1 Introduction

The term structure of interest rates for default-free discount bonds has been a topic covered in many papers. As a first approximation to its analysis, one-factor models were developed. These models assume that the movements of the yield curve are determined by a single state variable. This state variable is usually the instantaneous riskless interest rate and is modeled as a diffusion process. Examples of these models are Vasicek (1977), Dothan (1978) and Cox, Ingersoll and Ross (hereafter CIR) (1985b). An empirical comparison among them can be seen in Chan *et al* (1992).

In this type of models, the instantaneous returns on bonds of all maturities are perfectly correlated. Moreover, since the single state variable follows a Markov process, the whole term structure of interest rates may be derived from the current value of the instantaneous interest rate. Although these models are very tractable, a single state variable may be not sufficient to capture adequately the direction of future yield curve changes.

Some theoretical work employing one-factor models with jumps also exists. Ahn and Thompson (1988) extend the CIR model to accommodate jump effects in the day-to-day movements in interest rates, and they develop a bond pricing model. Das (1994a) is the first empirical study of a jump-diffusion model of interest rates. The estimation procedure, using weekly data, allows him to identify where jumps occur in the data. He is also able to estimate the jump arrival frequency, size and sign along with the parameters of the diffusion process. In a subsequent paper, Das (1994b) analyzes the role of jump-diffusion interest rates in the bond markets when allowing the distribution of stochastic jumps to be time-varying. Das and Foresi (1996) extend the Vasicek model with the addition of jumps which displace interest rates by discrete amounts, but do not change their central tendency. Finally, Moreno and Peña (1996) study the dynamic behavior of the term structure of Interbank interest rates and the pricing of options on interest rate derivative securities by positing a single factor model with jumps. They also perform a qualitative examination of the linkage between Monetary Authorities' interventions and jumps in daily data.

Multi-factor models of interest rates, which arise as an attempt to avoid the unrealistic features related to one-factor models and to explain a greater variety of term structure movements over time, assume the existence of more than one state variable in the term structure of interest rates. As a practical matter, the number of factors is usually restricted to a maximum of two. For instance, Richard (1978) and CIR(1985b) assume that bond prices depend on the expected short-term (instantaneous) real interest rate, R, and the expected short-term (instantaneous) inflation rate, π. Brennan and Schwartz (1979) use the instantaneous interest rate and the long-term rate as state variables. In a similar way, Schaefer and Schwartz (1984) consider a model based on the consol rate (the yield on a bond with infinite maturity) and the spread, the difference between the consol rate and the short rate. Heath, Jarrow and Morton (1992) use two unspecified factors that affect forward rates. These two factors can be interpreted as a "long-run" factor (it affects all maturity forward rates equally), and a spread between a "short" and a "long term" factor because it affects the short maturity forward rates more than long term rates. Finally, Longstaff and Schwartz (1992) develop a model in which the state variables are the short-term interest rate and the volatility of the short-term interest rate. Examples of two-factor models with jumps are Naik and Lee (1995) and Das and Foresi (1996). The former authors consider regime shifts that alter the mean of bond yields as well as the volatility of yield changes. The two state variables are the regime index, with discrete changes, and the deviation of the short rate from the mean rate for the current regime. On the other hand, Das and Foresi (1996) develop a model in which jumps change the conditional central tendency of interest rates.

More recently, Chen (1996) proposes a three-factor model in which the future short rate depends on 1) the current short rate, 2) the short-term mean of the short rate, and 3) the current volatility of the short rate. These three state variables are modeled as square root processes and a general formula for interest rate derivatives is obtained. Although this type of models is still in a very preliminary stage, the first results seem to be very promising.

In this paper a two-factor model of the term structure of interest rates is presented. As indicated above, most one-factor models use the short-term rate as the single state variable. We add the long-term rate as the second state variable. With both factors, we are able to explain not only the changes in the yield curve (short and long) end but also the intermediate movements of the yield curve using its extremities. Brennan and Schwartz (1979) have also dealt with these two factors. They assume that the long-term rate and the instantaneous rate follow a joint Gauss-Markov process and evaluate the ability of the model to price bonds of different maturities. The parameters of the stochastic processes followed by interest rates are estimated with data on Canadian interest rates and a sample of Canadian bonds is priced.

In the spirit of Schaefer and Schwartz (1984), we redefine variables and model default free discount bond prices as a function of time to maturity and two factors, the long-term interest rate and the spread (difference between the long-term rate and the short-term (instantaneous) riskless rate of interest). As interest rates have a tendency to be pulled back to a long-run level, a phenomenon known as mean reversion, we reflect this fact assuming that each factor follows an Ornstein-Uhlenbeck process. Using non-arbitrage conditions, a general bond pricing equation is derived and a closed-form expression for the prices of bonds of different maturities is computed.

The paper is organized as follows. Section 2 derives the basic valuation equation that prices of any default free discount bond must satisfy. In Section 3 we compute a closed-form expression for the price of a bond of any maturity. The implications for the properties of the term structure are analyzed in Section 4. In Section 5 a closed-form expression for interest rate derivatives prices is derived. We apply this formula to price bond options and options on a bond portfolio. Moreover, more complex options prices are also evaluated. Section 6 describes the basic characteristics of the empirical application in which we compare the accuracy of our model with a one-factor model. Section 7 summarizes the main conclusions.

2 The Bond Pricing Equation

In this section, we derive the partial differential equation that prices of bonds of different maturities must verify. This equation is an equilibrium relationship between the expected returns of bonds which differ only in their maturity.

The main assumption we make is that the price, at time t, of a default free discount bond that pays \$1 at maturity T depends only on the current values of a set of state variables (X_i) and time to maturity, $\tau = T - t$. Thus, our first problem concerns to the selection of the state variables which are relevant for the determination of these prices.

One possible alternative would be to use the short-term (instantaneous) interest rate and the long-term rate as state variables. Thus, we may explain the intermediate movements of the yield curve by means of its extreme values.

Although most previous studies use the short-term interest rate as one of the state variable, we redefine these variables and, similarly to Schaefer and Schwartz (1984), choose two factors: the long-term rate, denoted by L, and the spread, denoted by s, the difference between the long rate and the short-term rate, denoted by r.

This selection of state variables allows us to use the assumption that both variables are orthogonal. Empirical evidence that supports this assumption has been shown in several papers as that of Ayres and Barry (1980), Schaefer (1980), and Nelson and Schaefer (1983). Ayres and Barry (1980) propose that the correlations of changes in long rates and changes in spreads are close to zero and corroborate this assumption using data from the Salomon Brothers yield book from January, 1956 through August, 1978. Schaefer (1980) shows that this idea is consistent with Brennan and Schwartz's (1980) estimates. Finally, Nelson and Schaefer (1983) have also tested the Ayres and Barry's orthogonality proposition for notes and bonds from the CRSP Government Bond Tape during the period 1930-1979. Using orthogonal variables helps to simplify the computation of the closed-form solution for the fundamental bond pricing equation.

After choosing the state variables, we assume that their dynamics over time are given by the following stochastic differential equations:

$$\left\{ \begin{array}{lcl} ds & = & \beta_1(s,L)dt + \sigma_1(s,L)dw_1 \\ dL & = & \beta_2(s,L)dt + \sigma_2(s,L)dw_2 \end{array} \right. \tag{1}$$

where t denotes calendar time, and dw_1 and dw_2 are Wiener processes where $E[dw_1] = E[dw_2] = 0$, $dw_1^2 = dw_2^2 = dt$, and (by the orthogonality between these variables) $E[dw_1 dw_2] = 0$. $\beta_1(.)$ and $\beta_2(.)$ are the expected instantaneous rates of change in the state variables and $\sigma_1^2(.)$ and $\sigma_2^2(.)$ are the instantaneous variances of changes in these two variables.

Therefore, these two variables follow a joint markovian process. This assumption implies that the expected future values of these variables is determined exclusively by their present values.

Let $P(s,L,t,T) \equiv P(s,L,\tau)$ be the price, at time t, of a default free discount bond that pays \$1 at maturity $T = t + \tau$. The instantaneous percentage price change of this bond is given by the following stochastic differential equation

$$\frac{dP(s,L,t,T)}{P(s,L,t,T)} = \mu(s,L,t,T)dt + s_1(s,L,t,T)dw_1 + s_2(s,L,t,T)dw_2 \tag{2}$$

where $\mu(s,L,t,T)$ is the expected rate of return of the bond, and $s_1(s,L,t,T)$ and $s_2(s,L,t,T)$ are the unexpected variations in return due to the random changes in the state variables.

Applying Itô's Lemma and the equation (1), we obtain

$$\begin{aligned} dP(.) &= P_s ds + P_L dL + P_t dt + \frac{1}{2}P_{ss}(ds)^2 + \frac{1}{2}P_{LL}(dL)^2 \\ &= \Big[P_s\beta_1(.) + P_L\beta_2(.) + P_t + \frac{1}{2}\sigma_1^2(.)P_{ss} + \frac{1}{2}\sigma_2^2(.)P_{LL} \;\; dt \\ &+ \sigma_1(.)P_s dw_1 + \sigma_2(.)P_L dw_2 \Big] \end{aligned} \tag{3}$$

where

$$P_s = \frac{\partial P(.)}{\partial s}, \quad P_L = \frac{\partial P(.)}{\partial L}, \quad P_{ss} = \frac{\partial^2 P(.)}{\partial s^2}, \quad P_{LL} = \frac{\partial^2 P(.)}{\partial L^2}$$

Equating (2) to (3), we obtain

$$\mu(s,L,t,T) = \frac{1}{P}\Big[P_s\beta_1(.) + P_L\beta_2(.) + P_t + \frac{1}{2}\sigma_1^2(.)P_{ss} + \frac{1}{2}\sigma_2^2(.)P_{LL} \tag{4}$$

$$s_1(s,L,t,T) = \sigma_1(.)\frac{P_s}{P}, \quad s_2(s,L,t,T) = \sigma_2(.)\frac{P_L}{P} \Big] \tag{5}$$

Since there are two stochastic variables driving all bond prices, we can set up a hedge portfolio, consisting of bonds of three different maturities, that is instantaneously riskless. Thus, we consider the investment strategy consisting of a portfolio V with three discount bonds of (arbitrary) maturities T_1, T_2 and T_3. The proportions we invest in each bond are z_1, z_2 and z_3, respectively. Therefore, the rate of return of this portfolio is given by

$$\begin{aligned} &\frac{dV(s,L,t,T)}{V(s,l,t,T)} = \sum_{i=1}^{3} z_i \frac{dP(s,L,t,T_i)}{P(s,L,t,T_i)} \\ = &\sum_{i=1}^{3} z_i\mu(s,L,t,T_i)dt + \sum_{i=1}^{3} z_i s_1(s,L,\tau_i)dw_1 + \sum_{i=1}^{3} z_i s_2(s,L,t,T_i)dw_2 \end{aligned} \tag{6}$$

Now we choose the proportions invested in each bond, z_i, ($\sum_{i=1}^{3} z_i = 1$) in such a way that the uncertainty of the return of this portfolio disappears, that is, these proportions are chosen so that the coefficients of dw_i in the above equation are equal to zero:

$$\sum_{i=1}^{3} z_i s_1(s,L,t,T_i) = \sum_{i=1}^{3} z_i s_2(s,L,t,T_i) = 0 \tag{7}$$

Under no-arbitrage conditions, the expected rate of return of this portfolio must be equal to the instantaneous riskless rate of interest, that is

$$\sum_{i=1}^{3} z_i \mu(s, L, t, T_i) = r$$

or, equivalently

$$\sum_{i=1}^{3} (z_i \mu(s, L, t, T_i) - r) = 0 \tag{8}$$

The equations (7) and (8) form a linear homogeneous system of three equations and three unknowns (the portfolio proportions). This system has a non-zero solution if and only if the matrix

$$C = \begin{pmatrix} s_1(s, L, t, T_1) & s_1(s, L, t, T_2) & s_1(s, L, t, T_3) \\ s_2(s, L, t, T_1) & s_2(s, L, t, T_2) & s_2(s, L, t, T_3) \\ \mu(s, L, t, T_1) - r & \mu(s, L, t, T_2) - r & \mu(s, L, t, T_3) - r \end{pmatrix} \tag{9}$$

is singular. Hence, it must be verified that the rows of this matrix are linearly dependent. The coefficients of the linear relationship which links these rows do not depend on maturity because we have chosen arbitrarily the maturities of the three bonds of this portfolio.

Therefore, there is a vector $\lambda(s, L, t) = (\lambda_1(s, L, t), \lambda_2(s, L, t))$ independent of τ such that

$$\mu(s, L, t, T) - r = \lambda_1(s, L, t) s_1(s, L, t, T) + \lambda_2(s, L, t) s_2(s, L, t, T) \tag{10}$$

We have substituted T_i for T because, since we have chosen arbitrarily the maturities of the bonds to be included in the portfolio, then this equilibrium relationship for the expected rate of return on a bond is valid for all maturities. Equation (10) expresses the instantaneous risk premium (the difference between the expected rate of return on the bond and the riskless interest rate) as a sum of two components which are derived from the two sources of uncertainty, that is, the two state variables.

The coefficients of this linear combination, $\lambda_1(.)$ and $\lambda_2(.)$, can be interpreted as the market prices of the spread and long-term rate risk because $s_1(s, L, t, T)$ and $s_2(s, L, t, T)$ are the instantaneous standard deviations of the return on the bond derived from unexpected changes in both variables.

Substituting the expressions for $\mu(.)$, $s_1(.)$, and $s_2(.)$ given by (4) and (5) into (10) we get

$$\Big[P_s\beta_1(.) + P_L\beta_2(.) + \frac{1}{2}\sigma_1^2(.)P_{ss} + \frac{1}{2}\sigma_2^2(.)P_{LL} \;\; + P_t \\ = rP + \lambda_1(s,L,t)\sigma_1(.)P_s + \lambda_2(s,L,t)\sigma_2(.)P_L\Big] \qquad (11)$$

Rearranging terms, we obtain the partial differential equation that the price of a default free discount bond for all maturities must satisfy:

$$\frac{1}{2}[\sigma_1^2(.)P_{ss} + \sigma_2^2(.)P_{LL}] + [\beta_1(.) - \lambda_1(.)\sigma_1(.)]P_s \\ +[\beta_2(.) - \lambda_2(.)\sigma_2(.)]P_L + P_t - rP = 0 \qquad (12)$$

Given the stochastic process (1), assumed for the state variables, (12) is the fundamental equation for the pricing of default free discount bonds of different maturities which depend solely on the spread, s, the long-term interest rate, L, and the time to maturity, τ. In this equation we have the market prices of risk, λ_i, because our model solves for all bond prices relative to each other. The only way to tie down the prices is by means of the exogenous parameters, the market prices of risk.

The solution of the equation (12), subject to the terminal condition given by the payment to be received at maturity, that is, $P(s, L, 0) = 1$, $\forall s, L$, allows us to price discount bonds and, thereafter, infer the term structure of interest rates. This solution is carried out in the next section.

3 Pricing of Discount Bonds

In this section, closed-form expressions for default free discount bond prices for all maturities are computed from the fundamental valuation equation we have obtained in the previous section. Once obtained this solution, implications on the properties of the term structure are analyzed.

The coefficients of the bond pricing equation (12) are the market prices of state variables risk, $\lambda_i(.)$, and the parameters of the joint stochastic process (1) which is assumed for the spread and the long-term rate. In order to solve this valuation equation, we must make some assumptions about the market prices of risk and the dynamics of the state variables. Since a constant market price of risk implies strong restrictions on the preferences of investors, we establish the following:

Assumption 1 *The market price of each state variable risk is linear in this variable, that is*

$$\lambda_1(.) = a + bs, \quad \lambda_2(.) = c + dL \tag{13}$$

Assumption 2 *Each of the state variables follow a diffusion process*

$$\begin{cases} ds &= k_1(\mu_1 - s)dt + \sigma_1 dw_1 \\ dL &= k_2(\mu_2 - L)dt + \sigma_2 dw_2 \end{cases} \tag{14}$$

This process, known as Ornstein-Uhlenbeck process, has been used previously by Vasicek (1977). It has mean reversion - an important stylized fact that interest rates usually show - and constant variance. For each state variable, $k_i > 0$ is the coefficient of mean reversion which reflects the speed of adjustment of the variable towards its long-run mean value, μ_i, σ_i is the (constant) standard deviation of each variable and dw_i are standard Gauss-Wiener processes.

The stationary (or steady state) distribution of a stochastic process, if it exists, is obtained from a time-independent solution of the stochastic differential equations given by (14), that is, $s(t,w) = s(w)$, $L(t,w) = L(w)$. Following Malliaris and Brock (1988) (Section 2.9, pp. 106-108), we prove the existence (and compute the expression) of the stationary distributions for the state variables:

$$\alpha(s) = \frac{f(s)}{1 - F(0)} \tag{15}$$

$$\alpha(L) = \frac{g(L)}{1 - G(0)} \tag{16}$$

where $F(.)$ and $f(.)$ are, respectively, the distribution and density functions of a normal variable with mean μ_1 and standard deviation $\sigma_1 / \sqrt{2k_1}$. Analogously, $G(.)$ and $g(.)$ are, respectively, the distribution and density functions of a normal variable with mean μ_2 and standard deviation $\sigma_2 / \sqrt{2k_2}$.

Moreover, it can be proved (see Vasicek (1977)) that the conditional expectation and variance of the processes $\{s(u), u \geq t\}$, and $\{L(u), u \geq t\}$, given the current value of each variable, are

$$\begin{aligned} E_t[s(u)] &= \mu_1 + (s(t) - \mu_1)e^{-k_1(u-t)}, \quad u \geq t \\ V_t[s(u)] &= \frac{\sigma_1^2}{2k_1}(1 - e^{-2k_1(u-t)}), \quad u \geq t \end{aligned} \tag{17}$$

and

$$\begin{aligned} E_t[L(u)] &= \mu_2 + (L(t) - \mu_2)e^{-k_2(u-t)}, \quad u \geq t \\ V_t[L(u)] &= \frac{\sigma_2^2}{2k_2}(1 - e^{-2k_2(u-t)}), \quad u \geq t \end{aligned} \tag{18}$$

respectively.

It may be verified that, as k_i tends to infinity, the conditional mean of the state variable goes to μ_i and its variance vanishes. If k_i approaches to zero, the conditional mean goes to the current value of the factor and the variance to $\sigma_1^2(u - t)$.

Under Assumptions 1 and 2, we can rewrite the equation (12) as

$$\begin{aligned} &\frac{1}{2}[\sigma_1^2 P_{ss} + \sigma_2^2 P_{LL}] + [(k_1\mu_1 - a\sigma_1) - (k_1 + b\sigma_1)s]P_s \\ &+[(k_2\mu_2 - c\sigma_2) - (k_2 + d\sigma_2)L]P_L + P_t - (L + s)P = 0 \end{aligned} \tag{19}$$

or, equivalently

$$\frac{1}{2}\sigma_1^2 P_{ss} + q_1(\hat{\mu}_1 - s)P_s + \frac{1}{2}\sigma_2^2 P_{LL} + q_2(\hat{\mu}_2 - L)P_L + P_t - (L + s)P = 0 \tag{20}$$

subject to the terminal condition

$$P(s, L, T, T) = 1, \quad \forall s, L \tag{21}$$

where

$$\begin{cases} q_1 = k_1 + b\sigma_1, & \hat{\mu}_1 = (k_1\mu_1 - a\sigma_1)/q_1 \\ q_2 = k_2 + d\sigma_2, & \hat{\mu}_2 = (k_2\mu_2 - c\sigma_2)/q_2 \end{cases} \tag{22}$$

Solving the partial differential equation (20) we obtain the following proposition:

Proposition 1 *The value at time t of a discount bond that pays \$1 at time T, $P(s,L,t,T) \equiv P(s,L,\tau)$, is given by*

$$P(s,L,t,T) = A(\tau)e^{-B(\tau)s - C(\tau)L} \tag{23}$$

where $\tau = T - t$ and

$$\begin{array}{rcl} A(\tau) & = & A_1(\tau)A_2(\tau) \\ A_1(\tau) & = & \exp\left\{-\frac{\sigma_1^2}{4q_1}B^2(\tau) + s^*(B(\tau)-\tau)\right\} \\ A_2(\tau) & = & \exp\left\{-\frac{\sigma_2^2}{4q_2}C^2(\tau) + L^*(C(\tau)-\tau)\right\} \\ B(\tau) & = & \left(1-e^{-q_1\tau}\right)/q_1 \\ C(\tau) & = & \left(1-e^{-q_2\tau}\right)/q_2 \end{array} \tag{24}$$

with

$$\begin{array}{lll} q_1 = k_1 + b\sigma_1, & s^* = \hat{\mu}_1 - \sigma_1^2/(2q_1^2), & \hat{\mu}_1 = (k_1\mu_1 - a\sigma_1)/q_1 \\ q_2 = k_2 + d\sigma_2, & L^* = \hat{\mu}_2 - \sigma_2^2/(2q_2^2), & \hat{\mu}_2 = (k_2\mu_2 - c\sigma_2)/q_2 \end{array} \tag{25}$$

Proof:

The method of the separation of variables allows us to write the solution of the equation (20) subject to (21) as

$$P(s,L,t,T) = X(s,t,T)Z(L,t,T) \tag{26}$$

where $X(s,t,T)$ solves the equation

$$\frac{1}{2}\sigma_1^2 X_{ss} + q_1(\hat{\mu}_1 - s)X_s + X_t - sX = 0 \tag{27}$$

subject to the terminal condition

$$X(s,T,T) = 1, \quad \forall s \tag{28}$$

and $Z(L,t,T)$ is the solution of the equation

$$\frac{1}{2}\sigma_2^2 Z_{LL} + q_2(\hat{\mu}_2 - L)Z_L + Z_t - LZ = 0 \tag{29}$$

with terminal condition

$$Z(L,T,T) = 1, \quad \forall L \tag{30}$$

To solve Equation (27), we posit a solution of the type

$$X(s,t,T) = A_1(\tau)e^{-B(\tau)s} \tag{31}$$

where $\tau = T - t$. From (31), we have

$$\begin{aligned} X_s(.) &= -B(t,T)X(.), \quad X_{ss}(.) = B^2(t,T)X(.) \\ X_t(.) &= -\left[\frac{A_1'(\tau)}{A_1(\tau)} - B'(\tau)s\right]X(.) \end{aligned} \tag{32}$$

and, so, the equation (27) becomes

$$\frac{1}{2}\sigma_1^2 B^2(t,T) - q_1(\hat{\mu}_1 - s)B(t,T) - \left[\frac{A_1'(\tau)}{A_1(\tau)} - B'(\tau)s\right] - s = 0 \tag{33}$$

where, from (28), the terminal conditions are given by

$$A_1(0) = 1, \quad B(0) = 0 \tag{34}$$

Equation (33) is linear in the variable s and, therefore, it becomes null when the corresponding coefficients are equal to zero. Hence, this equation is equivalent to the following system of first-order differential equations

$$q_1 B(\tau) + B'(\tau) - 1 = 0 \tag{35}$$

$$\frac{1}{2}\sigma_1^2 B^2(\tau) - q_1\hat{\mu}_1 B(\tau) - \frac{A_1'(\tau)}{A_1(\tau)} = 0 \tag{36}$$

subject to the terminal conditions (34).

We first solve (35) with terminal condition $B(0) = 0$. Including this solution in (36), integrating this equation, and using the condition $A_1(0) = 1$, we obtain

$$\begin{aligned} B(\tau) &= \frac{1 - e^{-q_1\tau}}{q_1} \\ A_1(\tau) &= \exp\left\{-\frac{\sigma_1^2}{4q_1}B^2(\tau) + s^*(B(\tau) - \tau)\right\} \end{aligned} \tag{37}$$

where

$$s^* = \hat{\mu}_1 - \sigma_1^2/(2q_1^2) \tag{38}$$

Replacing (37) into (31), we obtain the final expression for $X(s,t,T)$. In a completely similar fashion, the solution of Equation (29) is given by

$$Z(L,t,T) = A_2(\tau)e^{-C(\tau)L} \tag{39}$$

where

$$\begin{aligned} A_2(\tau) &= \exp\left\{-\frac{\sigma_2^2}{4q_2}C^2(\tau) + L^*(C(\tau)-\tau)\right\} \\ C(\tau) &= \frac{1-e^{-q_2\tau}}{q_2} \end{aligned} \tag{40}$$

with

$$L^* = \hat{\mu}_2 - \sigma_2^2 / (2q_2^2) \tag{41}$$

Therefore, the final expression for $Z(L,t,T)$ is given by replacing (40) into (39). Including the final expressions for $X(L,t,T)$ and $Z(L,t,T)$ into (26), we obtain the closed-form formula for the default free discount bond prices for all maturities.

□

The two terms in equation (37) verify

$$\left\{ \begin{array}{llll} B(\tau) > 0, \forall\tau > 0, & B(0) = 0, & B(\infty) = 1/q_1 \\ B(\tau) - \tau < 0, \forall\tau > 0 & & \\ A_1(\tau) < 1, \forall\tau > 0, & A_1(0) = 1, & A_1(\infty) = 0 \end{array} \right. \tag{42}$$

and, analogously, the terms $A_2(\tau)$ and $C(\tau)$ in (40) satisfy

$$\left\{ \begin{array}{llll} C(\tau) > 0, \forall\tau > 0, & C(0) = 0, & C(\infty) = 1/q_2 \\ C(\tau) - \tau < 0, \forall\tau > 0 & & \\ A_2(\tau) < 1, \forall\tau > 0, & A_2(0) = 1, & A_2(\infty) = 0 \end{array} \right. \tag{43}$$

The discount bond price, $P(s,L,\tau)$, is a function of the two state variables, s, and L, and the time to maturity, $\tau = T - t$. It depends on the parameters of the joint Ornstein-Uhlenbeck process (k_1, μ_1, σ_1, k_2, μ_2, and σ_2) as well as on the market prices of risk.

Substituting $t = T$, into (23), it is easily checked that the maturity condition for the price bond, $P(s, L, 0) = 1, \forall s, L$, is satisfied. Moreover, using (42) and (43), it is derived that

$$P(0, 0, \tau) = A(\tau) = A_1(\tau) A_2(\tau) < 1, \quad \forall \tau > 0$$

The bond price also shows economically realistic features such as

$$\lim_{s \to \infty} P(s, L, \tau) = 0, \quad \lim_{L \to \infty} P(s, L, \tau) = 0, \quad \lim_{\tau \to \infty} P(s, L, \tau) = 0$$

that is, when any of the variables which affect the bond price (state variables or time to maturity) tends to infinity, the price converges to zero.

The bond price function is decreasing and convex in both factors because its partial derivatives with respect to s and L are negative

$$\begin{aligned} P_s(s, L, \tau) &= -B(\tau) P(s, L, \tau) < 0 \\ P_L(s, L, \tau) &= -C(\tau) P(s, L, \tau) < 0 \end{aligned} \tag{44}$$

and the second partial derivatives verify

$$\begin{aligned} P_{ss}(s, L, \tau) &= B^2(\tau) P(s, L, \tau) > 0 \\ P_{LL}(s, L, \tau) &= C^2(\tau) P(s, L, \tau) > 0 \\ P_{sL}(s, L, \tau) &= B(\tau) C(\tau) P(s, L, \tau) > 0 \end{aligned} \tag{45}$$

The bond price is decreasing with the time to maturity. To see this, we apply (23) and (24) to get

$$\begin{aligned} P_\tau(s, L, \tau) &= \left[\frac{A'(\tau)}{A(\tau)} - B'(\tau) s - C'(\tau) L \right] P(s, L, \tau) \\ &= \left[\frac{A_1'(\tau)}{A_1(\tau)} - B'(\tau) s + \frac{A_2'(\tau)}{A_2(\tau)} - C'(\tau) L \right] P(s, L, \tau) \end{aligned}$$

When analyzing the sign of these derivatives, we will consider only the two first derivatives since the proofs are exactly the same for $A_2'(\tau) / A_2(\tau)$ and $C'(\tau)$.

Using (36) and (37), and rearranging terms we obtain

$$\frac{A_1'(\tau)}{A_1(\tau)} = -q_1 B(\tau) \left[s^* + \frac{\sigma_1^2 e^{-q_1 \tau}}{2 q_1^2} \right] < 0, \quad \forall \tau > 0 \tag{46}$$

while, deriving in (37), we get

$$B'(\tau) = e^{-q_1\tau} > 0, \quad \forall \tau > 0 \tag{47}$$

As the proofs for $A_2'(\tau)/A_2(\tau)$ and $C'(\tau)$ are completely similar, we omit them and we have the result aforementioned.

4 Implications for the Term Structure of Interest Rates

After computing the closed-form expression for the bond price for any maturity, we obtain the term structure (and properties) of interest rates.

The forward interest rates, denoted by $f(s, L, t, T) \equiv f(s, L, \tau)$, at time t for the future period at date $T = t + \tau$ are given by

$$f(s, L, \tau) = \frac{P_t}{P} = -\frac{P_\tau}{P} = -\left[\frac{A_1'(\tau)}{A_1(\tau)} + \frac{A_2'(\tau)}{A_2(\tau)} - B'(t, T)s - C'(t, T)L\right] \tag{48}$$

which, from (46) and (47) are always positive.

Applying the equations (35)-(36) and their analogous for the variable $Z(.)$, and rearranging terms, it is verified that

$$\begin{aligned} f(s, L, \tau) &= r - q_1(s - \hat{\mu}_1)B(\tau) - q_2(L - \hat{\mu}_2)C(\tau) \\ &- \frac{1}{2}\left[\sigma_1^2 B^2(\tau) + \sigma_2^2 C^2(\tau)\right] \end{aligned} \tag{49}$$

Using (42) and (43), it may be checked that

$$\begin{aligned} f(s, L, 0) &= r(t), \quad \forall s, L \\ f(s, L, \infty) &= s^* + L^*, \quad \forall s, L \end{aligned}$$

that is, the forward rate curve starts at the current value of the spot rate and the forward rate on a very long period is independent of the current value of the two factors.

Derivating (48), it may be shown that, for a given maturity, the forward rate is a linear and increasing function of the two factors:

$$f_s(s, L, \tau) = B'(t, T) > 0$$
$$f_L(s, L, \tau) = C'(t, T) > 0$$

As depicted in Figure 1, the forward rate curve may present many shapes: increasing, decreasing or humped. From (48), we have

$$f_\tau(s,L,\tau) = [q_1(\hat{\mu}_1 - s) - \sigma_1^2 B(\tau)]B'(\tau) + [q_2(\hat{\mu}_2 - L) - \sigma_2^2 C(\tau)]C'(\tau) \quad (50)$$

The shape of this curve depends on its starting value. Thus, it increases with maturity if

$$\begin{cases} s < \hat{\mu}_1 - (\sigma_1/q_1)^2 \\ L < \hat{\mu}_2 - (\sigma_2/q_2)^2 \end{cases}$$

it decreases with maturity if

$$\begin{cases} s > \hat{\mu}_1 \\ L > \hat{\mu}_2 \end{cases}$$

and it is a humped curve in the remaining cases.

The bias of this curve, denoted as $bf(s,L,\tau)$, is given by the excess of the forward rate over the expected level of interest rates at time T when the bond matures, $E_t[r(T)]$. Applying (17), (18), and (49) leads to

$$\begin{aligned} bf(.) &= f(s,L,\tau) - E_t[r(T)] \\ &= (\hat{\mu}_1 - s)(1 - e^{-q_1\tau}) + (s - \mu_1)(1 - e^{-k_1\tau}) - \frac{1}{2}\sigma_1^2 B^2(\tau) \\ &+ (\hat{\mu}_2 - L)(1 - e^{-q_2\tau}) + (L - \mu_2)(1 - e^{-k_2\tau}) - \frac{1}{2}\sigma_2^2 C^2(\tau) \quad (51) \end{aligned}$$

It is verified that

$$bf(s,L,\infty) = (s^* + L^*) - (\mu_1 + \mu_2)$$

It may be shown that the bias of the forward curve is negative when

$$\begin{cases} a + b\mu_1 \geq 0 \\ c + d\mu_2 \geq 0 \end{cases}$$

it has a positive value if

$$\begin{cases} a + b\mu_1 \leq -\sigma_1/q_1 \\ c + d\mu_2 \leq -\sigma_2/q_2 \end{cases}$$

and it depends on the maturity of the bond in the remaining cases.

The yield on a bond that matures at time $T = t + \tau$, denoted by $Y(s,L,t,T) \equiv Y(s,L,\tau)$, is the continuously compounded rate of return related to such bond. It is implicitly defined as

$$P(s,L,\tau)e^{Y(s,L,\tau)\tau} = 1$$

or, equivalently

$$Y(s,L,\tau) = -\frac{1}{\tau}\ln(P(s,L,\tau)) \tag{52}$$

which in our model, from (23), becomes

$$\begin{aligned} Y(.) &= -\frac{\ln(A(\tau))}{\tau} + \frac{B(\tau)}{\tau}s + \frac{C(\tau)}{\tau}L \\ &= \frac{\sigma_1^2}{4q_1}\frac{B^2(\tau)}{\tau} + s^*\left[1 - \frac{B(\tau)}{\tau}\right] + \frac{B(\tau)}{\tau}s \\ &+ \frac{\sigma_2^2}{4q_2}\frac{C^2(\tau)}{\tau} + L^*\left[1 - \frac{C(\tau)}{\tau}\right] + \frac{C(\tau)}{\tau}L \end{aligned} \tag{53}$$

For fixed s and L, the shape of $Y(s,L,\tau)$ characterizes the term structure of interest rates, or yield curve, at time t. By applying the L'Hôpital's rule, it can be seen that

$$\lim_{\tau\to 0}\frac{B(\tau)}{\tau} = \lim_{\tau\to 0}\frac{C(\tau)}{\tau} = 1 \tag{54}$$

which implies

$$\lim_{\tau\to 0}\frac{B^2(\tau)}{\tau} = \lim_{\tau\to 0}\frac{C^2(\tau)}{\tau} = 0 \tag{55}$$

Using these results into (53), it may be verified that

$$\lim_{\tau\to 0} Y(s,L,\tau) = s(t) + L(t) = r(t)$$

that is, the yield curve starts at the current value of the spot rate.

As $B(\tau)$ and $C(\tau)$ are bounded functions of maturity (see (42) and (43)), they satisfy

$$\lim_{\tau\to\infty}\frac{B(\tau)}{\tau} = \lim_{\tau\to\infty}\frac{B^2(\tau)}{\tau} = \lim_{\tau\to\infty}\frac{C(\tau)}{\tau} = \lim_{\tau\to\infty}\frac{C^2(\tau)}{\tau} = 0 \tag{56}$$

and, so, the yield to maturity on a very long bond is

$$Y(s,L,\infty) = s^* + L^*$$

a quantity that is independent of the current value of the two factors.

Given a certain maturity, the yield on a bond is a linear and increasing function of the two factors, s, and L. Its partial derivatives with respect to these factors are

$$Y_s(s, L, \tau) = \frac{B(\tau)}{\tau} > 0$$
$$Y_L(s, L, \tau) = \frac{C(\tau)}{\tau} > 0$$

This curve may present different shapes. From (52), it is shown that

$$Y_\tau(s, L, t, T) = \frac{[f(s, L, t, T) - Y(s, L, t, T)]}{\tau} \tag{57}$$

The shape of this curve depends on the initial value, $r(t)$. If it is "small enough", the curve is increasing with maturity. In other cases, it is decreasing or humped.

The bias of the yield curve, denoted as $by(s, L, \tau)$ is given by the comparison between the yield of the bond and the expected level of interest rates in the period until maturity of the bond, $\frac{1}{\tau}\int_t^T E_t[r(u)]du$. Integrating (17) and (18), we obtain that

$$\frac{1}{\tau}\int_t^T E_t[r(u)]du = \mu_1 + (s - \mu_1)\frac{1 - e^{-k_1\tau}}{k_1\tau} + \mu_2 + (L - \mu_2)\frac{1 - e^{-k_2\tau}}{q_2\tau} \tag{58}$$

Subtracting (58) from (53), we get

$$\begin{aligned} by(s, L, \tau) &= Y(s, L, \tau) - \frac{1}{\tau}\int_t^T E_t[r(u)]du \\ &= (s^* - \mu_1)\left[1 - \frac{B(\tau)}{\tau}\right] + (s - \mu_1)\left[\frac{1 - e^{-q_1\tau}}{q_1\tau} - \frac{1 - e^{-k_1\tau}}{k_1\tau}\right] \\ &+ (L^* - \mu_2)\left[1 - \frac{C(\tau)}{\tau}\right] + (L - \mu_2)\left[\frac{1 - e^{-q_2\tau}}{q_2\tau} - \frac{1 - e^{-k_2\tau}}{k_2\tau}\right] \\ &+ \frac{\sigma_1^2}{4q_1}\frac{B^2(\tau)}{\tau} + \frac{\sigma_2^2}{4q_2}\frac{C^2(\tau)}{\tau} \end{aligned}$$

It may be verified that the bias of the yield curve is positive If

$$\begin{cases} s < \mu_1 < s^* \\ L < \mu_2 < L^* \end{cases}$$

and if

$$\begin{cases} a + b\mu_1 \geq 0 \\ c + d\mu_2 \geq 0 \end{cases}$$

then we have a negative bias.

The instantaneous term premium, $\pi(s, L, t, T) \equiv \pi(s, L, \tau)$, is defined as the excess of the expected return on the bond over the current spot rate. Substituting the expression (23) given by Proposition (1 into (5), we get the expressions for the unexpected variations in the return on the bond

$$\begin{aligned} s_1(s, L, t, T) &= s_1(s, L, \tau) = -\sigma_1 B(\tau) \\ s_2(s, L, t, T) &= s_2(s, L, \tau) = -\sigma_2 C(\tau) \end{aligned}$$

Including this equation into (10), it is obtained that the expected rate of return of the bond, $\mu(s, L, t, T) \equiv \mu(s, L, \tau)$, is

$$\mu(s, L, t, T) = r - \sigma_1(a + bs)B(\tau) - \sigma_2(c + dL)C(\tau)$$

Thus, the instantaneous term premium is given by

$$\pi(s, L, \tau) = -\sigma_1(a + bs)B(\tau) - \sigma_2(c + dL)C(\tau) \tag{59}$$

Therefore, the term premium is proportional to the unexpected variations in the return on the bond. Moreover, these unexpected variations are proportional to the standard deviation of the factors and are increasing with the time to maturity of the bond.

It is easily checked that

$$\begin{aligned} \pi(s, L, t, t) &= 0 \\ \pi(s, L, \infty) &= -\frac{\sigma_1(a + bs)}{q_1} - \frac{\sigma_2(c + dL)}{q_2} \end{aligned}$$

For a given maturity, the term premium is linear in both factors. From (59), it is verified that

$$\begin{aligned} \pi_s(s, L, \tau) &= -b\sigma_1 B(\tau) \\ \pi_L(s, L, \tau) &= -d\sigma_2 C(\tau) \\ \pi_\tau(s, L, \tau) &= -\sigma_1(a + bs)B'(\tau) - \sigma_2(c + dL)C'(\tau) \end{aligned}$$

Hence, the term premium increases with the factor $s(L)$ if $b(d) < 0$. It is a smooth function of time to maturity, τ, which increases (decreases) with time to maturity if the market prices of risk are negative (positive).

5 A Closed-Form Expression for Interest Rate Derivatives

The bond pricing equation (20), with the addition of the appropriate boundary conditions, allows us to derive closed-form expressions for other contingent claims. Thus, the price at time t, $U(s,L,t,T) \equiv U(s,L,\tau)$, of a security with terminal payoff $g(s_T, L_T)$ at time T, satisfies the following partial differential equation

$$\frac{1}{2}\sigma_1^2 U_{ss} + q_1(\hat{\mu}_1 - s)U_s + \frac{1}{2}\sigma_2^2 U_{LL} + q_2(\hat{\mu}_2 - L)U_L + U_t - (L+s)U = 0 \quad (60)$$

subject to the terminal condition[1]

$$U(s,L,T,T) = g(s_T, L_T) \quad (61)$$

Hence, we may use the solution to this equation (for a particular function $g(s,L)$) to obtain the prices of different interest rate derivatives. Solving the partial differential equation (60) with the terminal condition (61) leads to the following proposition:

Proposition 2 *Given the interest rate dynamics specified in Assumption 2, the value at time t, $U(s,L,t,T) \equiv U(s,L,\tau)$, of an interest rate derivative with the terminal payoff $g(s_T, L_T)$ at time T is*

$$U(s,L,t,T) = P(s,L,t,T)E[g(s',L')] \quad (62)$$

where

$$\begin{aligned} s' &\sim N(m_s(s,t,T) - q_{sY}(t,T), v_s^2(t,T)) \\ L' &\sim N(m_L(s,t,T) - q_{LY}(t,T), v_L^2(t,T)) \end{aligned} \quad (63)$$

with

$$m_s(s,t,u) \;=\; e^{-q_1(u-t)}s + \left(1 - e^{-q_1(u-t)}\right)\hat{\mu}_1$$

[1]Notice that, if $g \equiv 1$, we obtain the closed-form expression for discount bond prices of Section 3.

$$
\begin{aligned}
q_{sY}(t,u) &= \frac{\sigma_1^2\left(1-e^{q_1(u-t)}\right)^2}{2q_1^2} \\
v_s^2(t,u) &= \frac{\sigma_1^2}{2q_1}\left(1-e^{-2q_1(u-t)}\right) \\
m_L(L,t,u) &= = e^{-q_2(u-t)}L + \left(1-e^{-q_2(u-t)}\right)\hat{\mu}_2 \\
q_{LY}(t,u) &= \frac{\sigma_2^2\left(1-e^{q_2(u-t)}\right)^2}{2q_2^2} \\
v_L^2(t,u) &= \frac{\sigma_2^2}{2q_2}\left(1-e^{-2q_2(u-t)}\right) \qquad (64)
\end{aligned}
$$

Proof:

Let $\tilde{s}(t)$ and $\tilde{L}(t)$ be the "risk-neutral processes" defined by

$$
\begin{cases}
d\tilde{s} &= q_1(\hat{\mu}_1 - \tilde{s})dt + \sigma_1 dw_1 \\
d\tilde{L} &= q_2(\hat{\mu}_2 - \tilde{L})dt + \sigma_2 dw_2
\end{cases}
$$

and set $Y(t,u) = \int_t^u (\tilde{s}(v) + \tilde{L}(v))dv$.

Then, we can apply Friedman (1975) (Section 6, Theorem 5.3, p. 148) to obtain the solution to (60)-(61). This solution is given by[2]

$$
U(s,L,t,T) = E_{s,L,t}\left[g(\tilde{s}(T),\tilde{L}(T))e^{-Y(t,T)}\right] \qquad (65)
$$

Denoting by $p(s,L,t,u,s',L',Y)$ the joint probability density of the variable

$$
X(u) = [\tilde{s}(u), \tilde{L}(u), Y(t,u)]'
$$

conditional on $\tilde{s}(t) = s$, $\tilde{L}(t) = L$, (65) is equivalent to

$$
\begin{aligned}
U(s,L,t,T) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(s',L')e^{-Y}p(s,L,t,T,s',L',Y)dY\,ds'dL' \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(s,L,s',L',t,T)g(s',L')ds'dL' \qquad (66)
\end{aligned}
$$

where

$$
G(s,L,s',L',t,u) = \int_{-\infty}^{\infty} e^{-Y}p(s,L,t,u,s',L',Y)dY \qquad (67)
$$

[2] $E_{s,L,t}[.] = E[.|\tilde{s}(t) = s, \tilde{L}(t) = L]$. Similarly for variance and covariance terms.

Applying Arnold (1974) (Corollary 8.2.4, p. 130), we have

$$
\begin{aligned}
\tilde{s}(u) &= e^{-q_1(u-t)}\tilde{s}(t) + \int_t^u e^{-q_1(u-z)}[q_1\hat{\mu}_1 dz + \sigma_1 dw_1(z)] \\
\tilde{L}(u) &= e^{-q_2(u-t)}\tilde{L}(t) + \int_t^u e^{-q_2(u-z)}[q_2\hat{\mu}_2 dz + \sigma_2 dw_2(z)]
\end{aligned}
$$

Rearranging terms, we obtain

$$
\begin{aligned}
\tilde{s}(u) &= e^{-q_1(u-t)}[\tilde{s}(t) - s] + \hat{\mu}_1 + \sigma_1 \int_t^u e^{-q_1(u-z)} dw_1 \\
\tilde{L}(u) &= e^{-q_2(u-t)}[\tilde{L}(t) - L] + \hat{\mu}_2 + \sigma_2 \int_t^u e^{-q_2(u-z)} dw_2
\end{aligned}
$$

and, hence, it may be verified (see Arnold (1974), Section 8.3, pp. 134–136) that

$$
\begin{aligned}
m_s(s,t,u) &\equiv E_{s,L,t}[\tilde{s}(u)] = e^{-q_1(u-t)}s + \left(1 - e^{-q_1(u-t)}\right)\hat{\mu}_1 \\
v_s^2(t,u) &\equiv V_{s,L,t}[\tilde{s}(u)] = \frac{\sigma_1^2}{2q_1}\left(1 - e^{-2q_1(u-t)}\right) \\
m_L(L,t,u) &\equiv E_{s,L,t}[\tilde{L}(u)] = e^{-q_2\tau}L + \left(1 - e^{-q_2\tau}\right)\hat{\mu}_2 \\
v_L^2(t,u) &\equiv V_{s,L,t}[\tilde{L}(u)] = \frac{\sigma_2^2}{2q_2}\left(1 - e^{-2q_2(u-t)}\right)
\end{aligned}
$$

Applying Arnold (1974) (Theorem 8.2.12, p. 133), it may be verified that the variable $X(u)$ is trivariately normally distributed. A little algebra leads us to

$$
\begin{aligned}
E_{s,L,t}[Y(t,u)] &= \left[\hat{\mu}_1(u-t) + (s - \hat{\mu}_1)\frac{1 - e^{-q_1(u-t)}}{q_1}\right] \\
&\quad + \left[\hat{\mu}_2(u-t) + (L - \hat{\mu}_2)\frac{1 - e^{-q_2(u-t)}}{q_2}\right] \\
&\equiv n_{sY}(s,t,u) + n_{LY}(L,t,u) \qquad (68)
\end{aligned}
$$

Interchanging order of integration and applying Davidson (1994) (Section 30.3, pp. 503–509), we get

$$
V_{s,L,t}[Y(t,u)] = \left[\frac{\sigma_1^2}{2q_1^3}\left(4e^{-q_1(u-t)} - e^{-2q_1(u-t)} + 2q_1(u-t) - 3\right)\right]
$$

$$
\begin{aligned}
&+ && \left[\frac{\sigma_2^2}{2q_2^3}\left(4\,e^{-q_2(u-t)} - e^{-2q_2(u-t)} + 2q_2(u-t) - 3\right)\right] \\
&\equiv && v_{sY}^2(t,u) + v_{LY}^2(t,u) \\
q_{sY}(t,u) &\equiv && \mathrm{Cov}_{s,L,t}[\tilde{s}(u), Y(t,u)] = \frac{\sigma_1^2\left(1 - e^{q_1(u-t)}\right)^2}{2q_1^2} \\
q_{LY}(t,u) &\equiv && \mathrm{Cov}_{s,L,t}[\tilde{L}(u), Y(t,u)] = \frac{\sigma_2^2\left(1 - e^{q_2(u-t)}\right)^2}{2q_2^2}
\end{aligned}
$$

and, so, we obtain the distribution of the variable X:

$$X \sim N(\xi, V)$$

where

$$\xi = \begin{pmatrix} m_s \\ m_L \\ n_{sY} + n_{LY} \end{pmatrix}, \quad V = \begin{pmatrix} v_s^2 & 0 & q_{sY} \\ 0 & v_L^2 & q_{LY} \\ q_{sY} & q_{LY} & v_{sY}^2 + v_{LY}^2 \end{pmatrix}$$

Therefore, the joint probability density of $X(u)$ conditional on $\tilde{s}(t) = s$, $\tilde{L}(t) = L$ is given by

$$p(s, L, t, u, s', L', Y) = \frac{1}{(2\pi)^{3/2}|V|^{1/2}} \exp\left\{-\frac{1}{2}(X-\xi)'V^{-1}(X-\xi)\right\}$$

Replacing this expression into (67), we obtain

$$G(s, L, s', L', t, u) = \int_{-\infty}^{\infty} \frac{1}{(2\pi)^{3/2}|V|^{1/2}} e^{-Y} \exp\left\{-\frac{1}{2}(X-\xi)'V^{-1}(X-\xi)\right\} dY$$

which, after a tedious algebra, becomes

$$
\begin{aligned}
G(s, L, s', L', t, u) &= \exp\left\{\frac{1}{2}v_{sY}^2(t,u) - n_{sY}(s,t,u)\right\} f_1(s') \\
&\times \exp\left\{\frac{1}{2}v_{LY}^2(t,u) - n_{LY}(L,t,u)\right\} f_2(L')
\end{aligned}
$$

where $f_1(.)$ is the density function of a normal variable with mean $m_s - q_{sY}$ and standard deviation v_s. Similarly, $f_2(.)$ corresponds to a normal distribution with mean $m_L - q_{LY}$ and standard deviation v_L.

Substituting this expression into (66), we get

$$\begin{aligned} U(s, L, \tau) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left\{\frac{1}{2}v_{sY}^2 - n_{sY}\right\} \exp\left\{\frac{1}{2}v_{LY}^2 - n_{LY}\right\} \\ &\times\ f_1(s') f_2(L') g(s', L') ds' dL' \end{aligned} \tag{69}$$

If $g \equiv 1$ and $T = u$, equation (69) implies that

$$\begin{aligned} P(s, L, t, T) &= \exp\left\{\frac{1}{2}v_{sY}^2(t, T) - n_{sY}(s, t, T)\right\} \\ &\times\ \exp\left\{\frac{1}{2}v_{LY}^2(t, T) - n_{LY}(L, t, T)\right\} \end{aligned} \tag{70}$$

Replacing (70) into (69) leads to the final expression for the value, at time t, of the interest rate derivative.

$\square$

Next, we may use the closed-form expression given by this proposition, with the appropriate terminal payoff, $g(s_T, L_T)$, to obtain the prices of different interest rate derivatives. The following are several examples:

- European option on a zero-coupon bond.
- European option on a portfolio of bonds.
- Interest rate cap.
- Interest rate floor.
- Interest rate collar.
- Interest rate swap.
- Interest rate swaption.
- Compound option.
- "As you like it" option.
- Binary option.

(a) An **European call option on a zero-coupon bond** is the right, not the obligation, to buy a zero-coupon bond at fixed maturity date. Let K be the strike price of this option. If the option is exercised at expiration, T_c, the callholder pays K and receives a discount bond which matures at time $T_b > T_c$.

Equation (62) for the particular case

$$g(s, L) = P(s, L, T_c, T_b)$$

implies that the price, at time t, of the bond received by the callholder is given by

$$P(s, L, t, T_b) = P(s, L, t, T_c) E[P(s', L', T_c, T_b)]$$

If we define

$$\tilde{P} = P(s', L', T_c, T_b)$$

then, it is verified that

$$E[\tilde{P}] = \frac{P(s, L, t, T_b)}{P(s, L, t, T_c)} \tag{71}$$

Applying (70), we have

$$\begin{aligned} \tilde{P} &= \exp\left\{\frac{1}{2}[v_{sY}^2(T_c, T_b) + v_{LY}^2(T_c, T_b)]\right\} \\ &\times \exp\{-[n_{sY}(s', T_c, T_b) + n_{LY}(L', T_c, T_b)]\} \end{aligned}$$

Since $n_{sY}(s', T_c, T_b)$ and $n_{LY}(L', T_c, T_b)]$ are linear in s' and L' (see equation (68)), then $\tilde{P}$ is the exponential of a linear combination of two normal variables (see equation (63)) and, therefore, $\tilde{P}$ follows a lognormal distribution. Moreover, the coefficients of this linear combination imply that

$$\begin{aligned} \sigma_{\tilde{P}}^2 &\equiv \mathrm{Var}[\ln(\tilde{P})] \\ &= \left[\frac{1 - e^{-q_1(T_b - T_c)}}{q_1}\right]^2 v_s^2(t, T_c) + \left[\frac{1 - e^{-q_2(T_b - T_c)}}{q_2}\right]^2 v_L^2(t, T_c) \end{aligned}$$

The price at time t, $C(s, L, t, T_c; K, T_b)$, of the aforementioned call option follows from the equation (62) with the terminal condition

$$g(s, L) = C(s, L, T_c, T_c; K, T_b) = \max\{P(s, L, T_c, T_b) - K, 0\} \tag{72}$$

At this point, we require that $P(0,0,T_c,T_b) > K$. Otherwise, since the bond price is decreasing in both state variables, $P(s,L,T_c,T_b) < K$ for all s and L and the option will never be exercised.

Replacing (72) into (62) gives

$$C(s,L,t,T_c;K,T_b) = P(s,L,t,T_c)E[\tilde{Z}] \tag{73}$$

where

$$\tilde{Z} = \max\{\tilde{P} - K, 0\} = (\tilde{P} - K)I_{[K,\infty)}(\tilde{P})$$

and $I(.)$ is the indicator function defined as

$$I_{[K,\infty)}(x) = \begin{cases} 0 & \text{if} \quad x < K \\ 1 & \text{if} \quad x > K \end{cases}$$

Then

$$E[\tilde{Z}] = \int_k^\infty (\tilde{P} - K) f(\tilde{P}) d\tilde{P} \tag{74}$$

To compute this expectation, we define the new variable

$$v(\tilde{P}) = \frac{E[\ln(\tilde{P})] - \ln(\tilde{P})}{\sigma_{\tilde{P}}}$$

If we apply the relationship

$$\ln(E[\tilde{P}]) = E[\ln(\tilde{P})] + \frac{1}{2}V[\ln(\tilde{P})]$$

then (74) becomes

$$E[\tilde{Z}] = E[\tilde{P}]\Phi(h + \sigma_{\tilde{p}}) - K\Phi(h) \tag{75}$$

where $\Phi(.)$ denotes the distribution function of a standard normal variable and

$$h = v(K) = \frac{E[\ln(\tilde{P})] - \ln(K)}{\sigma_{\tilde{P}}}$$

Substituting (71) and (75) into (73), it follows that the final expression for the call price is

$$C(s,L,t,T_c;K,T_b) = P(s,L,t,T_b)\Phi(h + \sigma_{\tilde{p}}) - KP(s,L,t,T_c)\Phi(h) \tag{76}$$

There is a big likeness between this formula and the Black-Scholes expression for option prices. In both formulae, we have a random variable, $\tilde{P}$, the price of the underlying security at option expiration, that is lognormally distributed. The discount factor $P(s,L,t,T_c)$ is the analogous of $e^{-r(T_c-t)}$ and $\sigma_{\tilde{p}}^2$, the variance of the logarithm of $\tilde{P}$, is equivalent to $\sigma^2(T_c-t)$.

European put prices are obtained by call-put parity, that is

$$\text{call} - \text{put} = P(s,L,t,T_b) - KP(s,L,t,T_c) \tag{77}$$

(b) Equation (76) may be extended to obtain the price of an **European call option on a portfolio of N discount bonds**. Let K and T_c be the strike price and expiration of this option, respectively. The portfolio consists of N discount bonds and we invest a proportion α_i, $i=1,...,N$ in each T_b^i-maturity bond.

Let $C_\alpha(s,L,t,T_c;K,T_b^N)$ denote the price, at time t, of the portfolio option. Analogously to the bond option price, it follows that

$$C_\alpha(s,L,t,T_c;K,T_b^N) = P(s,L,t,T_c)E[\max\{\tilde{P}_\alpha - K, 0\}] \tag{78}$$

where

$$\tilde{P}_\alpha = \sum_i \alpha_i P(s',L',T_c,T_b^i) \tag{79}$$

and i is such that $T_c < T_b^i$.

Equations (71) and (79) imply that

$$E[\tilde{P}_\alpha] = \sum_i \alpha_i \frac{P(s,L,t,T_b^i)}{P(s,L,t,T_c)} \tag{80}$$

Let $\bar{s}$ and $\bar{L}$ be such that

$$K = \sum_i \alpha_i P(\bar{s},\bar{L},T_c,T_b^i) = \sum_i \alpha_i K_i \tag{81}$$

Since the bond price is decreasing in both factors, s and L, it follows that

$$\max\{\tilde{P}_\alpha - K, 0\}] = \sum_i \alpha_i \max\{P(s',L',T_c,T_b^i) - K_i, 0\} \tag{82}$$

Replacing this equality into (78), a little algebra leads to

$$C_\alpha(s, L, t, T_c; K, T_b^N) = \sum_i C_\alpha(s, L, t, T_c; K_i, T_b^i) \tag{83}$$

Hence, the call option on a N-bond portfolio is equivalent to a portfolio of call options with adequate strike prices, K_i. Moreover, since a coupon bond is a particular case of the above bond portfolio, this expression allows us to price any call option on coupon bonds. A similar argument applies to European put options on a portfolio of discount bonds

(c) An **interest rate cap** places a maximum amount on the interest payments made on a floating-rate loan. Thus, as shown in Figure 2, a cap guarantees that the rate charged on a loan at any given time will be the lesser of the prevailing rate and a certain level, known as the *cap rate*. Therefore, this financial instrument insures against the rate of interest on a floating rate rising above the cap rate.

We assume that interest payments are made at times $1, 2, ..., n$ from the beginning of the life of the cap. Let R_c and R_k ($k = 1, 2, ..., n$) be the cap rate and the prevailing interest rate at each payment time, respectively. Let $\$M$ be the principal of the loan. Then, at time $k + 1$, the writer of the cap is required to pay

$$M \max\{R_k - R_c, 0\}$$

This payoff is equivalent to

$$\frac{M}{1 + R_k} \max\{R_k - R_c, 0\}$$

at time k. Rearranging terms, this expression becomes

$$\max\left\{M - \frac{1 + R_c}{1 + R_k} M, 0\right\}$$

Therefore, this expression corresponds to the payment of a put option (named caplet) that expires at time k on a discount bond of maturity $k + 1$. The face value of the bond is $(1 + R_c)M$ and the strike price is M. As a cap is a sequence of such caplets, it can be interpreted as a portfolio of European put options on discount bonds.

(d) **Interest rate floors** can be defined analogously to caps. A floor places a lower limit on the interest rate to be charged (see Figure 2). Therefore, it provides insurance against a fall in interest rate below a certain level (*floor rate*). Similarly to interest rate caps, an interest rate floor is a portfolio of European call options on discount bonds. If the floor rate is R_f, then

$$\max\left\{\frac{1+R_f}{1+R_k}M - M, 0\right\} \tag{84}$$

is the terminal payoff to be used at each payment time.

(e) A **collar** is just a long position on a cap and a short position on a floor with the same settlement dates and reset intervals. Therefore, the price of the collar is the difference between the prices of these two derivatives.

(f) A **swap** is a private arrangement between two companies, A and B, to exchange a stream of cash flows in the future according to a prearranged formula. The most common type of swap is an *interest rate swap* in which B agrees to make A periodic interest payments at a fixed rate on a notional principal $\$M$ for a number of years. At the same time, B receives interest at a floating rate on the same notional principal for the same period of time. There is no exchange of principal amounts. Thus, a swap has the effect of transforming a fixed rate loan into a floating rate loan or vice versa. Usually, the two companies deal with a financial intermediary to arrange the swap.

We can assume, for valuation purposes only, that, at the end of its life, both companies pay one each other the notional principal $\$M$. Hence, the swap is an arrangement in which 1) Company B has lent the intermediary $\$M$ at a floating rate and 2) the intermediary has lent company B $\$M$ at a fixed rate. That is, the financial institution has sold a $\$M$ floating rate bond to company B and has purchased a $\$M$ fixed rate bond to company B. Therefore, an interest rate swap can be regarded as an agreement to exchange a fixed rate bond for a floating rate bond and, hence, the value of this swap is the difference between the values of these two bonds. Thus, assuming that the financial institution receives fixed payments and makes floating payments, and denoting by P_1 and P_2 the values of the fixed and floating rate bonds underlying the swap, respectively, the value of the swap, V_s is given by

$$V_s = P_1 - P_2$$

Analogously, if the financial institution is paying fixed and receiving floating, the value of the swap is

$$V_s = P_2 - P_1$$

(g) An **interest rate swaption** is an option on an interest rate swap. Thus, it gives the holder the right to enter into an interest rate swap for the strike price K at time $T < T_s$, time in which the swap expires. Therefore, it can be regarded as an option to exchange a fixed rate bond for a floating rate bond. Let $V(s, L, t, T_s)$ be the value at time t of this swap. The value of the call swaption can be obtained by letting

$$g(s, L) = \max\{V(s, L, T, T_s) - K, 0\}$$

in formula (72). A similar argument applies to put swaptions.

(h) A **compound option** is an option on an option. Therefore, it has two strike prices and two exercise dates, $T_1 < T_2$. We have four possible configurations: a call on a call, a call on a put, a put on a call, and a put on a put. The first two give the holder the right to buy the underlying option and the second two allow the holder to sell.

If we consider a call on a call, at time T_1, the holder of the compound option has the right to buy the underlying call option at the first strike price, K_1. This second call option gives the holder the right to buy, at time T_2, the underlying bond which matures at time T_3 for the second strike price, K_2.

This option will be exercised at T_1 if the value of the underlying option on that date, $C(s, L, T_1, T_2; K_2, T_3)$, is greater than the first strike price. Therefore, the terminal payoff at T_1 of the compound option is

$$g(s_{T_1}, L_{T_1}) = \max\{C(s, L, T_1, T_2; K_2, T_3) - K_1, 0\}$$

(i) An **"As you like it" option** is an option in which the holder, at time T_1, can buy either a call or a put. Thus, the value at this time of this option is

$$\max\{C, P\}$$

where C and P are the values of the underlying call and put, respectively.

We assume that both options are European, have the same strike price, K, and mature at time T_2. The underlying asset in the two options is a bond with maturity at time T_3. Using the call-put parity (77), we obtain that

$$\begin{aligned} \max\{C, P\} &= \max\{C, C + KP(s, L, T_1, T_2) - P(s, L, T_1, T_3)\} \\ &= C + \max\{KP(s, L, T_1, T_2) - P(s, L, T_1, T_3), 0\} \end{aligned}$$

Therefore, this option is a combination of 1) a call option with strike price K and maturity T_2 and 2) a put option with strike price $KP(s, L, T_1, T_2)$ and maturity T_1. So, it can be valuated using the formulas obtained for options on discount bonds.

If the underlying options differ in the strike price and time to maturity, the "as you like it" option is similar to the compound options that we have analyzed above.

(j) A **binary option** is an option with discontinuous payoffs. Two examples of this type of options are *cash or nothing call* and *asset or nothing call*.

A cash or nothing call pays out a predetermined fixed amount, Q, if the option is in-the-money at expiration, and zero otherwise. That is, it pays out nothing if the underlying bond price P ends up below the strike price K and pays out Q, if it ends up above the strike price. Let T_c and T_b be the expiration dates of the call option and the underlying bond, respectively. The terminal payoff of this option is given by

$$\frac{Q}{P(s, L, T_c, T_b) - K} \max\{P(s, L, T_c, T_b) - K, 0\}$$

An asset or nothing call pays out nothing if the bond price P ends up below the strike price K and pays an amount equal to the bond price if it ends up above the strike price. Therefore, its terminal payoff is given by

$$\frac{P(s, L, T_c, T_b)}{P(s, L, T_c, T_b) - K} \max\{P(s, L, T_c, T_b) - K, 0\}$$

where T_c and T_b are the maturity dates of the call option and the underlying bond, respectively.

Many other types of options can be priced using similar approaches. Similarly to the European options we have seen above, we can price **American**

options. This type of options can be exercised at any time up to the expiration date in contrast to European options that can only be exercised on the expiration date itself. We consider an American call option that has an exercise price of K and expires at time T_c. We assume that this option is written on a coupon bond paying a continuous dividend at a rate of $\alpha(t)$ and maturing at time $T_b > T_c$. Denoting by $P(s, L, t, T_b)$ the value at time t of the underlying bond, the price at time t, $V(s, L, t, T_c; K, T_b)$, of this American option follows from the equation (62) with the boundary conditions

$$V(s, L, T_c, T_c; K, T_b) = \max\{P(s, L, T_c, T_b) - K, 0\} \tag{85}$$

$$\lim_{(s,L)\to\mathcal{B}} V(s, L, t, T_c; K, T_b) = P(s, L, t, T_b) - K \tag{86}$$

$$\lim_{(s,L)\to\mathcal{B}} \frac{\partial V(s, L, t, T_c; K, T_b)}{\partial P(s, L, t, T_b)} = -1 \tag{87}$$

where $\mathcal{B}$ denotes the exercise region, which represents the bond price above which the American call is exercised optimally. Conditions (86) and (87) are called the "value matching" and the "supper contact" conditions, respectively.

6 Empirical Application

In this section, we describe the basic characteristics of empirical application. The spread, the difference between short and long-term interest rates, and the long-term rate are the state variables of the two-factor model. The instantaneous riskless interest rate and the long-term rate are approximated by the 1-day and 10-year interest rates, respectively.

Our database is given by daily interest rates and zero-coupon bond prices and was obtained from the Research Department, Bank of Spain[3]. The price data used consists of a cross-sectional time-series database of zero-coupon bonds for the period from 2 January 1991 to 29 December 1995. We consider ten maturities: 1, 7, and 15 days, 1, 3, and 6 months, and 1, 3, 5, and 10 years. Interest rates are expressed in annualized form and cover the same sample period, providing 1230 observations in total.

[3] See Nuñez (1995) for details on these data and the procedure they were estimated.

Plots of the spread and interest rate series as well as its first difference are provided in Figures 3 and 4. Both interest rate series increase in the period March-October 1992 and from June 1994 to March 1995 and decrease in the first semester of 1991, in the period from June to December 1993 (when they attain the minimum values which are close to 7%) and in the second semester of 1995. Short-term interest rates are larger than 10% until October 1993 while the long-term interest rates exceed this level in the whole period except from June 1993 through June 1994.

Focusing on the first difference of the variables, most of the changes in the short-term interest rates are smaller than 100 basis points. The highest changes in this variable (about 4%) are obtained at the second week of May 1993. On the other hand, changes in long-term rates are much smoother. These changes move into a narrower interval and are never bigger than 80 basis points. Therefore, changes in the spread are quite similar to changes in short-term interest rates. Thus, the spread does not usually rise (or fall) more than 1% except in the second week of May 1993 when we attain the extreme values of the changes in this variable which are close to 4%.

Tables I–III show summary statistics, correlation and autocorrelation structure for all the state variables entering the model. In short, Table I shows means, variances, extreme values, skewness and excess of kurtosis coefficients for the state variables. These numerical characteristics concerning to the location, dispersion and shape are computed for the data set through the entire sample period. The autocorrelation coefficient of first order, denoted by ρ_1, is included in this table.

For the two interest rates, the unconditional average interest rates are larger than 10%. The short-term rate is more volatile and moves into a wider interval than long-term rates do. On the other hand, the spread has a mean value very close to zero and ranges between -4% and 8%. The maximum (18.21%) and minimum (6.53%) short-term interest rates correspond to 13 May 1993 and 7 June 1994, respectively. Analogously, the long-term interest rates attain their extreme values (13.28% and 7.58%) at 6 February 1991 and 1 February 1994, respectively.

The correlation matrix (see Table II) shows the small correlation between the spread and the long-term rate. The autocorrelation coefficients of order j of the state variables are shown in Table III. These coefficients are near one and decay very slowly. Hence, the main characteristic of these data is the almost uniformly high degree of serial correlation.

Characteristics concerning to the first order differentiation of the original data are included in Tables IV-VI. We observe that mean changes in interest rates are negative but quite close to zero. Hence, a small decrease - in mean - in interest rates through the sample period is inferred. As mean decreases with maturity, we deduce that long-term rates go down less than short-term interest rates. Therefore, the mean value of changes in the spread is negative.

Changes in long-term interest rates are less dispersed than changes in short-term rates. Daily changes show a large kurtosis coefficient (indicative of fat tails in the distribution of the variables) though it decreases with maturity.

Table V reports the correlation coefficients among changes in the state variables. This table shows the small correlation between the change in the spread and the change in the long-term rate and, hence, suggests that our theoretical assumption about the state variables is empirically corroborated.

Table VI shows the increased stationarity in the data. The autocorrelation coefficients for the first difference of the data decay more quickly (in comparison with the variables in levels) and are negligible when lag is large enough. Since the first order autocorrelation coefficient, ρ_1, is negative, evidence of mean reversion in spread and interest rates is derived.

Next we present the empirical performance of the two-factor model in comparison with the one-factor model that assumes the short-term interest rate as the unique state variable. Similarly to the state variables of the two-factor model, we make the following assumptions:

Assumption 3 *The market price of the short-term interest rate risk is linear in this variable, that is*

$$\lambda_3(.) = e + fr \tag{88}$$

Assumption 4 *The short-term interest rate follows a Ornstein-Uhlenbeck process*

$$dr = k_3(\mu_3 - r)dt + \sigma_3 dw_3 \tag{89}$$

The quantity k_3 reflects the speed of adjustment of the short-term interest rate towards its long-run mean value, μ_3, σ_3 is the (constant) standard deviation of this state variable and dw_3 is a standard Gauss-Wiener process.

Under the one-factor model, the closed-form expression for the default free discount bond prices is given by:

$$P(s, L, t, T) = P(s, L, \tau) = A_3(\tau)e^{-D(\tau)r} \tag{90}$$

where

$$\begin{aligned} A_3(\tau) &= \exp\left\{-\frac{\sigma_3^2}{4q_3}D^2(\tau) + r^*(D(\tau)-\tau)\right\} \\ D(\tau) &= (1-e^{-q_3\tau})/q_3 \end{aligned} \tag{91}$$

with

$$q_3 = k_3 + f\sigma_3, \quad r^* = \hat{\mu}_3 - \sigma_3^2/(2q_3^2), \quad \hat{\mu}_3 = (k_3\mu_3 - e\sigma_3)/q_3 \tag{92}$$

Each state variable of the two-factor model, s and L, as well as the short-term interest rate, r, follow a Ornstein-Uhlenbeck process (see equations (14) and (89)). The diffusion parameters of these processes ($k_i, \mu_i, \sigma_i, i = 1,2,3$) are estimated by Hansen's Generalized Method of Moments[4]. The econometric specification in discrete time is

$$\begin{aligned} s_t - s_{t-1} = a_1 + b_1 s_{t-1} + \varepsilon_t^s, \quad \varepsilon_t^s &\sim IID(0,\sigma_1^2) \\ L_t - L_{t-1} = a_2 + b_2 L_{t-1} + \varepsilon_t^L, \quad \varepsilon_t^L &\sim IID(0,\sigma_2^2) \\ r_t - r_{t-1} = a_3 + b_3 r_{t-1} + \varepsilon_t^r, \quad \varepsilon_t^r &\sim IID(0,\sigma_3^2) \end{aligned}$$

with

$$\mathrm{Cov}(\varepsilon_t^s, \varepsilon_t^L) = \mathrm{Cov}(\varepsilon_t^s, \varepsilon_t^r) = \mathrm{Cov}(\varepsilon_t^L, \varepsilon_t^r) = 0$$

so that

$$\begin{aligned} k_1 &= -b_1, \qquad \mu_1 = -\frac{a_1}{b_1} \\ k_2 &= -b_2, \qquad \mu_2 = -\frac{a_2}{b_2} \\ k_3 &= -b_3, \qquad \mu_3 = -\frac{a_3}{b_3} \end{aligned}$$

The estimation results obtained for the whole period 1991-1995 are included in Table VII and show that the parameters b_i of the discrete time specification are significantly different from zero. Hence, the diffusion parameters k_i are also significantly different from zero and, so, there is evidence of mean reversion in interest rates and spread series. Both interest rates tend to a mean value close to 10%. The spread tends to a mean value close to zero and is the state variable with highest speed of mean reversion.

[4]For details on this procedure and its applications in this framework, see Moreno and Peña (1996).

After estimating the diffusion parameters of the processes followed by the three state variables of both models (the spread and the long-term rate in the two-factor model, the short-term rate in the one-factor model), we use the values of these parameters ($k_i, \mu_i, \sigma_i, i = 1, 2, 3$) to obtain the remaining parameters of equations (23) and (90).

Thus, following Das (1994a), we use the specifications

$$\begin{aligned} P &= P(q_1, q_2, s^*, L^* | k_1, k_2, \mu_1, \mu_2, \sigma_1, \sigma_2; s, L, \tau) + \varepsilon \\ P &= P(q_3, r^* | k_3, \mu_3, \sigma_3; r, \tau) + \varepsilon \end{aligned} \tag{93}$$

where P is the observed price of the discount bonds available at time t, $P(.)$ is the closed-form pricing equation for each model (see equations (23) and (90)) and ε is an error term.

We employ a panel of data consisting of a time series of yield curves containing a cross-section of zero-coupon bond prices to estimate the parameters ($q_i, i = 1, 2, 3, s^*, L^*, r^*$) of equation (93) for each day of the period 1991-1995. Thus, we have a data matrix with 1230 rows and 10 columns. The row i contains the (ten) zero-coupon bond prices available at time i. Each column includes the bond prices corresponding to a certain maturity: the first column contains the 1-day bond prices for each day, the second one includes the 7-day bond prices,..., and the last column provides the prices of bonds with 10 years to maturity.

For each day of the period 1991-1995, we estimate the non-linear equations (93). This estimation, when applied to the first equation, provides the parameters of the two-factor model (that is, q_1, q_2, s^*, and L^*) while estimating the second equation we obtain the parameters of the one-factor model, that is, q_3 and r^*. Estimation results for the daily parameters of the two models are portrayed in Table VIII. This table shows the average of the estimated values obtained for the full sample period and reflects that all the parameters are highly significant.

At this step, we can use the estimated parameters obtained from equation (93) in conjunction with equations (25) and (92) and Assumptions 1 and 3 to obtain explicitly the daily market prices of risk related to each state variable. A summary of these values is reported in Table IX. Panel A of this table includes the averages of market prices of risk for the period 1991-1995 and Panel B reports the average results when we divide this period year by year.

For the full period, we can observe that the market prices of risk for the three state variables are positive and significantly different from zero.

The highest mean value corresponds to the long-term interest rate while the lowest minimum value is related to the spread.

Dividing this period year by year, the parameters are also significantly different from zero. Dealing with the one-factor model, the mean market price of risk has a positive value in the three first years and reaches the largest values at 1993. The mean market price of risk of the spread is also negative in the last two years of the period that we have considered. On the other hand, the market price of risk related to the long-term interest rate takes a positive mean value in 1993 (when it attains its maximum values) and 1995 and is negative in the remaining years.

We can also use the parameters obtained from the estimation of the equation (93) and the estimates of the diffusion parameters to analyze the within and out-of-sample properties of both models.

The two competing models are evaluated first on within-sample data for 1991-1994 and then on out-of-sample forecasts for 1995. For each day of the period 1991-1994, the within-sample estimated data are obtained by including the (daily) estimated parameters and the estimated diffusion parameters in the equation (93). In order to generate k-step-ahead forecasts for the bond prices, for both models, the coefficient estimates are taken from time t. These estimates are used to generate the $t+k$-time forecast. This procedure is continued throughout the forecast sample until the last day of 1995.

Once obtained the within and out-of-sample forecasts, we compute the (within and out-of-sample) pricing errors of both models to compare one each other. We define, for time t, the error, e_t, and the percentage error, PE_t, as

$$e_t = P_t - \hat{P}_t$$

and

$$PE_t = \frac{P_t - \hat{P}_t}{P_t} 100$$

where P_t and $\hat{P}_t$ are, respectively, the observed and the estimated bond price, for time t, of the discount bond of a given maturity.

Pricing errors, in absolute and percentage terms, for both models, in the whole within-sample period are provided in Figures 5 and 6. Considering maturities up to 1 month, it can be seen - for both models - a very large pricing error in the second week of May 1993. This error coincides with a sharp change (mentioned earlier) in the short-term interest rates and in the spread. This large error is also found in the remaining maturities when dealing with the two-factor model.

The one-factor model, for maturities from 3 months to 3 years, overestimates the bond prices in 1991-1992, period in which short-term interest rate is greater than 10%. For these maturities, the largest positive errors, indicative of underpricing, occur in the first semester of 1994, period in which the short-term interest rates were smaller than 8%. For the longest maturity, 10 years, the one-factor model provides the opposite result: underpricing in 1991-1992 and overpricing from January, 1993 to June, 1994. Finally, a sight of these two figures does not suggest that the errors from the two-factor model follow a systematic pattern.

Denoting by N the number of days of the period (or subperiod) that we consider, we compute five different measures related to pricing errors in order to compare the performance of the one and two-factor models:

1. **Mean Error (ME)**. This measure gives an equal weight to the error of each day. If the errors are added together, positive values will offset negative values and the average error may be small, even though the daily errors may be substantial. It is defined as

$$ME = \frac{1}{N}\sum_{t=1}^{N} e_t = \frac{1}{N}\sum_{t=1}^{N}(P_t - \hat{P}_t)$$

2. **Mean Absolute Error (MAE)**. This measure is also known as mean absolute deviation. As the mean error, it also weights equally the error of each day but it does not offset the positive and negative values of the daily errors. Its expression is

$$MAE = \frac{1}{N}\sum_{t=1}^{N} |e_t| = \frac{1}{N}\sum_{t=1}^{N} |P_t - \hat{P}_t|$$

3. **Root Mean Squared Error (RMSE)**. It is one of the most commonly used measures of accuracy. It is supposed that the loss function is quadratic and its definition is

$$RMSE = \sqrt{\frac{1}{N}\sum_{t=1}^{N}(e_t)^2} = \sqrt{\frac{1}{N}\sum_{t=1}^{N}(P_t - \hat{P}_t)^2}$$

4. **Mean Percentage Absolute Error (MAPE)** This measure is similar to the mean absolute error but it weights each error by the actual value of each day. It is given by

$$MAPE = \frac{1}{N}\sum_{t=1}^{N}|PE_t|$$

5. **Root Mean Squared Percentage Error (RMSPE)**. It is similar to the root mean squared error and the daily errors are weighted by the actual values. Its expression is

$$RMSPE = \sqrt{\frac{1}{N}\sum_{t=1}^{N}(PE_t)^2}$$

These measures, using both models, are computed for the within and the out-of-sample periods as well as for different subperiods. The within and out-of-sample results are reported in Tables X-XII and Tables XIII-XVI, respectively.

Thus, the results for the whole within-sample period (1991-1994) are shown in Table X. For this period, the one-factor model overestimates the prices of bonds of maturities up to 6 months as well as the 10-year bond prices. On the other hand, the two-factor model underprices the bonds whose maturities range from 15 days to 1 year and the longest bonds.

We find that both models fit the data very well. Although it can be seen that the pricing error measures increase with time to maturity, the mean absolute percentage error (MAPE) from the one and two-factor models never exceeds 1.6% and 0.3%, respectively.

The MAE and the MAPE statistics indicate that the estimates from the two-factor model are more accurate than those from the one-factor model. Thus, the two-factor model estimates decrease these statistic relative to the one-factor model by more than half for bonds of maturities up to 1 year and by more than 80% for 3 and 10-year bonds.

This table also reflects that, based on a root mean squared error (RMSE or RMSPE) criterion, the two-factor model produces more accurate estimates, especially on 3 and 10-year bonds. Only in 1 and 3-month bonds, the one-factor model performs slightly better than the two-factor model.

Table XI includes the within-sample error measures for the year 1992. The one-factor model produces a slight overpricing for all the maturities, except for 10-year bonds. As in the period 1991-1994, the two-factor model overestimates slightly the prices of bonds of maturities up to six months.

All the statistics show the large increase in accuracy of the two-factor model relative to the one-factor model. It can be seen that, for all maturities but 5 years, the error measures from the one-factor model are more than three times those from the two-factor model. As with the full within-sample period, the largest improvements in accuracy are found in 3 and 10-year bonds where the statistics from the two-factor model are less than 20% than the ones obtained with the one-factor model.

We conclude the within-sample results with Table XII which contains the error measures, for each semester[5] of the period 1991-1994, for 10-year bonds. For these bonds, based on a MAPE criterion, the one-factor model performs quite well for the period 1991-1992 while its accuracy declines in the period June 1993 - June 1994. On the other hand, the two-factor model works better in all the semesters of the within-sample period, it fits specially well in the first semester of 1992 and in the second one of 1994, and its superiority over the one-factor model is specially high in the three last semesters.

The predictive power of both models is analyzed by studying one and five-step ahead forecasts of daily bond prices, in the year 1995, for each maturity. Summary statistics are reported in Tables XIII-XVI.

Tables XIII-XIV include the results for one-step-ahead forecasts while the last two tables contain the five-step-ahead forecasts. Thus, Table XIII shows that the predictive performance of both models is reasonably good although it deteriorates with time to maturity. Both models perform similarly on shorter maturities but, increasing the maturity of the bonds, the two-factor model forecasts better than the one-factor model. Thus, all the error measures are reduced by more than 20% when we consider bonds longer than three years.

Table XIV focuses on 10-year bonds and details its forecasts for each month of 1995. The one-factor model performs better in the second semester of this year when the MAPE is always smaller than 1%. Analogously, the MAPE statistic for the two-factor model is generally close to 0.5%. The performance of this model is specially well in the first term of 1995 when it reduces the error measures from the one-factor model by more than 40%.

The last two tables report the results that we have obtained with five-step-ahead forecasts for both models. Similarly to Table XIII, Table XV

[5]Other subperiods were analyzed and the conclusions do not change qualitatively.

includes the overall forecasts for all the bonds in the year 1995.

The predictive power decreases relative to one-step-ahead forecasts and it declines with time to maturity. In both models, the forecast errors, in percentage terms, are smaller than 1% for all maturities but for 10-year bonds. As before, both models perform similarly in maturities up to three months. The superiority of the two-factor model is weaker than in the previous forecasts but there is still an improvement of 10% in the longer bonds.

Finally, Table XVI includes the forecasting results, separating the year 1995 by quarters, for 6-month and 10-year bonds. For 6-month bonds, the MAPE statistic is smaller than 0.11% in all the subperiods. Both models reach the best forecasts in the second semester of 1995. The improvement of forecasting power of the two-factor model is about 10% in all the quarters. When considering 10-year bonds, the MAPE statistic ranges between 1% and 2.5%. Once again, the two-factor model produces more accurate forecasts and decreases the MAE and MAPE statistics in about 20% in the first quarter of this year. In the remaining quarters, the predictive improvement never exceed 11%.

7 Conclusion

We have presented a two-factor model of the term structure of interest rates. The main assumption is that the price of all default free discount bonds is a function of time to maturity and two state variables. These variables are the long-term interest rate and the spread (difference between the short-term (instantaneous) riskless rate and the long-term rate).

Assuming that both factors follow a joint Ornstein-Uhlenbeck process, we derived a general bond pricing equation which must be satisfied by the values of all default free discount bonds. After computing a closed-form expression for zero-coupon bond prices for any maturity, we examined its implications for the term structure of interest rates.

We also derived a closed-form solution for interest rate derivatives. This formula was applied to price European options on discount bonds. We showed the similarity between this expression and the one derived by Black-Scholes. Moreover, we extended this formula to options on discount bond portfolios. As a consequence, we are able to price any European option on coupon bonds.

We also illustrated how this formula can be used to price more complex types of options.

Finally, we presented the empirical performance of the two-factor model in comparison with a one-factor model that assumes the short-term interest rate as the single state variable. The diffusion parameters have been estimated by Hansen's Generalized Method of Moments and the results suggest evidence of mean reversion in interest rate and spread series. The remaining parameters were estimated by a cross-sectional technique that allowed us to identify the market prices of risk related to each state variable. For the full sample, we have shown that the market prices of risk for the three state variables are positive and significantly different from zero.

The two competing models were evaluated first on within-sample data for 1991-1994 and then on out-of-sample forecasts for 1995. Although both models fit the data very well, the error statistics indicate that, for all the bonds, the within-sample estimates from the two-factor model reduced the error measures relative to the one-factor model by more than 50%. Moreover, the largest improvements in accuracy are found in 3 and 10-year bonds in which all the statistics from the one-factor model are reduced by more than 75%.

The predictive power of both models has been analyzed by studying one and five-step ahead forecasts of daily bond prices, in the year 1995, for each maturity. Although the predictive performance of both models is reasonably good, the statistics show that the one-step-ahead forecasts from the two-factor model are always closer to the data than those from the one-factor model. Both models perform in a similar way on shorter maturities but, for longer bonds, all the error measures decrease more than 20%.

The predictive performance of five-step-ahead forecasts declines with regard to one-step-ahead forecasts, although both models still forecast quite well. These forecasts deteriorate with time to maturity but the forecast errors, in percentage terms, are smaller than 1% for most of the maturities. Although the superiority of the two-factor model over the one-factor model is weaker than in the previous forecasts, there still remains an improvement of 10% for the longer bonds.

Therefore, regardless of statistics used, the subperiods analyzed or the maturities considered, empirical evidence suggests that the two-factor model is more accurate (in both within and out-of-sample data) than the one-factor model.

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Table I. Summary Statistics of State Variables

This table provides summary statistics of the state variables. Means, standard deviations, extreme values, skewness coefficients, and excess kurtosis are computed from January 1991 through December 1995. Raw data is in percentage terms. The number of observations is denoted by n.

Variable	Spread	Long-term Rate	Short-term Rate
n	1230	1230	1230
Mean	0.09257	10.4467	10.5393
Standard Deviation	1.96963	1.0884	2.1808
Minimum	-4.078	7.5794	6.5306
Maximum	7.433	13.2838	18.2134
Skewness	-0.27139	-0.5503	0.16944
Excess of Kurtosis	-0.59905	0.39954	-0.84618
ρ_1	0.9842	0.9919	0.9861

Table II. Correlation Matrix of State Variables

This table provides correlation coefficients of the state variables. These coefficients are computed from January 1991 through December 1995. Raw data is in percentage terms.

Variable	Spread	Long-Term Rate	Short-Term Rate
Spread	1.0000		
Long-Term Rate	-0.0718	1.0000	
Short-Term Rate	0.8673	0.4342	1.0000

Table III. Correlation Structure of State Variables

This table shows correlation coefficients of order j, denoted by ρ_j, of the state variables. These coefficients are computed from January 1991 through December 1995. Raw data is in percentage terms.

	Spread	Long-Term Rate	Short-Term Rate
ρ_1	0.9842	0.9919	0.9861
ρ_2	0.9758	0.9854	0.9786
ρ_3	0.9718	0.9780	0.9745
ρ_4	0.9667	0.9718	0.9693
ρ_5	0.9625	0.9655	0.9647
ρ_6	0.9590	0.9591	0.9608
ρ_7	0.9534	0.9531	0.9554
ρ_8	0.9481	0.9468	0.9501
ρ_9	0.9439	0.9420	0.9457
ρ_{10}	0.9403	0.9361	0.9416
ρ_{11}	0.9376	0.9304	0.9380
ρ_{12}	0.9329	0.9247	0.9333

Table IV. Summary Statistics of Changes in State Variables

This table provides summary statistics of the changes in state variables. Means, standard deviations, extreme values, skewness coefficients, and excess kurtosis are computed from January 1991 through December 1995. Raw data is in percentage terms. The number of observations is denoted by n.

Variable	Spread	Long-term Rate	Short-term Rate
n	1229	1229	1229
Mean	-0.00165	-0.00284	-0.00449
Standard Deviation	0.34792	0.11608	0.3453
Minimum	-4.0344	-0.7715	-4.0929
Maximum	3.3687	0.8188	3.417
Skewness	-0.4659	-0.12488	-0.45456
Excess of Kurtosis	25.0356	8.70715	27.553
ρ_1	-0.2393	-0.1508	-0.2565

Table V. Correlation Matrix of Changes in State Variables

This table provides correlation coefficients of the changes in state variables. These coefficients are computed from January 1991 through December 1995. Raw data is in percentage terms.

Maturity	Spread	Long-Term Rate	Short-Term Rate
Spread	1.0000		
Long-Term Rate	-0.1891	1.0000	
Short-Term Rate	0.9439	0.1456	1.0000

Table VI. Correlation Structure of Changes in State Variables

This table shows correlation coefficients of order j, denoted by ρ_j, of the state variables. These coefficients are computed from January 1991 through December 1995. Raw data is in percentage terms.

	Spread	Long-Term Rate	Short-Term Rate
ρ_1	-0.2393	-0.1508	-0.2565
ρ_2	-0.1435	0.0356	-0.1415
ρ_3	0.0396	-0.0699	0.0451
ρ_4	-0.0312	-0.0099	-0.0289
ρ_5	-0.0261	0.0353	-0.0259
ρ_6	0.0693	-0.0577	0.0590
ρ_7	-0.0093	0.0920	-0.0030
ρ_8	-0.0374	-0.1064	-0.0281
ρ_9	-0.0157	0.0769	-0.0157
ρ_{10}	-0.0283	-0.0488	-0.0183
ρ_{11}	0.0625	-0.0010	0.0478
ρ_{12}	0.0497	0.0616	0.0460

Table VII. Estimates of the Diffusion Parameters

This table provides the parameter estimates (with t-values in parentheses) of the Vasicek processes followed by each state variable. The sample period is from January 1991 to December 1995. The parameters are estimated by means of the Generalized Method of Moments applied to the following equation

$$\begin{aligned} s_t - s_{t-1} &= a_1 + b_1 s_{t-1} + \varepsilon_t^s, \quad & \varepsilon_t^s &\sim IID(0, \sigma_1^2) \\ L_t - L_{t-1} &= a_2 + b_2 L_{t-1} + \varepsilon_t^L, \quad & \varepsilon_t^L &\sim IID(0, \sigma_2^2) \\ r_t - r_{t-1} &= a_3 + b_3 r_{t-1} + \varepsilon_t^r, \quad & \varepsilon_t^r &\sim IID(0, \sigma_3^2) \end{aligned}$$

with

$$\text{Cov}(\varepsilon_t^s, \varepsilon_t^L) = \text{Cov}(\varepsilon_t^s, \varepsilon_t^r) = \text{Cov}(\varepsilon_t^L, \varepsilon_t^r) = 0$$

Variable	a	b	k	μ	σ
Spread	-2.08×10^{-4}	-0.015447	0.015447	-0.01347	0.3467
	(-0.0210)	(-3.0756)	(3.0756)	(-0.021)	
Long-Term Rate	0.0732	-0.007287	0.007287	10.05747	0.1159
	(2.2968)	(-2.3988)	(2.3988)	(20.881)	
Short-Term Rate	0.13086	-0.01284	0.012841	10.19102	0.3443
	(2.6980)	(-2.8498)	(2.8498)	(13.155)	

Table VIII. Averages of Pure Cross-Sectional Regressions

This table contains the estimation results, for each day of the period 1991-1995, of the parameters $(q_i, i = 1,2,3, s^*, L^*, r^*)$ in the closed-form pricing equation for both models

$$P(s, L, t, T) = P(s, L, \tau) = A(\tau)e^{-B(\tau)s - C(\tau)L}$$

where

$$\begin{aligned} A(\tau) &= A_1(\tau)A_2(\tau) \\ A_1(\tau) &= \exp\left\{-\frac{\sigma_1^2}{4q_1}B^2(\tau) + s^*(B(\tau) - \tau)\right\} \\ A_2(\tau) &= \exp\left\{-\frac{\sigma_2^2}{4q_2}C^2(\tau) + L^*(C(\tau) - \tau)\right\} \\ B(\tau) &= (1 - e^{-q_1\tau})/q_1 \\ C(\tau) &= (1 - e^{-q_2\tau})/q_2 \end{aligned}$$

and

$$P(s, L, t, T) = P(s, L, \tau) = A_3(\tau)e^{-D(\tau)r}$$

where

$$\begin{aligned} A_3(\tau) &= \exp\left\{-\frac{\sigma_3^2}{4q_3}D^2(\tau) + r^*(D(\tau) - \tau)\right\} \\ D(\tau) &= (1 - e^{-q_3\tau})/q_3 \end{aligned}$$

Numbers in parentheses represent the average of the t-statistics of cross-sectional regressions. The numbers in square brackets [.] represent the standard deviation of the time series of parameter estimates.

One-Factor Model		**Two-Factor Model**			
q_3	r^*	q_1	q_2	s^*	L^*
1.8803	6.7638	0.4984	0.3909	-12.5420	0.0708
(47.32)	(583.88)	(65.24)	(52.02)	(-39.98)	(61.59)
[5.5197]	[10.8496]	[0.9849]	[0.6812]	[27.7878]	[30.2253]

Table IX. Averages of Market Prices of Risk

This table contains the estimation results, for each day of the period 1991-1995, of the market prices of risk (λ_i, $i = 1, 2, 3$) related to each state variable. Numbers in parentheses represent the average of the t-statistics of these estimates. The numbers in square brackets [.] represent the standard deviation of the time series of market prices of risk estimates.

Panel A: Period 1991-1995

One-Factor Model		Two-Factor Model	
	λ_3 (Short-term rate)	λ_1 (Spread)	λ_2 (Long-term rate)
	3.5178	0.2386	4.8419
	(34.54)	(5.09)	(15.22)
	[13.1444]	[12.2793]	[41.8299]

Panel B: Period 1991-1995, year by year

One-Factor Model		Two-Factor Model	
	λ_3 (Short-term rate)	λ_1 (Spread)	λ_2 (Long-term rate)
	2.4644	3.1513	-8.4789
1991	(73.75)	(46.88)	(-49.31)
	[5.9616]	[10.1866]	[32.8720]
	0.9249	1.8632	-6.9682
1992	(75.95)	(34.10)	(-30.41)
	[1.7058]	[5.6779]	[15.2471]
	20.3565	2.6534	37.8514
1993	(72.67)	(1.19)	(30.75)
	[15.5313]	[13.3915]	[59.6728]
	-0.5997	-0.7678	-1.7900
1994	(-11.00)	(-2.07)	(-29.76)
	[13.0031]	[18.0933]	[44.0095]
	-5.8946	-5.7401	2.8856
1995	(-38.70)	(-54.31)	(1.75)
	[3.2618]	[7.5040]	[20.7386]

Table X. Within-Sample Pricing Error Measures. 1991-1994

This table contains the within-sample pricing error measures of the one and two-factor models for the period 1991-1994. We consider zero-coupon bonds with face value of $1 and with maturities ranging from 1 day to 10 years. We have computed five different error measures: the mean error (ME), the mean absolute error (MAE), the root mean squared error (RMSE), the mean absolute percentage error (MAPE) and the root mean squared percentage error (RMSPE).

One-Factor Model					
Maturity	ME	MAE	RMSE	MAPE	RMSPE
1-day	-0.000000	0.000000	0.000000	0.000012	0.000020
7-day	-0.000002	0.000002	0.000005	0.000228	0.000463
15-day	-0.000005	0.000007	0.000016	0.000753	0.001570
1-month	-0.000017	0.000028	0.000052	0.002849	0.005281
3-month	-0.000078	0.000165	0.000256	0.016969	0.026337
6-month	-0.000116	0.000424	0.000742	0.044640	0.078232
1-year	0.000106	0.001154	0.001969	0.127859	0.219042
3-year	0.001170	0.003030	0.004746	0.402362	0.632349
5-year	0.000208	0.002157	0.003050	0.358169	0.508367
10-year	-0.003633	0.006202	0.010055	1.587365	2.506414
Two-Factor Model					
Maturity	ME	MAE	RMSE	MAPE	RMSPE
1-day	-0.000000	0.000000	0.000000	0.000010	0.000016
7-day	-0.000000	0.000001	0.000004	0.000106	0.000421
15-day	0.000000	0.000003	0.000015	0.000321	0.001519
1-month	0.000003	0.000012	0.000053	0.001239	0.005413
3-month	0.000034	0.000074	0.000273	0.007631	0.028254
6-month	0.000109	0.000200	0.000712	0.021114	0.075686
1-year	0.000241	0.000480	0.001457	0.053265	0.162536
3-year	-0.000081	0.000517	0.001156	0.070541	0.159114
5-year	-0.000481	0.001547	0.002887	0.256526	0.467754
10-year	0.000157	0.000606	0.001535	0.167148	0.432554

Table XI. Within-Sample Pricing Error Measures for the year 1992

This table contains the within-sample pricing error measures of the one and two-factor models for the year 1992. We consider zero-coupon bonds with face value of $1 and with maturities ranging from 1 day to 10 years. We have computed five different error measures: the mean error (ME), the mean absolute error (MAE), the root mean squared error (RMSE), the mean absolute percentage error (MAPE) and the root mean squared percentage error (RMSPE).

One-Factor Model					
Maturity	ME	MAE	RMSE	MAPE	RMSPE
1-day	-0.000000	0.000000	0.000000	0.000002	0.000003
7-day	-0.000001	0.000001	0.000001	0.000077	0.000149
15-day	-0.000003	0.000003	0.000006	0.000313	0.000640
1-month	-0.000014	0.000015	0.000030	0.001481	0.003054
3-month	-0.000114	0.000121	0.000246	0.012435	0.025345
6-month	-0.000382	0.000407	0.000818	0.043320	0.087150
1-year	-0.001071	0.001137	0.002253	0.129028	0.256560
3-year	-0.001963	0.001995	0.004337	0.293732	0.644732
5-year	-0.000032	0.001308	0.002251	0.238129	0.421463
10-year	0.001693	0.001890	0.004296	0.600565	1.396797
Two-Factor Model					
Maturity	ME	MAE	RMSE	MAPE	RMSPE
1-day	0.000000	0.000000	0.000000	0.000003	0.000005
7-day	0.000000	0.000000	0.000001	0.000035	0.000068
15-day	0.000000	0.000001	0.000002	0.000124	0.000230
1-month	0.000001	0.000005	0.000009	0.000533	0.000940
3-month	0.000005	0.000037	0.000062	0.003838	0.006357
6-month	0.000002	0.000109	0.000188	0.011595	0.019969
1-year	-0.000042	0.000305	0.000510	0.034493	0.057827
3-year	-0.000081	0.000349	0.000500	0.050313	0.072872
5-year	0.000208	0.001013	0.001542	0.181938	0.281741
10-year	-0.000061	0.000341	0.000605	0.103375	0.187527

Table XII. Within-Sample Pricing Error Measures for 10-year Bonds

This table contains the within-sample pricing error measures of the one and two-factor models for each semester of the period 1991-1994. We consider zero-coupon bonds with face value of $1 and with maturity of 10 years. We have computed five different error measures: the mean error (ME), the mean absolute error (MAE), the root mean squared error (RMSE), the mean absolute percentage error (MAPE) and the root mean squared percentage error (RMSPE).

One-Factor Model					
Period	ME	MAE	RMSE	MAPE	RMSPE
1991:I	0.000248	0.001060	0.002724	0.337539	0.852722
1991:II	0.000076	0.000794	0.001811	0.232777	0.529352
1992:I	-0.000053	0.000313	0.000522	0.087562	0.148913
1992:II	0.003498	0.003521	0.006103	1.130811	1.986138
1993:I	-0.004414	0.004420	0.007783	1.246517	2.191687
1993:II	-0.018630	0.018630	0.018928	4.503009	4.590118
1994:I	-0.014238	0.015052	0.017290	3.511278	4.022404
1994:II	0.005308	0.005308	0.005577	1.537958	1.621560
Two-Factor Model					
Period	ME	MAE	RMSE	MAPE	RMSPE
1991:I	-0.000107	0.000380	0.000814	0.119866	0.253813
1991:II	0.000388	0.000718	0.001193	0.209669	0.348161
1992:I	-0.000023	0.000261	0.000420	0.071953	0.115901
1992:II	-0.000099	0.000425	0.000750	0.135853	0.240062
1993:I	-0.000184	0.001061	0.003271	0.309411	0.963612
1993:II	0.000468	0.000765	0.001397	0.186202	0.341814
1994:I	0.000892	0.001025	0.001746	0.245455	0.414290
1994:II	-0.000123	0.000181	0.000385	0.050999	0.107501

Table XIII. Comparison of One-Step-Ahead Forecasts for the year 1995

This table contains the out-of-sample pricing error measures of the one and two-factor models for the year 1995. We compute one-step-ahead forecasts for prices of zero-coupon bonds with face value of $1 and with maturities ranging from 1 day to 10 years. We report five different error measures: the mean error (ME), the mean absolute error (MAE), the root mean squared error (RMSE), the mean absolute percentage error (MAPE) and the root mean squared percentage error (RMSPE).

One-Factor Model					
Maturity	ME	MAE	RMSE	MAPE	RMSPE
1-day	0.000000	0.000002	0.000004	0.000241	0.000353
7-day	0.000001	0.000016	0.000024	0.001648	0.002410
15-day	0.000003	0.000032	0.000047	0.003246	0.004740
1-month	0.000011	0.000068	0.000098	0.006831	0.009912
3-month	0.000074	0.000188	0.000264	0.019189	0.026946
6-month	0.000213	0.000345	0.000472	0.036095	0.049372
1-year	0.000423	0.000592	0.000782	0.065144	0.086047
3-year	-0.000827	0.001647	0.002221	0.225719	0.306270
5-year	-0.001998	0.002710	0.003500	0.467543	0.609406
10-year	0.002731	0.003469	0.004322	1.035002	1.300559
Two-Factor Model					
Maturity	ME	MAE	RMSE	MAPE	RMSPE
1-day	0.000000	0.000002	0.000004	0.000241	0.000353
7-day	0.000000	0.000016	0.000024	0.001648	0.002410
15-day	0.000000	0.000032	0.000047	0.003242	0.004738
1-month	-0.000001	0.000067	0.000098	0.006781	0.009874
3-month	-0.000016	0.000180	0.000255	0.018363	0.026064
6-month	-0.000065	0.000321	0.000436	0.033589	0.045608
1-year	-0.000209	0.000580	0.000767	0.063822	0.084462
3-year	-0.000239	0.001320	0.001790	0.180321	0.245460
5-year	0.000961	0.002096	0.002760	0.357493	0.473140
10-year	0.000177	0.002174	0.002889	0.641176	0.857203

Table XIV. Comparison of One-Step-Ahead Forecasts for 10-year Bonds

This table contains the out-of-sample pricing error measures of the one and two-factor models for each month of the year 1995. We compute one-step-ahead forecasts for prices of zero-coupon bonds with face value of $1 and with maturity of 10 years. We report five different error measures: the mean error (ME), the mean absolute error (MAE), the root mean squared error (RMSE), the mean absolute percentage error (MAPE) and the root mean squared percentage error (RMSPE).

One-Factor Model					
Period	ME	MAE	RMSE	MAPE	RMSPE
1995:I	0.006637	0.006637	0.007043	2.037704	2.161546
1995:II	0.005492	0.005492	0.005802	1.651017	1.743038
1995:III	0.003847	0.004455	0.005099	1.426704	1.632764
1995:IV	0.003368	0.003585	0.003896	1.131093	1.228890
1995:V	0.003515	0.004376	0.005721	1.296065	1.683145
1995:VI	0.002111	0.003443	0.004240	1.028990	1.263961
1995:VII	0.002515	0.003142	0.004035	0.924951	1.189217
1995:VIII	0.001255	0.002028	0.002312	0.579070	0.660393
1995:IX	0.000804	0.002788	0.003323	0.788667	0.941235
1995:X	0.000756	0.001676	0.002115	0.475194	0.599417
1995:XI	0.001623	0.002134	0.002423	0.585739	0.664042
1995:XII	0.000989	0.001852	0.002329	0.482712	0.608311
Two-Factor Model					
Period	ME	MAE	RMSE	MAPE	RMSPE
1995:I	-0.000097	0.001703	0.002181	0.524966	0.673928
1995:II	-0.000148	0.001449	0.001841	0.436042	0.554446
1995:III	-0.000615	0.002752	0.003596	0.885325	1.156787
1995:IV	0.000267	0.001483	0.001907	0.468220	0.603750
1995:V	0.000543	0.003592	0.004614	1.061447	1.351137
1995:VI	-0.000863	0.002836	0.003771	0.850663	1.135221
1995:VII	0.000135	0.002498	0.003163	0.735885	0.934982
1995:VIII	0.000309	0.001714	0.001953	0.489602	0.558105
1995:IX	0.000550	0.002790	0.003398	0.790182	0.963648
1995:X	0.000321	0.001597	0.002033	0.453225	0.576983
1995:XI	0.000914	0.001732	0.002053	0.475627	0.563572
1995:XII	0.001045	0.001725	0.002163	0.448974	0.564103

Table XV. Comparison of Five-Step-Ahead Forecasts for the year 1995

This table contains the out-of-sample pricing error measures of the one and two-factor models for the year 1995. We compute five-step-ahead forecasts for prices of zero-coupon bonds with face value of $1 and with maturities ranging from 1 day to 10 years. We report five different error measures: the mean error (ME), the mean absolute error (MAE), the root mean squared error (RMSE), the mean absolute percentage error (MAPE) and the root mean squared percentage error (RMSPE).

One-Factor Model					
Maturity	ME	MAE	RMSE	MAPE	RMSPE
1-day	0.000000	0.000004	0.000006	0.000425	0.000599
7-day	0.000001	0.000029	0.000041	0.002912	0.004107
15-day	0.000003	0.000057	0.000081	0.005764	0.008117
1-month	0.000013	0.000121	0.000170	0.012228	0.017145
3-month	0.000089	0.000338	0.000469	0.034582	0.047975
6-month	0.000265	0.000638	0.000871	0.066794	0.091163
1-year	0.000588	0.001208	0.001623	0.132807	0.178648
3-year	-0.000178	0.003421	0.004461	0.467336	0.613071
5-year	-0.001071	0.004838	0.006269	0.829320	1.085631
10-year	0.003801	0.006263	0.007752	1.843197	2.291128
Two-Factor Model					
Maturity	ME	MAE	RMSE	MAPE	RMSPE
1-day	0.000000	0.000004	0.000006	0.000425	0.000600
7-day	0.000000	0.000029	0.000041	0.002909	0.004112
15-day	0.000001	0.000057	0.000081	0.005751	0.008135
1-month	0.000001	0.000120	0.000171	0.012137	0.017204
3-month	-0.000003	0.000327	0.000469	0.033434	0.047936
6-month	-0.000017	0.000590	0.000857	0.061792	0.089773
1-year	-0.000052	0.001140	0.001600	0.125428	0.176266
3-year	0.000457	0.003296	0.004247	0.449916	0.582409
5-year	0.001967	0.004852	0.006077	0.826338	1.039909
10-year	0.001142	0.005498	0.006991	1.617977	2.074195

Table XVI. Comparison of Five-Step-Ahead Forecasts for 6-month and 10-year Bonds

This table contains the out-of-sample pricing error measures of the one and two-factor models for each quarter of the year 1995. We compute five-step-ahead forecasts for prices of zero-coupon bonds with face value of $1 and with maturity of 6 months and 10 years. We report five different error measures: the mean error (ME), the mean absolute error (MAE), the root mean squared error (RMSE), the mean absolute percentage error (MAPE) and the root mean squared percentage error (RMSPE).

Panel A: 6-month Bonds

One-Factor Model					
Period	ME	MAE	RMSE	MAPE	RMSPE
1995:I	0.000076	0.001038	0.001306	0.108704	0.136850
1995:II	0.000313	0.000619	0.000765	0.064922	0.080179
1995:III	0.000281	0.000406	0.000557	0.042469	0.058307
1995:IV	0.000397	0.000489	0.000651	0.051043	0.067861
Two-Factor Model					
Period	ME	MAE	RMSE	MAPE	RMSPE
1995:I	-0.000281	0.000992	0.001336	0.103954	0.140094
1995:II	-0.000056	0.000543	0.000718	0.056939	0.075213
1995:III	0.000016	0.000382	0.000508	0.039963	0.053153
1995:IV	0.000262	0.000442	0.000608	0.046100	0.063438

Panel B: 10-year Bonds

One-Factor Model					
Period	ME	MAE	RMSE	MAPE	RMSPE
1995:I	0.003895	0.006470	0.007815	2.004213	2.424683
1995:II	0.004386	0.007751	0.009623	2.337035	2.877908
1995:III	0.002966	0.005815	0.007382	1.665991	2.114848
1995:IV	0.004015	0.005018	0.005661	1.364009	1.531478
Two-Factor Model					
Period	ME	MAE	RMSE	MAPE	RMSPE
1995:I	-0.001815	0.005149	0.006909	1.612713	2.194463
1995:II	0.001280	0.006844	0.008548	2.063493	2.555043
1995:III	0.001610	0.005236	0.006751	1.498283	1.928249
1995:IV	0.003603	0.004780	0.005391	1.300279	1.460397

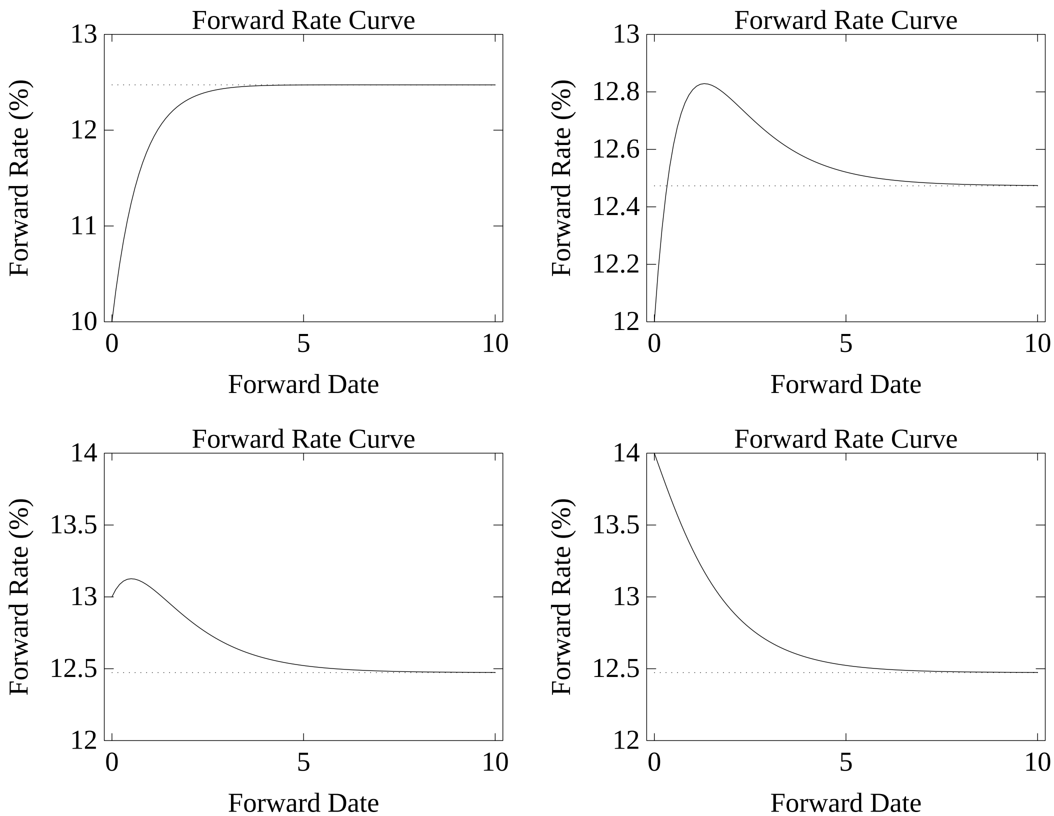


Figure 1: Forward Rate Curve.

The forward rates $f(s, L, t, T) \equiv f(s, L, \tau)$ at time t for the future period at date $T = t + \tau$ are given by

$$f(s, L, \tau) = r - q_1(s - \hat{\mu}_1)B(\tau) - q_2(L - \hat{\mu}_2)C(\tau) - \frac{1}{2}\left[\sigma_1^2 B^2(\tau) + \sigma_2^2 C^2(\tau)\right]$$

The parameter values correspond to 2 January 1991: $q_1 = 1.3456$, $\hat{\mu}_1 = 4.5924$, $\sigma_1 = 0.3467$, $q_2 = 0.744$, $\hat{\mu}_2 = 7.9259$, $\sigma_2 = 0.1159$. The two factor values, s and L, from left to right, top to bottom, are given by the vectors $(2, 8)$, $(2, 10)$, $(3, 10)$, and $(4, 10)$, respectively.

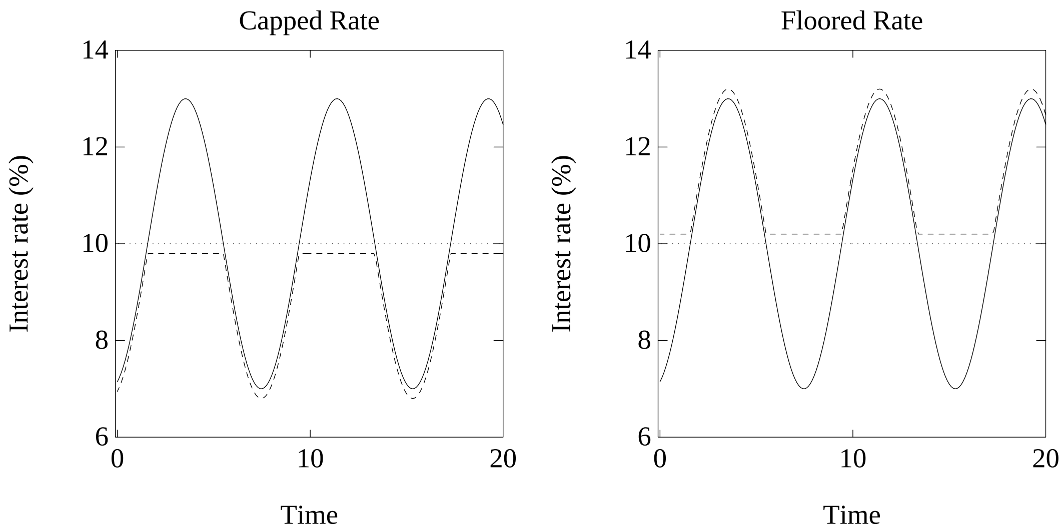


Figure 2: Borrower's Effective Interests Rates with Caps and Floors.

This figure depicts the floating interest rate (solid line), the cap rate (dotted line), and the effective capped interest rate (dashed line).

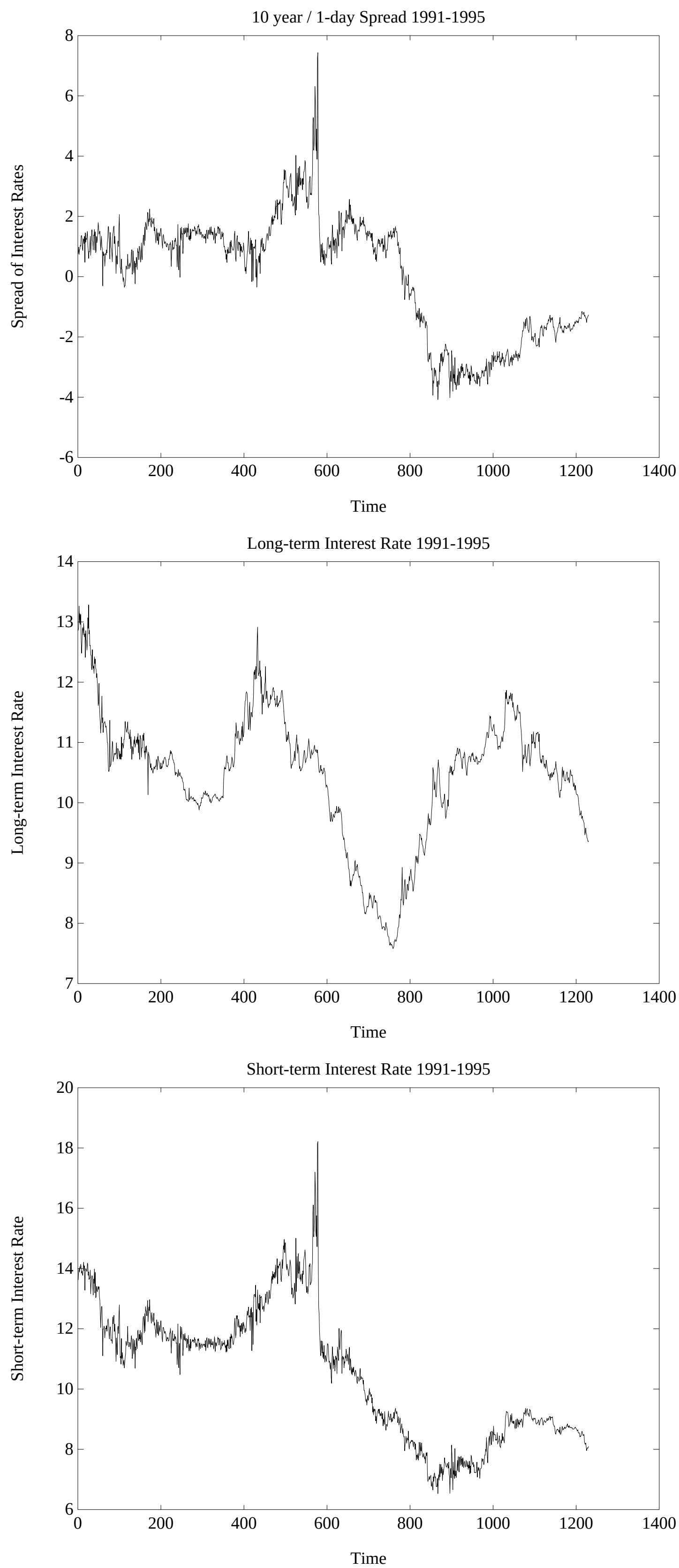


Figure 3: Plot of state variables.

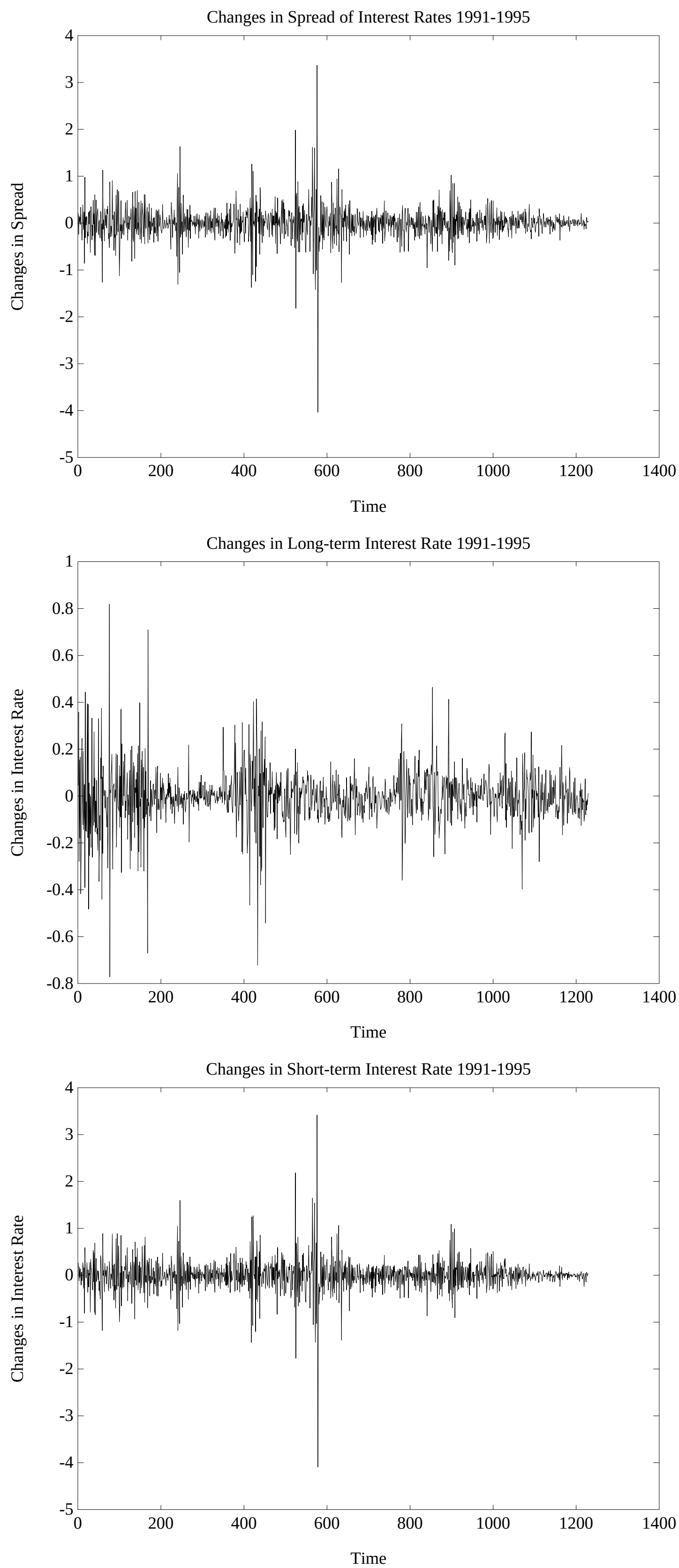
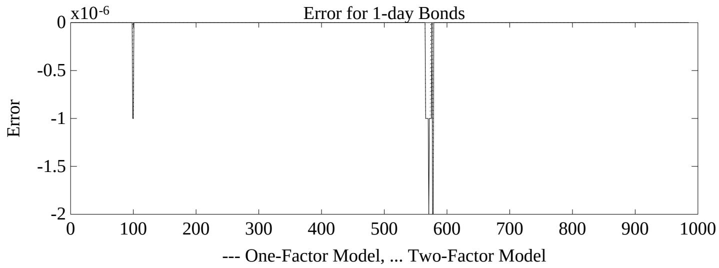
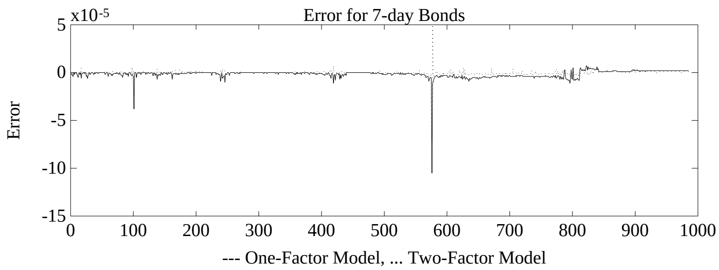
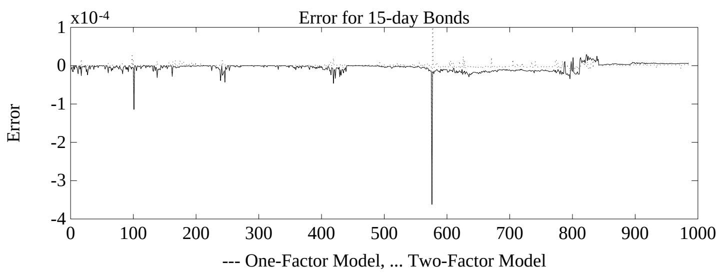
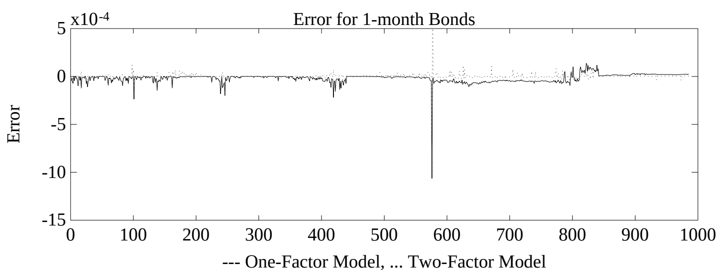


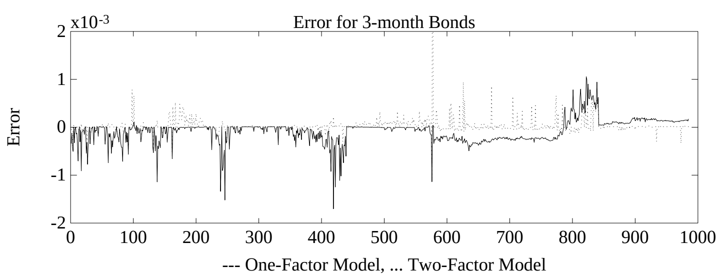
Figure 4: Plot of state variables in differences.

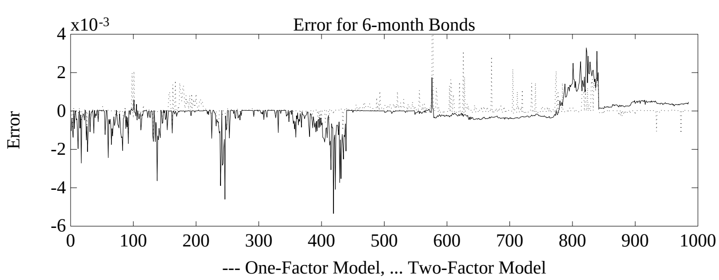

Error for 1-day Bonds
x10-6
Error
--- One-Factor Model, ... Two-Factor Model


Error for 7-day Bonds
x10-5
Error
--- One-Factor Model, ... Two-Factor Model


Error for 15-day Bonds
x10-4
Error
--- One-Factor Model, ... Two-Factor Model


Error for 1-month Bonds
x10-4
Error
--- One-Factor Model, ... Two-Factor Model


Error for 3-month Bonds
x10-3
Error
--- One-Factor Model, ... Two-Factor Model


Error for 6-month Bonds
x10-3
Error
--- One-Factor Model, ... Two-Factor Model

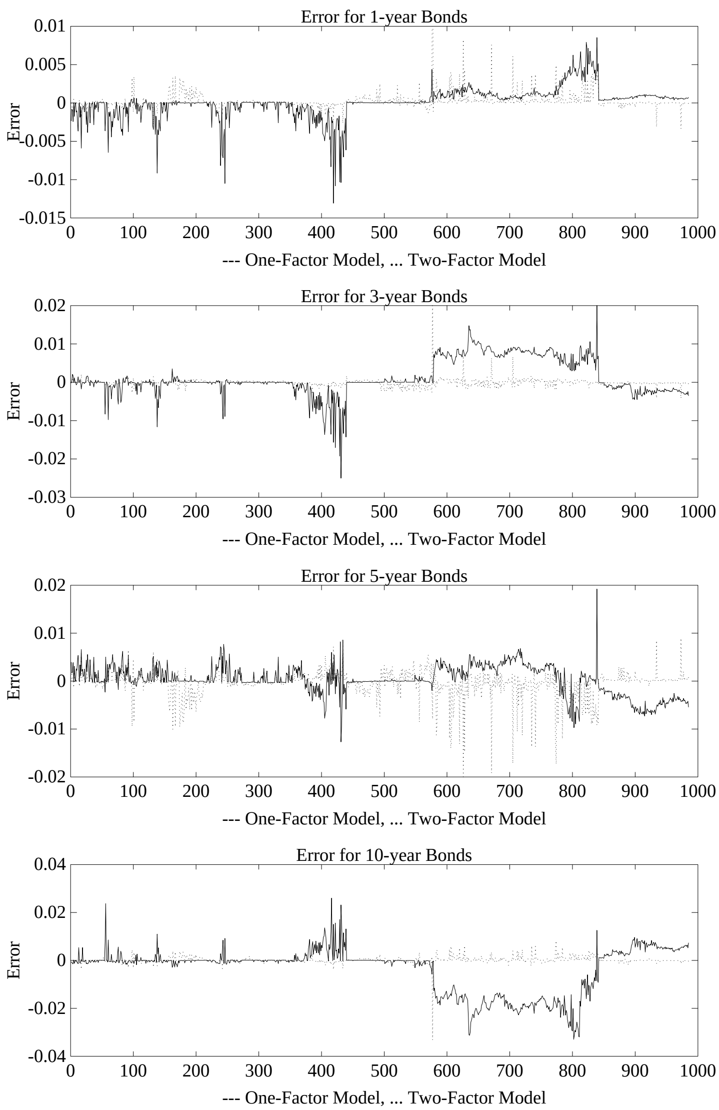
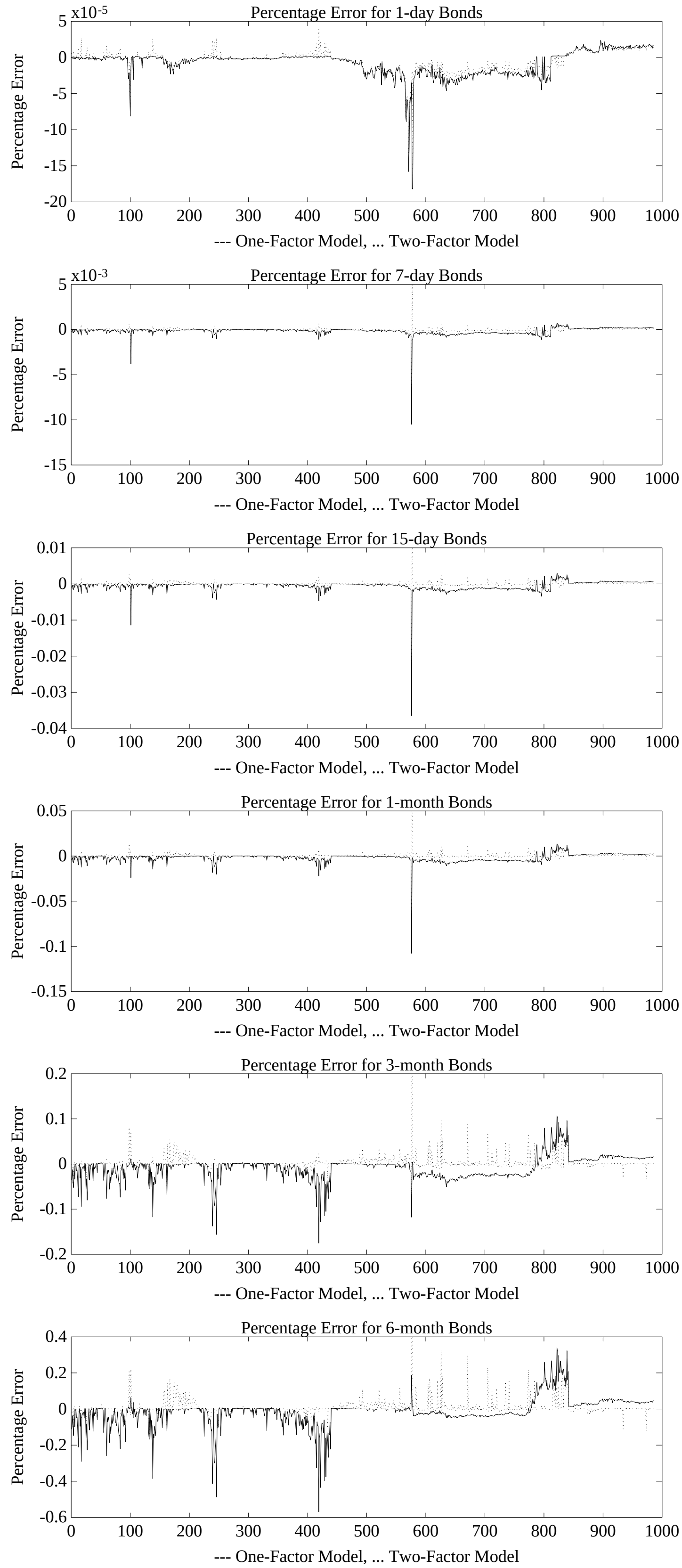


Figure 5: Within-Sample Errors of the One and Two-Factor Model.


Percentage Error for 1-day Bonds
x10-5
Percentage Error
--- One-Factor Model, ... Two-Factor Model
Percentage Error for 7-day Bonds
x10-3
Percentage Error
--- One-Factor Model, ... Two-Factor Model
Percentage Error for 15-day Bonds
Percentage Error
--- One-Factor Model, ... Two-Factor Model
Percentage Error for 1-month Bonds
Percentage Error
--- One-Factor Model, ... Two-Factor Model
Percentage Error for 3-month Bonds
Percentage Error
--- One-Factor Model, ... Two-Factor Model
Percentage Error for 6-month Bonds
Percentage Error
--- One-Factor Model, ... Two-Factor Model

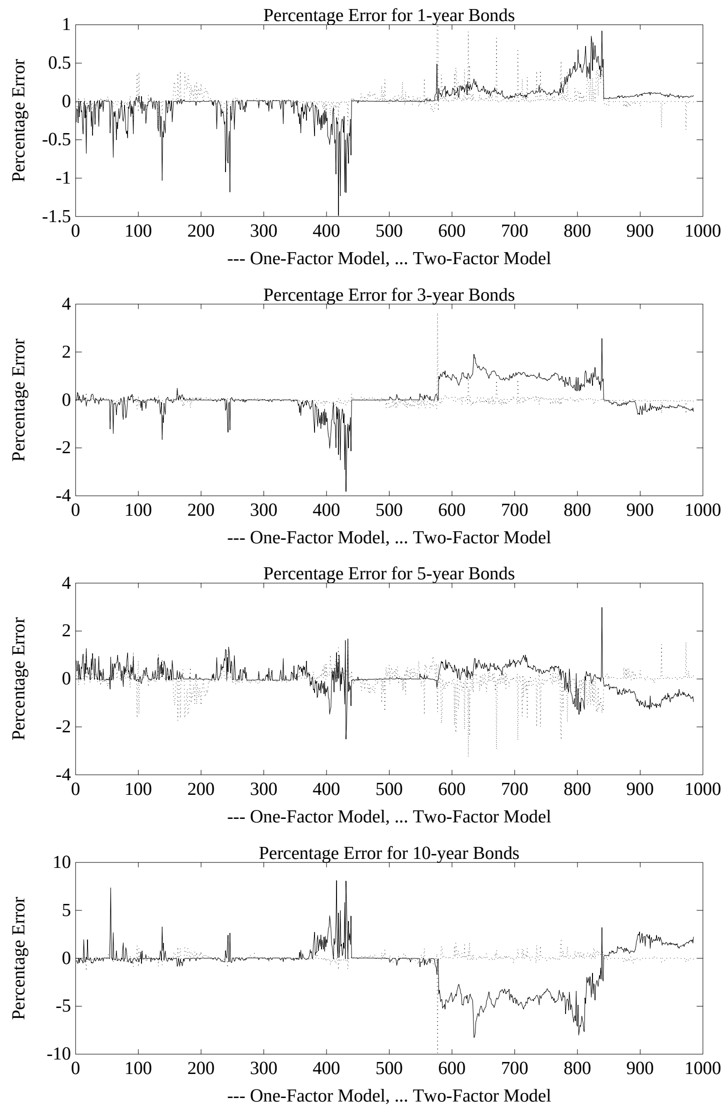


Figure 6: Within-Sample Percentage Errors of the One and Two-Factor Model.