Auctions of Licences and Market Structure

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Abstract

This paper studies sequential auctions of licences to operate in a market where those firms that obtain at least one licence then engage in a symmetric market game. I employ a new refinement of Nash equilibrium, the concept of *Markovian recursively undominated equilibrium.* The unique solution satisfies the following properties: (i) when several firms own licences before the auction (incumbents), new entrants buy licences in each stage, and (ii) when there is no more than one incumbent, either the single firm preempts entry altogether or entry occurs in every stage, depending on the parameter configuration.

1 Introduction

This paper studies auctions of licences to operate in a market. Since the outcome of an auction partly determines the market structure that will emerge after the auction, this setting seems appropriate to address interesting questions about entry preemption and the persistence of certain market structures in the face of entry. The relationship between auction procedures and resulting market structure was seriously addressed in practice by regulators and auction designers during the recent FCC auctions of radio spectrum bands. More generally, any auction of licences or productive capacity raises similar issues and can be addressed within the present analysis.

Entry preemption and the persistence of monopoly have received considerable attention in the Industrial Organization literature (Gilbert and Newberry, 1982; Dasgupta et al., 1983) . It has been argued that a monopoly will persist even when new capacity becomes available. Because in a non-cooperative environment aggregate industry profits decrease with entry, the total cost of entry to the incumbents is larger than the benefit to the entrant. If a single incumbent has to bear this entire cost, it will be willing to pay more to avoid entry than the potential entrants are willing to pay to accomplish it. Therefore, a monopolist is expected to engage in preemptive investment to avoid entry.

However, Krishna (1993) shows that this argument, initially formulated in a static framework, does not necessarily extend to the dynamic case. She studies the persistence of monopoly in the context of a sequential auction of capacity. For a large class of demand specifications, she singles out an equilibrium outcome in which entry occurs in every stage except the last. Her analysis, however, relies on a fundamentally *asymmetric* formulation: in the market that emerges after the auction, except for a price setting monopolist, all the remaining firms behave as price takers. It is apparent that this may not be consistent with Nash equilibrium in the post-auction market game unless the capacity units auctioned are *small* relative to the size of the market.

In this paper, I study sequential auctions of licences in the case in which a *symmetric* market structure emerges after the auction. I suppose that each firm that ends the auction with at least one licence will engage in a *symmetric* market game. For example, symmetric scenarios in which the post-auction market game is either a Cournot or a Bertrand game are consistent with our formulation. Note that the symmetry assumption implies that licences are *unrestricted*: a firm that has a single licence is entitled to sell as much output as it wishes; or equivalently, in an auction of capacity units this implies that the units auctioned are *large*: a firm that owns a single capacity unit can serve the whole market. Although in this model firms that own units do not need additional units to expand their production, they may still wish to obtain more units simply to preempt entry by other firms.

The analysis had to cope with two main obstacles: the enormous multiplicity of equilibria characteristic of many dynamic games and the presence of externalities. Rodríguez (1994) shows that the indeterminacy that arises in many sequential auctions cannot be effectively reduced by standard refinements of Nash equilibrium. The concept of *Markovian recursively undominated (MRU) equilibrium* developed in that paper is appropriate for the present application. Briefly, an MRU equilibrium is a Markovian equilibrium in which the players do not use locally dominated choices. I say that a choice prescribed by a strategy profile to a given player at a given stage and history is *locally dominated* at that stage and history if it is weakly dominated in the game obtained by substituting the subgames starting at the next stage by the corresponding payoff vectors under the profile in question. By requiring that the players do not employ *locally dominated* choices, this concept imposes a strong strategic stability requirement locally at each stage and conditional on each history. However, this requirement is consistent with recursive construction of a solution, which results in considerable simplification of the analysis.

Externalities are present here since a firm's payoff depends not only on whether this firm obtains a licence to enter the market, but also on the total number of firms that manage to enter. Although these externalities may result in non-uniqueness, I identify conditions on the rate at which profits decrease with entry that eliminate the indeterminacy and are satisfied in an interesting class of economic environments.

A brief summary of the main results follows. The MRU allocation is unique and independent of whether the auction is a first or a second price sequential auction. When there is more than one incumbent firm before the auction, new entrants will buy licences in every stage. On the other hand, when there is at most one initial incumbent, either a single firm preempts entry completely or entry occurs in each stage. Which case will prevail depends on the sign of a parameter that will, in equilibrium, be equal to the net benefit to a monopolist of complete preemption.

These results rely on the assumption that in the market that emerges after the auction profits decrease with entry, but the rate at which they decrease is *not too large*. In particular, this assumption is satisfied when firms have a constant returns technology, the market demand is concave and the post-auction market game is a symmetric Cournot game. In fact, even sharper results are available in this particular case. I show that when the number of units auctioned is large enough, entry will necessarily occur in every stage, a result that contrasts with the persistence of monopoly that prevails in the static case. It suggests that a government can always approximate a competitive solution by auctioning a sufficiently large number of licences. However, I also show that a revenue maximizing seller will in many cases prefer to offer only the largest number of units that is consistent with complete preemtion by a monopolist.

Another scenario amenable to treatment by these methods is the case of firms with a constant returns technology that engage in a symmetric Bertrand game after the auction. In this case, however, the net benefit of complete pre-emption is always positive so that the completely preemptive outcome prevails, as in the static case.

The paper is organized as follows. In section 2 I lay out the model and introduce the solution concept. In section 3 I analyze a Cournot game that is consistent with the basic assumptions of the model about the market that emerges after the auction. Section 4 derives the main results about the auction. Most proofs are included in the appendix.

2 The Model

In this section I lay out the model of sequential auctions of licences and introduce the solution concept that will be used.

Suppose that t licences to operate in a market are auctioned sequentially among n firms. Two alternative auction procedures will be considered: sequential first price auctions and sequential second price auctions. Firms that own licences before the auction are called incumbents. Although firms that already have a licence do not need more licences to increase their operation, they may still want to buy additional licences to prevent other firms from entering the market. Any firm that owns at least one licence after the auction ends has the right to operate without restrictions in the market that emerges. Note that not only the market game that is expected to be played among incumbents and new entrants determines the bidders' valuations, but also the outcome of the auction itself partly determines the ensuing market structure.

2.1 Notation and Basic Assumptions

I model the auction as a multistage game of complete information and observable histories. Let $N = \{1,..,n\}$ be the set of players (firms) and $T = \{1,..,t\}$ be the set of units auctioned sequentially, where $n > t$. I assume that the bidding space B is a set of nonnegative multiples of a *small* money unit u.[1] Let $b_{ir} \in B$ denote player i's bid in the rth stage, $r \in T$. At each stage, all the players bid simultaneously and the corresponding unit is allocated to one of the highest bidders; a random tie-breaker selects the winner in case of a tie. Let $a_r \in N$ denote the winner of the rth unit and p_r the price he pays. In the case of a *Sequential First Price Auction* ($SFPA$), p_r equals the largest bid for the rth unit; in the case of a *Sequential Second Price Auction* ($SSPA$), it equals the second largest bid.

Denote $b^r = (b_{i1},..,b_{i,r-1})$ and $a^r = (a_{i1},..,a_{i,r-1})$. Thus, the history up to the rth stage is $h_r = (b^r, a^r)$. Let H_r denote the set of all the histories up to r and define $H = \cap_{\tau \in T} H_\tau$. I assume that histories are publicly observable. Each history $h_r \in H_r$ determines an allocation of units a^r and a set $I_r = I_r(a^r)$ of players that own at least one unit at the beginning of stage r. In particular, I_1 is the set of incumbent firms prior to the auction and I_{t+1} is the composition of the industry after the auction. I assume without loss of generality that there is at least one firm in the market before the auction starts, i.e.:$|I_1| \geq 1$.

Let w_{ir} denote the amount paid by player i in the rth stage. Also denote $w_i^r = (w_{i1},..,w_{i,r-1})$ and $w^r = (w_i^r)_{i \in N}$. For a given allocation (a^{t+1}, w^{t+1}), player i's payoff is represented by the quasi-linear function $u_i(a^{t+1}, w_i^{t+1}) = \rho_i(a^{t+1}) - \sum_{\tau \in T} w_{i\tau}$, where the revenue $\rho_i(a^{t+1})$ is induced by the outcome of the market game that follows the auction.

I assume that players who own at least one unit engage in the symmetric Nash equilibrium of a *symmetric* market game. This presupposes that licences are *unrestricted*: a firm that has a single licence is entitled to sell as much output as it wishes. On the other hand, players that do not own any licence after the auction ends will not be able to enter the industry. Consequently, the equilibrium profits of each firm in the industry can be written as a function $\pi(|I_{t+1}|)$ of the total

[1] Except for a qualification made in footnote 2 for the case of first price auctions, all the results in this paper extend to the case in which $B = \Re_+$.

number $|I_{t+1}|$ of firms in the industry , so we have that $\rho_i(a^{t+1}) = \pi(|I_{t+1}(a^{t+1})|)$ when $i \in I_{t+1}(a^{t+1})$, and $\rho_i(a^{t+1}) = 0$ otherwise. We assume that the function π satisfies the following conditions:

$$(A_1) \quad m\pi(m) > (m+1)\pi(m+1) \quad m = 1, 2, ...$$
$$(A_2) \quad 2\pi(m+1) > \pi(m) \qquad m = 2, 3, ...$$

Note that assumption (A_1) requires that total industry profits do not increase with entry, and that (A_2) puts an upper bound on the rate at which individual firms' profits decrease with entry. In section 2 we will see that these assumptions are actually satisfied by an important class of economic environments.

2.2 Solution Concept

In this subsection I define the solution concept used for the analysis of the sequential auction. A more complete discussion of this topic is found in Rodriguez [7]. It is shown in there that the standard refinements of Nash equilibrium fail to select a plausible and determinate solution for finite multistage auctions even in simple cases. The concept of MRU equilibrium is developed there to provide a tractable analytical tool and preserve the predictive power of the model. Using the logic of backwards induction, this concept provides a rather simple extension to a dynamic setting of the concept of Nash equilibrium in undominated strategies which is often applied to static auctions.

Before I provide a precise definition of the solution concept, I introduce some related concepts. A *behavior strategy* for player i is a map $\beta_i : H \to \Delta(B)$, where $\Delta(B)$ denotes the set of probability distributions on the bidding space B. Let $u_{ir}(h_r;\beta)$ denote player i's expected payoff gross of payments prior to stage r and conditional on the history h_r, provided that the players use the profile $\beta = (\beta_i)_{i \in N}$ in the subgame determined by h_r. More explicitly, $u_{ir}(h_r;\beta) = E_\beta\left[u_i(\mathbf{a}^{t+1}, \mathbf{w}_i^{t+1}) \mid h_r\right] + \sum_{\tau=1}^{r-1} w_{i\tau}$, where E_β is the expectation operator with respect to the distribution induced by β. In particular, if the players select bids $b_r = (b_{ir})_{i \in N}$ at the history h_r and bid according to the profile β after the stage r, their expected payoffs (gross of payments prior to r) are $u_{i,r+1}(h_r, b_r; \beta) - w_{ir}$, for each $i \in N$. Given β and h_r, these payoffs define a static game where the bidders' strategies are their bids at r. I refer to this game as the *local auction* of the rth unit at h_r given β. I say that a bid for the rth unit is *locally dominated at h_r given β* if it is weakly dominated in the local auction of the rth unit at h_r given β. I also say that a subgame perfect Nash equilibrium β is *recursively undominated* if none of the choices prescribed (with positive probability) by β to any player is locally dominated at its corresponding stage and history given β.

By ruling out locally dominated strategies, this restriction imposes a strong local strategic stability requirement at each history. Moreover, since a local auction at h_r given β depends on β only through the local strategies prescribed for subsequent stages, this concept is recursive in nature, which simplifies the analysis considerably. Existence of a subgame perfect equilibrium that is recursively undominated for the games considered in this paper is a direct consequence of the existence of a Nash equilibrium in undominated strategies for finite games since each local auction is a finite game (Van Damme, 1991).

Moreover, I will restrict my attention to equilibria that are Markovian, a condition often used in the literature and that seems rather natural in the present context. I should say that although this condition simplifies somewhat the treatment, it is actually not restrictive in the present case.

The *payoff relevant partition* of the set of histories H is the partition of H into sets that contain all the histories that define *strategically equivalent subgames.* A behavior strategy is *Markovian* if it is measurable with respect to the payoff relevant partition (i.e.: $\beta_i(h) = \beta_i(h')$ whenever h and h' define strategically equivalent subgames). A *Markovian equilibrium* is a Nash equilibrium in which all the players employ Markovian strategies. In our case, the Markovian assumption implies that histories that project the same allocation of units determine strategically equivalent subgames. To see this, consider two histories $\hat{h}_r$ and $\tilde{h}_r$ that project the same allocation of units, i.e.: $\hat{a}^r = \tilde{a}^r$. Note that for every allocation $(a_r, .., a_t; w_{ir}, .., w_{it})_{i \in N}$ we have that $u_i(\hat{a}^r, a_r, .., a_t; \hat{w}_i^r, w_{ir}, .., w_{it}) + \sum_{\tau=1}^{r-1} \hat{w}_{i\tau} = u_i(\tilde{a}^r, a_r, .., a_t; \tilde{w}_i^r, w_{ir}, .., w_{it}) + \sum_{\tau=1}^{r-1} \tilde{w}_{i\tau}$. In other words, for every $i \in N$, u_i induces the same preference ordering among allocations for the subgames determined by $\hat{h}_r$ and $\tilde{h}_r$. Thus, both subgames are strategically equivalent and all the players will behave identically in each of them at every Markovian equilibrium.

The Markovian assumption allows us to define an array of recursive valuations associated to a strategy profile. Given the profile β, I define player i's *recursive valuations* for the rth unit at h_r as $v_{ij}\,(h_r; \beta) = u_{i,r+1}(h_r, a_r{=}i; \beta) - u_{i,r+1}(h_r, a_r{=}j; \beta)$. This can be interpreted as the amount that player i has to pay for the rth unit at h_r in order to be indifferent between obtaining the unit and allowing player j to obtain it instead, provided that all the players use the profile β after period r. In section 4 we will see that recursive valuations provide a convenient characterization of local auctions and locally dominated choices.

Now, I can define the solution concept that will be employed in the analysis of the sequential auction. A subgame perfect Nash equilibrium of the sequential auction that is both Markovian and recursively undominated will be called an **MRU equilibrium**, which is short for *Markovian recursively undominated equilibrium.* An allocation determined by an MRU equilibrium will be called an *MRU allocation.* Existence of an MRU equilibrium for multistage games with observed actions is a trivial consequence of the existence of a subgame perfect equilibrium that is recursively undominated.

3 The Post-Auction Market Game

In the preceding section I have characterized the market that emerges after the auction in terms of a profit function π that satisfies conditions A_1 and A_2. Here I address the question of whether such a profit function is consistent with the equilibrium correspondence of any reasonable market game. More specifically, I investigate conditions under which the Nash equilibria of a symmetric Cournot game satisfy A_1 and A_2.

Consider an industry in which m firms produce a homogeneous good using the same constant returns to scale technology. Their cost of production is represented by the function $c(q) = cq$, for some constant $c \geq 0$. Suppose that they can sell their total output Q at a price consistent with the inverse demand function $p(Q)$. I assume that this function is bounded, non-increasing and continuously differentiable in $(0, \infty)$. Moreover, $p(0) > 0$ and $p(\bar{Q}) = 0$ for some $\bar{Q} > 0$. It also satisfies either one of the following assumptions:

(M_1) $p'(Q) + Qp''(Q) < 0$ for $Q \in (0, \bar{Q})$.

(M_2) the restriction of p to $[0, \bar{Q}]$ is a concave function.

I suppose that firms engage in a *Cournot game*. It is well known that there is a unique Cournot equilibrium for the preceding problem (Friedman, 1982). Let q_m denote the equilibrium output of an individual firm, indexed by the number m of firms in the industry. Denote $Q_m = mq_m$ and $\pi(m) = (p(Q_m) - c)\, q_m$. The following result describes some basic properties of the unique Cournot equilibrium.

Lemma 1 *Suppose that the inverse demand function satisfies condition M_1. Then the unique Cournot equilibrium must satisfy the following properties:*

(i) $Q_m < Q_{m+1}$ $\quad m = 1, 2, ..$

(ii) π *satisfies condition* A_1.

(iii) $2\pi(m+1) > \pi(m)$ $\quad m = 3, 4, ...$

We conclude that the concavity of total revenues required by condition M_1 not only ensures that assumption A_1 holds, but also ensures that the conditions required by assumption A_2 are satisfied for $m > 2$. To guarantee that these conditions are also satisfied for $m = 2$ we may need a stronger assumption. The following example shows that conditions A_1 and A_2 are satisfied when the demand function is linear.

Example 1 (Linear Demand) *Suppose that* $p(Q) = a - bQ$. *In this case, equation (2) can be easily solved to obtain* $q_m = \frac{a-c}{b(1+m)}$ *and* $\pi(m) = (1/b)\frac{a-c}{1+m}^2$. *Moreover,* $m\pi(m) - (m+1)\pi(m+1) = b^2K(m)^2\left[m(m+1)-1\right] > 0$ *for* $m \geq 1$ *and* $\pi(m) - 2\pi(m+1) = bA(m)(2-m^2) < 0$ *for* $m > 1$ *since* $K(m) = \left(\frac{a-c}{b(m+1)(m+2)}\right) > 0$. *Thus, conditions* A_1 *and* A_2 *hold in this case.*

In fact, it can be shown that all concave demand functions satisfy both A_1 and A_2. Since M_2 implies M_1, that A_1 is satisfied follows from the preceding lemma. Consider the case of A_2. Suppose that p is strictly concave. Let p^l denote the tangent to p at the quantity Q_m. Clearly, p^l is a linear demand function and the corresponding Cournot equilibrium satisfies $Q^l = Q_m$ and $\pi^l(m) = \pi(m)$. Moreover, since total output increases with entry by Lemma 1, we know that both Q_{m+1} and Q^l are larger than Q_m. The strict concavity of p implies both that p^l is more elastic than p at each quantity $Q > Q_m$ and also that the elasticity of p decreases as the output increases. We must conclude that firms produce less at the Cournot equilibrium corresponding to the demand function that is less elastic at the relevant range. Thus, $Q_{m+1} < Q^l$. But since total profits decrease in Q at outputs larger than the monopoly level Q_1, we must also conclude that $\pi(m+1) > \pi^l(m+1)$. Consequently, $\pi(m) - 2\pi(m+1) < \pi^l(m) - 2\pi^l(m+1) < 0$ for $m > 1$, where the second inequality follows from example 1. This argument establishes somewhat informally the first statement of the following lemma (A more detailed proof is included in the appendix).

Lemma 2 *Suppose that the inverse demand function satisfies condition* M_2. *Then, at the unique Cournot equilibrium,* $\pi(m)$ *satisfies both* A_1 *and* A_2. *Moreover,* $\pi(1) < \left(\frac{1+m}{2}\right)^2 \pi(m)$, *for* $m = 2, 3, \ldots$

4 The Auction

In this section I examine in detail the auction of licences. Subsection 4.1 focuses on the role of externalities in the context of a single-unit auction. Section 4.2 includes the main results of the paper. Proposition 1 provides a characterization of the MRU allocation for sequential auctions that satisfy assumptions A_1 and A_2. Moreover, sharper implications are drawn for the case in which the post-auction market game is a Cournot game. Extension to the case of a Bertrand game is provided in a final remark.

4.1 Preliminary Remark: Externalities

A particularity of the auctions that we study here is that some firms' payoffs may exhibit externalities. The bidders' payoffs in the auction are induced by their expected profits in the ensuing market game and, typically, these profits depend on the total number of firms in the market. Therefore, each bidder's payoff depends not only on whether he obtains a licence to operate in the market, but also on the total number of firms that own licences. For instance, if at some point during the auction several firms own licences, each one of these incumbent firms would rather have other incumbents buy the remaining licences than allowing a reduction of profits due to new entry. As a result of these externalities, some of the incumbents' valuations may not be uniquely defined: they may depend on whether they expect another incumbent or a new entrant to be the highest bidder for the unit in question. An example illustrates this matter.

Example 2 (Static Second Price Auction) *A single licence is sold using a second price auction.* $|I_1|$ *incumbent firms and* $n-|I_1|$ *potential new entrants participate in the auction. Firms that own at least one licence are expected to engage in a Cournot game after the auction. If a new entrant obtains the licence the post auction market will be shared by* $|I_1|+1$ *firms, each one making* $\pi(|I_1|+1)$ *in profits. Since potential entrants obtain zero profits if they fail to enter the market, their valuation for the licence is unambiguously defined as* $v_i = \pi(|I_1|+1)$*, where* $i \notin I_1$*. The case of incumbents is somewhat different. No matter which incumbent obtains the licence, the total number of firms in the market remains the same, so each one of them makes* $\pi(|I_1|)$ *in profits. Thus, if the incumbent firm* i *believes that the new entrant* j *will be the highest bidder, he will attach to the licence a value of* $v_{ij} = \pi(|I_1|) - \pi(|I_1|+1) > 0$*, according to the expected loss of profits due to entry. However, if* i *believes that the highest bidder will be another incumbent* k*, he will value the licence in* $v_{ik} = \pi(|I_1|) - \pi(|I_1|) = 0$*. Consequently, when* $|I_1| > 1$*, unlike the new entrants' valuations, the incumbents' valuations are not uniquely defined.*

I introduce the following concepts and observations to facilitate the analysis of the role of externalities in our problem.

A *single-unit auction with externalities* is defined by a *valuations-matrix* $(v_{ij})_{i\in N,\, j\in N\setminus\{i\}}$. Also denote $\underline{v}_i = \min_{j\in N\setminus\{i\}} v_{ij}$ and $\bar{v}_i = \max_{j\in N\setminus\{i\}} v_{ij}$. The following lemma provides an immediate characterization of the sets of undominated strategies for each bidder, both in the case of a *first price auction (FPA)* and of a *second price auction (SPA)*.

Lemma 3 *Consider a static auction with valuations-matrix* $(v_{ij})_{i\in N,\, j\in N\setminus\{i\}}$*. Then (i) in the case of a* FPA*, bids* $b \geq \bar{v}_i$ *are weakly dominated for player* i*; and (ii) in the case of a* SPA*, bids* $b < \underline{v}_i$ *and bids* $b > \bar{v}_i$ *are weakly dominated for player* i*.*

Since bidders' valuations may not be uniquely defined due to the presence of externalities, Lemma 1 implies that undominated outcomes may be non-unique. However, one can single out a case in which externalities do not preclude uniqueness of the undominated Nash equilibrium allocation.

Given a valuations-matrix $(v_{ij})_{i\in N,\, j\in N\setminus\{i\}}$, relabel the players so that $\bar{v}_1 \geq \bar{v}_2 \geq ... \geq \bar{v}_n$. I say that the valuations-matrix displays **Determinate High Values** if, for every $i \in N$, $\bar{v}_i \geq \bar{v}_2$ implies that $\underline{v}_i = \bar{v}_i$. In this case, we can write $v_1 = v_{1j}$ and $v_2 = v_{2j}$. For simplicity, I assume that $\overline{v}_1, \overline{v}_2 \in B$. The following result is an immediate consequence of Lemma 3.

Lemma 4 *Consider a single-unit auction whose valuations-matrix displays Determinate High Values. Denote $V_1 = \{i \in N : \bar{v}_i = \bar{v}_1\}$. Then, independently of whether the auction is a FPA or a SPA, at every undominated Nash equilibrium, each $i \in V_1$ participates in the tie-breaker and the winner pays a price $p \cong v_2$, an approximation that becomes exact as the money unit u tends to* 0.[2]

In Example 2 the unique undominated choice for a new entrant i consists in bidding $v_i = \pi(|I_1| + 1)$. When $|I_1| = 1$, the only incumbent, say bidder j, knows that his failure to obtain the licence would necessarily result in entry, so his valuation also is uniquely defined by $v_j = \pi(|I_1|) - \pi(|I_1| + 1)$. Since in this case all the valuations are uniquely defined and $v_j - v_i = \pi(|I_1|) - 2\pi(|I_1| + 1) > 0$ by assumption A_1, we conclude that the initial monopolist buys the licence at a price $p_1 = \pi(|I_1| + 1)$.

The strategic problem of the incumbents is somewhat different when $|I_1| > 0$. Since incumbent's valuations are not uniquely defined in this case, Lemma 3 implies that all the bids between 0 and $\pi(|I_1|) - \pi(|I_1| + 1)$ are undominated for each incumbent. Note that due to the presence of externalities the undominated Nash equilibrium allocation may be non-unique. For instance, when $I_1 = \{1, 2\}$ and $\pi(2) > 2\pi(3)$, the new entrants' valuations are between the two possible valuations of each incumbent. A public good problem between the incumbents arises: an incumbent will buy the licence and preempt entry if and only if the other incumbent abstains from doing it himself. Consequently, there are two undominated Nash equilibrium allocations, a different incumbent buying the unit at a price equal to $\pi(3)$ in each one of them. However, condition A_2 rules out the possibility of non-uniqueness by ensuring that the uniquely defined valuation of the potential new entrants is larger than both possible valuations of the incumbents, so the valuations exhibit Determinate High Values in this case. According to Lemma 4 all the undominated Nash equilibria result in the same allocation: a tie-breaker decides which entrant gets the licence and the winner pays an amount equal to $\pi(3)$.

4.2 The Sequential Auction: Main Results

In this subsection I examine in detail the sequential auctions of licences emphasizing the connection between the outcome of the auction and the resulting market

[2]When $B = \Re_+$, the equilibrium price is exactly equal to v_2 in the case of a SPA. In the case of a FPA, undominated Nash equilibria may not exist. However, at every undominated *ϵ-equilibrium*, the allocation is identical to the one of the corresponding SPA.

structure.

The preceding discussion of Example 2 provides a simple illustration of some fundamental insights. The entry preemption paradigm claims that a monopolist always has incentive to preempt entry by buying either licences or capacity. It argues that since total profits decrease with entry because of the appearance of competition in the market, the decrease in the monopolist's profits due to entry is larger than the increase in the profits of the new entrants. Consequently, the monopolist will outbid potential entrants in any static, once-and-for-all sale. However, the persistence of monopoly does not generally extend to the case of markets where other forms of imperfect competition prevail. As we have seen in the context of Example 2, a Cournot oligopoly may allow entry in the auction of a single licence. Although entry reduces the aggregate profits of the oligopoly, unlike the monopoly case these profits are not appropriated by a single oligopolist; therefore, none of them receives the full blow of the reduction. In fact, if profits do not decrease too fast, the decrease in profits expected by each incumbent firm if entry occurs may be smaller than the increase expected by new entrants, so entry will occur in this case.

Since the concept of MRU equilibrium reduces the analysis of the sequential auction to the one of a recursively defined sequence of single-unit auctions, the preceding observations about single unit auctions are directly relevant to the analysis of the more general case. The following remarks highlight the relationship between static and dynamic cases.

First, note that the *instability of an oligopoly* facing entry extends to the dynamic case: the last stage of a sequential auction is in fact a single-unit auction involving an oligopoly, and according to Example 2, entry will occur as long as condition A_2 is satisfied. In fact, a simple recursive argument shows that entry occurs in every stage. We conclude that if *some* competition is present, it will *expand.*

The *question of persistence or stability of a monopoly* that faces entry in a dynamic setting is somewhat more involved. However, an immediate insight follows directly from the preceding paragraph: if entry ever occurs we will be dealing with an auction that involves an oligopoly thereafter, so entry will continue to occur thereafter. Potential entrants know that were they to succeed in obtaining a licence in stage s, entry would continue to occur and $t-s+2$ firms would end up in the market after the auction. Thus, they are willing to offer the amount $\pi(t-s+2)$ for the sth unit. Consequently, if a monopolist that has already preempted entry up to the rth stage wishes to preempt entry completely, he has to pay an amount equal to $\sum_{s=r}^{t}\pi(t-s+2)=\sum_{\tau=2}^{t-r+2}\pi(\tau)$ for the remaining licences. Moreover, since a monopolist that preempts entry completely will make an amount $\pi(1)$ in gross profits, we conclude that the benefit of complete preemption is $\pi(1)-\sum_{\tau=2}^{t-r+2}\pi(\tau)$. On the other hand, if the monopolist allows entry in the rth stage, entry will continue thereafter and he will end up making $\pi(t-r+2)$ in profits. Restricting our attention to the alternative between buying the remaining units or allowing entry immediately, we can talk unambiguously of the net benefit to a monopolist that has already preempted r units of preempting the remaining units, which we define as the difference between the benefits associated to the preceding alternative, say

$$\Delta_t(r) = \pi(1) - \pi(t-r+2) - \sum_{\tau=2}^{t-r+2} \pi(\tau)$$

Indeed, that alternative reflects the choices available to a single incumbent in equilibrium. The following result shows that, generically, the outcome of the sequential auction will be a monopoly if and only if both $|I_1| = 1$ and the *net benefit of complete preemption* $\Delta_t(1)$ is positive. Otherwise, entry in each stage should be expected.

Proposition 1 *Consider the auction games described in section 2. Both for a SFPA and for the corresponding SSPA, the MRU allocation is unique and satisfies the following properties:*

(i) Suppose that both $|I_1| = 1$ *and* $\Delta_t(1) > 0$. *Then* $|I_{t+1}| = 1$ *and* $p_\tau \cong \pi(t-\tau+2)$ *for all* $\tau \in T$.

(ii) Suppose that either $|I_1| > 1$ *or* $\Delta_t(1) < 0$. *Then* $|I_{t+1}| = |I_1| + t$ *and* $p_\tau \cong \pi(|I_1| + t)$, *for all* $\tau \in T$.

(iii) Suppose that $|I_1| = 1$ *and* $\Delta_t(1) = 0$. *Then there is a tie-breaker for the first unit. If* $i \in I_1$ *is selected we are in case (i). Otherwise, we are in case (ii).*

The following example illustrates the last result.

Example 3 (Two-Stage Second Price Auction) *We consider the same environment as in Example 2 except that now two licences are auctioned sequentially. Suppose that* $I_1 = \{1\}$. *To find the MRU allocation we proceed recursively. Since the last stage is a static auction, we know from Example 2 that if firm* 1 *obtains the first licence, it will also obtain the second one at a price* $p_2 = \pi(2)$. *Thus, firm* 1*'s expected payoff gross of the first period's payment is* $\pi(1) - \pi(2)$ *if it buys the first licence. We also know that if entry occurs in the first stage, entry will occur again in the second one as a consequence of assumption* A_2. *Thus, if a new entrant* i *buys the first licence, both* i *and* 1 *have an expected payoff gross of first period payments equal to* $\pi(3)$. *These observations imply that the recursive valuation of the first unit to firm* 1 *is* $v_1(h_1) = \pi(1) - \pi(2) - \pi(3)$. *Now consider the case of a potential entrant* i. *If he obtains the first unit, entry will occur again in the second stage, so* i*'s payoff gross of the first period's payment is* $\pi(3)$. *If* i *fails to obtain the first unit two scenarios are possible: either firm* 1 *gets it or* *some other entrant gets it. In the first case, firm* 1 *will preempt entry completely so firm i ends up with no licence at all and zero profits. Finally, if some other entrant j gets the first unit, firm i has a positive probability of getting the second unit at a price* $p_2 = \pi(3)$. *However, since entry occurs twice in this case, firm i's expect payoff gross of the first period's payment will be* $\pi(3) - \pi(3) = 0$ *anyway. We conclude that the recursive valuation of the first unit to a potential entrant i is* $v_i(h_1) = \pi(3)$. *Since all the recursive valuations are uniquely defined, they are the only locally undominated bids for each bidder in the first stage according to Lemma 4 . Moreover, note that* $v_1(h_1) - v_i(h_1) = \pi(1) - \pi(2) - 2\pi(3) = \Delta_2(1)$. *Generically, either one of two cases may arise. When* $\Delta_2(1) > 0$ *firm* 1 *is the high value player so it obtains both units at prices* $p_1 = \pi(3)$ *and* $p_2 = \pi(2)$. *When* $\Delta_2(1) < 0$ *all the potential entrants are high value players so the tie breaker decides which one obtains the first licence at price* $p_1 = \pi(3)$. *According with Example 2, in the second stage, another new entrant obtains the second unit at the same price.*

Proposition 1 extends and qualifies the entry preemption insights derived from a static formulation. It confirms the idea that oligopolies would allow entry but severely qualifies the presumption that a monopolist would always have an incentive to avoid entry. It should be stressed that the limits of the analysis are clearly associated to the validity of assumptions A_1 and, particularly, A_2. However, as we have seen in section 2, the assumptions are satisfied by some interesting market games.

In fact, in the particular case in which the post-auction market game is the symmetric *Cournot game* described in that section, somewhat sharper predictions can be obtained. Consider an environment in which a single monopolist faces a large number of potential entrants. We know that if a single unit is auctioned, the monopolist always preempts entry. The question is whether the monopolist would ever abandon his preemptive behavior to allow entry if a sufficiently large number of units is offered. The following corollary of Proposition 1 shows that if a sufficiently large number of licences is auctioned, new entry occurs in every stage. Let $\mathcal{N}$ denote the set of the natural numbers, and also denote $t^0 = \sup\{t \in \mathcal{N} : \Delta_t(1) > 0\}$.

Corollary 1 *Consider the case in which* $N = \mathcal{N}$ *and* $|I_1| = 1$. *Suppose that the post-auction market game is the Cournot game described in section 2 and that the market demand satisfies assumption* M_2. *Then*

(i) If $t \leq t^0$, then $|I_{t+1}| = 1$. Otherwise, $|I_{t+1}| = |I_1| + t$.

(ii) Suppose also that p satisfies assumption M_2. Then $t^0 < \infty$.

Proof. (i) Just notice that $\Delta_{t+1}(1) - \Delta_t(1) = \pi(t+1) - 2\Delta(t+2) < 0$ by A_2, and then apply Proposition 1.

(ii) Lemma 2 implies that $\pi(1) < \left(\frac{1+r}{2}\right)^2 \pi(r)$ for $r = 2, 3, ...$ Thus, using the fact that $\sum_{\tau=1}^{\infty} \left(\frac{1}{\tau}\right)^2 \cong 1.6$, we have that $\sum_{\tau=2}^{\infty} \pi(\tau) > 4\pi(1) \sum_{\tau=2}^{\infty} \left(\frac{1}{1+\tau}\right)^2 \cong 1.6\,\pi(1)$. But then $\sum_{\tau=2}^{\infty} \pi(\tau) > \pi(1)$, so we conclude that $\Delta_t(1) < 0$ for t large enough. □

This result contrasts sharply with the static case. It suggests that by auctioning sequentially a sufficiently large number of licences, a government can ensure the competitiveness of the resulting market. But will the government or, more generally, a seller actually do that? Suppose that the seller aims at maximizing his own revenue. Let R_t denote the seller's revenue associated to the MRU allocation of the sequential auction of t units and let $\hat{t}$ denote the number of units that maximizes that revenue. The following characterization of this revenue is an immediate consequence of Proposition 1 and Corollary 1:

$$\begin{array}{ll} R_t = \sum_{\tau=2}^{t+1} \pi(\tau) & \text{when } t \leq t^0 \\ R_t = t\pi(t+1) & \text{when } t > t^0 \end{array} \tag{1}$$

Examination of the expression (6) leads to the following result:

Corollary 2 *Consider the case in which $N = \mathcal{N}$ and $|I_1| = 1$. Suppose that the post-auction market game is the Cournot game described in section 2 and that the market demand satisfies assumption M_2. Then*

(i) In general, $\hat{t} \geq \max\{2, t^0\}$. Moreover, $t^0 > 2$ implies that $\hat{t} = t^0$.

(ii) Suppose that $\pi(3) > (t/2)\,\pi(t+1)$ for $t = 3, 4, ...$ Then $\hat{t} = \max\{2, t^0\}$.

(iii) In the *linear demand* case, $\hat{t} = t^0 = 2$.

Proof.

(i) First note that $R_t - R_{t+1} = -\pi(t+2) < 0$ for $t < t^0$. Thus, $\hat{t} \geq t^0$. Also note that $R_1 - R_2 = \pi(2) - 2\pi(3) < 0$ when $t^0 = 1$. Thus, $\hat{t} \geq 2$. Finally, suppose that $t^0 > 2$. Then, for $t > t^0$, we have that

$$\begin{aligned} R_{t^0} - R_t &= \sum_{\tau=2}^{t^0+1} \pi(\tau) - t\pi(t+1) \\ &\geq \left(\frac{t^0+1}{2}\right) \pi(t^0+1) + \sum_{\tau=3}^{t^0+1} \pi(\tau) - t\pi(t+1) \\ &> (t^0+1)\,\pi(t^0+1) - t\pi(t+1) > 0 \end{aligned}$$

where the inequalities follow by A_1. We conclude that $\hat{t} = t^0$.

(ii) In view of part (i) we only need to consider the case of $t^0 \leq 2 \leq t$. Note that by assumption $R_2 - R_t = 2\pi(3) - t\pi(t+1) > 0$ when $t^0 = 1$, and $R_2 - R_t =$

$\pi(2) + \pi(3) - t\pi(t+1) \geq 2\pi(3) - t\pi(t+1) > 0$ when $t^0 = 2$. The conclusion follows.
(iii) It follows directly from Example 1 and part (ii). $\square$

In other words, this result identifies cases in which a revenue maximizing seller prefers to sell less than the number of licences or capacity units that would be consistent with new entry. In particular, this will be the case whenever the largest number of units consistent with the monopoly solution, i.e.: t_0, is larger than two. In fact, even when that number is equal to two, the seller may choose not to encourage competition, an example of which is the linear demand case. Thus, although it is always possible to encourage competition by auctioning a sufficiently large number of units, the possibility that complete preemption occurs should not be neglected.

4.3 Remark: The Bertrand Case

Let us briefly consider the case in which the post-auction market game is a symmetric Bertrand game with constant returns to scale technology. In particular, if a single incumbent preempts entry completely, it will set prices so as to obtain the monopoly level of profits $\pi(1) > 0$. On the other hand, if entry occurs, price competition will drive total industry profits to zero, i.e.: $\pi(m) = 0$ for $m > 1$.

Although this case does not exactly satisfy assumptions A_1 and A_2, it is a limiting case of the class of problems that satisfy these assumptions. Nonetheless, the characterization of the MRU allocation of the corresponding sequential auction is almost immediate in the present case. First note that if there is more than one incumbent at the beginning of stage r, i.e.: $|I_r| > 1$, both incumbents and potential entrants expect to make zero profits in whatsoever market emerges from the auctions. Consequently, everybody's recursive valuation must be zero, so the tie breaker decides who obtains the licence at a price equal to zero. Suppose now that $I_r = \{1\}$. Arguing recursively, suppose that if firm 1 buys the rth unit it will also buy the remaining units at a price approximately equal to zero, obtaining expected profits equal to $\pi(1)$. Since firm 1's expected profits are zero if an entrant obtains the rth unit, we must conclude that its recursive valuation for the rth unit is $\pi(1)$. But then firm 1 will also obtain the rth unit at a price approximately equal to zero. The following result follows.

Proposition 2 *Consider the case $|I_1| = 1$. Suppose that the post-auction market game is a symmetric Bertrand game with constant returns to scale technology. Then a single firm preempts entry completely.*

5 Conclusion

Many of the insights about entry preemption and persistence of monopoly that had initially been formulated in a static framework were reexamined here in a dynamic setting through an analysis of sequential auctions of licences. The findings extend and partly contrast with those initial insights. When the post-auction market game is a symmetric Cournot game, entry will occur repeatedly as long as a sufficiently large number of units is auctioned. However, a monopoly could still persist if for some reason the seller does not offer a number of units large enough. One possible reason is that in some cases to sell a large number of units is not revenue maximizing. The persistence of monopoly also extends to the dynamic case when the post-auction market game is a Bertrand game, which is only a limiting case of the class of problems studied here. These findings also qualify and extend the results of Krishna [4] to the case in which the post-auction market game is symmetric.

I use a new solution concept that makes the analysis remarkably simple and provides a solution that is both independent of whether the selling procedure is a first or a second price sequential auction and completely determinate as long as externalities play a limited role. Although the assumption that externalities are small appears compelling in the class of economic environments studied here, future research should extend the analysis of sequential auctions to environments involving externalities, non-separability and incomplete information.

Appendix

Proof of Lemma 1.
(i) Optimality of the firms' decisions and symmetry require that the following first order condition be satisfied by the equilibrium output:

$$p'(Q_m)Q_m + m[p(Q_m) - c] = 0 \quad for\ m = 1, 2, .. \tag{2}$$

This implies that $[p'(Q_{m+1})Q_{m+1} + mp(Q_{m+1})] - [p'(Q_m)Q_m + mp(Q_m)] = c - p(Q_{m+1}) < 0$. By condition M_1, $p'(Q)Q+mp(Q)$ is decreasing in Q. We conclude that $Q_m < Q_{m+1}$.
(ii) Denote $\Pi(Q) = [p(Q) - c]\, Q$ and note that $\Pi(Q_m) = m\pi(m)$, for $m = 1, 2, ...$ Equation (2) implies that $\Pi'(Q_1) = p'(Q_1)Q_1 + p(Q_1) - c = 0$. Moreover, (M_1) implies that $\Pi''(Q) = p''(Q)Q + 2p'(Q) < 0$. Thus, part (ii) follows from part (i).
(iii) Equation (2) implies that $\pi(m) = (p(Q_m) - c)\, q_m = -p'(Q_m)q_m^2$. Thus, we can write

$$\begin{aligned} \pi(m) - 2\pi(m+1) &= 2p'(Q_{m+1})q_{m+1}^2 - p'(Q_m)q_m^2 \\ &= 2(1+m)^{-2}p'(Q_{m+1})Q_{m+1}^2 - m^{-2}p'(Q_m)Q_m^2 \end{aligned} \tag{3}$$

Note that M_1 implies that

$$(d/dQ)\left[p'(Q)Q^2\right] = \left[p''(Q)Q + 2p'(Q)\right]\ Q < 0 \tag{4}$$

Since $Q_m < Q_{m+1}$ and $2(1+m)^{-2} > m^{-2}$ for $m > 2$, we conclude from (3) and (4) that $\pi(m) - 2\pi(m+1) < 0$ for $m > 2$, as desired. □

Proof of Lemma 2. Since M_2 is stronger than M_1, $\pi(m)$ satisfies A_1 by Lemma 1(ii). Moreover, Lemma 1(iii) implies that the inequality in A_2 is satisfied for $m > 2$. It only remains to show that this inequality also is verified for $m = 2$. First note that

$$\pi\,(2) - 2\pi\,(3) = \left(p(Q_2) - p(Q_3)\right) q_2 + \left(p(Q_3) - c\right)\left(q_2 - 2q_3\right) \tag{5}$$

Concavity of p implies that $p(Q_2) - p(Q_3) \leq p'(Q_3)(Q_2 - Q_3)$. Combining with equation (2), we obtain that $p(Q_2) - p(Q_3) \leq q_3^{-1}\left(c - p(Q_3)\right)\left(2q_2 - 3q_3\right)$. Substituting this inequality into (5), we conclude that

$$\pi\,(2) - 2\pi\,(3) \leq -\left(2/q_3\right)\left(p(Q_3) - c\right)\left(q_2 - q_3\right)^2 < 0$$

as desired.

To establish the last statement, note that

$$4\pi\,(1) - (1+m)^2\,\pi\,(m) = 4q_1\left(p(Q_1) - p(Q_m)\right) + \left(p(Q_m) - c\right)\left(4q_1 - (1+m)^2 q_m\right) \tag{6}$$

As above, $p(Q_1) - p(Q_m) \leq q_m^{-1}\left(c - p(Q_m)\right)\left(q_1 - mq_m\right)$ by concavity of p and equation (2). Substituting this into (6), we conclude that

$$4\pi(1) - (1+m)^2\pi(m) \leq -\left(1/q_m\right)\left(p(Q_m) - c\right)\left(2q_1 - (1+m)q_m\right)^2 < 0$$

as desired. □

Proof of Lemma 3. (i) Consider bids $b \geq \bar{v}_i$ and $\hat{b} < \bar{v}_i$. If the choices by the remaining players and the outcome of the tie-breaker are such that either i wins with $\hat{b}$ or loses with b, then i weakly prefers $\hat{b}$ to b. On the other hand, suppose that i wins with b but loses with $\hat{b}$. Since i makes zero expected profits with $\hat{b}$ and negative expected profits with b, he strictly prefers $\hat{b}$ to b. Part (i) follows. (ii) Suppose that $b > \bar{v}_i$. Player i is indifferent between bidding $\bar{v}_i$ or b conditional on the fact that either both bids win or both lose. On the other hand, if the choices of the remaining players and the tie-breaker are such that b wins and $\bar{v}_i$ loses, i pays a price larger than $\bar{v}_i$ and thus, he makes negative profits, as opposed to the zero profits made (conditionally) by a bid of $\bar{v}_i$. When $b < \underline{v}_i$ the argument is analogous. □

Proof of Lemma 4. (i) Lemma 3(i) implies that $b_i < \bar{v}_i$ with probability 1, for every $i \in N$. Let $\underline{b}_1$ denote the lowest bid that 1 uses with positive probability. If $\underline{b}_1 < \bar{v}_2$ with positive probability, then player 2 will rather outbid $\underline{b}_1$ with probability 1. But then $\underline{b}_1$ makes 0 profits and is not an optimal choice for player 1. The contradiction shows that $\underline{b}_1 = \bar{v}_2$. (ii) follows directly from Lemma 3(ii). □

Proof of Proposition 1. Equivalently, we show that both for a $SFPA$ and for the corresponding $SSPA$, the MRU allocation is unique and satisfies the following properties at each $h_s \in H_s$ and for every $s \in T$:

(a) Suppose that both $|I_s| = 1$ and $\Delta_t(s) > 0$. Then $|I_{t+1}| = 1$ and $p_\tau \cong \pi(t - \tau + 2)$, for all $\tau \geq s$.

(b) Suppose that either $|I_s| > 1$ or $\Delta_t(s) < 0$. Then $|I_{t+1}| = |I_s| + t - s + 1$ and $p_\tau \cong \pi(|I_s| + t - s + 1)$, for all $\tau \geq s$.

(To simplify the notation I have omitted the non-generic case in which $\Delta_t(1) = 0$, although the extension is immediate. I have also omitted reference to the strategy β in $u_{ir}(h_r, \beta)$ and $v_{ij}(h_r, \beta)$). The proof of this result is recursive. The main steps are contained in the following two lemmas.

Lemma 5 *Suppose that the conditions (a)-(b) are satisfied after the rth stage. Then the recursive valuations at stage r satisfy the following conditions:*

(i) Suppose that both $|I_r| = 1$ and $\Delta_t(r+1) > 0$. Then

$$\begin{array}{lll} v_{ij}(h_r) = \pi(1) - \sum_{\tau=2}^{t-r+2} \pi(\tau) & \textit{when} & i \in I_r \\ v_{ij}(h_r) = \pi(t - r + 2) & \textit{when} & i \notin I_r \end{array}$$

(ii) Suppose that either $|I_r| > 1$ or $\Delta_t(r+1) < 0$. Then

$$\begin{array}{lll} v_{ij}(h_r) = 0 & \textit{when} & i \in I_r \textit{ and } j \in I_r \\ v_{ij}(h_r) = \pi(|I_r| + t - r) - \pi(|I_r| + t - r + 1) & \textit{when} & i \in I_r \textit{ and } j \notin I_r \\ v_{ij}(h_r) = \pi(|I_r| + t - r + 1) & \textit{when} & i \notin I_r \end{array}$$

Proof. First suppose that both $|I_r| = 1$ and $\Delta_t(r+1) > 0$. If $a_r \in I_r$ then $I_{r+1} = I_r$. Thus, as a consequence of the hypothesis $|I_{t+1}| = 1$ and $p_\tau \cong \pi(t - \tau + 2)$ for all $\tau \geq r$. Similarly, if $a_r \notin I_r$, then $|I_{t+1}| = |I_r| + t - r + 1$ and $a_\tau \in I_r$ and $p_\tau \cong \pi(|I_r| + t - r + 1)$ for all $\tau \geq r$. Thus,

$$\begin{array}{ll} u_{i,r+1}(h_{r+1}) = \pi(1) - \sum_{\tau=2}^{t-r+1} \pi(\tau) & \text{if both } i \in I_r \text{ and } a_r \in I_r \\ u_{i,r+1}(h_{r+1}) = \pi(t - r + 2) & \text{if either } a_r = i \notin I_r \text{ or both } i \in I_r \text{ and } a_r \notin I_r \\ u_{i,r+1}(h_{r+1}) = 0 & \text{if both } a_r \neq i \text{ and } i \notin I_r \end{array}$$

Part (i) follows by substituting in the definition of the recursive values at r.

Suppose now that either $|I_r| > 1$ or $\Delta_t(r+1) < 0$. The hypothesis implies that $u_{i,r+1}(h_{r+1}) = \pi(|I_{r+1}| + t - r)$. Again, part (ii) follows by substituting in the definition. □

Lemma 6 *Suppose that the recursive values at r associated to an MRU allocation satisfy conditions (i) and (ii) of Lemma 5. Then the MRU allocation must satisfy conditions (a) and (b) at every $h_r \in H_r$.*

Proof. Suppose that both $|I_r| = 1$ and $\Delta_t(r) > 0$. Since $\Delta_t(r) - \Delta_t(r+1) = \pi(t-r+1) - 2\pi(t-r+2) < 0$ for $r < t$ by A_2, we have that $\Delta_t(r+1) > 0$. Thus, by condition (i) of Lemma 5 the recursive values at r display Determinate High Values and also $v_i(h_r) - v_j(h_r) = \Delta_t(r) > 0$ when $i \in I_r$ and $j \notin I_r$. Using Lemma 4 to analyze the local auction at r, we conclude that $a_r \in I_r$ and $p_r \cong \pi(t-r+2)$. This establishes part (i).

Suppose now that either $|I_r| > 1$ or $\Delta_t(r) < 0$. Note that $r < t$ since $\Delta_t(t) = \pi(1) - 2\pi(2) > 0$ by A_1. Since $\Delta_t(\tau)$ is increasing in τ we only need to consider the following cases. When both $|I_r| = 1$ and $\Delta_t(r+1) > 0 > \Delta_t(r)$, condition (i) of Lemma 5 implies that the recursive values at r display Determinate High Values and that $v_i(h_r) - v_j(h_r) = \Delta_t(r) < 0$ if $i \in I_r$ and $j \notin I_r$. Thus, in this case Lemma 4 implies that $a_r \notin I_r$ and $p_r \cong \pi(|I_r| + t - r + 1)$, as desired. Finally, when either $|I_r| > 1$ or $\Delta_t(r+1) < 0$, condition (ii) of Lemma 5 and assumption A_2 imply that, for $i \in I_r$ and $j \notin I_r$, we have that $\bar{v}_i(h_r) - v_j(h_r) = \pi(|I_r| + t - r) - 2\pi(|I_r| + t - r + 1) < 0$. Since by assumption $n > t$, we conclude that the recursive values at r display Determinate High Values. Thus, Lemma 4 implies that $a_r \notin I_r$ and $p_r \cong \pi(|I_r| + t - r + 1)$, as desired. □

To complete the proof of Proposition 1, note that Lemma 5 shows that if the conditions (a)-(b) are satisfied after the rth stage then the recursive valuations at r associated with the MRU allocation must satisfy the conditions (i) and (ii) of Lemma 5. But then Lemma 6 implies that the conditions (a)-(b) are satisfied also at r, as desired. □

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