1 Introduction

Fixed-income markets have experienced, two decades ago, a great increase in the volatility of the assets which these markets deal with[1]. Because of that, the academics and, in general, the participants in these market have developed and implemented tools and techniques in order to manage the risk derived from interest rates. In general, the price of a bond portfolio is influenced by many factors: the level and the volatility of the interest rates, changes in the shape of the yield curve, default probability and liquidity changes. In particular, we will consider default-free securities and deep and liquid markets. We will focus on two types of risk that depend on the changes that the yield curve can show:

- *"Market risk"*. It is caused by changes in the level of the interest rates. Hence, this risk is associated to **parallel changes** (changes with similar size in all the maturities) in the yield curve.
- *"Yield curve risk"*. It is derived from **non-parallel changes**, that is, changes in the shape of the yield curve. It can be decomposed into two terms:
  - Changes in the slope: Changes in the short-term interest rates have different size in relation to changes in long-term interest rates. The slope of the yield curve decreases (increases) if the short-term interest rates change more (less) than interest rates corresponding to longer maturities.
  - Changes in the curvature: Changes in the extremes of the yield curve are similar but with different direction when compared to changes in intermediate maturities.

The classic solution to manage the market risk is based on the duration in order to immunize the corresponding bond portfolio. Duration reflects the sensitivity of the asset price when interest rates change. Therefore, the duration establishes a relationship between the interest rates volatility and changes in bond yields.

[1]This fact is illustrated in Nelson and Schaefer (1983) and Smithson and Smith (1995).

Immunization allows us to set up and manage a bond portfolio in such a way that this portfolio reaches a predetermined goal: to mimic a certain index, to guarantee a set of future payments[2] or to obtain a certain return.

Generally speaking, the methodology used when immunizing a bond portfolio is based on equating the durations of this bond portfolio and the asset to be replicated. The main assumption of duration is that the yield of different securities changes in the same direction and size. Therefore, it is assumed that yield curve changes are parallel or, in other words, there is a uniform increase (or decrease) in all interest rates (see Macaulay (1938)). Thus, duration estimates the change in the bond price when the yield curve changes in a parallel fashion and, in that case, it can be an appropriate tool to manage the market risk. In this line, several papers as Fisher and Weil (1971), Bierwag *et al* (1981), Bierwag *et al* (1983a) and Brennan and Schwartz (1983) show how this measure can reduce this type of risk. Moreover, Bierwag and Khang (1979) show that, when there is only an uncertainty source affecting the yield curve, a coupon-bond portfolio can be immunized by means of the duration. Several papers as Ingersoll (1983), Nelson and Schaefer (1983) and Brennan and Schwartz (1983) show that duration shows better performance than more sophisticated methods.

Results obtained with duration can be improved by means of convexity. Duration can be interpreted as a first order approximation to the change in the bond price. Then, it is a valid measure only when the yield curve shows small changes and, in general, duration undervalues the expected gain (or loss) when interest rates change. On the other hand, convexity can be seen as a second order term, less important than duration but with an important influence if changes in the yield curve are large enough. Therefore, convexity can correct the estimation, obtained with duration, of the change in the bond price when interest rates change in a large amount.

Existence of non-parallel movements[3] in the yield curve limits the use of duration. These limitations are illustrated in papers as Ingersoll *et al* (1978), Cox *et al* (1979), Hilliard and Jordan (1992) and D'Eclessia and Zenios (1994). These papers reflect that the results obtained from duration are worse than the ones provided by alternative techniques.

[2]This is the goal which the immunization was established and developed initially by Redington (1952).

[3]Empirical evidence of these movements can be seen in Jones (1991), Litterman and Scheinkman (1991), Zhang (1993) and Knez *et al* (1994).

Several duration measures related to the non-parallel movements in the yield curve have been proposed and tested empirically in different papers. Thus, in Bierwag *et al* (1983b), several additive and multiplicative stochastic processes and their durations are evaluated. Garbade (1985) deals with immunization when there are changes in the slope of the yield curve. Gultekin and Rogalski (1984), Elton *et al* (1988) and Elton *et al* (1990) develop and test empirically duration measures based on multi-factor models. Klaffky *et al* (1992) proposes two duration measures (*"reshaping durations"*) to reflect changes in the extremes of the yield curve. Ho (1992) proposes the *"key rate durations"* based on the changes of the interest rates corresponding to certain maturities. Reitano (1992, 1996) deals with non parallel changes and develops a duration vector (*"directional durations"*) that indicates the direction of these changes. Finally, Ibáñez (1997) proposes a new immunization strategy based on a linear dispersion measure and proves that any dispersion measure reflects the immunization risk. Moreover, this author computes the *maximin* portfolios in models when immunization is not feasible and in hedging models with two periods.

The main characteristic of this set of measures is that they are not derived from a model of the term structure of interest rates but they are arbitrarily specified. Several exceptions can be highlighted. Thus, Ingersoll *et al* (1978), Cox *et al* (1979) and Chen (1996) develop duration measures based on the model presented in Cox *et al* (1985b). Recently, Wu (1996) presents and tests empirically duration measures based on Vasicek (1977) and Cox *et al* (1985b).

The goal of this paper is to define and apply duration measures based on the two-factor continuous-time model presented in Moreno (1997). Thus, we can analyze the behavior of a bond portfolio in relation to different changes in the yield curve. Therefore, we can solve the limitations of the duration when dealing with non-parallel changes in the yield curve.

The limitations of conventional duration are illustrated by a numerical example included in the final appendix. In this example, two bond portfolios with the same modified duration are presented. These two portfolios have a different yield and convexity. With no changes in the yield curve, it seems clear that the best strategy is to buy the portfolio with higher yield and to sell the portfolio with lower yield. However, the different convexity of these portfolios suggests different investment alternatives if interest rates show certain changes. In this appendix, it is shown that, in spite of having the same modified duration, the relative behavior, as deduced from the difference in yields, of these two portfolios depends on the magnitude and the type of change in yields. Therefore, the main conclusion of this example is that measures previously used such as yield, duration, or convexity of a bond portfolio do not provide adequate information about the performance of this bond portfolio when the yield curve changes. In fact, the final yield obtained from the portfolio depends on the size and the type of the change that interest rates show.

Similarly to Chen (1996), we will generalize the conventional duration measures and will obtain several measures of "generalized duration" that will reflect the changes in the stochastic factors of our model. Once obtained these measures, we will analyze their application to the computation of the hedging ratios that allows us to immunize a bond portfolio by using options on bonds. Moreover, we will analyze if these new measures can be used to manage the interest rate risk derived from changes (parallel and/or in the slope) in the yield curve.

This paper is organized as follows. Section 2 presents the two-factor model in which this paper is based on. This model provides a theoretical framework in which we can develop a method for managing the risks derived from non-parallel changes in the yield curve. In Section 3 conventional duration and convexity are extended using the two-factor model presented in the previous section. Hedging ratios, applicable to immunize a bond portfolio, are computed in Section 4. In Section 5 we focus on the hedging problem. Section 6 analyzes the hedging associated to changes in the slope of the yield curve. In Section 7 it is shown how the new measures solve the limitations related to conventional duration. In the final appendix, a numerical example illustrates the limitations of the conventional duration. Results regarding to this example are included in Tables I-VI. Finally, Tables VII-XII include the results obtained with the proposal of solution of these limitations presented in this paper. Finally, Section 8 summarizes the main conclusions of this paper.

2 The model

In this section we present briefly the two-factor continuous-time model for the term structure of interest rates that has been proposed and analyzed more deeply in Moreno (1997).

The main assumption made by this model is that the price, at time t, of a default-free discount bond that pays \$1 at maturity T depends only on the time to maturity and two state variables: the long-term interest rate, denoted by L, and the spread, denoted by s, equal to the difference between the short-term (instantaneous) riskless interest rate, r, and the long-term interest rate. This selection of state variables allows us to use the assumption that both variables are orthogonal[4].

After choosing the state variables, we assume that their dynamics over time are given by the following stochastic differential equations:

$$\begin{cases} ds &= \beta_1(s,L)dt + \sigma_1(s,L)dw_1 \\ dL &= \beta_2(s,L)dt + \sigma_2(s,L)dw_2 \end{cases} \tag{1}$$

where t denotes calendar time, and dw_1 and dw_2 are Wiener processes where $E[dw_1] = E[dw_2] = 0$, $dw_1^2 = dw_2^2 = dt$, and (by the orthogonality between these variables) $E[dw_1 dw_2] = 0$. $\beta_1(.)$ and $\beta_2(.)$ are the expected instantaneous rates of change in the state variables and $\sigma_1^2(.)$ and $\sigma_2^2(.)$ are the instantaneous variances of changes in these two variables.

Let $P(s,L,t,T) \equiv P(s,L,\tau)$ be the price, at time t, of a default-free discount bond that pays \$1 at maturity $T = t + \tau$. Applying Itô's Lemma, setting up a hedge portfolio, consisting of bonds of three different maturities that is instantaneously riskless and assuming no-arbitrage conditions, we obtain the partial differential equation that the price of a default-free discount bond for all maturities must satisfy:

$$\begin{aligned} &\frac{1}{2}[\sigma_1^2(.)P_{ss} + \sigma_2^2(.)P_{LL}] + [\beta_1(.) - \lambda_1(.)\sigma_1(.)]P_s \\ &+[\beta_2(.) - \lambda_2(.)\sigma_2(.)]P_L + P_t - rP = 0 \end{aligned} \tag{2}$$

Given the stochastic process (1), assumed for the state variables, (2) is the fundamental equation for the pricing of default-free discount bonds

[4]Empirical evidence that supports this assumption has been shown in several papers as that of Ayres and Barry (1980), Schaefer (1980), and Nelson and Schaefer (1983).

of different maturities which depend solely on the spread, s, the long-term interest rate, L, and the time to maturity, τ. In this equation we have the market prices of risk, λ_i, because our model solves for all bond prices relative to each other. The only way to tie down the prices is by means of these exogenous parameters, the market prices of risk.

The solution of the equation (2), subject to the terminal condition given by the payment to be received by the bondholder at maturity, that is, $P(s, L, 0) = 1$, $\forall s, L$, allows us to price discount bonds and, thereafter, infer the term structure of interest rates. In order to solve this valuation equation, we must make some assumptions about the market prices of risk and the dynamics of the state variables. Since a constant market price of risk implies strong restrictions on the preferences of investors, we establish the following:

Assumption 1 *The market price of each state variable risk is linear in this variable, that is*

$$\lambda_1(.) = a + bs, \quad \lambda_2(.) = c + dL \tag{3}$$

Assumption 2 *Each of the state variables follow a diffusion process*

$$\begin{cases} ds &= k_1(\mu_1 - s)dt + \sigma_1 dw_1 \\ dL &= k_2(\mu_2 - L)dt + \sigma_2 dw_2 \end{cases} \tag{4}$$

This process, known as Ornstein-Uhlenbeck process, has been used previously by Vasicek (1977). It has mean reversion - an important stylized fact that interest rates usually show - and constant variance. For each state variable, $k_i > 0$ is the coefficient of mean reversion which reflects the speed of adjustment of the variable towards its long-run mean value, μ_i, σ_i is the (constant) standard deviation of each variable and dw_i are standard Gauss-Wiener processes.

Under Assumptions 1 and 2, we can rewrite the equation (2) as

$$\frac{1}{2}\sigma_1^2 P_{ss} + q_1(\hat{\mu}_1 - s)P_s + \frac{1}{2}\sigma_2^2 P_{LL} + q_2(\hat{\mu}_2 - L)P_L + P_t - (L + s)P = 0 \tag{5}$$

subject to the terminal condition

$$P(s, L, T, T) = 1, \forall s, L \tag{6}$$

where

$$\begin{cases} q_1 = k_1 + b\sigma_1, & \hat{\mu}_1 = (k_1\mu_1 - a\sigma_1)/q_1 \\ q_2 = k_2 + d\sigma_2, & \hat{\mu}_2 = (k_2\mu_2 - c\sigma_2)/q_2 \end{cases} \tag{7}$$

Solving the partial differential equation (5) we obtain the following proposition:

Proposition 1 *The value at time t of a discount bond*[5] *that pays \$1 at time T, $P(s, L, t, T) \equiv P(s, L, \tau)$, is given by*

$$P(s, L, \tau) = A(\tau)e^{-B(\tau)s - C(\tau)L} \tag{8}$$

where $\tau = T - t$ and

$$\begin{aligned} A(\tau) &= A_1(\tau)A_2(\tau) \\ A_1(\tau) &= \exp\left\{-\frac{\sigma_1^2}{4q_1}B^2(\tau) + s^*(B(\tau) - \tau)\right\} \\ A_2(\tau) &= \exp\left\{-\frac{\sigma_2^2}{4q_2}C^2(\tau) + L^*(C(\tau) - \tau)\right\} \\ B(\tau) &= (1 - e^{-q_1\tau})/q_1 \\ C(\tau) &= (1 - e^{-q_2\tau})/q_2 \end{aligned} \tag{9}$$

with

$$\begin{aligned} q_1 = k_1 + b\sigma_1, \quad & s^* = \hat{\mu}_1 - \sigma_1^2/(2q_1^2), \quad \hat{\mu}_1 = (k_1\mu_1 - a\sigma_1)/q_1 \\ q_2 = k_2 + d\sigma_2, \quad & L^* = \hat{\mu}_2 - \sigma_2^2/(2q_2^2), \quad \hat{\mu}_2 = (k_2\mu_2 - c\sigma_2)/q_2 \end{aligned} \tag{10}$$

3 Generalized Duration and Convexity

In this section, we will generalize the concepts of duration and convexity taking into account the two-factor model for the analysis of the term structure of the interest rates that it has been presented previously. Thus, we can measure the interest rate risk with respect to these two stochastic factors and, hence, we are able to reflect the effects of changes in interest rates on the yield of a bond or a bond portfolio.

[5] Many other types of interest rates derivatives were priced by solving the valuation equation with the appropriate terminal condition. For details, see Moreno (1997).

We remember briefly the definitions of conventional duration and convexity of a coupon bond. Macaulay (1938) defines the duration of a coupon bond as a weighted average of the coupons paid by this bond:

$$\sum_{t=1}^{n} t \frac{CF_t/(1+y)^t}{P}$$

where CF_t is the coupon paid by the bond at time t, $(t = 1, 2, ..., n)$, y is the yield prevailing at these periods and P is the bond price. Dividing this expression by the number of payments per year, we obtain the Macaulay duration in years.

Macaulay duration indicates the volatility of the bond price because it is verified that

$$\begin{aligned} & \text{(Approximate) percentage change in the bond price} = \\ - & \left(\frac{1}{1+y}\right) \times \text{Macaulay duration} \times \text{Yield change} \end{aligned}$$

This equation can be rewritten as

$$\begin{aligned} & \text{(Approximate) percentage change in the bond price} = \\ - & \text{(Modified) duration} \times \text{Yield change} \end{aligned}$$

Thus, the modified duration indicates the percentage price change of a bond per 100-basis-point change in yield. For example, the price of a bond with modified duration equal to D changes approximately $D\%$ if its yield changes 100 basis points.

For small changes in yields, modified duration gives a good approximation of the percentage change in bond price. However, this measures does not capture the effect of the convexity of a bond on its price when yields change by more than a small amount. Thus, the estimation of the percentage change in bond price may be improved by using convexity, a measure that can reflect the curvature of the bond pricing equation. The convexity of a coupon bond is defined as

$$\sum_{t=1}^{n} t(t+1) \frac{CF_t/(1+y)^t}{(1+y)^2 P}$$

To convert the convexity from periods to years, this expression must be divided by the square of the number of payments per year.

Let $P(s,L,t,T) \equiv P(s,L,\tau)$ be the price, at time t, of a default-free zero-coupon bond that pays \$1 at maturity, $T = t + \tau$. This price is given by the expression

$$P(s,L,t,T) = P(t,T) = e^{-(T-t)Y(s,L,t,T)} \tag{11}$$

where $Y(s,L,t,T) \equiv Y(s,L,\tau)$, is the (continuously compound) yield related to this bond, that is, the yield to maturity.

Applying Itô's lemma, the instantaneous change in bond price is given by the following stochastic differential equation:

$$dP(.) = P_s ds + P_L dL + P_t dt + \frac{1}{2}P_{ss}(ds)^2 + \frac{1}{2}P_{LL}(dL)^2 \tag{12}$$

where

$$P_s = \frac{\partial P(.)}{\partial s}, \quad P_L = \frac{\partial P(.)}{\partial L}, \quad P_t = \frac{\partial P(.)}{\partial t}, \quad P_{ss} = \frac{\partial^2 P(.)}{\partial s^2}, \quad P_{LL} = \frac{\partial^2 P(.)}{\partial L^2}$$

Using the expression of these partial derivatives and the dynamics of the state variables (see equation (8) and Assumption 2), the equation (12) can be rewritten as

$$\begin{aligned} dP(t,T) &= [P_s k_1(\mu_1 - s) + P_L k_2(\mu_2 - L) + P_t + \frac{1}{2}P_{ss}\sigma_1^2 + \frac{1}{2}P_{LL}\sigma_2^2]dt \\ &- (T-t)P(t,T)\left[\frac{\partial Y(t,T)}{\partial s}\sigma_1 dw_1 + \frac{\partial Y(t,T)}{\partial L}\sigma_2 dw_2\right] \\ &= \mu_p(.)dt \\ &- (T-t)P(t,T)\left[\frac{\partial Y(t,T)}{\partial s}\sigma_1 dw_1 + \frac{\partial Y(t,T)}{\partial L}\sigma_2 dw_2\right] \end{aligned} \tag{13}$$

Next, we consider a coupon bond paying n coupons c_i at times t_i, $i = 1, 2, \ldots, n$. This bond has a nominal value equal to \$1 and matures at time $T = t_n$. This bond can be interpreted as a portfolio of n discount bonds: $n-1$ zero-coupon bonds mature at times t_i, $i = 1, \ldots, n-1$ and their face values are equal to c_i, $i = 1, \ldots, n-1$ and a zero-coupon bond that matures at time $T = t_n$ with a nominal value equal to $1 + c_n$, nominal value of the coupon bond plus the last coupon paid by this bond.

Let $P^*(s, L, t, T) \equiv P^*(t, T)$ be the price, at time t, of this coupon bond. Then

$$P^*(t,T) = \sum_{i=1}^{n} c_i P(t, t_i)$$

Using equation (13), it is verified that

$$\begin{aligned}
dP^*(t,T) &= \sum_{i=1}^{n} c_i \left(\mu_p(.)dt - (t_i - t)P(t,t_i) \left[\frac{\partial Y(t,t_i)}{\partial s}\sigma_1 dw_1 + \frac{\partial Y(t,t_i)}{\partial L}\sigma_2 dw_2 \right] \right) \\
&= \sum_{i=1}^{n} c_i \mu_p(.)dt \\
&- \sum_{i=1}^{n} c_i (t_i - t) P(t,t_i) \frac{\partial Y(t,t_i)}{\partial s}\sigma_1 dw_1 \\
&- \sum_{i=1}^{n} c_i (t_i - t) P(t,t_i) \frac{\partial Y(t,t_i)}{\partial L}\sigma_2 dw_2
\end{aligned}$$

Therefore, the instantaneous percentage change in the price of this coupon bond is given by the expression

$$\begin{aligned}
\frac{dP^*(t,T)}{P^*(t,T)} &= \frac{1}{P^*(t,T)} \sum_{i=1}^{n} c_i \mu_p(.)dt \\
&- \frac{1}{P^*(t,T)} \sum_{i=1}^{n} c_i (t_i - t) P(t,t_i) \frac{\partial Y(t,t_i)}{\partial s}\sigma_1 dw_1 \\
&- \frac{1}{P^*(t,T)} \sum_{i=1}^{n} c_i (t_i - t) P(t,t_i) \frac{\partial Y(t,t_i)}{\partial L}\sigma_2 dw_2
\end{aligned}$$

that can be rewritten as

$$\frac{dP^*(t,T)}{P^*(t,T)} = \frac{1}{P^*(t,T)} \sum_{i=1}^{n} c_i \mu_p(.)dt - D_s \sigma_1 dw_1 - D_L \sigma_2 dw_2$$

where

$$\begin{aligned}
D_s &= \frac{1}{P^*(t,T)} \sum_{i=1}^{n} c_i (t_i - t) P(t,t_i) \frac{\partial Y(t,t_i)}{\partial s} \\
D_L &= \frac{1}{P^*(t,T)} \sum_{i=1}^{n} c_i (t_i - t) P(t,t_i) \frac{\partial Y(t,t_i)}{\partial L}
\end{aligned}$$

The parameters D_s and D_L represent the "generalized duration" measures and reflect the sensitivity of the bond yield to changes in the factors s and L. This "generalized duration" is different from the conventional duration in two characteristics:

- There are two duration measures, one for each factor. With the conventional duration, there is just one duration measure, that reflects the sensitivity of the bond yield to changes in the single factor.
- There is an additional term
$$\frac{\partial Y(t,t_i)}{\partial s}, \quad \frac{\partial Y(t,t_i)}{\partial L}, \quad i = 1, 2, \ldots, n$$
that reflects the sensitivity of the yield to maturity to changes in each factor.

Now, we can define the measures of "generalized duration":

Definition (Generalized Duration) *The "generalized durations" D_s and D_L of a bond that pays n coupons c_i at times t_i, $i = 1, 2, \ldots, n$ with respect to the factors s and L are given by the expressions*

$$D_s = \frac{1}{P^*(t,T)} \sum_{i=1}^{n} c_i(t_i - t)P(t,t_i)\frac{\partial Y(t,t_i)}{\partial s}$$
$$D_L = \frac{1}{P^*(t,T)} \sum_{i=1}^{n} c_i(t_i - t)P(t,t_i)\frac{\partial Y(t,t_i)}{\partial L}$$

where $P(t,t_i)$ is the price, at time t, of a zero-coupon bond that matures at time t_i (see Proposition 1)

For a zero-coupon bond, we have

$$P^*(t,T) = P(t,T), \quad c_1 = c_2 = c_{n-1} = 0, \quad c_n = 1$$

Therefore, the "generalized durations" D_s and D_L of a zero-coupon bond with respect to the factors s and L are given by

$$\begin{aligned} D_s &= \frac{1}{P(t,T)}(T-t)P(t,T)\frac{\partial Y(t,T)}{\partial s} = (T-t)\frac{\partial Y(t,T)}{\partial s} \\ D_L &= \frac{1}{P(t,T)}(T-t)P(t,T)\frac{\partial Y(t,T)}{\partial L} = (T-t)\frac{\partial Y(t,T)}{\partial L} \qquad (14) \end{aligned}$$

Replacing the equations (8) and (11) in (14), we obtain that the final expression for the measures of "generalized duration" corresponding to a zero-coupon bond with respect to the factors s and L are given by

$$D_s = B(t,T) = B(\tau) \tag{15}$$

$$D_L = C(t,T) = C(\tau) \tag{16}$$

where $B(\tau)$ and $C(\tau)$ are the terms included in the final expression that we have obtained for the zero-coupon bond pricing formula (see Proposition 1).

Hence, $B(\tau)$ and $C(\tau)$ indicate the sensitivity of a zero-coupon bond in relation to the risk derived from interest rate changes. Thus, $B(\tau)$, the duration with respect to the spread, reflects the sensitivity of a zero-coupon bond to spread changes. Therefore, $B(\tau)$ assesses the influence of the changes in the slope of the yield curve on the bond price. Analogously, $C(\tau)$, the duration with respect to the long-term rate, reflects the influence of parallel changes in the yield curve on the price of this zero-coupon bond.

Therefore, both measures of "generalized duration" can be useful to deal with the interest rate risk derived from changes in the level and in the slope of the yield curve. Once quantified the behavior of the bond portfolio associated to these changes, these measures can be an adequate tool for portfolio management. Thus, the investors who want to immunize their portfolios with respect to these changes must manage the portfolios in such a way that their measures of "generalized duration" equate the ones of the asset to be replicated. If these investors believe that interest rates will fall (increase), the composition of the portfolio must be changed in order to increase (decrease) its "generalized duration". A similar argument runs if they expect an increase or decrease in the spread.

Convexity is a measure that can complement the results provided by duration. Convexity corrects the estimation (obtained from duration) of the bond price change when interest rates change in a large amount. Now, we will generalize the convexity to take into account the changes that can experience the two stochastic factors of our model.

Definition (Generalized Convexity) *The "generalized convexities" δ_s and δ_L of a bond that pays n coupons c_i at time t_i, $i = 1, 2, \ldots, n$ with respect to the factors s and L are given by the expressions*

$$\delta_s = \frac{1}{P^*(t,T)} \sum_{i=1}^{n} c_i (t_i - t) P(t, t_i) \frac{\partial^2 Y(t, t_i)}{\partial s^2}$$

$$\delta_L = \frac{1}{P^*(t,T)} \sum_{i=1}^{n} c_i(t_i - t) P(t,t_i) \frac{\partial^2 Y(t,t_i)}{\partial L^2}$$

where $P(t,t_i)$ is the price, at time t, of a zero-coupon bond that matures at time t_i (see Proposition 1)

Hence, the "generalized convexities" δ_s and δ_L of a zero-coupon bond with respect to the factors s and L are given by

$$\begin{aligned} \delta_s &= \frac{1}{P(t,T)} \frac{\partial^2 P(t,T)}{\partial s^2} = B^2(t,T) = B^2(\tau) \\ \delta_L &= \frac{1}{P(t,T)} \frac{\partial^2 P(t,T)}{\partial L^2} = C^2(t,T) = C^2(\tau) \end{aligned} \tag{17}$$

4 Hedging Ratios

An alternative technique to duration as a tool of managing interest rate risk may be performed by means of options on bonds. As duration measures the sensitivity of a present value to changes in interest rates, this measure can be applicable not only to bonds but it can be extended to options. Thus, it is possible to define measures of the sensitivity of different interest rate derivatives with respect to several factors and, after that, to construct the corresponding hedging strategy.

We consider an European call option on a zero-coupon bond. Let K be the strike price of this option. It this option is exercised at expiration, T_c, the callholder pays K and receives a discount bond which matures at time $T_b > T_c$.

The price at time t, $C(s,L,t,T_c;K,T_b)$, of this call option (see Moreno (1997), Section 3.5) is given by

$$C(s,L,t,T_c;K,T_b) = P(t,T_b)\Phi(h + \sigma_{\tilde{P}}) - KP(t,T_c)\Phi(h) \tag{18}$$

where $P(t,T_i)$ is the price, at time t, of a zero-coupon bond that matures at time T_i (see Proposition 1), $\Phi(.)$ denotes the distribution function of a standard normal variable and

$$h = \frac{\ln(P(t,T_b)) - \ln(KP(t,T_c))}{\sigma_{\tilde{P}}} - \frac{1}{2}\sigma_{\tilde{P}}$$

Derivating the equation (18), we obtain that the generalized duration of this call option to spread changes (an indicative measure of the change in the call price to changes in this factor) is given by

$$\begin{aligned}\frac{\partial C(.)}{\partial s} &= P_s(t,T_b)\Phi(h+\sigma_{\tilde{p}}) + P(t,T_b)\Phi_s(h+\sigma_{\tilde{p}}) \\ &- KP_s(t,T_c)\Phi(h) - KP(t,T_c)\Phi_s(h) \\ &= -B(t,T_b)P(t,T_b)\Phi(h+\sigma_{\tilde{p}}) + P(t,T_b)\Phi_s(h+\sigma_{\tilde{p}}) \\ &+ KB(t,T_c)P(t,T_c)\Phi(h) - KP(t,T_c)\Phi_s(h) \qquad (19)\end{aligned}$$

Applying the chain's rule, it follows that

$$\begin{aligned}\Phi_s(h+\sigma_{\tilde{p}}) &= \frac{\partial\Phi(h+\sigma_{\tilde{p}})}{\partial(h+\sigma_{\tilde{p}})}\frac{\partial(h+\sigma_{\tilde{p}})}{\partial s} = f(h+\sigma_{\tilde{p}})\frac{B(t,T_c)-B(t,T_b)}{\sigma_{\tilde{p}}} \\ \Phi_s(h) &= f(h)\frac{B(t,T_c)-B(t,T_b)}{\sigma_{\tilde{p}}}\end{aligned}$$

Therefore, equation (19) can be rewritten as

$$\begin{aligned}\frac{\partial C(.)}{\partial s} &= -B(t,T_b)P(t,T_b)\Phi(h+\sigma_{\tilde{p}}) + P(t,T_b)f(h+\sigma_{\tilde{p}})\frac{B(t,T_c)-B(t,T_b)}{\sigma_{\tilde{p}}} \\ &+ KB(t,T_c)P(t,T_c)\Phi(h) - KP(t,T_c)f(h)\frac{B(t,T_c)-B(t,T_b)}{\sigma_{\tilde{p}}} \\ &= [B(t,T_c)-B(t,T_b)]\left[P(t,T_b)\frac{f(h+\sigma_{\tilde{p}})}{\sigma_{\tilde{p}}} - KP(t,T_c)\left(\frac{f(h)}{\sigma_{\tilde{p}}} - \Phi(h)\right)\right. \\ &- \left. B(t,T_b)C(t,T_c;K,T_b)\right] \qquad (20)\end{aligned}$$

Analogously, the generalized duration of the call option to changes in the long-term interest rate is given by

$$\begin{aligned}\frac{\partial C(.)}{\partial L} &= [C(t,T_c)-C(t,T_b)]\left[P(t,T_b)\frac{f(h+\sigma_{\tilde{p}})}{\sigma_{\tilde{p}}} - KP(t,T_c)\left(\frac{f(h)}{\sigma_{\tilde{p}}} - \Phi(h)\right)\right. \\ &- \left. C(t,T_b)C(t,T_c;K,T_b)\right]\end{aligned}$$

These measures of "generalized duration" of the option allow us to obtain the hedging ratio corresponding to this option. We will apply the following relationship[6] that links the generalized durations of the call option and its underlying asset (a zero-coupon bond):

$$\begin{aligned}\text{Gener. duration of the option} &= \text{Elasticity of the option} \\ &\times \text{Gener. duration of the bond} \quad (21)\end{aligned}$$

where

$$\text{Elasticity of the option} = \frac{P(t,T_b)}{C(t,T_c;K,T_b)} \frac{\partial C(t,T_c;K,T_b)}{\partial P(t,T_b)} \tag{22}$$

The elasticity of the option is the product of two terms. The first term is the ratio of the price of the bond (underlying asset) to the price of the option and may be interpreted as the "leverage" of this option. The second term reflects the influence of a change in the underlying asset price on the option price and it is the hedging ratio. Therefore, from equations (15), (21) and (22), it is deduced that the hedging ratio ("delta" of the option) is equal to

$$\Delta = \frac{\partial C(.)}{\partial s} \frac{1}{B(t,T_b)} \frac{C(t,T_c;K,T_b)}{P(t,T_b)}$$

where $\partial C(.)/\partial s$ is given by equation (20).

5 The Hedging Problem

Generally speaking, the hedging problem is related to the construction of a trading (buy or sell) strategy from certain assets in order to replicate the value of a target security. In particular, the goal may be to obtain a set of cash-flows such that we can guarantee a set of future payments (this problem is known as *"asset/liability management"*). Once the "replicating" strategy of the target security is constructed, hedging the risk of this target security is reached by selling this "replicating" strategy. In other words, hedging one position requires the synthetic construction of the opposite position.

Let $T(s, L, t)$ be the price, at time t, of the target security to be hedged. This price is given by the solution of the partial differential equation with

[6] See Fabozzi (1993), Chap. 15.

the appropriate terminal condition that reflects the payment to be received at the maturity of the asset (see Moreno (1997), Section 3.5).

We will make the following assumptions:

- There exist two traded assets, U_1 and U_2, whose market values, at time t, are given by

$$\begin{aligned} U_1 &= U_1(s, L, t) \\ U_2 &= U_2(s, L, t) \end{aligned}$$

- The price of the target-asset and of these two assets is at least once differentiable with respect to s and L.

- The changes in the price of these assets when the two factors of our model change are linearly independent:

$$\begin{vmatrix} U_{1s} & U_{2s} \\ U_{1L} & U_{2L} \end{vmatrix} \neq 0$$

Under these conditions, we set up a hedging portfolio with one share of the target security T, x_1 shares of the asset U_1 and x_2 shares of the asset U_2. The market value, at time t, of this portfolio is given by

$$V(s, L, t) = T(s, L, t) + x_1 U_1(s, L, t) + x_2 U_2(s, L, t)$$

This portfolio is perfectly hedged in relation to changes in the two factors s and L if its market value does not change when these two factors change. Therefore, it must be verified that

$$\frac{\partial V(s, L, t)}{\partial s} = \frac{\partial V(s, L, t)}{\partial L} = 0$$

Thus, we obtain the following system of equations

$$\begin{cases} T_s(s, L, t) + x_1 U_{1s}(s, L, t) + x_2 U_{2s}(s, L, t) = 0 \\ T_L(s, L, t) + x_1 U_{1L}(s, L, t) + x_2 U_{2L}(s, L, t) = 0 \end{cases}$$

that can be rewritten as

$$\begin{cases} x_1 U_{1s}(s, L, t) + x_2 U_{2s}(s, L, t) = -T_s(s, L, t) \\ x_1 U_{1L}(s, L, t) + x_2 U_{2L}(s, L, t) = -T_L(s, L, t) \end{cases}$$

This system of equations, because of the last assumption made previously, has a unique solution $(x_1(s,L,t), x_2(s,L,t))$ that reflects the shares of the assets $U_1(.)$ and $U_2(.)$ to be included in the hedging portfolio in order to hedge the target security. This solution is give by the Cramer's rule applied to the system of equations:

$$x_1 = \frac{\begin{vmatrix} -T_s(.) & U_{2s}(.) \\ -T_L(.) & U_{2L}(.) \end{vmatrix}}{\begin{vmatrix} U_{1s}(.) & U_{2s}(.) \\ U_{1L}(.) & U_{2L}(.) \end{vmatrix}}, \quad x_2 = \frac{\begin{vmatrix} U_{1s}(.) & -T_s(.) \\ U_{1L}(.) & -T_L(.) \end{vmatrix}}{\begin{vmatrix} U_{1s}(.) & U_{2s}(.) \\ U_{1L}(.) & U_{2L}(.) \end{vmatrix}}$$

where $T_s(.)$ and $T_L(.)$ are the measures of "generalized duration" of the target security, $U_{1s}(.)$ and $U_{1L}(.)$ are the "generalized durations" of the security U_1 and $U_{2s}(.)$ and $U_{2L}(.)$ are the "generalized durations" of the security U_2.

6 Hedging Yield Curve Risk

In this section, we will use the measures of "generalized duration" we have defined above to perform a hedging strategy against an specific type of risk, the risk derived from "non-parallel" changes in the yield curve.

We consider an investor that must pay a certain amount, $\$M$, at time m. At the current time, t_0, the wealth of this investor is equal to the present value - calculated according to the term structure prevailing at time t_0, $Y(s_0, L_0, t_0, t) \equiv Y(t_0, t)$ - of the amount $\$M$ to be paid in the future.

The goal of the investor is to use his current wealth to buy a portfolio of n bonds. The bonds available in the market have the following characteristics:

- Principal amount of each bond: \$1
- Bond price: $P_1(.), P_2(.), ..., P_n(.)$
- Bond maturity: $T^1, T^2, ..., T^n$
- Coupon flow: $c_i(t), t \leq T^i$, $i = 1, 2, \ldots, n$

The price, at time m, of each bond is

$$\begin{aligned}
V_1(s_0, L_0) &= \int_{t_0}^{T^1} c_1(t)\exp\{-Y(m,t)(t-m)\}dt + \exp\{-Y(m,T^1)(T^1-m)\} \\
V_2(s_0, L_0) &= \int_{t_0}^{T^2} c_2(t)\exp\{-Y(m,t)(t-m)\}dt + \exp\{-Y(m,T^2)(T^2-m)\} \\
&\dots \\
V_n(s_0, L_0) &= \int_{t_0}^{T^n} c_n(t)\exp\{-Y(m,t)(t-m)\}dt + \exp\{-Y(m,T^n)(T^n-m)\}
\end{aligned}$$

Let x_1, x_2,..., x_n be the proportions of the portfolio that we invest in each bond. Then, the value, at time m, of the portfolio consisting of n bonds is

$$\begin{aligned}
V_p(s_0, L_0) &= \sum_{i=1}^{n} x_i V_i(s_0, L_0) \\
&= \sum_{i=1}^{n} x_i \left[\int_{t_0}^{T^i} c_i(t)\exp\{-Y(m,t)(t-m)\}dt + \exp\{-Y(m,T^i)(T^i-m)\}\right]
\end{aligned}$$

That is the value of this bond portfolio if the yield curve available at time t_0 does not change during the time interval $[t_0, m]$. But, from now, it is assumed that, because of changes in the two stochastic factors of our model, the yield curve changes instantaneously after the bond portfolio is bought. The new term structure is given by $Y(\hat{s}, \hat{L}, t_0, t) \equiv \hat{Y}(t_0, t)$. Considering this new term structure, the value - at time m - of the i-th bond is given by

$$V_i(\hat{s}, \hat{L}) = \int_{t_0}^{T^i} c_i(t)\exp\{-\hat{Y}(m,t)(t-m)\}dt + \exp\{-\hat{Y}(m,T^i)(T^i-m)\} \quad (23)$$

and, therefore, the market value, at time m, of this bond portfolio is

$$\begin{aligned}
V_p(\hat{s}, \hat{L}) &= \sum_{i=1}^{n} x_i V_i(\hat{s}, \hat{L}) \\
&= \sum_{i=1}^{n} x_i \left[\int_{t_0}^{T^i} c_i(t)\exp\{-\hat{Y}(m,t)(t-m)\}dt + \exp\{-\hat{Y}(m,T^i)(T^i-m)\}\right]
\end{aligned}$$

This new market value must be large enough to guarantee the payment of the amount $\$M$ at time m. The goal of hedging is to make sure that the lowest future value of this bond portfolio (as a function of the values $\hat{s}$ and $\hat{L}$) is equal or greater than $\$M$.

Therefore, the "immunized portfolio", denoted as $x^* = (x_1^*, x_2^*, \ldots, x_n^*)$, must make the optimal value of

$$\begin{aligned} \min_{s,L} \quad & \left\{ \sum_{i=1}^{n} x_i^* V_i(\hat{s}, \hat{L}) \right\} \\ \text{s.t} \quad & \sum_{i=1}^{n} x_i^* P_i = \exp\{-Y(t_0, m)(m - t_0)\} M \end{aligned}$$

equal to $\$M$. The constraint of this problem is the budget constraint of the investor: the price of the "immunized portfolio" must be equal to the wealth of the investor at time t_0.

Hence, the "immunized portfolio" $x^* = (x_1^*, x_2^*, ..., x_n^*)$ is the solution of the problem

$$\begin{aligned} \max_{x \geq 0} \quad & \left\{ \min_{s,L} \left\{ \sum_{i=1}^{n} x_i V_i(\hat{s}, \hat{L}) \right\} \right\} \\ \text{s.t} \quad & \sum_{i=1}^{n} x_i P_i = \exp\{-Y(t_0, m)(m - t_0)\} M \end{aligned}$$

Using the equations (11) and (23) and the closed-form expression for the bond price, we can deduce that the "immunized portfolio" $x^* = (x_1^*, x_2^*, ..., x_n^*)$ solves the problem

$$\begin{aligned} \max_{x \geq 0} \quad & \{ \min_{s,L} \sum_{i=1}^{n} x_i \int_{t_0}^{T^i} c_i(t) \exp\{\ln(A(t-m)) - B(t-m)\hat{s} - C(t-m)\hat{L}\} dt \\ & + \sum_{i=1}^{n} x_i \exp\{\ln(A(T^i - m)) - B(T^i - m)\hat{s} - C(T^i - m)\hat{L}\} \ \} \\ \text{s.t} \quad & \sum_{i=1}^{n} x_i P_i = \exp\{-Y(t_0, m)(m - t_0)\} M \end{aligned}$$

7 A Proposal of Solution for the Limitations of the Conventional Duration

In this section we illustrate numerically how the new measures of "generalized duration" D_s and D_L can be applied to mitigate the limitations of the conventional duration. In the example shown below we have two portfolios with the same "generalized durations" with respect to the spread and the long-term interest rate. Both portfolios differ in yield and convexity. We will see that, because of the equality between their "generalized durations", the relative behavior of both portfolios does not depend on the magnitude nor the type of changes that the yield curve shows.

The portfolio 1 consists of bonds A, B and C and the portfolio 2 includes only the bond D. The characteristics of these bonds, whose nominal value is equal to 100 and whose coupons are paid semesterly, are given by the following table:

Bond	A	B	C	D
Coupon (%)	5.5	10	12	9
Maturity (years)	5	15	20	10
Yield (%)	5.5	10	12	9
Duration with respect to s (D_s)	0.5085	0.5499	0.6344	0.5346
Duration with respect to L (D_L)	0.8714	0.9342	1.0750	0.9123
Convexity with respect to s (δ_s)	0.3686	0.3924	0.4519	0.3826
Convexity with respect to L (δ_L)	1.1002	1.1644	1.3354	1.1439

The proportions of the bonds A, B and C to be included in the portfolio 1 are chosen to equate the "generalized durations" D_s and D_L (per a 100-basis-points change in yield) and the market values of both portfolios. Thus, we obtain the following system of equations:

$$\begin{cases} x_A D_s^A + x_B D_s^B + x_C D_s^C &= D_s^D \\ x_A D_L^A + x_B D_L^B + x_C D_L^C &= D_L^D \\ x_A + x_B + x_C &= 1 \end{cases} \tag{24}$$

where x_j, $j = A, B, C$, is the proportion invested on each bond and where D_i^j, $i = s, L$, $j = A, B, C$, represents the "generalized duration" of the j-th bond with respect to the i-th factor. Solving this system of equations, we obtain that the proportions of the bonds A, B and C are 59.930%, 11.203% and 28.866%, respectively.

Now we can compute the "generalized convexities" and the yield of both portfolios. To compute the values associated to the portfolio 1, we use the weighted average (where the weights are given by the proportion of the portfolio invested on each bond) of the convexity and the yield of the bonds included in this portfolio. The results obtained are the following:

	Portfolio 1	Portfolio 2
Duration with respect to s (D_s)	0.5346	0.5346
Duration with respect to L (D_L)	0.9123	0.9123
Duration with respect to r (D_r)	2.8476	3.1155
Convexity with respect to s (δ_s)	0.3848	0.3826
Convexity with respect to L (δ_L)	1.1451	1.1439
Yield (%)	7.5272	9

These results corroborate that both portfolios have the same "generalized durations" with respect to the spread and to the long-term interest rate. Moreover, it is shown that the "generalized duration" with respect to the short-term interest rate and the yield of the portfolio 2 are greater than the ones of the portfolio 1. On the other hand, the"generalized convexities" of the portfolio 2 are slightly lower than the ones of the portfolio 1. The difference between the yields of the two portfolios suggests that the best strategy to be followed consists of buying the portfolio 2 and selling the portfolio 1. Thus, we can obtain a gain of 147.2 basis points. However, the greater "generalized convexities" of the portfolio 1 suggest that this portfolio would provide a greater yield (in comparison with the portfolio 2) if there would be certain shifts in the yield curve.

From now, we will assume that there is a shift in the yield curve instantaneously after the acquisition of these portfolios and we will analyze the relative behavior, measured by the difference between the yields obtained by both portfolios in a certain investment horizon (six months). In short, we will analyze three types of shifts in the yield curve:

- Parallel shift: interest rates with different maturities change by the same magnitude and with the same direction.
- Twist in the slope of the yield curve: interest rates with different maturities shift in the same direction but with different magnitude. We consider two alternatives:
  - Changes in short-term (long-term) interest rates are equal to changes in intermediate-term interest rates plus (less) a certain amount. Under this assumption, the slope of the yield curve decreases.

– Changes in short-term (long-term) interest rates are equal to changes in intermediate-term interest rates less (plus) a certain amount. Therefore, there is an increase in the slope of the yield curve.

As a consequence of our investment, we will obtain a certain accumulated value for each portfolio at the end of the investment horizon. This accumulated value is equal to the sum of three terms: the coupons paid by the bonds included in each portfolio, the interests generated from these coupons and the market value of the bonds included in each portfolio. Changes in interest rates have two effects on this accumulated value and, hence, on the yield of each portfolio. The first effect of this change is derived from the amount of money generated from the coupons and we will refer it as "reinvestment risk". The second effect is related to the market value of the bond and it represents the "price risk".

If interest rates increase, the coupons we have received generate a bigger amount of money but the market value of the bond decreases. The opposite situation happens if interest rates fall. Therefore, the final value generated by each portfolio - in comparison with the value obtained if there are no changes in interest rates - will depend on the changes in interest rates and on the effects of both types of risk.

In our case, the investment horizon is equal to six months. Therefore, we do not need to make any assumption on the reinvestment rate of the coupons because each bond is sold at the end of the investment horizon, when we receive the first (and last) coupon.

Table VII shows the accumulated value we obtain six months later and the yield (in annual terms) obtained with each one of the bonds included in the portfolio 1 when there is a parallel change in the yield curve. The first column of this table includes the magnitude, equal for the three bonds, of the change in interest rates. In this case, the accumulated value is equal to the market value of the bond plus the coupon we have received. The value of this coupon, paid by the bonds A, B and C, is equal to 2.75, 5 and 6, respectively. We can observe that, when interest rates are increasing, the yield provided by each bond is decreasing. The higher yield is provided by the bond C because it corresponds to the longest maturity. On the other hand, the bond A, corresponding to the shortest maturity, shows the narrower interval of yields.

Table VIII includes the accumulated value and the yield provided by each portfolio. The accumulated value of the portfolio 1 is the weighted average of the accumulated values (see the previous table) that we have obtained with the two bonds included in it. Analogously to the Table VII, the first column shows the change in the yield curve. The last column shows the difference between the yields of both portfolios. In short, this difference is computed as "yield of the portfolio 2" minus "yield of the portfolio 1". Thus, a negative (positive) value means that the portfolio 1 generates a higher (lower) than the portfolio 2.

The strategy to be followed with these portfolios depends on the type of shift in the yield curve. As indicated by the last column of this table, the portfolio 1 will outperform the portfolio 2 if interest rates increase. On the other hand, we can see that the portfolio 2 provides a greater yield than the portfolio 1 when interest rates fall.

This result is a consequence of a feature we have highlighted previously: the "generalized duration" with respect to the short-term interest rate of the portfolio 1 is lower than the one of the portfolio 2. Therefore, an increase (decrease) in interest rates implies a greater decrease (increase) in the accumulated value obtained from this portfolio in comparison with the decrease (increase) occurred in the accumulated value of the portfolio 1. Therefore, this portfolio performs better (worse) than the portfolio 2. Other consequence of the difference of "generalized durations" with respect to the short-term interest rate is that the additional yield obtained in each case increases monotonically with respect to the magnitude of the change in interest rates: the longer the change in interest rates, the bigger the influence on the final yield provided by each portfolio.

After that, we will analyze the effects of a change in the slope of the yield curve. The results obtained with the two alternatives we consider are included in Tables IX-X and Tables XI-XII, respectively.

In the first non-parallel change to be analyzed, it is assumed a flattening (decrease in the slope) of the yield curve. In short, it is assumed that the change in the interest rate related to the maturity of the bond A (C) is equal to the change in interest rate corresponding to the bond D plus (minus) 100 basis points whereas the change in the interest rate corresponding to 15 years (bond B) is equal to the change in the interest rate corresponding to the bond D minus 50 basis points. The results we obtain for the three bonds included in the portfolio 1 are shown in Table IX while Table X includes the results obtained with the two portfolios.

Analogously to Table VII, Table IX shows the influence of the change in interest rates on the accumulated value and on the yield of each one of the bonds included in the portfolio 1. The first column shows the change in interest rates related to 10 years, the maturity of the bond D included in the portfolio 2. The results are similar to the ones obtained with a parallel change: an increase in interest rates leads to a decrease in the bond yield. Moreover, the highest yields are obtained for the bond with the longest maturity while the bonds with shortest maturity show a narrower range of values.

Similarly to Table VIII, Table X includes the accumulated value and the final yield obtained from each portfolio. Analogously to Table IX, the first column indicates the change in 10-years interest rates. The last column of this table shows the difference between the yields of both portfolios. It has been analyzed a different type of change in the yield curve but the last column of this table shows analogous results to the ones obtained with the parallel change: the portfolio 1 outperforms the portfolio 2 if interest rates rise and the opposite situation occurs if interest rates fall. As in the previous change, this behavior is because of the portfolio 1 shows less "generalized duration" with respect to the short-term interest rate than the portfolio 2. In percentage terms, an increase in interest rates harms more to the portfolio 2 while a decrease in interest rates benefits more to the portfolio 1. Analogously to the previous change, the difference between the"generalized durations" with respect to the short-term interest rate implies that the additional gain provided from each portfolio is monotonic with respect to the magnitude of the change in interest rates.

The second non-parallel change reflects an increase in the slope of the yield curve. The change in the interest rate associated to the shortest (longest) maturity is equal to the change in the interest rate related to the bond D minus (plus) 100 basis points while the change in 15-year interest rates (bond B) is equal to the change corresponding to the bond D plus 50 basis points. The results obtained for the three bonds included in the portfolio 1 are shown in Table XI while Table XII contains the results obtained for both portfolios.

Table XI shows the changes occurred in the accumulated value and in the yield of each bond. Analogously to the table IX, the first column contains the change of 10-year interest rates, the maturity of the bond D. The results are similar to the obtained with the two previous cases: an increase in interest rates decreases the final yield of each bond and the highest yields are obtained with the longest bond.

The accumulated value and the yield of each portfolio are included in the Table XII. Once again, the last column of this table shows similar results to the obtained in the previous changes: the portfolio 2 outperforms the portfolio 1 if interest rates fall and conversely if interest rates rise. Moreover, as in the above changes, this aspect is a consequence of the differences in generalized durations with respect to the short-term interest rate. It is also verified that the additional yield of the portfolio increases if we consider higher changes in interest rates.

Therefore, the main conclusion of this example is that, after analyzing three types of changes in the yield curve, the measures of "generalized duration" inform appropriately about the future behavior of a portfolio with respect to unexpected changes in the yield curve. It has been corroborated that, independently of the magnitude and the type of change in interest rates, the portfolio 1 provides a higher (lower) yield in comparison with the portfolio 2 whenever interest rates rise (fall). Moreover, we have also shown that the additional gain obtained in each case is monotonic with respect to the magnitude of the change in the yield curve. These two characteristics are because of the lower "generalized duration" with respect to the short-term interest rate shown by the portfolio 1: if interest rates rise, the accumulated value of each portfolio decreases less than the accumulated value of the portfolio 2 and, hence, a higher yield is obtained. The opposite situation runs if interest rates fall.

This example illustrates a fact with important practical consequences for the management of fixed income portfolios: given a certain portfolio, it is possible to build a second portfolio with the same " generalized durations" with respect to the spread and to the long-term interest rate. These equal "generalized durations" guarantee that the relative behavior of these two portfolios does not depend on the type (and magnitude) of the future change in the yield curve. That is, with a change in the level and/or in the slope of the yield curve, the portfolio with lower (higher) "generalized duration" with respect to the short-term interest rates is better than the other one if interest rates increase (fall). Therefore, if we expect an increase (decrease) in interest rates, we must choose the portfolio with lower (higher) "generalized duration" with respect to the short-term interest rate.

If the expectations on the movements in interest rates change over time, we can change (1) the election between the available portfolios and/or (2) the composition of these portfolios.

8 Conclusions

Interest rate risk is associated to changes in the yield curve. We have two types of risk, the "market risk", related to the parallel changes in this curve and the "yield curve risk", derived from changes in the slope and/or in the curvature of the yield curve. The classic solution to manage the risk market is based on conventional duration but it remains unresolved the problem of managing the second type of risk.

Trying to manage this type of risk, this paper has presented and applied new measures of "generalized duration". These measures are based on a certain two-factor continuous-time model for the term structure of interest that has been briefly presented. This model was initially proposed in Moreno (1997) and allows to generalize the conventional duration and convexity in order to obtain measures of "generalized" duration and convexity.

This new measure of "generalized duration" has been used to compute the hedging ratios. These ratios allow us to immunize a bond portfolio by means of options on bonds. Moreover, we have described how these measures can be used to manage the interest rate risk with respect to changes (parallel or in the slope) in the yield curve.

Finally, we have shown a numerical example that illustrates how the new measures presented in this paper can help us to deal with the limitations of previous and more conventional techniques. The problem associated to conventional duration is that this measure does not provide adequate information about the future behavior of a portfolio when the yield curve changes in a non-parallel way.

The example we have presented is based on two portfolios with the same "generalized durations" with respect to the spread and to the long-term interest rate. Analyzing three different changes in the yield curve (a parallel change and two types of change in the slope), it has been corroborated that the new measures of "generalized durations" do provide adequate information on the future behavior of a portfolio with respect to unexpected changes in the yield curve. Thus, we have shown that, independently of the magnitude and the type of change in interest rates, the portfolio with lower "generalized duration" with respect to the short-term interest rate provides a higher (lower) yield if interest rates rise (fall). The additional gain we obtain is monotonic with respect to the magnitude of the change in the yield curve. In other words, if interest rate increase, the accumulated value of the portfolio with lower “generalized duration” with respect to the short-term interest rate decreases less than the accumulated value of the alternative portfolio and, hence, it is obtained a higher yield.

As a consequence, these new measures of “generalized duration” can be a useful tool when managing fixed income portfolios: independently of the type of change (and of the magnitude of this change) in interest rates, the relevant characteristics to determine the future behavior of a fixed income portfolio are the “generalized duration” with respect to the short-term interest rate and the expectations on the movements (increase or decrease) in interest rates. Thus, with two alternative portfolios that have the same measures of “generalized duration” with respect to the spread and to the long-term interest rate, the best portfolio is the one with lower (higher) “generalized duration” with respect to the short-term interest rate if interest rates rise (fall). Therefore, depending on the expectations about the future movements (increase or decrease) of interest rates, we must choose the appropriate portfolio (with lower or higher “generalized duration” with respect to the short-term interest rate).

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Appendix:
Limitations of the Conventional Duration

In this appendix we illustrate numerically the limitations of the conventional duration as an informative measure of the changes in bond (or bond portfolio) price when yield changes. Moreover, this appendix also includes the results obtained with the proposal of solution that has been presented in this paper.

In the example shown in this appendix, two bond portfolios with the same modified duration will be considered. Both portfolios differ in their yield and convexity. A priori, the best strategy may be to buy the portfolio with higher yield and to sell the portfolio with lower yield. However, the different convexity in both portfolios suggests that other investment alternative would be possible if there were certain changes in the yield curve. We will see that, in spite of having the same modified duration, the relative future behavior of these portfolios depends on the magnitude and the type of change in the yield curve.

We consider two bond portfolios: the portfolio 1 includes the bonds A and B and the portfolio 2 consists of only bond C. These three bonds have a nominal value equal to 100, pay semesterly a coupon and their characteristics are given by the following table:

Bond	Coupon (%)	Maturity (years)	Yield (%)	Modified Duration	Convexity
A	7	5	7	4.1583	20.9592
B	9.75	20	9.75	8.7284	120.7668
C	9	10	9	6.5039	56.3576

The proportions of the bonds A and B to be included in portfolio 1 are chosen to equate the modified durations, per a 100-basis-point change in yield, and the market values of both portfolios. Analogously to equation (24), we obtain the following system of equations

$$\begin{cases} x_A D_r^A + x_B D_r^B & = & D_r^C \\ x_A + x_B & = & 1 \end{cases}$$

where x_j, $j = A, B$, are the proportions invested in each bond whereas D_r^j, $j = A, B$, represents the "generalized duration" of the bond j with respect to the short-term interest rate. Solving this system of equations, we obtain that the proportions invested in bonds A and B are 48.674% and 51.326%, respectively. Using these weights, we obtain the modified duration, the convexity and the yield of both portfolios. These values are as follows:

	Modified Duration	Convexity	Yield (%)
Portfolio 1	6.5039	72.1864	8.4114
Portfolio 2	6.5039	56.3576	9

These results show that the yield (convexity) of the portfolio 2 is higher (lower) than the one of the portfolio 1 and suggest that the portfolio 1 can generate a higher yield than the portfolio 2 if there were certain shifts in the yield curve.

We will assume a shift in the yield curve and we will analyze the relative behavior of both portfolios. Analogously to the section 7, we will assume that (1) this shift is instantaneously after the acquisition of the portfolio, (2) there can be three types of shifts: a parallel shift and two alternative changes in the slope of the yield curve, and (3) the investment horizon is equal to six months.

Table I shows the accumulated value obtained at the end of the investment horizon and the yield (in annual terms) provided by each bond when there is a parallel change in the yield curve. The first column of this table includes the magnitude, equal for all the maturities, of the change in interest rates. We can see that, when interest rates are rising, the final yield provided by each bond is increasing. The more extreme values correspond to the bond B because it has the longest maturity while the bond A, with the shortest maturity, shows the narrower range of yields.

Table II contains the results obtained with both portfolios. The accumulated value of the portfolio 1 is the weighted average of the accumulated values provided by the two bonds included in this portfolio. The first column, analogously to Table I, shows the magnitude of the change in the yield curve. The last column contains the difference between the yield of both portfolios ("yield of the portfolio 2" minus "yield of the portfolio 1").

As indicated by the last column of this table, the portfolio 1 outperforms the portfolio 2 if interest rates change more than 200 basis points. The additional yield obtained from the portfolio 1 increases monotonically with respect to the magnitude of the change in interest rates. The superior performance of the portfolio 1, when interest rates change in a large amount, is because of its higher convexity.

On the other hand, we see that the portfolio 2 provides a higher yield than the portfolio 1 when interest rates change less than 200 basis points. In this case, we have a parallel and "small enough" change in interest rates. Because this is the main assumption of the conventional duration, we obtain the intuitive and expected result: if two portfolios show the same duration, the better investment alternative is the portfolio with higher yield when there are no changes in interest rates.

After this first change in the yield curve, we analyze the consequences of two different changes in the slope of the yield curve. Specifically, the first alternative we analyze assumes that (1) the change in the interest rate related to the bond A is equal to the change in the interest rate corresponding to the intermediate maturity (bond C) plus 25 basis points and (2) the change in the interest rate associated to the longest maturity (bond B) is equal to the change in 10-year interest rates minus 25 basis points. Therefore, this change indicates a flattening (decrease in the slope) of the yield curve. The results obtained for the three available bonds and the two portfolios are included in Tables III-IV.

The first column in Table III indicates the magnitude of the change in 10-year interest rates, the intermediate maturity we are considering. Analogously to the parallel change, it is seen that (1) an increase in interest rates leads to a decrease in the yield of the bond and (2) the highest yields are obtained with the longest bond.

Similarly to Table II, Table IV contains the results obtained with both portfolios. The last column of this table shows the difference between the final yields provided by the two portfolios. Because of all the values included in this column are negative, it is concluded that, independently of the magnitude of the change in interest rates, the portfolio 1 always outperforms the portfolio 2. Its additional yield, in comparison to the portfolio 1, depends on the magnitude of the change in interest rates: the bigger the decrease in interest rates, the higher improvement in yield.

The second non-parallel change we consider indicates an increase in the slope of the yield curve. The change in the interest rate related to the shortest maturity (bond A) is equal to the change in (the intermediate maturity) bond C's yield minus 25 basis points whereas that on bond B will be equal to the change in bond C's yield plus 25 basis points. The results obtained for the three bonds and both portfolios are contained in Tables V-VI.

Table V shows the changes in the accumulated value and in the yield of each bond. We obtain similar results to the previous cases: an increase in interest rates implies a decrease in the yield of each bond and the more extreme yields correspond to the bond with longest maturity.

The accumulated value and the yield of each portfolio are included in Table VI. The difference between the yields of both portfolios shows that the portfolio 1 outperforms the portfolio 2 if the yield on bond C rises more than 300 basis points or fall more than 350 basis points. For lower changes in this yield, the portfolio 2 provides a higher yield than the portfolio 1.

Table I. Relative behavior of three bonds with respect to a parallel change in the yield curve

This table contains the accumulated value (market price of the bond plus its coupon) and the yield (in annual terms) provided by three bonds when the yield curve changes in a parallel fashion (interest rates with different maturities change in the same magnitude, indicated in the first column). The nominal value of each bond is equal to 100. The value of the coupon (paid semesterly) is equal to 3.5, 4.875 and 4.5, respectively. The maturity of the three bonds is equal to 5, 20 and 10 years, respectively. The investment horizon is equal to six months.

	Bond A		**Bond B**		**Bond C**	
Yield Change	Accum. Value	Yield (%)	Accum. Value	Yield (%)	Accum. Value	Yield (%)
5	86.495	-27.008	73.089	-53.820	78.661	-42.677
4.5	88.028	-23.943	75.452	-49.095	80.802	-38.395
4	89.595	-20.808	77.959	-44.080	83.030	-33.938
3.5	91.198	-17.602	80.624	-38.751	85.349	-29.301
3	92.838	-14.323	83.458	-33.083	87.762	-24.474
2.5	94.515	-10.969	86.475	-27.048	90.275	-19.448
2	96.231	-7.537	89.690	-20.619	92.892	-14.215
1.5	97.986	-4.027	93.119	-13.761	95.617	-8.764
1	99.782	-0.435	96.780	-6.439	98.457	-3.085
0.5	101.619	3.239	100.691	1.383	101.416	2.832
0	103.5	7	104.875	9.75	104.5	9
-0.5	105.424	10.848	109.353	18.706	107.714	15.429
-1	107.393	14.786	114.152	28.304	111.066	22.133
-1.5	109.408	18.816	119.298	38.597	114.562	29.125
-2	111.470	22.941	124.822	49.645	118.209	36.419
-2.5	113.582	27.164	130.758	61.516	122.014	44.029
-3	115.743	31.486	137.140	74.280	125.985	51.971
-3.5	117.955	35.911	144.009	88.019	130.130	60.261
-4	120.221	40.442	151.409	102.818	134.457	68.915
-4.5	122.540	45.080	159.387	118.774	138.976	77.953
-5	124.915	49.830	167.996	135.992	143.696	87.392

Table II. Relative behavior of two portfolios with respect to a parallel change in the yield curve

This table contains the accumulated value (market price of the bond plus its coupon) and the yield (in annual terms) provided by two portfolios when the yield curve changes in a parallel fashion (interest rates with different maturities change in the same magnitude, indicated in the first column). The portfolio 1 includes the bonds A and B in proportions 48.674% and 51.326%, respectively. The portfolio 2 includes the bond C. The investment horizon is equal to six months.

	Portfolio 1		**Portfolio 2**		
Yield change	Accum. Value	Yield (%)	Accum. Value	Yield (%)	Difference (%)
5	79.615	-40.769	78.661	-42.677	-1.908
4.5	81.573	-36.853	80.802	-38.395	-1.541
4	83.623	-32.753	83.030	-33.938	-1.185
3.5	85.771	-28.457	85.349	-29.301	-0.843
3	88.023	-23.952	87.762	-24.474	-0.522
2.5	90.388	-19.222	90.275	-19.448	-0.226
2	92.874	-14.251	92.892	-14.215	0.036
1.5	95.488	-9.023	95.617	-8.764	0.258
1	98.241	-3.517	98.457	-3.085	0.431
0.5	101.143	2.286	101.416	2.832	0.545
0	104.205	8.411	104.5	9	0.588
-0.5	107.440	14.881	107.714	15.429	0.547
-1	110.862	21.724	111.066	22.133	0.409
-1.5	114.484	28.969	114.562	29.125	0.156
-2	118.323	36.647	118.209	36.419	-0.228
-2.5	122.397	44.795	122.014	44.029	-0.765
-3	126.725	53.451	125.985	51.971	-1.479
-3.5	131.328	62.656	130.130	60.261	-2.395
-4	136.228	72.457	134.457	68.915	-3.542
-4.5	141.452	82.904	138.976	77.953	-4.951
-5	147.026	94.053	143.696	87.392	-6.661

Table III. Relative behavior of three bonds with respect to a decrease in the slope of the yield curve

This table contains the accumulated value (market price of the bond plus its coupon) and the yield (in annual terms) provided by three bonds when there is a flattening of the yield curve. The nominal value of each bond is equal to 100. The value of the coupon (paid semesterly) is equal to 3.5, 4.875 and 4.5, respectively. The maturity of the three bonds is equal to 5, 20 and 10 years, respectively. The investment horizon is equal to six months. The first column includes the change in the 10-year interest rates. Moreover, it is verified that

Change in the 5 (20)-year interest rates = Change in the 10-year interest rates + (−) 0.25%

	Bond A		**Bond B**		**Bond C**	
Yield Change	Accum. Value	Yield (%)	Accum. Value	Yield (%)	Accum. Value	Yield (%)
5	85.742	-28.515	74.253	-51.492	78.661	-42.677
4.5	87.257	-25.484	76.687	-46.625	80.802	-38.395
4	88.807	-22.384	79.271	-41.456	83.030	-33.938
3.5	90.392	-19.214	82.019	-35.961	85.349	-29.301
3	92.013	-15.972	84.943	-30.113	87.762	-24.474
2.5	93.672	-12.655	88.057	-23.885	90.275	-19.448
2	95.368	-9.263	91.377	-17.245	92.892	-14.215
1.5	97.103	-5.792	94.919	-10.160	95.617	-8.764
1	98.879	-2.241	98.703	-2.593	98.457	-3.085
0.5	100.695	1.391	102.747	5.495	101.416	2.832
0	102.554	5.108	107.075	14.151	104.5	9
-0.5	104.456	8.912	111.711	23.422	107.714	15.429
-1	106.402	12.805	116.680	33.360	111.066	22.133
-1.5	108.394	16.789	122.011	44.022	114.562	29.125
-2	110.433	20.867	127.736	55.473	118.209	36.419
-2.5	112.520	25.040	133.890	67.781	122.014	44.029
-3	114.656	29.312	140.511	81.023	125.985	51.971
-3.5	116.843	33.686	147.640	95.280	130.130	60.261
-4	119.081	38.163	155.323	110.646	134.457	68.915
-4.5	121.373	42.747	163.609	127.219	138.976	77.953
-5	123.720	47.441	172.555	145.110	143.696	87.392

Table IV. Relative behavior of two portfolios with respect to a decrease in the slope of the yield curve

This table contains the accumulated value (market price of the bond plus its coupon) and the yield (in annual terms) provided by two portfolios when there is a flattening of the yield curve. The portfolio 1 includes the bonds A and B in proportions 48.674% and 51.326%, respectively. The portfolio 2 includes the bond C. The investment horizon is equal to six months. The first column includes the change in the 10-year interest rates. Moreover, it is verified that

Change in the 5 (20)-year interest rates = Change in the 10-year interest rates + (−) 0.25%

	Portfolio 1		**Portfolio 2**		
Yield Change	Accum. Value	Yield (%)	Accum. Value	Yield (%)	Difference (%)
5	79.845	-40.308	78.661	-42.677	-2.369
4.5	81.832	-36.335	80.802	-38.395	-2.059
4	83.913	-32.173	83.030	-33.938	-1.765
3.5	86.094	-27.810	85.349	-29.301	-1.491
3	88.384	-23.230	87.762	-24.474	-1.243
2.5	90.790	-18.419	90.275	-19.448	-1.029
2	93.319	-13.360	92.892	-14.215	-0.855
1.5	95.982	-8.034	95.617	-8.764	-0.729
1	98.788	-2.422	98.457	-3.085	-0.663
0.5	101.749	3.498	101.416	2.832	-0.665
0	104.875	9.75	104.5	9	-0.750
-0.5	108.179	16.359	107.714	15.429	-0.930
-1	111.677	23.355	111.066	22.133	-1.221
-1.5	115.383	30.767	114.562	29.125	-1.641
-2	119.314	38.629	118.209	36.419	-2.209
-2.5	123.489	46.978	122.014	44.029	-2.948
-3	127.926	55.853	125.985	51.971	-3.882
-3.5	132.650	65.300	130.130	60.261	-5.039
-4	137.682	75.365	134.457	68.915	-6.450
-4.5	143.051	86.103	138.976	77.953	-8.150
-5	148.785	97.570	143.696	87.392	-10.178

Table V. Relative behavior of three bonds with respect to a increase in the slope of the yield curve

This table contains the accumulated value (market price of the bond plus its coupon) and the yield (in annual terms) provided by three bonds when there is a steepening of the yield curve. The nominal value of each bond is equal to 100. The value of the coupon (paid semesterly) is equal to 3.5, 4.875 and 4.5, respectively. The maturity of the three bonds is equal to 5, 20 and 10 years, respectively. The investment horizon is equal to six months. the first column includes the change in the 10-year interest rates. Moreover, it is verified that

Change in the 5 (20)-year interest rates = Change in the 10-year interest rates − (+) 0.25%

	Bond A		Bond B		Bond C	
Yield Change	Accum. Value	Yield (%)	Accum. Value	Yield (%)	Accum. Value	Yield (%)
5	87.257	-25.484	71.960	-56.079	78.661	-42.677
4.5	88.807	-22.384	74.253	-51.492	80.802	-38.395
4	90.392	-19.214	76.687	-46.625	83.030	-33.938
3.5	92.013	-15.972	79.271	-41.456	85.349	-29.301
3	93.672	-12.655	82.019	-35.961	87.762	-24.474
2.5	95.368	-9.263	84.943	-30.113	90.275	-19.448
2	97.103	-5.792	88.057	-23.885	92.892	-14.215
1.5	98.879	-2.241	91.377	-17.245	95.617	-8.764
1	100.695	1.391	94.919	-10.160	98.457	-3.085
0.5	102.554	5.108	98.703	-2.593	101.416	2.832
0	104.456	8.912	102.747	5.495	104.5	9
-0.5	106.402	12.805	107.075	14.151	107.714	15.429
-1	108.394	16.789	111.711	23.422	111.066	22.133
-1.5	110.433	20.867	116.680	33.360	114.562	29.125
-2	112.520	25.040	122.011	44.022	118.209	36.419
-2.5	114.656	29.312	127.736	55.473	122.014	44.029
-3	116.843	33.686	133.890	67.781	125.985	51.971
-3.5	119.081	38.163	140.511	81.023	130.130	60.261
-4	121.373	42.747	147.640	95.280	134.457	68.915
-4.5	123.720	47.441	155.323	110.646	138.976	77.953
-5	126.123	52.247	163.609	127.219	143.696	87.392

Table VI. Relative behavior of two portfolios with respect to a increase in the slope of the yield curve

This table contains the accumulated value (market price of the bond plus its coupon) and the yield (in annual terms) provided by two portfolios when there is a steepening of the yield curve. The portfolio 1 includes the bonds A and B in proportions 48.674% and 51.326%, respectively. The portfolio 2 includes the bond C. The investment horizon is equal to six months. The first column includes the change in the 10-year interest rates. Moreover, it is verified that

Change in the 5 (20)-year interest rates = Change in the 10-year interest rates − (+) 0.25%

	Portfolio 1		**Portfolio 2**		
Yield Change	Accum. Value	Yield (%)	Accum. Value	Yield (%)	Difference (%)
5	79.406	-41.187	78.661	-42.677	-1.490
4.5	81.337	-37.324	80.802	-38.395	-1.070
4	83.358	-33.283	83.030	-33.938	-0.654
3.5	85.473	-29.052	85.349	-29.301	-0.248
3	87.691	-24.617	87.762	-24.474	0.143
2.5	90.017	-19.964	90.275	-19.448	0.516
2	92.460	-15.078	92.892	-14.215	0.863
1.5	95.028	-9.942	95.617	-8.764	1.178
1	97.731	-4.537	98.457	-3.085	1.452
0.5	100.577	1.155	101.416	2.832	1.676
0	103.579	7.159	104.5	9	1.840
-0.5	106.748	13.496	107.714	15.429	1.933
-1	110.096	20.193	111.066	22.133	1.940
-1.5	113.639	27.279	114.562	29.125	1.846
-2	117.391	34.783	118.209	36.419	1.636
-2.5	121.370	42.740	122.014	44.029	1.289
-3	125.593	51.186	125.985	51.971	0.785
-3.5	130.080	60.161	130.130	60.261	0.099
-4	134.855	69.710	134.457	68.915	-0.795
-4.5	139.940	79.881	138.976	77.953	-1.928
-5	145.363	90.727	143.696	87.392	-3.335

Table VII. Relative behavior of the three bonds included in the portfolio 1 with respect to a parallel change in the yield curve

This table contains the accumulated value (market price of the bond plus its coupon) and the yield (in annual terms) provided by the three bonds included in the portfolio 1 when there is parallel change in the yield curve (interest rates with different maturities change in the same magnitude, indicated in the first column). The nominal value of each bond is equal to 100. The value of the coupon (paid semesterly) is equal to 2.75, 5 and 6, respectively. The maturity of the three bonds is equal to 5, 15 and 20 years, respectively. The investment horizon is equal to six months.

	Bond A		**Bond B**		**Bond C**	
Yield Change	Accum. Value	Yield (%)	Accum. Value	Yield (%)	Accum. Value	Yield (%)
5	85.176	-29.646	75.759	-48.480	77.809	-44.381
4.5	86.757	-26.485	78.042	-43.915	79.966	-40.067
4	88.374	-23.250	80.444	-39.110	82.242	-35.514
3.5	90.029	-19.940	82.974	-34.051	84.648	-30.703
3	91.722	-16.554	85.638	-28.722	87.191	-25.616
2.5	93.455	-13.088	88.447	-23.105	89.883	-20.233
2	95.229	-9.541	91.409	-17.181	92.735	-14.529
1.5	97.044	-5.911	94.534	-10.931	95.758	-8.482
1	98.901	-2.196	97.833	-4.333	98.967	-2.064
0.5	100.803	1.606	101.317	2.635	102.376	4.752
0	102.75	5.5	105	10	106	12
-0.5	104.742	9.485	108.892	17.785	109.856	19.713
-1	106.782	13.565	113.010	26.021	113.964	27.928
-1.5	108.871	17.743	117.369	34.738	118.343	36.687
-2	111.010	22.020	121.983	43.967	123.017	46.034
-2.5	113.200	26.401	126.872	53.744	128.008	56.016
-3	115.443	30.887	132.053	64.107	133.344	66.688
-3.5	117.740	35.481	137.547	75.095	139.054	78.108
-4	120.093	40.186	143.376	86.753	145.168	90.337
-4.5	122.502	45.005	149.563	99.127	151.723	103.447
-5	124.971	49.942	156.133	112.267	158.756	117.512

Table VIII. Relative behavior of two portfolios with respect to a parallel change in the yield curve

This table contains the accumulated value (market price of the bond plus its coupon) and the yield (in annual terms) provided by two portfolios when the yield curve changes in a parallel fashion (interest rates with different maturities change in the same magnitude, indicated in the first column). The portfolio 1 includes the bonds A, B and C in proportions 59.930%, 11.203% and 28.866%, respectively. The portfolio 2 includes the bond D. The investment horizon is equal to six months.

	Portfolio 1		**Portfolio 2**		
Yield Change	Accum. Value	Yield (%)	Accum. Value	Yield (%)	Difference (%)
5	81.632	-36.734	78.661	-42.677	-5.943
4.5	83.480	-33.038	80.802	-38.395	-5.356
4	85.398	-29.202	83.030	-33.938	-4.736
3.5	87.390	-25.219	85.349	-29.301	-4.081
3	89.459	-21.081	87.762	-24.474	-3.392
2.5	91.609	-16.780	90.275	-19.448	-2.668
2	93.847	-12.305	92.892	-14.215	-1.909
1.5	96.175	-7.648	95.617	-8.764	-1.115
1	98.600	-2.798	98.457	-3.085	-0.287
0.5	101.128	2.256	101.416	2.832	0.576
0	103.763	7.527	104.5	9	1.472
-0.5	106.513	13.027	107.714	15.429	2.402
-1	109.385	18.770	111.066	22.133	3.363
-1.5	112.385	24.771	114.562	29.125	4.354
-2	115.523	31.046	118.209	36.419	5.373
-2.5	118.806	37.612	122.014	44.029	6.417
-3	122.243	44.487	125.985	51.971	7.483
-3.5	125.846	51.692	130.130	60.261	8.568
-4	129.623	59.247	134.457	68.915	9.668
-4.5	133.588	67.176	138.976	77.953	10.776
-5	137.751	75.503	143.696	87.392	11.888

Table IX. Relative behavior of the three bonds included in the portfolio 1 with respect to a decrease in the slope of the yield curve

This table contains the accumulated value (market price of the bond plus its coupon) and the yield (in annual terms) provided by the three bonds included in the portfolio 1 when there is a flattening of the yield curve. The nominal value of each bond is equal to 100. The value of the coupon (paid semesterly) is equal to 2.75, 5 and 6, respectively. The maturity of the three bonds is equal to 5, 15 and 20 years, respectively. The investment horizon is equal to six months. The first column includes the change in the 10-year interest rates. Moreover, it is verified that

Change in the 5 (15) [20]-year interest rates = Change in the 10-year interest rates +1% (−0.5%) [−1%]

	Bond A		Bond B		Bond C	
Yield Change	Accum. Value	Yield (%)	Accum. Value	Yield (%)	Accum. Value	Yield (%)
5	82.121	-35.757	78.042	-43.915	82.242	-35.514
4.5	83.631	-32.737	80.444	-39.110	84.648	-30.703
4	85.176	-29.646	82.974	-34.051	87.191	-25.616
3.5	86.757	-26.485	85.638	-28.722	89.883	-20.233
3	88.374	-23.250	88.447	-23.105	92.735	-14.529
2.5	90.029	-19.940	91.409	-17.181	95.758	-8.482
2	91.722	-16.554	94.534	-10.931	98.967	-2.064
1.5	93.455	-13.088	97.833	-4.333	102.376	4.752
1	95.229	-9.541	101.317	2.635	106	12
0.5	97.044	-5.911	105	10	109.856	19.713
0	98.901	-2.196	108.892	17.785	113.964	27.928
-0.5	100.803	1.606	113.010	26.021	118.343	36.687
-1	102.75	5.5	117.369	34.738	123.017	46.034
-1.5	104.742	9.485	121.983	43.967	128.008	56.016
-2	106.782	13.565	126.872	53.744	133.344	66.688
-2.5	108.871	17.743	132.053	64.107	139.054	78.108
-3	111.010	22.020	137.547	75.095	145.168	90.337
-3.5	113.200	26.401	143.376	86.753	151.723	103.447
-4	115.443	30.887	149.563	99.127	158.756	117.512
-4.5	117.740	35.481	156.133	112.267	166.308	132.616
-5	120.093	40.186	163.113	126.227	174.424	148.849

Table X. Relative behavior of two portfolios with respect to a decrease in the slope of the yield curve

This table contains the accumulated value (market price of the bond plus its coupon) and the yield (in annual terms) provided by two portfolios when there is a flattening of the yield curve. The portfolio 1 includes the bonds A, B and C in proportions 59.930%, 11.203% and 28.866%, respectively. The portfolio 2 includes the bond D. The investment horizon is equal to six months. The first column includes the change in the 10-year interest rates. Moreover, it is verified that

Change in the 5 (15) [20]-year interest rates = Change in the 10-year interest rates +1% (−0.5%) [−1%]

	Portfolio 1		**Portfolio 2**		
Yield Change	Accum. Value	Yield (%)	Accum. Value	Yield (%)	Difference (%)
5	80.957	-38.085	78.661	-42.677	-4.592
4.5	82.825	-34.349	80.802	-38.395	-4.045
4	84.766	-30.466	83.030	-33.938	-3.471
3.5	86.784	-26.430	85.349	-29.301	-2.870
3	88.884	-22.231	87.762	-24.474	-2.242
2.5	91.069	-17.860	90.275	-19.448	-1.588
2	93.346	-13.307	92.892	-14.215	-0.907
1.5	95.718	-8.562	95.617	-8.764	-0.201
1	98.193	-3.612	98.457	-3.085	0.527
0.5	100.776	1.552	101.416	2.832	1.279
0	103.473	6.946	104.5	9	2.053
-0.5	106.292	12.584	107.714	15.429	2.844
-1	109.240	18.481	111.066	22.133	3.652
-1.5	112.326	24.652	114.562	29.125	4.473
-2	115.557	31.115	118.209	36.419	5.304
-2.5	118.944	37.889	122.014	44.029	6.139
-3	122.497	44.995	125.985	51.971	6.975
-3.5	126.227	52.454	130.130	60.261	7.806
-4	130.145	60.290	134.457	68.915	8.624
-4.5	134.264	68.528	138.976	77.953	9.424
-5	138.598	77.197	143.696	87.392	10.194

Table XI. Relative behavior of the three bonds included in the portfolio 1 with respect to a increase in the slope of the yield curve

This table contains the accumulated value (market price of the bond plus its coupon) and the yield (in annual terms) provided by the three bonds included in the portfolio 1 when there is a steepening of the yield curve. The nominal value of each bond is equal to 100. The value of the coupon (paid semesterly) is equal to 2.75, 5 and 6, respectively. The maturity of the three bonds is equal to 5, 15 and 20 years, respectively. The investment horizon is equal to six months. The first column includes the change in the 10-year interest rates. Moreover, it is verified that

Change in the 5 (15) [20]-year interest rates = Change in the 10-year interest rates -1% (+0.5%) [+1%]

	Bond A		Bond B		Bond C	
Yield Change	Accum. Value	Yield (%)	Accum. Value	Yield (%)	Accum. Value	Yield (%)
5	88.374	-23.250	73.589	-52.821	73.823	-52.353
4.5	90.029	-19.940	75.759	-48.480	75.764	-48.471
4	91.722	-16.554	78.042	-43.915	77.809	-44.381
3.5	93.455	-13.088	80.444	-39.110	79.966	-40.067
3	95.229	-9.541	82.974	-34.051	82.242	-35.514
2.5	97.044	-5.911	85.638	-28.722	84.648	-30.703
2	98.901	-2.196	88.447	-23.105	87.191	-25.616
1.5	100.803	1.606	91.409	-17.181	89.883	-20.233
1	102.75	5.5	94.534	-10.931	92.735	-14.529
0.5	104.742	9.485	97.833	-4.333	95.758	-8.482
0	106.782	13.565	101.317	2.635	98.967	-2.064
-0.5	108.871	17.743	105	10	102.376	4.752
-1	111.010	22.020	108.892	17.785	106	12
-1.5	113.200	26.401	113.010	26.021	109.856	19.713
-2	115.443	30.887	117.369	34.738	113.964	27.928
-2.5	117.740	35.481	121.983	43.967	118.343	36.687
-3	120.093	40.186	126.872	53.744	123.017	46.034
-3.5	122.502	45.005	132.053	64.107	128.008	56.016
-4	124.971	49.942	137.547	75.095	133.344	66.688
-4.5	—	—	143.376	86.753	139.054	78.108
-5	—	—	149.563	99.127	145.168	90.337

Table XII. Relative behavior of two portfolios with respect to an increase in the slope of the yield curve

This table contains the accumulated value (market price of the bond plus its coupon) and the yield (in annual terms) provided by two portfolios when there is a steepening of the yield curve. The portfolio 1 includes the bonds A, B and C in proportions 59.930%, 11.203% and 28.866%, respectively. The portfolio 2 includes the bond D. The investment horizon is equal to six months. The first column includes the change in the 10-year interest rates. Moreover, it is verified that

Change in the 5 (15) [20]-year interest rates = Change in the 10-year interest rates −1% (+0.5%) [+1%]

	Portfolio 1		**Portfolio 2**		
Yield Change	Accum. Value	Yield (%)	Accum. Value	Yield (%)	Difference (%)
5	82.476	-35.047	78.661	-42.677	-7.630
4.5	84.312	-31.375	80.802	-38.395	-7.019
4	86.215	-27.569	83.030	-33.938	-6.369
3.5	88.188	-23.622	85.349	-29.301	-5.678
3	90.236	-19.526	87.762	-24.474	-4.947
2.5	92.363	-15.273	90.275	-19.448	-4.174
2	94.572	-10.855	92.892	-14.215	-3.359
1.5	96.868	-6.263	95.617	-8.764	-2.500
1	99.256	-1.487	98.457	-3.085	-1.598
0.5	101.741	3.483	101.416	2.832	-0.651
0	104.329	8.659	104.5	9	0.340
-0.5	107.026	14.052	107.714	15.429	1.377
-1	109.837	19.675	111.066	22.133	2.458
-1.5	112.771	25.542	114.562	29.125	3.583
-2	115.833	31.667	118.209	36.419	4.752
-2.5	119.032	38.065	122.014	44.029	5.963
-3	122.377	44.755	125.985	51.971	7.216
-3.5	125.876	51.753	130.130	60.261	8.507
-4	129.539	59.079	134.457	68.915	9.835
-4.5	—	—	138.976	77.953	—
-5	—	—	143.696	87.392	—