, \pi_l^\lambda - \bar{\mu}_l\, \pi_l^\mu + \alpha\, (v^1 + \cdots + v^l) \ge -c\, l, \quad \text{for } l \ge 1 : p_l \\
& \pi_l^\lambda - v^{l+1} \ge 0, \quad \text{for } l \ge 0 : \lambda_l \\
& \pi_l^\mu + v^l \ge 1, \quad \text{for } l \ge 1 : \mu_l \\
& \pi_l^\lambda, \pi_l^\mu \ge 0.
\end{aligned}
\tag{18}
$$

Now, from Lemma 8 it is easily seen that, for any threshold value $b \ge 0$, we can define a *feasible* solution $\{(v^{l,b}, \pi_l^{\lambda,b}, \pi_{l+1}^{\mu,b})\}_{l=0}^\infty$ for dual LP (18), by letting $v^{l,b} = \tilde{v}^{l,b}$,

$$\pi_l^{\lambda,b} = \tilde{v}^{l+1,b}\, 1\{l < b\}$$

and

$$\pi_{l+1}^{\mu,b} = 1 - \tilde{v}^{l+1,b}.$$

On the other hand, we further know that $\tilde{V}^{0,b}, \{(\tilde{p}_l^{k,b}, \tilde{\lambda}_l^{k,b}, \mu_{l+1}^{k,b})\}_{l=0}^\infty$ and $\tilde{V}^{0,u}, \{(\tilde{p}_l^{k,u}, \tilde{\lambda}_l^{k,u}, \mu_{l+1}^{k,u})\}_{l=0}^\infty$ are feasible solutions for primal LP (17), for any policy $u \in \mathcal{U}$.

These facts lead to our LP Value Decomposition Theorem, presented next, which gives an exact relation between the value function $\tilde{V}^{k,u}$, under policy $u \in \mathcal{U}$, and the value $\tilde{V}^{k,b}$ under the b-threshold policy.

Theorem 5 (LP Value Decomposition) *For any admissible policy $u \in \mathcal{U}$, initial state $L(0) = k \ge 0$ and threshold value $b \ge 0$, the following identities hold:*
(a)

$$\tilde{V}^{k,u} = \tilde{V}^{k,b} + \sum_{l=b}^\infty \tilde{v}^{l+1,b}\, \tilde{\lambda}_l^{k,u} + \sum_{0 \le l < b} \tilde{v}^{l+1,b}\, (\tilde{\lambda}_l^{k,u} - \bar{\lambda}_l\, \tilde{p}_l^{k,u}) + \sum_{l=1}^\infty (1 - \tilde{v}^{l,b})\, (\tilde{\mu}_l^{k,u} - \bar{\mu}_l\, \tilde{p}_l^{k,u});$$

(b)

$$\tilde{\mu}^{k,u} = \tilde{\mu}^{k,b} + \sum_{l=b}^\infty \Delta\tilde{\mu}^{l+1,b}\, \tilde{\lambda}_l^{k,u} + \sum_{0 \le l < b} \Delta\tilde{\mu}^{l+1,b}\, (\tilde{\lambda}_l^{k,u} - \bar{\lambda}_l\, \tilde{p}_l^{k,u}) + \sum_{l=1}^\infty (1 - \Delta\tilde{\mu}^{l,b})\, (\tilde{\mu}_l^{k,u} - \bar{\mu}_l\, \tilde{p}_l^{k,u});$$

(c)

$$\tilde{L}^{k,u} = \tilde{L}^{k,b} + \sum_{l=b}^\infty \Delta\tilde{L}^{l+1,b}\, \tilde{\lambda}_l^{k,u} + \sum_{0 \le l < b} \Delta\tilde{L}^{l+1,b}\, (\tilde{\lambda}_l^{k,u} - \bar{\lambda}_l\, \tilde{p}_l^{k,u}) - \sum_{l=1}^\infty \Delta\tilde{L}^{l,b}\, (\tilde{\mu}_l^{k,u} - \bar{\mu}_l\, \tilde{p}_l^{k,u}).$$

Proof
(a) We have

$$\sum_{l=0}^\infty \pi_l^{\lambda,b}\, (\tilde{\lambda}_l^{k,u} - \bar{\lambda}_l\, \tilde{p}_l^{k,u})$$

$$
\begin{aligned}
+\sum_{l=1}^{\infty} \pi_l^{\mu,b}\,(\tilde{\mu}_l^{k,u} - \bar{\mu}_l\,\tilde{p}_l^{k,u}) &= \sum_{l=0}^{\infty} \pi_l^{\lambda,b}\,(\tilde{\lambda}_l^{k,u} - \bar{\lambda}_l\,\tilde{p}_l^{k,u}) + \sum_{l=1}^{\infty} \pi_l^{\mu,b}\,(\tilde{\mu}_l^{k,u} - \bar{\mu}_l\,\tilde{p}_l^{k,u}) \\
&\quad + \tilde{V}^{0,b}\left[\sum_{l=0}^{\infty} \tilde{p}_l^{k,u} - 1\right] \\
&\quad + \sum_{l=0}^{\infty} \tilde{v}^{l+1,b}\left[\tilde{\mu}_{l+1}^{k,u} - \tilde{\lambda}_l^{k,u} + \alpha \sum_{m=l+1}^{\infty} \tilde{p}_m^{k,u} - \alpha\, 1\{l < k\}\right] \\
&= \tilde{p}_0^{k,u}\left[\tilde{V}^{0,b} - \bar{\lambda}_0\,\pi_0^{\lambda,b}\right] \\
&\quad + \sum_{l=1}^{\infty} \tilde{p}_l^{k,u}\left[\tilde{V}^{0,b} - \bar{\lambda}_l\,\pi_l^{\lambda,b} - \bar{\mu}_l\,\pi_l^{\mu,b} + \alpha \sum_{m=1}^{l} \tilde{v}^{m,b} + c\,k\right] \\
&\quad + \sum_{l=0}^{\infty} \tilde{\lambda}_l^{k,u}\,(\pi_l^{\lambda,b} - \tilde{v}^{l+1,b}) + \sum_{l=1}^{\infty} \tilde{\mu}^{l,u}\,(\pi_l^{\mu,b} + \tilde{v}^{l,b} - 1) \\
&\quad - \tilde{V}^{k,b} + \sum_{l=1}^{\infty} \tilde{\mu}^{l,u} - c \sum_{l=1}^{\infty} k\,\tilde{p}^{l,u} \qquad (19) \\
&= -\sum_{l=b}^{\infty} \tilde{v}^{l+1,b}\,\tilde{\lambda}_l^{k,u} - \tilde{V}^{k,b} + \tilde{V}^{k,u}, \qquad (20)
\end{aligned}
$$

which is the required identity. Notice that identity (19) follows by rearranging terms and noticing

$$\tilde{V}^{0,b} + \alpha \sum_{l=0}^{\infty} \tilde{v}^{l+1,b}\, 1\{l < k\} = \tilde{V}^{k,b},$$

and final identity (20) follows from

$$\tilde{V}^{0,b} - \bar{\lambda}_0\,\pi_0^{\lambda,b} = 0$$

and

$$\tilde{V}^{0,b} - \bar{\lambda}_l\,\pi_l^{\lambda,b} - \bar{\mu}_l\,\pi_l^{\mu,b} + \alpha \sum_{m=1}^{l} \tilde{v}^{m,b} + c\,k = 0,$$

which is a consequence of Lemma 7(a).

Parts (b) and (c) follow directly from (a). □

Theorem 5 gives directly recursions relating throughput and WIP performance measures under successive threshold policies, as stated in our next result. Let function $g(b)$ be as defined in (30).

Corollary 6 (Throughput-WIP Recursions and Invariance) *For any initial state* $L(0) = k \geq 0$ *and threshold value* $b \geq 0$,
(a)[Throughput Recursion]

$$
\begin{aligned}
\tilde{\mu}^{k,b+1} &= \tilde{\mu}^{k,b} + \Delta\tilde{\mu}^{b+1,b}\,\tilde{\lambda}_b^{k,b+1} \\
&= \tilde{\mu}^{k,b} + \frac{\bar{\lambda}\,\bar{\mu}_b\,g(b)}{\bar{\mu}_{b+1} + \alpha}\,\Delta\tilde{\mu}^{b+1,b+1}\,\tilde{p}_b^{k,b+1};
\end{aligned}
$$

(b)[WIP Recursion]

$$
\begin{aligned}
\tilde{L}^{k,b+1} &= \tilde{L}^{k,b} + \Delta\tilde{L}^{b+1,b}\,\tilde{\lambda}_b^{k,b+1} \\
&= \tilde{L}^{k,b} + \frac{\bar{\lambda}_b\,\bar{\mu}_b\,g(b)}{\bar{\mu}_{b+1} + \alpha}\,\Delta\tilde{L}^{b+1,b+1}\,\tilde{p}_b^{k,b+1};
\end{aligned}
$$

(c)[Invariance] *suppose Assumptions 2 and 3 hold; then, for* $k \geq 0$, $\tilde{\mu}^{k,b+1} > \tilde{\mu}^{k,b}$ *and* $\tilde{L}^{k,b+1} > \tilde{L}^{k,b}$*; furthermore,*

$$\frac{\Delta\tilde{\mu}^{b+1,b+1}}{\Delta\tilde{L}^{b+1,b+1}} = \frac{\tilde{\mu}^{k,b+1} - \tilde{\mu}^{k,b}}{\tilde{L}^{k,b+1} - \tilde{L}^{k,b}}.$$

Proof
(a, b) The first identity in both (a) and (b) follows directly from Theorem 5. The second identity follows from equation (32) in Appendix A.

(c) In Appendix A it is shown that, under Assumptions 2 and 3, $g(b) > 0$, $\Delta\tilde{\mu}^{b,b} > 0$ and $\Delta\tilde{L}^{b,b} > 0$, for $b \geq 1$ (see Lemma 9). The invariance property stated follows directly from these results, together with parts (a, b) and the fact that $\tilde{p}_b^{k,b+1} > 0$, which is easily seen to hold from its probabilistic interpretation. □

The sufficient optimality conditions of Chen and Yao (1990) follow directly from Theorem 5: They are simply the LP optimality conditions for *basic feasible solution* $\tilde{V}^{0,b}, \{(\tilde{p}_l^{k,b}, \tilde{\lambda}_l^{k,b}, \mu_{l+1}^{k,b})\}_{l=0}^{\infty}$ of LP (17), namely, the reduced costs of non-basic variables are nonpositive.

Corollary 7 (Optimality Conditions; Chen and Yao (1990)) *Suppose for a given* $b^* \geq 0$ *the marginal value function of the* b^**-threshold policy satisfies the following conditions:*
(i) $0 \leq \tilde{v}^{l,b^*} \leq 1$, *for* $1 \leq l \leq b^*$*;*
(ii) $\tilde{v}^{k,b^*} \leq 0$, *for* $l \geq b^* + 1$*;*
or, equivalently, the cost rate $c > 0$ *satisfies*
(i') $\frac{\Delta\tilde{\mu}^{l,b^*-1}}{\Delta\tilde{L}^{l,b^*}} \leq c \leq \frac{\Delta\tilde{\mu}^{l,b^*}}{\Delta\tilde{L}^{l,b^*}}$, *for* $1 \leq l \leq b^*$*;*
(ii') $\frac{\Delta\tilde{\mu}^{l,b^*}}{\Delta\tilde{L}^{l,b^*}} \leq c$, *for* $l \geq b^* + 1$.
Then the b^**-threshold policy is optimal for control problem (12) .*

Proof
Let $u \in \mathcal{U}$ be an arbitrary admissible policy. Then we have, for a b^* satisfying (i) and (ii),

$$\begin{aligned}\tilde{V}^{k,u} &= \tilde{V}^{k,b^*} + \sum_{l=b}^{\infty} \tilde{v}^{l+1,b^*}\,\tilde{\lambda}_l^{k,u} + \sum_{l=0}^{b^*-1} \tilde{v}^{l+1,b^*}\,(\tilde{\lambda}_l^{k,u} - \bar{\lambda}_l\,\tilde{p}_l^{k,u}) + \sum_{l=1}^{\infty}(1 - \tilde{v}^{l,b^*})\,(\tilde{\mu}_l^{k,u} - \bar{\mu}_l\,\tilde{p}_l^{k,u}) \\ &\leq \tilde{V}^{k,b^*},\end{aligned}$$

where the identity follows by Theorem 5(a), and the inequality follows from conditions (i), (ii) and Lemma 16(a, b).

Conditions (i', ii') simply reformulate (i, ii), taking into account the definition of $\tilde{v}^{k,b}$. □

We present next another optimality condition, which is needed to prove part of the Threshold Property for this model (Theorem 6(b.1)).

Corollary 8 (Optimality Condition II) *Suppose for a given* $c < 0$,

$$\tilde{v}^{l,0} \geq 1, \qquad \textit{for } l \geq 1;$$

or, equivalently,

$$\frac{1 - \Delta\tilde{\mu}^{l,0}}{\Delta\tilde{L}^{l,0}} \leq -c, \qquad \textit{for } l \geq 1.$$

Then, optimal control problem $\min\{\tilde{\mu}^{k,u} - c\,\tilde{L}^{k,u} : u \in \mathcal{U}\}$ *is solved optimally by the* 0*-threshold policy.*

Proof
Letting $b = 0$ in Theorem 5(a) we have, for each policy $u \in \mathcal{U}$,

$$\begin{aligned}\tilde{V}^{k,u} &= \tilde{V}^{k,0} + \sum_{l=0}^{\infty} \tilde{v}^{l+1,0}\,\tilde{\lambda}_l^{k,u} + \sum_{l=1}^{\infty}(1 - \tilde{v}^{l,0})\,(\tilde{\mu}_l^{k,u} - \bar{\mu}_l\,\tilde{p}_l^{k,u}) \\ &\geq \tilde{V}^{k,0},\end{aligned}$$

where the last inequality follows from the hypothesis that $\tilde{v}^{l,0} \geq 1$ for $l \geq 1$. □

Having now at our disposal the sufficient optimality conditions given in Corollaries 7 and 8, we proceed next to apply them to verify the optimality of threshold policies for the critical cost parameter values, as stated in Theorem 6(b).

Proposition 3 *Suppose Assumptions 2 and 3 hold. Then,*
(a) $\frac{\Delta\tilde{\mu}^{l,b-1}}{\Delta\tilde{L}^{l,b}} < c_\alpha^b < \frac{\Delta\tilde{\mu}^{l,b}}{\Delta\tilde{L}^{l,b}}$, *for* $1 \leq l < b$*;*
(b) $\frac{\Delta\tilde{\mu}^{l,b}}{\Delta\tilde{L}^{l,b}} < c_\alpha^b$, *for* $l \geq b + 1$;
(c) $\frac{1-\Delta\tilde{\mu}^{l,0}}{\Delta\tilde{L}^{l,0}} = \alpha = -c_\alpha^{k,0}$, *for* $k, l \geq 1$.

Proof
Part (a) follows from Lemma 15 and Lemma 4(a).

Part (b) follows from Lemma 4(b).

Part (c) follows from Lemma 3(b, c). □

Now, putting together the sufficient optimality conditions in Corollaries 7 and 8 with the results in Proposition 3 yields directly part (b) of the Threshold Property in Theorem 6.

Corollary 9 *Suppose Assumptions 2 and 3 hold. Then, for* any *admissible policy* $u \in \mathcal{U}$,
(a) $\tilde{\mu}^{k,u} - c_\alpha^{k,0}\, \tilde{L}^{k,u} \geq \tilde{V}^{k,0}(c_\alpha^{k,0})$;
(b) $\tilde{\mu}^{k,u} - c_\alpha^{b}\, \tilde{L}^{k,u} \leq \tilde{V}^{k,b}(c_\alpha^{b})$, *for* $b \geq 1$.

7 The time-average case

We show in this section that the time-average version of time-discounted optimal control problem (12) can be cast and analyzed in the LP framework of Section 2 in a straightforward way, based on our analysis of the time-discounted case.

We consider now the time-average throughput and WIP performance measures

$$\bar{\mu}^u = \lim_{T\to\infty} \frac{1}{T} \int_0^T E_u\left[\mu(t)\right]$$

and

$$\bar{L}^u = \lim_{T\to\infty} \frac{1}{T} \int_0^T E_u\left[L(t)\right],$$

respectively, where $u \in \mathcal{U}$.

The key property we shall apply is that time-average performance measures correspond to the limits of time-discounted measures as the discount factor α vanishes, by elementary *Tauberian theorems*: For any $L(0) = k \geq 0$ and $u \in \mathcal{U}$ we have

$$\bar{\mu}^u = \lim_{\alpha\to 0} \tilde{\mu}^{k,u}$$

and

$$\bar{L}^u = \lim_{\alpha\to 0} \tilde{L}^{k,u}.$$

We further define the limiting marginal performance measures

$$\Delta\bar{\mu}^{k,u} = \lim_{\alpha\to 0} \Delta\tilde{\mu}^{k,u}$$

and

$$\Delta\bar{L}^{k,u} = \lim_{\alpha\to 0} \Delta\tilde{L}^{k,u}, \qquad \text{for } k \geq 0,\ u \in \mathcal{U}.$$

We are interested in solving the time-average optimal control problem

$$\bar{V}^*(c) = \max\left\{\bar{\mu}^u - c\,\bar{L}^u : u \in \mathcal{U}\right\}, \tag{21}$$

for each $c > 0$.

Since our analysis of the time-average case follows directly from our analysis of the time-discounted case, letting $\alpha \to 0$, we state the results without proof.

Let us define the *critical cost parameters* c_0^0 and $\boldsymbol{c}_0 = \{c_0^b\}_{b\geq 1}$ by setting $c_0^0 = 0$ and

$$c_0^b = \lim_{\alpha\to 0} c_\alpha^b = \lim_{\alpha\to 0} \frac{\Delta\tilde{\mu}^{b,b}}{\Delta\tilde{L}^{b,b}} \qquad \text{for } b \geq 1. \tag{22}$$

The recursion shown in Figure 2 is also valid for computing the c_0^b's, just by letting $\alpha = 0$.

We present next the main result of this section, which says that, under Assumptions 2 and 3, $(\bar{\mu}^u, \bar{L}^u)$ satisfies the Threshold Property.

Theorem 6 (Threshold Property: Time-average case) *Suppose capacity limits satisfy Assumptions 2 and 3. Then, performance pair* $(\bar{\mu}^u, \bar{L}^u)$ *satisfies the Threshold Property:*
(a) *the performance pairs of threshold policies,* $\{(\bar{\mu}^b, \bar{L}^b)\}_{b\geq 0}$*, satisfy Assumption 1;*
(b) *under* any *admissible policy* $u \in \mathcal{U}$,
(b.1) $\bar{\mu}^u = \bar{\mu}^u - c_0^0\, \bar{L}^u \geq \bar{V}^0(c_0^0) = 0$;
(b.2) $\bar{\mu}^u - c_0^b\, \bar{L}^u \leq \bar{V}^b(c_0^b) = \bar{\mu}^b - c_0^b\, \bar{L}^b$, *for* $b \geq 1$.

Let us consider the model's time-average throughput-WIP achievable performance region

$$\mathcal{X} = \{(\bar{\mu}^u, \bar{L}^u) : u \in \mathcal{U}\},$$

and the polygon

$$\mathcal{P}(c_0^0, \boldsymbol{c}_0, \boldsymbol{d}) = \left\{(\mu, L) \geq \mathbf{0} : \mu - c_0^0\, L \geq d^0 \quad \text{and} \quad \mu - c_0^b\, L \leq d^b, \quad \text{for } b \geq 1\right\},$$

where sequence $\boldsymbol{d} = \{d_\alpha^b\}_{b \geq 0}$ is defined, analogously as in (3), by

$$d^b = \begin{cases} 0 & \text{if } b = 0 \\ \bar{\mu} - c_0^b\, \bar{L} & \text{if } b \geq 1. \end{cases}$$

Since the Threshold Property holds, Theorem 1 applies, giving an exact semi-infinite LP formulation of optimal control problem (21).

Corollary 10 (Semi-infinite LP Formulation) *Under Assumptions 2 and 3, $\mathcal{P}(c_0^0, \boldsymbol{c}_0, \boldsymbol{d})$ is a threshold polygon, whose vertex set includes $\{(\bar{\mu}^b, \bar{L}^b), b \geq 0\}$. Furthermore,*

$$\bar{V}^*(c) = \max\{\mu - c\, L : (\mu, L) \in \mathcal{P}(c_0^0, \boldsymbol{c}_0, \boldsymbol{d})\}.$$

Furthermore, Corollary 1 of the LP framework yields the characterization of the optimal threshold policy we present next. Let us define the *critical threshold function* $b^*(\cdot)$ by

$$b^*(c) = \min\left\{b \geq 0 : c_0^{b+1} \leq c\right\}, \qquad \text{for } c > 0.$$

Corollary 11 (Threshold Optimality) *Under Assumptions 2 and 3 on capacity limits, control problem (21) is solved optimally by the $b^*(c)$-threshold policy, for $c > 0$.*

In addition, Corollary 2 gives an analytical characterization of the *Min WIP* function $\bar{L}_{\min}(\cdot)$, defined by

$$\bar{L}_{\min}(\mu) = \min\{\bar{L}^u : \bar{\mu}^u = \mu, u \in \mathcal{U}\} = \min\{L : (\mu, L) \in \mathcal{X}\}.$$

Corollary 12 (Min WIP Characterization)

$$\bar{L}_{\min}(\mu) = \bar{L}^{b-1} + \frac{1}{c_0^b}(\mu - \bar{\mu}^{b-1}), \qquad \textit{for } \mu \in \left[\bar{\mu}^{b-1}, \bar{\mu}^b\right],\ b \geq 1.$$

A Performance analysis of threshold policies

We present in this Appendix a number of technical lemmas on basic identities and inequalities satisfied by throughput-WIP pairs under threshold policies, and which are needed in our proof of the Threshold Property for the model analyzed in this paper.

Basic identities

Let $\bar{\lambda}_k^b = 1\{k < b\}\, \bar{\lambda}_k$ denote the input intensity under the b-threshold policy when there are k customers in the system. We further define,, for notational convenience, $\tilde{V}^{-1,b} = \tilde{\mu}^{-1,b} = \tilde{L}^{-1,b} = \tilde{v}^{0,b} = \Delta\tilde{\mu}^{0,b} = \Delta\tilde{L}^{0,b} = 0$.

Our next two results are elementary, and were given by Chen and Yao (1990), so we state them without proof. They lay the groundwork for analyzing threshold policies.

Lemma 7 (Value Function Recursion) *For any threshold value $b \geq 0$, the following recursions hold:*
(a) $(\bar{\lambda}_k^b + \bar{\mu}_k + \alpha)\, \tilde{V}^{k,b} = \bar{\lambda}_k^b\, \tilde{V}^{k+1,b} + \bar{\mu}_k\, \tilde{V}^{k-1,b} + \alpha\, (\bar{\mu}_k - c\, k)$, *for $k \geq 0$;*
(b) $(\bar{\lambda}_k^b + \bar{\mu}_k + \alpha)\, \tilde{\mu}^{k,b} = \bar{\lambda}_k^b\, \tilde{\mu}^{k+1,b} + \bar{\mu}_k\, \tilde{\mu}^{k-1,b} + \alpha\, \bar{\mu}_k$, *for $k \geq 0$;*
(c) $(\bar{\lambda}_k^b + \bar{\mu}_k + \alpha)\, \tilde{L}^{k,b} = \bar{\lambda}_k^b\, \tilde{L}^{k+1,b} + \bar{\mu}_k\, \tilde{L}^{k-1,b} + \alpha\, k$, *for $k \geq 0$.*

Lemma 8 (Marginal Value Function Recursion) *For any threshold value $b \geq 0$, the following recursions hold:*
(a) $(\bar{\lambda}_{k-1}^b + \bar{\mu}_k + \alpha)\, \tilde{v}^{k,b} = \bar{\lambda}_k^b\, \tilde{v}^{k+1,b} + \bar{\mu}_{k-1}\, \tilde{v}^{k-1,b} + \Delta\bar{\mu}_k - c$, *for $k \geq 1$;*
(b) $(\bar{\lambda}_{k-1}^b + \bar{\mu}_k + \alpha)\, \Delta\tilde{\mu}^{k,b} = \bar{\lambda}_k^b\, \Delta\tilde{\mu}^{k+1,b} + \bar{\mu}_{k-1}\, \Delta\tilde{\mu}^{k-1,b} + \Delta\bar{\mu}_k$, *for $k \geq 1$;*
(c) $(\bar{\lambda}_{k-1}^b + \bar{\mu}_k + \alpha)\, \Delta\tilde{L}^{k,b} = \bar{\lambda}_k^b\, \Delta\tilde{L}^{k+1,b} + \bar{\mu}_{k-1}\, \Delta\tilde{L}^{k-1,b} + 1$, *for $k \geq 1$.*

The following discussion and notation is also adapted from Chen and Yao's analysis. For a threshold value $b \geq 1$, let us write $\tilde{\boldsymbol{v}}^b = (\tilde{v}^{1,b}, \ldots, \tilde{v}^{b,b})'$. By Lemma 8(a) we have $\tilde{\boldsymbol{v}}^b = \boldsymbol{B}^b \, \tilde{\boldsymbol{v}}^b + \boldsymbol{h}^b$, where

$$\boldsymbol{h}^b = \left(\frac{\Delta \bar{\mu}_1 - c}{\bar{\lambda}_0 + \bar{\mu}_1 + \alpha}, \ldots, \frac{\Delta \bar{\mu}_b - c}{\bar{\lambda}_{b-1} + \bar{\mu}_b + \alpha} \right)' \tag{23}$$

and $\boldsymbol{B}^b$ is the tri-diagonal matrix defined by

$$\boldsymbol{B}^b = \begin{pmatrix} 0 & \frac{\bar{\lambda}_1}{\lambda_0 + \bar{\mu}_1 + \alpha} & & & \\ \frac{\bar{\mu}_1}{\bar{\lambda}_1 + \bar{\mu}_2 + \alpha} & 0 & \frac{\bar{\lambda}_2}{\bar{\lambda}_1 + \bar{\mu}_2 + \alpha} & & \\ & \ddots & \ddots & \ddots & \\ & & \frac{\bar{\mu}_{b-2}}{\bar{\lambda}_{b-2} + \bar{\mu}_{b-1} + \alpha} & 0 & \frac{\bar{\lambda}_{b-1}}{\bar{\lambda}_{b-2} + \bar{\mu}_{b-1} + \alpha} \\ & & 0 & \frac{\bar{\mu}_{b-1}}{\bar{\lambda}_{b-1} + \bar{\mu}_b + \alpha} & 0 \end{pmatrix},$$

for $b \geq 2$, and $\boldsymbol{B}^1 = 0$. Chen and Yao (1990) show that, if

$$\Delta \bar{\lambda}_k \leq \Delta \bar{\mu}_k, \qquad \text{for } k \geq 1, \tag{24}$$

then the spectral radius of $\boldsymbol{B}^b$ is strictly less than one, and hence the inverse matrix $(\boldsymbol{I} - \boldsymbol{B}^b)^{-1} > \boldsymbol{0}$ ($\boldsymbol{I}$ is the identity matrix). Since (24) follows from our Assumptions 2 and 3 we thus have $(\boldsymbol{I} - \boldsymbol{B}^b)^{-1} > \boldsymbol{0}$, and

$$\tilde{\boldsymbol{v}}^b = (\boldsymbol{I} - \boldsymbol{B}^b)^{-1} \, \boldsymbol{h}^b. \tag{25}$$

Let $\hat{\boldsymbol{v}}^{b+1}$ be the first b components of $\tilde{\boldsymbol{v}}^{b+1}$, i.e., $\hat{\boldsymbol{v}}^{b+1} = (\tilde{v}^{1,b+1}, \ldots, \tilde{v}^{b,b+1})'$. We then have

$$\hat{\boldsymbol{v}}^{b+1} = \boldsymbol{B}^b \, \hat{\boldsymbol{v}}^{b+1} + \hat{\boldsymbol{h}}^{b+1}, \tag{26}$$

where

$$\hat{\boldsymbol{h}}^{b+1} = \boldsymbol{h}^b + \frac{\bar{\lambda}_b}{\bar{\lambda}_{b-1} + \bar{\mu}_b + \alpha} \, \tilde{v}^{b+1,b+1} \boldsymbol{e}_b, \tag{27}$$

and $\boldsymbol{e}_b = (0, \ldots, 0, 1)'$. Therefore, by (25) and (26),

$$\hat{\boldsymbol{v}}^{b+1} - \tilde{\boldsymbol{v}}^b = (\boldsymbol{I} - \boldsymbol{B}^b)^{-1} \, (\hat{\boldsymbol{h}}^{b+1} - \boldsymbol{h}^b). \tag{28}$$

It can be verified that the (b,b)th element of $(\boldsymbol{I} - \boldsymbol{B}^b)^{-1}$ is $\det(\boldsymbol{I} - \boldsymbol{B}^{b-1}) / \det(\boldsymbol{I} - \boldsymbol{B}^b)$. Hence the bth component of $\hat{\boldsymbol{v}}^{b+1} - \tilde{\boldsymbol{v}}^b$ is

$$\tilde{v}^{b,b+1} - \tilde{v}^{b,b} = \frac{\bar{\lambda}_b}{\bar{\lambda}_{b-1} + \bar{\mu}_b + \alpha} \frac{\det(\boldsymbol{I} - \boldsymbol{B}^{b-1})}{\det(\boldsymbol{I} - \boldsymbol{B}^b)} \tilde{v}^{b+1,b+1}. \tag{29}$$

Letting

$$g(b) = \frac{\bar{\lambda}_b + \bar{\mu}_{b+1} + \alpha}{\bar{\mu}_b} - \frac{\bar{\lambda}_b}{\bar{\lambda}_{b-1} + \bar{\mu}_b + \alpha} \frac{\det(\boldsymbol{I} - \boldsymbol{B}^{b-1})}{\det(\boldsymbol{I} - \boldsymbol{B}^b)} = \frac{\bar{\lambda}_b + \bar{\mu}_{b+1} + \alpha}{\bar{\mu}_b} \frac{\det(\boldsymbol{I} - \boldsymbol{B}^{b+1})}{\det(\boldsymbol{I} - \boldsymbol{B}^b)} > 0 \tag{30}$$

and $d(b) = \det(\boldsymbol{I} - \boldsymbol{B}^b) / \det(\boldsymbol{I} - \boldsymbol{B}^{b-1})$, with $d(1) = 1$, Chen and Yao show that the recursions given in Figure 2 hold and, furthermore,

$$g(b) \geq 1 + \frac{\Delta \bar{\mu}_{b+1} + \alpha}{\bar{\mu}_b} \geq 1 + \frac{\alpha}{\bar{\mu}_b}. \tag{31}$$

In addition, from the identities (obtained from Lemma 8),

$$(\bar{\mu}_{b+1} + \alpha) \, \tilde{v}^{b+1,b} = \bar{\mu}_b \, \tilde{v}^{b,b} + \Delta \bar{\mu}_{b+1} - c$$

$$(\bar{\lambda}_b + \bar{\mu}_{b+1} + \alpha) \, \tilde{v}^{b+1,b+1} = \bar{\mu}_b \, \tilde{v}^{b,b+1} + \Delta \bar{\mu}_{b+1} - c,$$

and relation (29), the following identity follows (corresponding to Chen and Yao's equation (5.16a)):

$$\tilde{v}^{b+1,b} = \frac{\bar{\mu}_b \, g(b)}{\bar{\mu}_{b+1} + \alpha} \, \tilde{v}^{b+1,b+1}. \tag{32}$$

Our next result corresponds to equation (6.5) in Chen and Yao.

Input: $\alpha \geq 0, \{\bar{\lambda}_k\}_{k=0}^{\infty} > \mathbf{0}, \{\bar{\mu}_k\}_{k=1}^{\infty} > \mathbf{0}$.
Output:$\{c_\alpha^b\}_{b=1}^{\infty}$

$$
\begin{aligned}
(\Delta\tilde{\mu}^{1,1}, \Delta\tilde{L}^{1,1}) &= \frac{1}{\bar{\lambda}_0 + \bar{\mu}_1 + \alpha}\,(\bar{\mu}_1, 1) \\
c_\alpha^1 &= \frac{\Delta\tilde{\mu}^{1,1}}{\Delta\tilde{L}^{1,1}} = \bar{\mu}_1 \\
d(1) &= 1 \\
\text{for } b \geq 1: & \\
c_\alpha^{b+1} &= \frac{\Delta\tilde{\mu}^{b,b} + \frac{\Delta\bar{\mu}_{b+1}}{\bar{\mu}_b}}{\Delta\tilde{L}^{b,b} + \frac{1}{\bar{\mu}_b}} \\
g(b) &= 1 + \frac{1}{\bar{\mu}_b}\left[\bar{\lambda}_b + \Delta\bar{\mu}_{b+1} + \alpha - \frac{\bar{\lambda}_b\,\bar{\mu}_b}{\bar{\lambda}_{b-1} + \bar{\mu}_b + \alpha}\,\frac{1}{d(b)}\right] \\
(\Delta\tilde{\mu}^{b+1,b+1}, \Delta\tilde{L}^{b+1,b+1}) &= \frac{1}{g(b)}\left[(\Delta\tilde{\mu}^{b,b}, \Delta\tilde{L}^{b,b}) + \frac{1}{\bar{\mu}_b}\,(\Delta\bar{\mu}_{b+1}, 1)\right] \\
\eta(b) &= \frac{\bar{\lambda}_b\,\bar{\mu}_b}{(\bar{\lambda}_{b-1} + \bar{\mu}_b + \alpha)\,(\bar{\lambda}_b + \bar{\mu}_{b+1} + \alpha)} \\
d(b+1) &= 1 - \frac{\eta(b)}{d(b)}
\end{aligned}
$$

Figure 2: Critical cost parameters computation.

Lemma 9 *Suppose Assumptions 2 and 3 hold. Then,*
(a)

$$g(b)\,\tilde{v}^{b+1,b+1} = \tilde{v}^{b,b} + \frac{\Delta\bar{\mu}_{b+1} - c}{\bar{\mu}_b}, \qquad \textit{for } b \geq 1,$$

with $\tilde{v}^{1,1} = (\bar{\mu}_1 - c)/(\bar{\lambda}_0 + \bar{\mu}_1 + \alpha)$.
(b) *Sequence* $\{\Delta\tilde{\mu}^{b,b}\}_{b\geq 1}$ *is positive, and satisfies*

$$g(b)\,\Delta\tilde{\mu}^{b+1,b+1} = \Delta\tilde{\mu}^{b,b} + \frac{\Delta\bar{\mu}_{b+1}}{\bar{\mu}_b}, \qquad \textit{for } b \geq 1,$$

with $\Delta\tilde{\mu}^{1,1} = (\bar{\mu}_1)/(\bar{\lambda}_0 + \bar{\mu}_1 + \alpha)$;
(c) *Sequence* $\{\Delta\tilde{L}^{b,b}\}_{b\geq 1}$ *is positive, and satisfies*

$$g(b)\,\Delta\tilde{L}^{b+1,b+1} = \Delta\tilde{L}^{b,b} + \frac{1}{\bar{\mu}_b}, \qquad \textit{for } b \geq 1,$$

with $\Delta\tilde{L}^{1,1} = 1/(\bar{\lambda}_0 + \bar{\mu}_1 + \alpha)$.

Proof
(a) By Lemma 8(a) we have

$$(\bar{\lambda}_b + \bar{\mu}_{b+1} + \alpha)\,\tilde{v}^{b+1,b+1} = \bar{\mu}_b\,\tilde{v}^{b,b+1} + \Delta\bar{\mu}_{b+1} - c, \tag{33}$$

$$(\bar{\mu}_{b+1} + \alpha)\,\tilde{v}^{b+1,b} = \bar{\mu}_b\,\tilde{v}^{b,b} + \Delta\bar{\mu}_{b+1} - c. \tag{34}$$

Solving for $\tilde{v}^{b,b}$ and $\tilde{v}^{b,b+1}$ in these equations, and substituting in (29) yields the result.

The identities in parts (b) and (c) follow from (a). The result that $\Delta\tilde{\mu}^{b,b} > 0$ and $\Delta\tilde{L}^{b,b} > 0$, for $b \geq 1$, follows by induction on b, since $g(b) > 0$. □

The following result is precisely Lemma 6.1 in Chen and Yao (1990).

Lemma 10 *Under Assumptions 2 and 3,*
(a) $\lim_{b\to\infty} g(b) = g < \infty$ *and* $g \geq 1 + \lim_{b\to\infty} \frac{\alpha}{\bar{\mu}_b}$;
(b) *if* $\lim_{b\to\infty} \bar{\mu}_b = \infty$, *then* $\lim_{b\to\infty} \bar{\mu}_b\,[g(b) - 1] = \alpha$.

The next lemma adapts and extends Theorem 6.2 of Chen and Yao.

Lemma 11 *Suppose Assumptions 2 and 3 hold;*
(a) *if* $\lim_{b\to\infty} \bar{\mu}_b = \bar{\mu}_\infty < \infty$, *then*
(a.1) $\lim_{b\to\infty} \Delta\tilde{\mu}^{b,b} = 0$;
(a.2) $\liminf_{b\to\infty} \Delta\tilde{L}^{b,b} > 0$;
(b) $\lim_{b\to\infty} \frac{\Delta\tilde{\mu}^{b,b}}{\Delta\tilde{L}^{b,b}} = 0$.

Proof
(a.1) By Lemma 9(b) and Proposition 15(b) we have

$$0 < \Delta\tilde{\mu}^{b,b} < 1, \qquad \text{for } b \geq 1.$$

Hence, $0 \leq \limsup_{b\to\infty} \Delta\tilde{\mu}^{b,b} \leq 1 < \infty$ and, taking lim sup on both sides of

$$g(b)\, \Delta\tilde{\mu}^{b+1,b+1} = \Delta\tilde{\mu}^{b,b} + \frac{\Delta\bar{\mu}_b}{\bar{\mu}_b},$$

(which holds by Lemma 9(b)) yields

$$g \limsup_{b\to\infty} \Delta\tilde{\mu}^{b,b} = \limsup_{b\to\infty} \Delta\tilde{\mu}^{b,b},$$

(remember $\lim_{b\to\infty} \Delta\bar{\mu}_b = 0$, by Assumption 2), where $g = \lim_{b\to\infty} g(b)$.

By Lemma 10(a), $g \geq 1 + \alpha/\bar{\mu}_\infty > 1$. It thus follows that $\limsup_{b\to\infty} \Delta\tilde{\mu}^{b,b} = 0$ which, together with $0 \leq \Delta\tilde{\mu}^{b,b}$ proves the result.

(a.2) By Lemma 9(c) we know that $\Delta\tilde{L}^{b,b} > 0$ for each $b \geq 1$, and hence $\liminf_{b\to\infty} \Delta\tilde{L}^{b,b} \geq 0$. Hence, taking lim inf on both sides of

$$g(b)\, \Delta\tilde{L}^{b+1,b+1} = \Delta\tilde{L}^{b,b} + \frac{1}{\bar{\mu}_b}, \tag{35}$$

(which holds by Lemma 9(c)) yields

$$g \liminf_{b\to\infty} \Delta\tilde{L}^{b,b} = \liminf_{b\to\infty} \Delta\tilde{L}^{b,b} + \frac{1}{\bar{\mu}_\infty}.$$

Since $g > 1$ and $\liminf_{b\to\infty} \Delta\tilde{L}^{b,b} \geq 0$ the result follows.

(b) This result corresponds to Theorem 6.2 in Chen and Yao's paper. What follows mirrors their proof: In the case $\lim_{b\to\infty} \bar{\mu}_b < \infty$, the result follows from part (a).

Consider now the case $\lim_{b\to\infty} \bar{\mu}_b < \infty$. Suppose the result did not hold. Since, by Lemma 31(b, c), we have $\Delta\tilde{\mu}^{b,b}, \Delta\tilde{L}^{b,b} > 0$, there would exist $c > 0$ such that $\Delta\tilde{\mu}^{b,b} - c\,\Delta\tilde{L}^{b,b} > 0$, for all $b \geq 1$. Now, by definition of $\tilde{v}^{b,b}$, this condition translates into $\tilde{v}^{b,b} > 0$, for all $b \geq 1$. Rewriting now Lemma 31(a) as

$$\bar{\mu}_b \left[\tilde{v}^{b,b} - \tilde{v}^{b+1,b+1}\right] = \bar{\mu}_b\, [g(b) - 1]\, \tilde{v}^{b+1,b+1} + c - \Delta\bar{\mu}_{b+1},$$

and taking lim inf on both sides of this identity (using Lemma 10(b)), we obtain

$$\begin{aligned} \liminf_{b\to\infty} \bar{\mu}_b \left[\tilde{v}^{b,b} - \tilde{v}^{b+1,b+1}\right] &= \alpha \liminf_{b\to\infty} \tilde{v}^{b,b} + c \\ &\geq c > 0. \end{aligned}$$

Therefore, it would be $\tilde{v}^{b,b} > \tilde{v}^{b+1,b+1}$ for b large enough and, since we assumed $\tilde{v}^{b,b} > 0$, it would follow that $\lim_{b\to\infty} \tilde{v}^{b,b} < \infty$, and hence the telescopic series $\sum_{b=1}^{\infty} \left[\tilde{v}^{b,b} - \tilde{v}^{b+1,b+1}\right]$ would converge. But this leads to a contradiction, as in the proof of part (b), completing the proof. □

We present next a new technical Lemma which is essential to our proof of Lemma 15 below.

Lemma 12 *Suppose Assumptions 2 and 3 hold. Then, for any $b \geq 2$ and $1 \leq k \leq b-1$ there exist numbers $\gamma^{k,b}, \delta^{k,b} \geq 0$, with $\gamma^{k,b} + \delta^{k,b} < 1$, such that*

$$\Delta\tilde{\mu}^{k,b} = \gamma^{k,b}\, \Delta\tilde{\mu}^{k+1,b} + \delta^{k,b}.$$

Proof
Our proof is by induction on k.

Case (I) $k=1$. We have, by Lemma 8(b),

$$\Delta\tilde{\mu}^{1,b} = \frac{\bar{\lambda}_1}{\bar{\lambda}_0 + \bar{\mu}_1 + \alpha}\,\Delta\tilde{\mu}^{2,b} + \frac{\bar{\mu}_1}{\bar{\lambda}_0 + \bar{\mu}_1 + \alpha},$$

whence the result holds for $k=1$, since $\bar{\lambda}_1 \leq \bar{\lambda}_0$. For $b=2$, this proves the result.

Case (II) $1 \leq k \leq b-1$, where $b \geq 3$. We apply induction on k. The case $k=1$ was proven above. Assume now the result holds for some $1 \leq k \leq b-2$. Let thus $\gamma^{k,b}$ and $\delta^{k,b}$ be nonnegative numbers, with $\gamma^{k,b} + \delta^{k,b} < 1$, satisfying

$$\Delta\tilde{\mu}^{k,b} = \gamma^{k,b}\,\Delta\tilde{\mu}^{k+1,b} + \delta^{k,b}$$

Now, again by Lemma 8(b),

$$(\bar{\lambda}_k + \bar{\mu}_{k+1} + \alpha)\,\Delta\tilde{\mu}^{k+1,b} = \bar{\lambda}_{k+1}\,\Delta\tilde{\mu}^{k+2,b} + \bar{\mu}_k\,\Delta\tilde{\mu}^{k,b} + \Delta\bar{\mu}_{k+1}.$$

Substituting for $\Delta\tilde{\mu}^{k,b}$ in this last identity we obtain

$$(\bar{\lambda}_k + \bar{\mu}_{k+1} - \bar{\mu}_k\,\gamma^{k,b} + \alpha)\,\Delta\tilde{\mu}^{k+1,b} = \bar{\lambda}_{k+1}\,\Delta\tilde{\mu}^{k+2,b} + (\bar{\mu}_k\,\delta^{k,b} + \Delta\bar{\mu}_{k+1}).$$

Now, let us define

$$\gamma^{k+1,b} = \frac{\bar{\lambda}_{k+1}}{\bar{\lambda}_k + \bar{\mu}_{k+1} - \bar{\mu}_k\,\gamma^{k,b} + \alpha},$$

and

$$\delta^{k+1,b} = \frac{\bar{\mu}_k\,\delta^{k,b} + \Delta\bar{\mu}_{k+1}}{\bar{\lambda}_k + \bar{\mu}_{k+1} - \bar{\mu}_k\,\gamma^{k,b} + \alpha}.$$

Now, under Assumptions 2 and 3 it is clear that $\gamma^{k+1,b}, \delta^{k+1,b} \geq 0$, and, furthermore,

$$\bar{\mu}_k\,(\delta^{k,b} + \gamma^{k,b} - 1) < 0 < -\Delta\bar{\lambda}_{k+1} + \alpha.$$

These inequalities imply (by adding $\bar{\lambda}_{k+1} + \bar{\mu}_{k+1}$ on both sides) that

$$\bar{\lambda}_{k+1} + \bar{\mu}_k\,\delta^{k,b} + \Delta\bar{\mu}_{k+1} < \bar{\lambda}_k + \bar{\mu}_{k+1} - \bar{\mu}_k\,\gamma^{k,b} + \alpha,$$

i.e., $\delta^{k+1,b} + \gamma^{k+1,b} < 1$, which completes the induction proof. □

Basic inequalities

The following elementary result will simplify our subsequent proofs.

Lemma 13 *Let $a, b, c, d, p, q > 0$. Then the following inequalities hold:*
(a) *if $a > p$ and $b > q$, then*

$$\frac{a-p}{b-q} < \frac{a}{b} \iff \frac{a}{b} < \frac{p}{q} \iff \frac{a-p}{b-q} < \frac{p}{q};$$

(b)

$$\frac{a+p}{b+q} < \frac{a}{b} \qquad \textit{if and only if} \qquad \frac{p}{q} < \frac{a}{b}.$$

Lemma 14 *Suppose Assumptions 2 and 3 hold. Then,*
(a)

$$\frac{\Delta\tilde{\mu}^{1,1}}{\Delta\tilde{L}^{1,1}} = \bar{\mu}_1;$$

(b) *for $b \geq 2$ and $1 \leq k \leq b$,*

$$\Delta\bar{\mu}_b < \frac{\Delta\tilde{\mu}^{k,b}}{\Delta\tilde{L}^{k,b}} \leq \bar{\mu}_1.$$

Proof
(a) This part follows directly from Lemma 9(b, c).

(b) Let $1 \leq k \leq b$. Let us write $\boldsymbol{A} = (a_{kl}) = (\boldsymbol{I} - \boldsymbol{B}^b)^{-1}$, and $d_k = \bar{\lambda}_{k-1} + \bar{\mu}_k + \alpha$. Then, from (25), (23) and

$$\tilde{v}_k^b = \Delta\tilde{\mu}^{k,b} - c\,\Delta\tilde{L}^{k,b},$$

it follows that

$$\Delta\tilde{\mu}^{k,b} = \frac{a_{k1}}{d_1}\,\Delta\bar{\mu}_1 + \cdots + \frac{a_{kb}}{d_b}\,\Delta\bar{\mu}_b$$

and

$$\Delta\tilde{L}^{k,b} = \frac{a_{k1}}{d_1} + \cdots + \frac{a_{kb}}{d_b}.$$

Now, we know that, under Assumptions 2 and 3, $\boldsymbol{A}$ is a positive matrix. Therefore, $\Delta\tilde{\mu}^{k,b}/\Delta\tilde{L}^{k,b}$ is a *convex combination* of $\Delta\bar{\mu}_1, \ldots, \Delta\bar{\mu}_b$. The result now follows from Assumption 2(i) and the fact that $a_{b1}\,\bar{\mu}_1 > 0$. □

Lemma 15 *Suppose Assumptions 2 and 3 hold. Then, for* $b \geq 1$,
(a)

$$\Delta\tilde{\mu}^{b,b} \leq \frac{\bar{\mu}_b}{\bar{\lambda}_{b-1} + \bar{\mu}_b + \alpha}.$$

(b)

$$\Delta\tilde{\mu}^{k,b} < 1, \quad \textit{for } 1 \leq k \leq b.$$

Proof
(a) We apply induction on $b \geq 1$. The case $b = 1$ follows from the identity

$$\Delta\tilde{\mu}^{1,1} = \frac{\bar{\mu}_1}{\bar{\lambda}_0 + \bar{\mu}_1 + \alpha}.$$

Suppose now that, for a given $b \geq 1$,

$$\Delta\tilde{\mu}^{b,b} \leq \frac{\bar{\mu}_b}{\bar{\lambda}_{b-1} + \bar{\mu}_b + \alpha}.$$

Now, we have, by Lemma 9(b), the induction hypothesis and inequality (31), respectively,

$$\begin{aligned} g(b)\,\Delta\tilde{\mu}^{b+1,b+1} &= \Delta\tilde{\mu}^{b,b} + \frac{\Delta\bar{\mu}_{b+1}}{\bar{\mu}_b} && (36)\\ &\leq \frac{\bar{\mu}_b}{\bar{\lambda}_{b-1} + \bar{\mu}_b + \alpha} + \frac{\Delta\bar{\mu}_{b+1}}{\bar{\mu}_b} && (37)\\ &\leq g(b)\,\frac{\bar{\mu}_{b+1}}{\bar{\lambda}_b + \bar{\mu}_{b+1} + \alpha}, && (38) \end{aligned}$$

which, since $g(b) > 0$, completes the induction proof of (a).

(b) By backwards induction on $k = 1 \ldots b$. The case $k = b$ follows from part (a).

Suppose now the result holds for some k with $2 \leq k \leq b$, i.e., $\Delta\tilde{\mu}^{k,b} < 1$. We notice that, by Lemma 12, there exist numbers $\gamma^{k-1,b}, \delta^{k-1,b} \geq 0$, with $\gamma^{k-1,b} + \delta^{k-1,b} < 1$, such that

$$\Delta\tilde{\mu}^{k-1,b} = \gamma^{k-1,b}\,\Delta\tilde{\mu}^{k,b} + \delta^{k-1,b}.$$

Therefore, by the induction hypothesis and the fact that $\gamma^{k-1,b} + \delta^{k-1,b} < 1$, respectively, we have

$$\begin{aligned} \Delta\tilde{\mu}^{k-1,b} &= \gamma^{k-1,b}\,\Delta\tilde{\mu}^{k,b} + \delta^{k-1,b}\\ &\leq (\gamma^{k-1,b} + \delta^{k-1,b})\\ &< 1, \end{aligned}$$

which completes the induction proof. □

Our next result presents the linear constraints satisfied by auxiliary performance measures.

Lemma 16 *The following linear constraints hold, for any policy* $u \in \mathcal{U}$, *initial state* $L(0) = k$ *and* $l \geq 0$:
(a)[Input capacity constraints]

$$\tilde{\lambda}_l^{k,u} - \bar{\lambda}_l\,\tilde{p}_l^{k,u} \leq 0;$$

(b)[Output capacity constraints]

$$\tilde{\mu}_l^{k,u} - \bar{\mu}_l\, \tilde{p}_l^{k,u} \le 0;$$

(c)[Flow balance constraints]

$$\tilde{\mu}_{l+1}^{k,u} - \tilde{\lambda}_l^{k,u} + \alpha \sum_{m>l} \tilde{p}_m^{k,u} = \alpha\, 1\{l < k\};$$

(d)[Probability constraint]

$$\sum_{l=0}^{\infty} \tilde{p}_l^{k,u} = 1.$$

Proof
First we notice that the auxiliary performance variables can be represented equivalently as follows: Let τ_α be a random time, distributed as an exponential random variable with rate α, independent of the system evolution. Then, we can write

$$\tilde{p}_l^{k,u} = P_u\{L(\tau_\alpha) = l | L(0) = k\},$$

$$\tilde{\lambda}_l^{k,u} = E\left[\lambda(\tau_\alpha)\, 1\{L(\tau_\alpha) = l\} | L(0) = k\}\right]$$

and

$$\tilde{\mu}_l^{k,u} = E\left[\mu(\tau_\alpha)\, 1\{L(\tau_\alpha) = l\} | L(0) = k\}\right].$$

(a)-(d) follow now easily as follows:

(a) We have, since $\lambda(\tau_\alpha)\, 1\{L(\tau_\alpha) = l\} \le \bar{\lambda}_k\, 1\{L(\tau_\alpha) = l\}$,

$$\begin{aligned} \tilde{\lambda}_l^{k,u} &= E\left[\lambda(\tau_\alpha)\, 1\{L(\tau_\alpha) = l\} | L(0) = k\}\right] \\ &\le \bar{\lambda}_k\, P_u\{L(\tau_\alpha) = l | L(0) = k\} = \bar{\lambda}_k\, \tilde{p}_l^{k,u}. \end{aligned}$$

(b) Similarly as in part (a), since $\mu(\tau_\alpha)\, 1\{L(\tau_\alpha) = l\} \le \bar{\mu}_k\, 1\{L(\tau_\alpha) = l\}$,

$$\begin{aligned} \tilde{\mu}_l^{k,u} &= E\left[\mu(\tau_\alpha)\, 1\{L(\tau_\alpha) = l\} | L(0) = k\}\right] \\ &\le \bar{\mu}_k\, P_u\{L(\tau_\alpha) = l | L(0) = k\} = \bar{\mu}_k\, \tilde{p}_l^{k,u}. \end{aligned}$$

(c) This is a flow balance identity on the system state (number-in-system). For each time $t \ge 0$ let $A_l(t)$ (resp. $D_l(t)$) denote the cumulative number of upward transitions from state k into $k+1$ (resp. downward transitions from state l into $l-1$) up to and including time t. We consider $A_l(0) = D_l(0) = 0$. Now, we can write the flow balance identity for state $l \ge 0$ at the random time τ_α as

$$A_l(\tau_\alpha) + 1\{L(0) > l\} = D_{l+1}(\tau_\alpha) + 1\{L(\tau_\alpha) > l\}.$$

Taking expectations in this identity, with respect to policy u and under initial state $L(0) = k$ gives the equality constraint in (c), since $\alpha\, E_u[A_l(\tau_\alpha) | L(0) = k] = \tilde{\lambda}_l^{k,u}$, $\alpha\, E_u[D_l(\tau_\alpha) | L(0) = k] = \tilde{\mu}_l^{k,u}$, and $\alpha\, E_u[1\{L(\tau_\alpha) = k\} | L(0) = k] = \tilde{p}_l^{k,u}$.

(d) This follows since $\{\tilde{p}_l^{k,u}\}_{l\ge 0} = \{P_u\{L(\tau_\alpha) = l\}_{l \ge 0}$ is a probability measure. □

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