What's in a *u*?

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Abstract

We revisit the long-lasting debate about the meaning of the utility function used in the standard Expected Utility (EU) model. Despite the common view that EU forces risk aversion and diminishing marginal utility of wealth to be pegged to one another, here we show that this is not the case. Marginal utility for money is an input into risk attitude, but it is not its sole determinant. The attitude towards 'pure risk' is also a contributing factor, and it is independent from the former. We discuss several theoretical implications of this result, for the following topics: (i) non-neutral risk attitudes for profit maximizing firms; (ii) risk-aversion over time lotteries in the presence of discounting; (iii) the equity premium puzzle. We also discuss matters of identification: (i) for firms; (ii) via *proxies*; (iii) via standard MLE-methods under parametric restrictions; and (iv) cross-context elicitation in multi-dimensional settings, and its relationship with the methods and results from the psychology literature.

Keywords: utility function – – risk-aversion – marginal utility
JEL Codes: C72; C91; C92; D80; D91.

1 Introduction

The meaning of the (Bernoulli) utility function used in the standard Expected Utility representation has long been debated, and this debate has typically been viewed as one of preferred interpretation of the object. On one side, a classical and natural interpretation is associated with Bernoulli ([1738] 1954): the marginal utility captures a notion of the marginal value, i.e., of pleasure or satisfaction (in a real sense, 'utility') of the additional unit. This interpretation is commonly held by economists and psychologists, and is in line with the compelling intuition that the reason we prefer 1 billion dollars for sure to a 50-50 chance of getting 2 billion dollars or nothing is that the value of what we can buy with that first billion is significantly higher than what we can buy with the second. In other words, risk-aversion in Expected Utility (EU) is driven by the diminishing marginal utility (value). On the other side, the orthodox decision theorists' viewpoint (e.g., Friedman and Savage, 1952) cautions against this interpretation. It takes the utility function as a representation of risk attitude, and nothing more.[1]

Regardless of one's view of these interpretations, it is generally accepted that a notion of marginal value (utility) and risk attitude are inexorably tied together in the Expected Utility representation. In fact, this view is so well-established that it has motivated the development of now widely-used alternatives to EU which aim to separate them. This is summarized in Yaari (1987)'s well-known quote:

> "In expected utility theory, the agent's attitude towards risk and the agent's attitude towards wealth are forever bonded together." [The reason this is problematic is that] "at the level of fundamental principles, risk aversion and diminishing marginal utility of wealth, which are synonymous under expected utility theory, are horses of different colors."

In this paper we focus on understanding how this debate can be resolved, and what can formally be said about the meaning of the utility function as an abstract concept. Our first result may appear striking: despite decades (if not centuries) of conventional wisdom, it is *not* the case that risk aversion and diminishing marginal value of wealth are pinned together in EU. While marginal value of wealth is certainly a factor in risk aversion, it is not its sole determinant.

These theoretical insights emerge from a *thought experiment*, within which we can define and separate 'pure risk' attitude from the marginal utility of money under certainty.[2] In particular, the utility function u can be viewed as the composition of two functions, $u = g \circ f$, where the curvature of f represents the marginal value of wealth, while g is a

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[1]This debate goes back to the 1950s. For a thorough historical perspective, see Moscati (2018, 2023).

[2]As we will discuss, there is a conceptual similarity between the logic of our thought experiment and the properties of an *ideal* state space to separately identify beliefs and utility in Savage (1954). Issues of *actual* identification are postponed to Section 4, where we discuss a few methods and their connection with Tsakas (forth.)'s solution to the classical problem of identification of beliefs with state-dependent utility.

CARA transformation, whose parameter represents the 'pure risk' attitude (Section 2).[3] Thus, two individuals could have identical values of wealth, but different risk-attitudes (in the standard sense of which gambles they accept), because they differ in how much they like or dislike risk, perhaps due to different personalities. By the same token, two individuals could have the same 'dislike' of risk, but value money in different ways, which in turn implies that they will have different risk attitudes over money. Hence, despite the well-known and common view that EU forces risk aversion and diminishing marginal utility of wealth to be pegged to one another, this is in fact not the case. Even within EU, horses of different colors are horses of different colors.

But it is not just that the commonly held interpretation is inaccurate and should be revised for the sake of interpretation alone. As we show in Section 3, once expected utility is viewed through the perspective of our approach, it also allows us to provide a solution to classical puzzles in the literature, such as the willingness to gamble or buy insurance, even at high wealth levels (e.g., Friedman and Savage, 1948), as well as to accommodate, within an expected utility framework, non-neutral risk-attitudes for profit-maximizing firms (another famous white whale of the classical literature – e.g., Yaari (1987), and Footnote 14 below). Furthermore, in a dynamic setting, we show that our approach enables us to reconcile, within an expected utility setting, exponential discounting in the certainty space, with risk-aversion over time lotteries, a theme that has recently gained a great deal of attention thanks to the seminal work of DeJarnette, Dillenberger, Gottlieb, and Ortoleva (2020). In this context, our approach provides foundation to a model recently put forward by Dillenberger, Gottlieb, and Ortoleva (forth.) to address this issue, albeit coming from a different perspective. Finally, we show how the separation between diminishing marginal utility of money and 'pure risk' attitude has important implications for the analysis of saving and investment decisions, which may provide novel insights (from *within* expected-utility) about Mehra and Prescott (1985)'s celebrated Equity Premium Puzzle.

The conceptual significance of the results and insights of Sections 2 and 3 stands independent of the issue of separately identifying the two components of the utility function in practice. Nonetheless, identifying these functions is itself a relevant exercise, both for understanding preferences and for enabling sharper predictions. We turn to this exercise in Section 4, and discuss three methods of identification with the corresponding datasets. We first consider profit-maximizing firms, where the identification exercise follows naturally from a concrete implementation of our thought experiment. We then consider individuals more generally, and provide the formal conditions on the observable dataset under which identification can be conducted via a *proxy* (i.e., a commodity that satisfies suitably defined properties, similar to Tsakas, forth.). We also suggest an example of a proxy that can be used in practice. Lastly, we discuss identification under specific parametric assumptions, of the kind that is most frequent in empirical and experimental work (cf. Gill and Prowse

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[3]A large literature has modeled explicitly the distinction between certain and risk preferences. But, following Yaari (1987), this has not been achieved within a EU setting (e.g., Abdellaoui et al., 2007 and references therein). Within this literature, the closest work in spirit is perhaps Schmidt and Zank (2022).

(2012)).

Finally, in Section 5 we show that the separation between 'pure risk' and marginal utility sheds light on important questions that arise in the context of risk when multiple commodities are involved.[4] In particular, we explain why risk attitude elicited in one domain may not correlate with risk attitude elicited in other domains, and show how our approach can solve this problem and lead to more accurate predictions *across* domains. We also show how eliciting risk attitudes over multiple domains may enable a better identification of the 'pure risk' parameter, lead to a stable ordering of individuals' risk attitudes that is portable *across* domains. In doing so, we also contribute to the discussion concerning the distinction between the standard choice-based economics methodology to elicit risk attitudes and the method often used in psychology, which involves questions that refer to a wide range of domains and forms of behavior (e.g., Frey et al., 2017). Our discussion formally connects the two approaches, and shows how they can be combined more effectively to understand and predict behavior.

The concluding section discusses how this work could be further developed, both theoretically and in combination with neuroeconomics research (see, for instance, Glimcher and Rustichini (2004); Camerer (2008); Rustichini (2009) for classic references, and Glimcher and Tymula (2023) for recent work), and various directions for future research.

2 Theoretical framework

In the standard approach to risk, within the von-Neumann-Morgenstern (vNM) framework, it is customary to consider preferences $\succsim^*$ over simple lotteries over the real line, $p, q \in \Delta(\mathbb{R})$ that satisfy the standard axioms (weak order, the Archimidean property and the independence axiom). Letting $m \in \mathbb{R}$ denote the quantity of money, vNM's representation theorem ensures that there exists a utility function $u^* : \mathbb{R} \to \mathbb{R}$ such that $p \succsim^* q$ if and only if $\sum p(m)u^*(m) \geq \sum q(m)u^*(m)$, and u^* is unique up to positive affine transformations.

2.1 A Conceptual Yardstick

In the setting above, suppose that two onlookers (scientists) observed data on the agent's choices and concluded, for instance, that he is risk-averse over m, so that his utility is concave. But suppose that they disagree on the meaning of this finding: The first observer argues that the curvature of u^*, and hence the attitude towards risk, is merely a consequence of the agent's decreasing 'marginal utility for money'. The second onlooker instead argues that this concave function represents the agent's dislike of risk alone, and has nothing to do with his valuation of m. In particular, he argues that nothing can be

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[4]For a carefully conducted analysis on risk attitude in multiple domains, see Ke and Zhang (2024), which introduce the hierarchical expected utility representation. We view this important contribution in this regard as complementary to that of our paper, where we remain strictly within the EU setting.

concluded concerning what each extra unit of money yields, in utility terms.[5]

Given the data at their disposal, how could these onlookers settle this argument, if not through debates over one's own introspection? As is usually the case with economics, we need a way of analyzing *trade-offs* to make further claims. In fact, the notion of 'marginal utility of money' itself is inherently one of comparison: a comparison of money and some concept of its value, however the latter is determined for the individual. That is, there must be at least an abstract *yardstick* of comparison, first to make sense of this debate, and second to settle it.[6]

Here we shall consider, as a pure thought experiment, which properties must such an idealized *yardstick* satisfy, and what this will imply for the interpretation of the utility function u^*. We set aside for the moment questions of measurement and identification, which are separate from the debate at hand, and will return to them in Section 4.

2.1.1   A Yardstick under Certainty

If such a yardstick existed, call it y, it would effectively be a second commodity, besides money, which could be traded-off against m to identify the marginal rate of substitution (MRS) in the certainty space, and hence irrespectively from the agent's attitude towards risk. This MRS would effectively pin down the utility of the agent for each unit of m, in terms of the chosen unit of measure, the yardstick.

Having elicited such an MRS, for any $m \in \mathbb{R}$, we could identify the quantity of the y-commodity that would make the agent indifferent between receiving m and that specific quantity of y. Let $f : \mathbb{R} \to \mathbb{R}$ denote such a function, with $f(m)$ being the yardstick-equivalent of m elicited from these preferences.

Formally, in this setting, the agent's *preferences* are defined over a richer commodity space, which includes both m and y. That is, let the space of outcomes be $Z = \mathbb{R}^2$, with typical element $z = (m, y) \in Z$, and $\succsim^c$ denote a weak order, strictly increasing in both components. For simplicity, we also maintain throughout that $\succsim^c$ is continuous. Then, for a 'utility of money' $f(m)$ expressed *in terms of the yardstick*, and for y to effectively have this role, it must be the case that, up to increasing transformations, the representation of $(\succsim^c, Z)$ takes the form $f(m) + y$. That is: $(m, y) \succsim^c (m', y')$ if and only if $f(m) + y \geq f(m') + y'$. If not, commodity y would not be a *yardstick*, with respect to

which the utility of m is expressed.[7] We therefore provide the following definition of an ideal yardstick under certainty.

Definition 1 (Yardstick under Certainty). *If the y-commodity is an ideal* yardstick under certainty*, then the preferences over the certain space $Z = \mathbb{R}^2$, $(\succsim^c, Z)$, are such that there exists a strictly increasing function $f : \mathbb{R} \to \mathbb{R}$ such that $(m, y) \succ (m', y')$ if and only if $f(m) + y > f(m') + y'$.*

Note that even under this thought experiment, we cannot yet make any statements concerning the utility function u^* of the EU-representation in the baseline setting, because f need not correspond to this function. It only captures the value of m, in units of the yardstick y, absent risk considerations.

2.1.2 From certainty to risk

Now, let us extend the domain of preferences to a risk setting, i.e. to account for simple lotteries $p, q \in \Delta(Z)$, maintaining all the standard vNM axioms, and let $(\succsim, \Delta(Z))$ denote the corresponding preference system. Again, we let δ_z denote the degenerate lottery over outcome $z = (m, y) \in Z$, and write $(p^m, y) \in \Delta(Z)$ to denote a lottery that is degenerate over the y-component and that randomizes over m according to $p^m \in \Delta(\mathbb{R})$.

First, to ensure that $(\succsim, \Delta(Z))$ naturally embeds the preferences over the certain domain, $(\succsim^c, Z)$, we maintain that for any (m, y) and (m', y'), it holds that $\delta_{(m,y)} \succsim \delta_{(m',y')}$ if and only if $(m, y) \succsim^c (m', y')$. Second, for commodity y to play the role of an ideal yardstick, *and nothing more*, there should be no interaction between the level of the yardstick and the risk preferences (i.e., preferences over lotteries over m.) This means that the preference system must satisfy the following *yardstick neutrality* condition: for any $y, y' \in \mathbb{R}$ and for any $p^m, q^m \in \Delta(\mathbb{R})$, $(p^m, y) \succsim (q^m, y)$ if and only if $(p^m, y') \succsim (q^m, y')$.[8]

In summary, besides the standard vNM axioms, the preference system $(\succsim^c, \Delta(Z))$ must satisfy the following properties:

1. **A Yardstick under Certainty:** there exists a $f : \mathbb{R} \to \mathbb{R}$ that is strictly increasing and such that $\delta_{(m,y)} \succ \delta_{(m',y')}$ if and only if $f(m) + y > f(m') + y'$.

2. **Yardstick Neutrality:** for any $y, y' \in \mathbb{R}$ and for any $p^m, q^m \in \Delta(\mathbb{R})$, $(p^m, y) \succsim (q^m, y)$ if and only if $(p^m, y') \succsim (q^m, y')$.

The first property follows directly from Definition 1 and the fact that $\succsim$ is an extension of $\succsim^c$. As for Yardstick Neutrality, it ensures that the addition of the ideal y-commodity,

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[7]While we find it more transparent to provide this property in terms of functions, it can clearly be expressed in terms of preferences too: $(m, y) \succsim^c (m', y')$ iff $(m, y + t) \succsim^c (m', y' + t)$, where $t \in \mathbb{R}$.

[8]This is analogous to assuming state independence, which serves to avoid an interaction in the preferences between states and outcomes. Thus, just as with state independence, where the *ideal* state space to use is one that does not interact with preferences (cf. Savage, 1954), the same holds here: the ideal yardstick does not interact with preferences over money. In Section 4.2 we will discuss how in practice the yardstick can be replaced by a *proxy* commodity, selected to approach this ideal benchmark, and the relationship with Tsakas (forth.)'s method of belief identification with state-dependent preferences.

per se, does not alter the set of preferences over monetary lotteries that could be expressed in the baseline (one-dimensional) setting $(\succsim^*, \Delta(\mathbb{R}_+))$. This is key to ultimately be able to draw a meaningful connection between the Bernoulli utility in the EU representation of $(\succsim, \Delta(Z))$, and the baseline u^* obtained in the standard one-dimensional setting. In fact, if Yardstick Neutrality did not hold, then it would mean that the baseline vNM preferences (and, hence, their representation) would be affected by an omitted variable problem, and their meaning would not be clear. Yardstick Neutrality therefore is both a natural and a necessary property to hold, within the standard vNM framework.

Proposition 1. *Under the maintained assumptions, $u : Z \to \mathbb{R}$ provides an EU representation of $(\succsim, \Delta(Z))$ if and only if there exists $g : \mathbb{R} \to \mathbb{R}$ such that $u(m, y) = g(f(m) + y)$ for all $(m, y) \in Z$, and which can only be one of the following functional forms (up to positive affine transformations): either (i) $g(x) = x$, or (ii) $g(x) = \frac{1 - e^{-\alpha x}}{\alpha}$, with $\alpha \neq 0$.*

The logic of this result is simple: from the assumptions on the certain preferences, it follows that any strictly increasing transformation of $f(m) + y$ represents $(\succsim^c, Z)$. Extending these preferences to $\Delta(Z)$, with the vNM axioms, ensures that there exists a $g : \mathbb{R} \to \mathbb{R}$ (unique now up to positive *affine* transformations) such that $p \succsim q$ if and only if $\sum p(m, y) g(f(m) + y) \geq \sum q(m, y) g(f(m) + y)$. Then, yardstick neutrality further ensures that preferences over monetary lotteries are invariant to y, and hence $g(f(\cdot))$ is an affine transformation of $g(f(\cdot) + y)$ for each y. It follows that $g : \mathbb{R} \to \mathbb{R}$ exhibits constant absolute risk-aversion (CARA).

Yardstick neutrality also allows us to define preferences over monetary lotteries alone, $(\succsim^m, \Delta(\mathbb{R}))$: for any y, say that $p^m \succsim_y q^m$ if and only if $(p^m, y) \succsim (q^m, y)$, and let $\succsim^m$ coincide with the $\succsim_y$-ordering for $y = 0$. (Under yardstick neutrality, $\succsim_y = \succsim_0$ for all y). With this, we can relate the u obtained in Proposition 1, as part of the representation of the preferences $(\succsim, \Delta(Z))$, with the u^* obtained from the representation of the standard one-dimensional preferences, $(\succsim^*, \Delta(\mathbb{R}))$, when $\succsim^m = \succsim^*$. Formally:

Proposition 2. *$u^* : \mathbb{R} \to \mathbb{R}$ is a (strictly increasing) Bernoulli utility associated with the EU-representation of some preference system $(\succsim^*, \Delta(\mathbb{R}))$ if and only if there exists $(\succsim, \Delta(Z))$ that satisfies the maintained axioms, with $\succsim^m = \succsim^*$, and a utility $u : \mathbb{R}^2 \to \mathbb{R}$ that represents it in the sense of Proposition 1, such that (i) $u^*(m) = u(m, 0) = g(f(m))$ for all $m \in \mathbb{R}$ and (ii) $u^*(\cdot)$ is a positive affine transformation of $u(\cdot, y)$ for all y.*

The first part of Proposition 2 says that, under the maintained assumptions, (including, in particular, yardstick neutrality), the set of Bernoulli utility functions generated in the two-dimensional setting is exactly the same as that generated in the standard (one-dimensional) setting for preferences over risk: The addition of the yardstick neither restricts nor enlarges the set of possible Bernoulli utilities. The second part states the implication of yardstick neutrality discussed above, now in the space of the representations: the vNM preferences over monetary lotteries are invariant to y, and hence if u

represents $(\succsim, \Delta(Z))$ in the sense of Prop. 1, then $u(m, 0)$ (and, thus, $u^*(m)$) is an affine transformation of $u(m, y)$ for any y.

In light of this result, in the following we will simply write $u(m)$, to refer interchangeably to either $u^*(m)$ or $u(m, 0)$, and refer to it as the Bernoulli utility function, which by Proposition 1 is understood to be of the form $u = g \circ f$, where g is a CARA transformation. Then, we write $g(x) = \frac{1 - e^{-\alpha x}}{\alpha}$ if $\alpha \neq 0$, and $g(x) = e^\alpha x$ otherwise, so that $\alpha \in \mathbb{R}$ denotes the (constant) coefficient of absolute risk aversion of g, with $\alpha = 0$ for risk-neutrality. Furthermore, for the case where f and u are twice differentiable (given the CARA property of g, f is differentiable if and only if u is), let $\alpha_u(m) = -\frac{u''(m)}{u'(m)}$ and $\alpha_f(m) = -\frac{f''(m)}{f'(m)}$ denote the Arrow-Pratt coefficients of absolute risk-aversion at m, for u and f, respectively. Then, the following result is immediate:

Corollary 1. *Under the maintained assumptions, if u is a Bernoulli utility function of the EU-representation, and if it is differentiable, then, for each $m \in \mathbb{R}$,*

$$\alpha_u(m) = \alpha \cdot f'(m) + \alpha_f(m). \tag{1}$$

Our exercise and results may appear reminiscent of Epstein and Zin (2013), in that, while the functional form we obtain is of course clearly distinct, it uses an enriched space. But note that our exercise is entirely within EU, rather than a generalization of a recursive utility setting (Kreps and Porteus, 1978). Moreover, while Epstein and Zin (2013) focus on the separation between risk attitude and intertemporal preferences, we study instead the connection between risk attitude and the utility over the good (money). For this reason, our enrichment is of a different nature from the Epstein-Zin exercise.

2.2 Discussion and Extensions

We discuss next the interpretation and some implications of our results, before turning to a discussion of some relaxations of the maintained assumptions.

2.2.1 On the Interpretation of f and g

By construction, the f function represents the value of money, m, in units of the yardstick y, as elicited in the *certainty domain*, i.e. purely on the basis of preferences $(\succsim^c, Z)$. The function g instead, is pinned down by the agent's of preferences over lotteries, and hence it is purely about risk. The u in the EU-representation, is the composition of these two functions: $u = g \circ f$. As usual, the decision maker's *risk-attitude over money* is expressed by the curvature of the u function, and its curvature therefore depends on the curvatures of the two functions, 'value of money' function f and 'pure risk function' g. Corollary 1 formalizes this idea, providing the decomposition of the Arrow-Pratt index of risk aversion of u in terms of the indices for g and f, respectively. The former is constant in m, due to

the result that g is CARA, and we will refer to it as the *'pure risk' parameter*.[9]

With this, *risk-aversion* coincides with *diminishing marginal utility for money* if and only if $\alpha = 0$. In this case, the agent is neutral about risk, per se, and his overall risk-attitude (the curvature of u) is entirely due to his value for money (the curvature of f).[10] At the opposite extreme, even if f is linear (as it would, for instance, in the important special case of a profit-maximizing firm, which we discuss in the next section), the agent could still exhibit risk aversion (or risk seeking-ness), depending on the value of α.

More broadly, if g is not linear, then there is a wedge between the curvature of u and that of f. Take, for instance, an agent with an everywhere decreasing marginal utility for money, who therefore exhibits a globally concave f. This agent may still be risk-loving overall, if his 'pure risk' attitude is such that g is sufficiently convex. Alternatively, take another agent, with exactly the same preferences under certainty (and, hence, the same f), but who is averse to 'pure' risk. Then, his u would be more concave than what dictated by the concavity of f alone, since it would reflect both the curvature due to the decreasing marginal utility of money, and that of g, which only reflects the agent's dislike for risk, net of his preferences over money.

In summary, *diminishing marginal utility for money* (i.e., concavity of f) is a *reason for* risk-aversion, but it need not suffice for it, nor need it be its sole determinant. The attitude towards *pure risk*, as represented by the 'pure risk' parameter α, also contributes. Thus, despite the well-known and common view that EU forces risk aversion and diminishing marginal utility of wealth to be pegged to one another, this is in fact not the case. Even within EU, horses of different colors are horses of different colors.

2.2.2 From u to f, and the St. Petersburg Paradox revisited

While questions of identification are left to Section 4, here we make the following observations concerning partial inferences that can be made. Since the function g is of CARA form, one of the following three must hold: i) it is bounded above if $\alpha > 0$, ii) it is bounded below if $\alpha < 0$, or iii) it unbounded if $\alpha = 0$. This means that if u is assumed to have a specific parametric form that is unbounded on one side or another, than we can make partial inferences on the α parameter. Specifically: if u is unbounded below, then it must be that $\alpha \geq 0$ so that the agent cannot be (pure) risk loving; if it is unbounded above, then $\alpha \leq 0$ and he cannot be (pure) risk averse; if it is unbounded both above and below, then g must be linear and he must be (pure) risk neutral.

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[9]Interestingly, in a very different exercise that focuses on utility versus welfare, and using a distinct set of properties, Dietrich (2025) also arrives to a CARA parameter of 'intrinsic risk attitude.' Outside of Expected Utility (and, specifically, within prospect theory), the literature has followed Yaari (1987) in distinguishing between riskless utility and risk attitude through probability weighting functions; see Abdellaoui, Barrios, and Wakker (2007) for a discussion and literature review.

[10]This case corresponds to Bernoulli ([1738] 1954)'s original theory, or to the view of the first of the 'onlookers' above, according to which risk-aversion arises *because* of decreasing marginal utility. The second onlooker, in contrast, is more agnostic and does not care to explain *why* risk-aversion arises: the u simply serves as a tool to make predictions, as in Friedman's 'choice-generating function' (see Footnote 5).

Let us now return to the famous St. Petersburg Paradox, which initially led Bernoulli to the Expected Utility formulation in the first place.[11] The paradox is the following: suppose a fair coin is flipped, and if it lands heads then the game ends and the agent receives 2 ducats (or euros), otherwise it is flipped again and if it lands heads than the agent receives 4 ducats, and so forth (reference). It has infinite expected value, but most individuals would not be willing to pay an infinite amount for it. In proposing an EU formulation, Bernoulli further proposed using the functional form $u(x) = log(x)$. However, since it is unbounded, it has itself been criticized for allowing the emergence of a variation of the same paradox, in which the agent would still be willing to pay an infinite amount. The proposed solution was precisely to resort to utility functions that are bounded above (CRRA functions with a risk attitude higher than that of the log function, as commonly used in practice, satisfy this property).

From the viewpoint of our approach, the reason why the (adjusted) St. Petersburg Paradox would still be an issue with an unbounded u such as the log function is precisely that it forces g to be linear, and hence it does *not* allow for any aversion with respect to 'pure risk', and it forces f to be log. The proposed solution of a bounded above u, such as a CRRA function with a high enough risk attitude, implies that $\alpha \geq 0$, and allows for aversion to pure risk. It also allows for f to be bounded above, which the log u does not.

2.2.3 On Uniqueness

It is worth noting that, while the vNM u is of course unique up to positive affine transformations, and so is the g in the representation in Propositions 1 and 2. The f is instead *unique*, and hence it is *cardinal* in the classical, Pre-Samuelsonian sense of the word (as opposed to the notion that has become standard since von Neumann and Morgenstern (1954); cf. Moscati, 2018). This is natural, since f represents the marginal utility of money in terms of the yardstick: as we will further clarify in Section 4, the choice of the yardstick pins down the units in which the MRS is expressed, and given such a choice, the marginal utility of money is cardinal in the classical sense. In contrast, the 'pure risk' attitude is cardinal in the sense of being unique up to positive affine transformations.[12]

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[11] A version of the paradox was initially put forward by Nicolas Bernoulli, with the version that became 'definitive' being proposed by Gabriel Cramer. The EU resolution was proposed by Daniel Bernoulli, who favored the log utility representation. (Cramer's favored square root representation suffers from the same unboundedness issue as the log function). See Moscati (2023) for a detailed discussion.

[12] In his critical account of the history of expected utility theory, Moscati (2018, 2023) distinguishes the classical notion of cardinality from von Neumann and Morgenstern's, which he refers to as 'scale-invariance'. It is interesting that the uniqueness(-es) in our result reflect, in a formal sense, the two main historical perspectives: the latter notion applies to the g function, which captures the 'pure risk' attitude, whereas the f function, which in a sense formalizes Bernoulli's rationale for risk-aversion, is cardinal in the sense that he and other classics meant it.

2.2.4 Multiple commodities

The logic above clearly extends to multi-dimensional settings, i.e. where preferences in the certainty space are defined over multiple commodities.[13] Specifically, let $X \subseteq \mathbb{R}^n$ denote the commodity space with typical element $x = (x_1, ..., x_n) \in \mathbb{R}^n$, and let $(\succsim^c, X)$ denote the preferences in the certainty space. As standard, these preferences could be identified in the usual way, with a utility representation unique up to increasing transformations. Then, for the *ideal yardstick* y (which, as above, should satisfy its key defining property: here $(x, y) \succsim^c (x', y')$ if and only if $(x, y + t) \succsim^c (x', y' + t)$), the MRS between the utility index and such y would pin down the function $f : X \to \mathbb{R}$, i.e. the utility under certainty. Then, under yardstick neutrality (which is formulated exactly as above, provided that Z is redefined as $Z = X \times \mathbb{R}$, one obtains results analogous to Propositions 1 and 2, so that the overall $u : X \to \mathbb{R}$ in the vNM representation can be written as $u(x) = g(f(x))$, where $g : \mathbb{R} \to \mathbb{R}$ is a CARA transformation with parameter $\alpha \in \mathbb{R}$.

3 So What Else?

The results above shed light on several important open questions, of both conceptual and practical significance. At a minimum, they enable us to reconcile seemingly contrasting views that are based on insights over different domain of preferences. For instance, it is typically argued that the marginal utility of money should become constant at sufficiently high levels of wealth. Yet, even billionaires buy insurance, or others perhaps gamble instead, in ways that would be inconsistent with a Bernoulli utility function that is approximately linear (e.g., Friedman and Savage, 1948). As discussed, these two views can be reconciled within our framework: depending on the α parameter, the agent could still exhibit risk aversion (or risk seeking-ness), even if the value of wealth were perfectly linear. Hence, insights based on observations made on the certainty space, need not coincide with, nor translate to, unique implications in the risky domain. There are two conceptually distinct forces at play.

In this section we discuss a few further implications of these general observations, specifically for three important problems: first, we discuss the possibility of accommodating, within the expected utility framework, non-neutral risk-attitudes for profit-maximizing firms; second, we discuss how, in the context of intertemporal choice, the results above speak to the issue of how to separate curvature of the 'within period' utility from risk-attitude over time lotteries (DeJarnette et al., 2020; Dillenberger et al., forth.); finally, we discuss some implications for saving and investment decisions, and for the *equity premium puzzle* (Mehra and Prescott, 1985).

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[13]In order to keep the focus on the distinction between 'pure risk' and certainty preferences, we do not dwell here upon the important question of risk and risk-aversion within a multi-dimensional settings, which has been recently studied in Ke and Zhang (2024).

3.1 Firms and Risk

Consider first the case in which the agent is a profit-maximizing firm. The thought exercise of the previous section becomes a simple and very concrete in this scenario: the unit of account (say, US dollars for a US firm) *is* the yardstick. Hence, if m denotes its money holdings in US dollars, the MRS between m and y under certainty in this case is the identity function. That is, $f(m) = m$. It is both standard and uncontroversial that these are the (neoclassical) firm's preferences under certainty.

It has also been customary to jump from this uncontroversial observation about the firm's preferences under certainty, to the conclusion that a profit-maximizing firm under expected utility must necessarily be risk-neutral. While this confusion seems natural from the viewpoint of our first onlooker (according to which risk aversion stems from a decreasing marginal utility of money), it strikes as peculiar from the viewpoint of our second onlooker – according to which the u in the vNM representation has no connection with the underlying preferences under certainty (cf. footnotes 5 and 10) – and which embodies a widely held decision-theoretic view at least since Friedman and Savage (1952). For the case of firms, the common practice clearly departs from the agnostic view of a mere index to represent risk attitude: for firms, a stronger position has been taken that the latter can be derived from the preferences under certainty. In fact, this view is so deep-rooted that it has lead economists to accept a model which is clearly at odds with the evidence that several firms do buy insurance.[14]

To be clear, this $g(\cdot)$ transformation should be applied to the standard profit function (say, for a monopolist, $\pi(q) = P(q) \cdot q - C(q)$, where q denotes the quantity, and $P(\cdot)$ and $C(\cdot)$ the inverse demand and cost functions, respectively). In the face of uncertainty, the firm here would maximize $\mathbb{E}[g(\pi)]$, not necessarily $\mathbb{E}[\pi]$. Since $g(\cdot)$ is an increasing transformation, this change in the objective function has no bearing on choices under certainty, but if $\alpha \neq 0$, it does affect choices under uncertainty (e.g., in portfolio choice problems, if costs or revenues are stochastic from the firm's viewpoint, etc.).

As an example, consider a monopolist facing a stochastic demand, where the price associated with output q is equal to $P(q) + \epsilon$ or $P(q) - \epsilon$, with equal probability. In this case, the optimal choice for a risk-neutral firm would be the same, independent of the magnitude of $\epsilon \geq 0$, and hence it would coincide with the optimal quantity under certainty (i.e., for $\epsilon = 0$), q^*. That a 'real world' monopolist would be indifferent over any magnitude of $\epsilon \geq 0$, and that it would not react to it in this setting, as entailed by risk-neutrality, seems contrary to common sense (cf., footnote 14). But, as explained, a

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[14]Yaari (1987), for instance, took it as an extra motivation for venturing outside of expected utility, with his 'dual theory' of choice under risk, with probability weighting: "Under the dual theory, maximization of a linear function of profits can be entertained simultaneously with risk aversion. How often has the desire to retain profit maximization led to contrived arguments about firms' risk neutrality?" But, as discussed above, a profit-maximizing firm must also be risk-neutral only if the g function is linear. In general, even a profit-maximizing firm may have a pure risk parameter $\alpha \neq 0$, in which case its optimization problem in the risky space is captured by $g(m) = \frac{1-e^{-\alpha m}}{\alpha}$.

profit-maximizing firm need not be risk-neutral. For any 'pure risk' parameter $\alpha \neq 0$, the optimal choice in this setting would respond to the magnitude of the demand shocks.[15]

Overall, this discussion makes a few points. First and foremost is the observation that a profit-maximizing firm need not be risk-neutral, even within expected utility. The second is that in this context the yardstick (dollars) is natural and easily observed. As a consequence, the pure risk parameter, α, is directly identified here from standard choice data under risk. (We will return to this point in Section 4.) Lastly, if it is true that a profit-maximizing firm need not be risk neutral, it is also true that its risk attitude can only take the CARA form. Hence, while for individuals a CARA utility function is often viewed as unrealistic (if perhaps convenient in applications), here it is the *only* appropriate functional form, under expected utility.[16]

3.2 Risk Preferences over Time Lotteries

Another setting where the results in Section 2 have immediate implications is given by intertemporal choice problems. For instance, let the multiple commodities in Section 2.2.4 consist of consumption levels in different periods. That is, the commodity space is $X = \times_{t=1,...}\mathbb{R}_+$, with typical element $x = (c_1, c_2, ...)$, where c_t denotes consumption in period t. In this setting, it is standard to assume the following specification for preferences under certainty: for any $x = (c_1, c_2, ...)$, $f(x) = \sum_{t=1,2,...} \beta^{t-1}v(c_t)$, where $v(\cdot)$ is the within-period utility function, and $\beta \in (0.1)$ the discount factor. Then, our general representation $u = g \circ f$ in this case takes the following form: for any $x = (c_1, c_2, ...) \in X$,

$$u(x) = g\left(\sum_{t=1,2,...} \beta^{t-1}v(c_t)\right), \tag{2}$$

where $g(\cdot)$ is a CARA transformation. The standard model, where risk-attitudes are entirely driven by the curvature of v, once again obtains for the special case where g is linear. However, as a number of recent influential papers have pointed out (e.g., DeJarnette et al. (2020); Dillenberger et al. (forth.); see also Strzalecki (2024), and references therein), the standard model has somewhat disappointing implications when it comes to risk-attitude over *time-lotteries* (i.e., lotteries that pay a fixed prize at a random time). Intuitively, due to the convexity of the exponential discounting with respect to t, the standard model implies that the agent must be risk-seeking over time lotteries, which clashes both with our introspection and with a substantial body of experimental evidence.

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[15]For example, with non-decreasing marginal costs, the optimal choice $q^*(\alpha, \epsilon)$ would be such that $q^*(0, \epsilon) = q^*$ under risk neutrality, but for any $\alpha > 0$ (resp., $\alpha < 0$) $q^*(\alpha, \epsilon)$ would be decreasing (resp., increasing) in ϵ, and such that $q^*(\alpha, 0) = q^*$ for any α. Moreover, for any $\epsilon > 0$, $q^*(\alpha, \epsilon)$ is strictly decreasing in α, and $lim_{\alpha \to \infty}q^*(\alpha, \epsilon) = q^-$ and $lim_{\alpha \to -\infty}q^*(\alpha, \epsilon) = q^+$, where q^- and q^+ denote the optimal choices conditional on the realization of $-\epsilon$ and $+\epsilon$, respectively.

[16]We note that since the utility function is equal to the composition of g and f, the former being CARA in general does not imply that the overall utility is CARA, but it is when f is linear, as for the case of firms.

If the 'pure risk' parameter of the $g(\cdot)$ transformation is sufficiently high, however, it is easy to see that one can retain the exponential discounting in the certainty space (for instance, to maintain both the tractability and cogency of the standard model in terms of saving decisions, consumption smoothing, etc.), while at the same time accommodating risk-aversion over time lotteries. Interestingly, the functional form in eq. (2) is exactly the same as recently put forward by Dillenberger et al. (forth.), albeit coming from a different perspective. In this sense, our results applied to this setting provide an alternative 'foundation' to the arguments in Dillenberger et al. (forth.), with one added qualification, which is that our arguments suggest that the g transformation be CARA (a restriction that need not hold in the model of Dillenberger et al. (forth.).[17]

This extra property of course imposes more structure on the preferences, for instance by entailing a specific relationship between the coefficient of risk-aversions over *money-lotteries* (i.e., lotteries with random prize paid at a fixed time), across different time periods. Specifically, letting $\alpha_v(m_t) = -\frac{v''(m_t)}{v'(m_t)}$, and letting $\alpha_t(m_t)$ denote the coefficient of absolute risk-aversion over money-lotteries that pay out at time t, evaluated at period $t = 1$, when the period-t money holding is m_t, then the following holds:

$$\alpha_t(m_t) = \alpha \cdot \beta^{t-1} v'(m_t) + \alpha_v(m_t). \tag{3}$$

Thus, an interesting implication of this model, which could be immediately obtained from this equation, is that if both g and v are concave, then the agent becomes progressively less risk-averse as the horizon of the money-lotteries is postponed (the opposite is true if v is concave and g convex).

3.3 Investement and Savings

In this section we discuss some implications of the above findings to saving and investment choices, and we show that they also play a role in another classical economics question, namely the Equity Premium Puzzle (Mehra and Prescott, 1985). For simplicity, let us consider the intertemporal setting above, but with two periods only, 'today' and 'tomorrow', with corresponding consumption levels denoted by c_t and c_{t+1}. Thus, preferences in the certainty space are such that $f(x) = v(c_t) + \beta v(c_{t+1})$, and as standard within this branch of the literature we also assume that the within-period utility function, $v : \mathbb{R}_+ \to \mathbb{R}$, is CRRA with parameter γ. Then, the overall preferences in the risky domain are

$$\mathbb{E}[u(c_t, c_{t+1})] = \mathbb{E}_t\Big[g\Big(v(c_t) + \beta v(c_{t+1})\Big)\Big] \tag{4}$$

where v is CRRA and g is CARA:

$$v(c) = \frac{c^{1-\gamma}}{1-\gamma} \text{ with } \gamma > 0; \text{ and } g(z) = \frac{1 - e^{-\alpha z}}{\alpha} \text{ if } \alpha \neq 0, \text{ and } g(z) = z \text{ if } \alpha = 0$$

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[17]See also Apesteguia et al. (2019) for a discussion of this form, and Proposition 5 in particular for a family of models separating risk and time in different ways).

The standard model once again obtains with $\alpha = 0$. There, the γ parameter therefore is the sole determinant of both the agents' savings under certainty, i.e. their 'consumption smoothing' motives, and their attitudes towards risk, and hence their investment decisions over risky asset. Standard estimations of the CRRA parameter, based on typical preferences over (within period) money lotteries, normally yield values in the range of $[1, 4]$. As is well-known, values of γ in this range entail substantially lower risk-premia for risky investments than are observed empirically. This is the celebrated Equity Premium Puzzle (Mehra and Prescott, 1985).

As we explain next, if $\alpha \neq 0$, then the expected utility model may account for larger risk-premia, even holding constant the agents' risk aversion over (within period) money lotteries. The intuition is that the g function plays no role when considering risk-free saving decisions, since it cancels out from the Euler equation associated with the optimal saving problem. The agents' consumption smoothing motive therefore is solely driven by the curvature of the v function, i.e. by the γ parameter. Preferences over risky assets, in contrast, depend on *both* the γ and α parameters. Hence, in the determination of the 'equity premium', the latter parameter only matters for the returns of the risky assets; the returns of risk-free assets are only linked to γ.[18]

We illustrate this point within a simple example, similar to the monopolist facing a stochastic demand that we discussed in Section 3.1. In particular, we consider an agent choosing the optimal level of savings/investment, first in risk-free bonds that pay a fixed return R_f, and then in a risky asset. The agent therefore in risk. Formally, letting y denote current income, the agent solves the following problem:

$$\max_{s \in (0,y)} \mathbb{E}_t \Big[g\big(v(c_t) + \beta v(c_{t+1}) \big) \Big]$$
$$\text{subj.to: } c_t^1 = y - s$$
$$c_{t+1}^1 = (1 + R)s$$

In the first setting, the agent invests in a *risk-free asset*, that pays a fixed return $R = R_f > 0$. In the second setting, the agent investment in a *risky asset*, that pays a random interest rate $R = \tilde{R}$ where

$$\widetilde{R} = \begin{cases} R^+ := \bar{R} + \delta & Pr = \frac{\varepsilon}{2} \\ \bar{R} & Pr = 1 - \varepsilon \\ R^- := \bar{R} - \delta & Pr = \frac{\varepsilon}{2} \end{cases}$$

Notice that here, for a fixed $\delta > 0$, increments in risk are parametrized by the value of

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[18]By continuity, the same logic also applies to the case where one considers the risk-premium compared to a low-risk (though not completely riskless) asset. Intuitively, the role of the g function becomes smaller, and eventually vanishes, as we approach the risk-free benchmark we discussed above.

ε. In particular, increments in ε keep the mean of $\widetilde{R}$ unchanged, $\mathbb{E}[\widetilde{R}] = \bar{R}$, while taking probability mass from the center towards the tails.

Letting $c_f = (c_t^f, c_{t+1}^f)$ and $c_r = (c_t^r, c_{t+1}^r)$ denote, respectively, the optimal consumption levels in the risk-free and risk-asset problems, we let $R^*(\alpha, \gamma, \varepsilon, \delta)$ denote the value of the mean return of the risky asset, $\bar{R}$, given which $\mathbb{E}[u(c^f)] = \mathbb{E}[u(c^r)]$. Within this simple example, one could therefore define the 'equity premium' as follows:

$$EP(\alpha, \gamma, \varepsilon, \delta) := R^*(\alpha, \gamma, \varepsilon, \delta) - R_f. \tag{5}$$

Applying the formula for the AP-index of risk aversion in (6) for the current consumption, c_t, evaluated at the optimal level in the risk-free setting, which we normalize to one (i.e., $c_t^f \equiv 1$), we obtain that the overall coefficient of risk aversion in this case is $AP_t(c_t^f) = \alpha + \gamma$. Thus, to vary the individual curvatures of g and v functions, while keeping the overall risk-attitude over (within period) money lotteries constant and within the typical range obtained from standard estimations, it suffices to impose that the sum of the two parameters is constant and equal to k, for some $k \in [1, 4]$: namely, $\alpha + \gamma = k$. For a fixed value of k (and holding ε and δ constant), the function $\Lambda(\alpha) := R^*(\alpha, \alpha - k, \varepsilon, \delta)$ therefore describes how the equity premium is affected by shifting some of the curvature from the g to the v function, keeping constant the overall risk-attitude over money lotteries. The following result summarizes a few key implications of our model:

Proposition 3. *For any k and $\alpha \in [0.k]$, $\frac{\partial \Lambda}{\partial \alpha}(\alpha)$ is continuously differentiable, such that $\frac{\partial \Lambda}{\partial \alpha}(\alpha) = \frac{\partial R^*}{\partial \alpha}(\alpha) - \frac{\partial R^*}{\partial \gamma}(\alpha)$, strictly positive at $\alpha = 0$, and strictly negative at $\alpha = k$.*

To understand the significance of this result, recall that the standard model corresponds to the case where $\alpha = 0$ and $\gamma = k$ (with $k \in [1, 4]$, according to the standard estimates). Hence, the result that $\frac{\partial \Lambda}{\partial \alpha}(0) > 0$ means that, holding constant the risk-attitude over money lotteries, moving from the standard model to one where $\alpha > 0$ increases the equity premium entailed by the the EU model: shifting some of the curvature from the v to the g function, holding risk-attitude constant, increases the model's ability to account for a higher equity premium. This effect, however, is not monotonic. At the opposite extreme of the relevant range, the opposite is true: if all the curvature in the model is due to the g function (if $\alpha = k$, then $\gamma = 0$ and hence v is linear), then the equity premium entailed by the model increases if some curvature is moved from g to v. The reason is that a parameter of $\gamma = 0$ shuts down the consumption smoothing motive.

The overall behavior of the Λ function therefore is driven by a tension between 'consumption smoothing' and the attitude towards 'pure risk', or in other words between precautionary savings and hedging in the face of risk. Separately accounting for the α and γ parameters may therefore shed new light on these classical concepts. Needless to say, a proper quantitative assessment of the extent to which these observations may provide an expected-utility explanation to the equity premium puzzle requires a careful empirical exercise, which is beyond the scope of this section. Nonetheless, we stress that this analysis

is fully within the vNM framework, and hence the insights that we just discussed do not rely on any deviation from the classical expected utility theory.

3.4 Discussion

The applications discussed in this section are only some of the theoretical implications of our main results, but there are several others. For instance, in relation to our earlier work on EU models of reference-dependent preferences (Alaoui and Penta, forth.), the results presented here imply that the reversals of risk attitude that occur in each of the representations of that paper (which include, among others, the standard S-shape utility function with *loss aversion* and *diminishing sensitivity*, as well as Genicot and Ray (2017)'s *aspiration model* and Diecidue and Van De Ven (2008)'s utility with a *discontinuity*), must all be coming from the f function, or in other words they must all be due to reference-dependence effects in the certainty domain.[19] That is, unless one posits that 'pure risk' preferences may feature reference dependence with respect to the yardstick itself, in which case one may want to consider a weakening of yardstick neutrality.

Also note that the separation between f and g need not apply to EU only. Aside from its historical role and the central position it still occupies within economics, we focus on EU because the conceptual points we raised are *most transparent* within it. That is because there is 'only u' within EU (as opposed to other components of the representations in other theories; see, e.g., footnote 3). But extending our approach beyond EU, for instance in combination with other known mechanisms that may contribute to addressing the equity premium puzzle, is clearly a promising direction for future research. Furthermore, as discussed in Alaoui, Penta, and Troccoli-Moretti (2025), the separation we identify between f and g may also suggest novel ways of departing from expected utility, to shed new light on classical 'paradoxes' (e.g., Rabin, 2000) as well as to explain new ones (cf. Alaoui, Penta, and Troccoli-Moretti, 2025).

These are only some examples of further theoretical implications of our results, and there are many other directions to explore. These developments are clearly beyond the scope of this paper, but we think they are a promising direction for future research.

4 Identification

Our theoretical exercise thus far have relied on the use of a *conceptual yardstick*, not necessarily an actual one. This thought experiment serves to make the point that the 'two horses of different colors' live together within expected utility. This theoretical insight is valid independent of matters of identification. But, from the viewpoint of identification, that thought experiment also serves as a benchmark to clarify what kind of data would be

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[19]For an empirical analysis of such expected utility models of reference dependence, see Alaoui, Hervy, Kariv, and Penta (2025), where we perform individual estimation of such preferences in the context of portfolio choices of Arrow securities. Our results highlight a great deal of individual heterogeneity and show a striking predictive power of these models.

best suited to separately identify the value of wealth (function f) from pure risk aversion (parameter α): if we had the yardstick readily available and agreed upon, as is the case with actual physical yardsticks (or meter sticks) to measure distance, then it would be straightforward to identify the parameters. But in practice, it is perhaps unclear at this stage which commodity can attain the benchmark conditions for a yardstick, or even how it can be verified that such a commodity fulfills the desired properties. In this section we will discuss how this can be done, and how f and α can be identified from choice data.

4.1 Identification for firms

Consider the case of the profit-maximizing firm discussed in Section 3.1, which is both important in itself, and useful to introduce a few conceptual points. There, the yardstick y is the home currency (e.g., dollars), and we let m denote the quantity of money held in a possibly different currency (e.g., euros). Then, letting d denote the exchange rate, in the certain space the firm maximizes $dm + y$. Under risk, as discussed, it maximizes either $u(m, y) = \frac{1 - e^{-\alpha(dm+y)}}{\alpha}$ with $\alpha \neq 0$, or $dm + y$ (in which case we will say that $\alpha = 0$ ($d = 1$ if m is money held in home currency).

If d is observable, then α can be identified in standard way, from choice data over lotteries over m. But note that the firm's overall risk attitude over m (the foreign currency) now depends on both d and α. The overall Arrow-Pratt coefficient for m, for instance, will be equal to $\alpha_u = \alpha d$.[20] This is intuitive: for instance, consider a gamble between 10 and 0 units of the foreign currency, m. In terms of the firm's own currency, this is effectively a gamble between \$10d and \$0. So, even though the function f here is linear in m, the pure risk coefficient is unique only up to the multiplicative constant $1/d$. Clearly, if d is observable, this does not create any practical issue. Otherwise, only the product αd can be jointly identified from lotteries over m; not α and d separately.

In the case of the firm, the availability of an obvious yardstick also allows a clean identification for more complicated cases, too, where the firm may also have other other objectives besides profits, such as environmental concerns, Corporate Social Responsibility, etc. Letting x denote the variable the firm is concerned with, then the f function could be identified from the MRS between x and money (in own currency), under certainty. In this case, f would not necessarily be linear, but the α parameter could still be identified from lotteries over own currency, or from lotteries over x, adapting the formula in Corollary 1 to x. Namely, $\alpha_u(x) = \alpha f'(x) + \alpha_f(x)$.

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[20]The Arrow-Pratt indices are obviously invariant to positive affine transformations of the utility function, but they are not necessarily invariant to rescaling the units of account. For instance, let $\hat{m} = d \cdot m$ for some $d > 0$. Then, if u is of the CRRA family, then it holds that $\alpha_u(\hat{m}) = \alpha_u(m)$. But if u is of the CARA family, then $\alpha_u(\hat{m}) = d \cdot \alpha_u(m)$.

4.2   Identification via Proxy

Turning to more general decision-makers (i.e. not necessarily profit maximizers), our first method consists of finding (or constructing) a reliable yardstick, or a proxy for it, to achieve identification to the extent possible. For this exercise, we return to the multiple goods case introduced in Section 2.2.4. In particular, let $x = (x_1, x_2)$, and assume that f is (strictly) increasing in each dimension. Suppose that we are primarily interested in good 1 (e.g., x_1 represents quantity of money, as in most of the paper), while good 2 is the 'candidate proxy' for the yardstick. Under the maintained assumptions on the yardstick, it must be that $u(x, y) = g(f(x) + y)$, where $g : \mathbb{R} \to \mathbb{R}$ is a CARA transformation with parameter $\alpha \in \mathbb{R}$, as above. Now, for good 2 to be an appropriate substitute for the yardstick, it would need to satisfy a quasilinearity property with respect to good 1, but also with respect to the yardstick itself:

Full Quasilinearity (FQ): Good 2 satisfies *full quasilinearity* if, for any x_1, x_2, x_1', $x_2', y, y', t \in \mathbb{R}$, $\delta_{x_1, x_2, y} \succsim \delta_{x_1', x_2', y'}$ if and only if $\delta_{x_1, x_2 + t, y} \succsim \delta_{x_1', x_2' + t, y'}$.

If this property holds, under the maintained assumptions on the yardstick, then the $f(\cdot)$ function above must be such that $f(x) = f_1(x_1) + d_2 x_2$ for some function f_1 and constant $d_2 > 0$. Thus, the overall utility function for (x, y) must take the form $u(x, y) = g(f_1(x_1) + d_2 x_2 + y)$. Hence, if one could find a commodity that satisfies FQ, then it would serve as a suitable proxy for the yardstick, and be effectively equivalent to it, up to a multiplicative constant $d_2 > 0$ (just like the exchange rate in the firm example above).

The issue with FQ, however, is that it cannot be directly tested with data that only involve commodities 1 and 2. But the following two implications of FQ are testable with data that only involve commodities 1 and 2:

Proxy Neutrality (PN): Good 2 satisfies satisfies *proxy neutrality* if, for any $x_2, x_2', y \in \mathbb{R}$, and for any $p^{x_1}, q^{x_1} \in \Delta(\mathbb{R})$, $(p^{x_1}, x_2, y) \succsim (q^{x_1}, x_2, y)$ if and only if $(p^{x_1}, x_2', y) \succsim (q^{x_1}, x_2', y)$.

Proxy Quasilinearity (PQ): Good 2 satisfies satisfies *partial (proxy) quasilinearity* if, $\delta_{x_1, x_2, y} \succsim \delta_{x_1', x_2', y}$ iff $\delta_{x_1, x_2 + t, y} \succsim \delta_{x_1', x_2' + t, y}$ for all $x_1, x_2, x_1', x_2', y, t \in \mathbb{R}$.

That PN is implied by FQ follows from the representation of the u function that holds under FQ (namely, $u(x, y) = g(f_1(x_1) + d_2 x_2 + y)$), since g being CARA implies that the risk attitudes on the x_1 component is not affected by the level of x_2 in this representation. Interestingly, while FQ only refers to the certainty space, PN is about preferences over lotteries over good 1. This connection from the certain to the uncertain space occurs thanks to the yardstick, which serves as a bridge between the two (FQ connects certain preferences over x to y, which in turn is connected to preferences over the uncertain space via yardstick neutrality).

PQ instead is clearly a weakening of FQ, as it characterizes the strongest implications of FQ that are testable in the certain space with data over x_1 and x_2 alone. PQ has a straightforward interpretation: under PQ, good 2 is only quasilinear with respect to good 1, not necessarily with respect to the yardstick, y.

Note that since both PN and PQ are implied by FQ, then both conditions also provide a way of falsifying FQ indirectly, from choice over lotteries over x_1 and x_2. A commodity that satisfies both PN and PQ therefore is *viable*, as choice data do not rule out that it might be a suitable proxy, in the sense that they do not falsify FQ.

Now, suppose that there are multiple goods beyond x_1, which could take the role of x_2 in the discussion above. Say that there is a set $\mathcal{K} = \{1, ..., K\}$ of candidate proxies, over which preferences are increasing, each of which is a *viable proxy* in the sense that it satisfies both PN and PQ. If commodity k also satisfies FQ, then we say that it is an *suitable proxy*.[21] As we show next, under the maintained assumptions on the yardstick, the following holds: if FQ holds for some $k^* \in \mathcal{K}$, then *any* candidate proxy $k \in \mathcal{K}$ satisfies FQ if and only if it satisfies PN and PQ. Putting everything together, we thus obtain the following result:

Proposition 4 (Detecting a Proxy). *Under the maintained assumptions:*
  1. If a suitable proxy exists, any viable proxy is suitable.
  2. If no proxy is viable, then no suitable proxy exists.

Note that this means that, if a suitable proxy exists, then it can be detected by testing FQ from choice data over x, via PN and PQ. From now on, we will assume that an ideal proxy exists. Then, from the discussion above, if good 2 satisfies both PN and PQ, then the representation of $u(x, y)$ involves three distinct objects: the function $f_1 : \mathbb{R} \to \mathbb{R}$, the scalar $d_2 > 0$, and CARA parameter $\alpha \in \mathbb{R}$ of the g function. Note, however, that such a representation involves y as well, and hence it contains more information than what could be gathered based on data on (x_1, x_2) alone. Hence, we introduce the following definition:

Definition 2. *Let good 2 be a* suitable proxy. *Then we say that (f_1, d_2, α) observationally represents $\succsim$ if, for all $p^x, q^x \in \Delta(\mathbb{R}^2)$ and $y \in \mathbb{R}$, $(p^x, y) \succsim (q^x, y)$ iff $\mathbb{E}_{p^x} u(x, y) \geq \mathbb{E}_{q^x} u(x, y)$, where the utility function u takes the form $u(x, y) = g(f_1(x) + d_2 x_2 + y)$ with CARA function $g(\cdot)$ having parameter α.*

In other words, this notion of *observational representation* allows only for inference on preferences from the available data on x_1 and x_2, without allowing the unobserved y to be varied. As we show next, under the maintained assumptions, the three parameters of the utility representation are identifiable from standard choice data up to a single multiplicative constant (plus an additive one for f_1), in the following sense:

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[21]To be clear, letting x_2^k denote the quantity of good $k \in \mathcal{K}$, we say that commodity k satisfies FQ (or PQ or PN) if the definition of FQ (resp., PQ or PN) given above holds using quantity x_2^k in the role of x_2.

Proposition 5 (Identification via Proxy). *Under the maintained assumptions, if commodity 2 is a* viable proxy, *and (f_1, d_2, α) and (f_1', d_2', α') both observationally represent $\succsim$ then there exists an $a > 0$ and $b \in \mathbb{R}$ s.t. $f_1' = af_1 + b$, $d_2' = ad_2$ and $\alpha' = \alpha/a$.*

In other words, we can identify the curvature of f_1 and the sign of α. Such an exercise is particularly useful for inferences on behavior and risk-attitude across domains. We will return to this point in Section 5 below.

In the absence of a 'natural' commodity that can serve as a viable proxy, one could think of constructing an artificial commodity that would plausibly satisfy properties PN and PQ. Consider the following example:[22]

Example 1 (An Artificial Proxy). Take a unit mass of anonymous third parties, whose marginal utilities are constant over some good, let's call it *Token*. Then, $x_2 \in \mathbb{R}_+$ could denote the fraction of such individuals that receive a fixed amount of tokens (say, one).[23] Under the assumption that the decision maker's preferences are increasing in such a number, and that under anonymity PQ holds, then PN amounts to assuming that the value of x_2 is orthogonal to the decision maker's preferences over lotteries of x_1. □

Proposition 5 also implies that, in order to *uniquely* identify α, we would require at least one data point on y. This is for similar reasons to those discussed in Section 4.1, about firms: in this case, without such a data point, we have no way of distinguishing between (f_1, d_2, α) and $(af_1 + b, ad_2, \alpha/a)$ (for $a > 0$) because we have no way of knowing what the correct exchange rate between the proxy and the yardstick is. However, if we could have, say one indifference condition involving y, then we could establish the proper exchange rate (for example, if $(x_1, x_2, 0) \sim^c (x_1, 0, y)$ for some x_1, x_2 and y, then $d_2 = y/x_2$), and hence pin down $(f_1 + b, d_2, \alpha)$ uniquely (aside for the additive constant b), which can of course be normalized to 0).

Alternatively, and perhaps more relevantly for empirical work, suppose that we do not have such a data point, but instead are willing to say that there is a viable proxy that is common to different individuals, and that its relationship with the yardstick is identical across individuals (i.e., we are willing to assume a common normalization). Then, one would still be able to perform interpersonal comparisons of both the α parameters and of the f_1 functions: for any choice of d_2, the α parameter and f_1 function would be uniquely identified, and hence their ordering across agents would be unaffected by the particular choice of d_2, as long as it is common to all agents.

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[22]We thank Elias Tsakas for inspiring us this example.

[23]Note that no randomization is involved in this setting. In particular, for $x_2 \in (0, 1)$, it is not the case that a given agent receives one token with probability x_2, and nothing otherwise. Rather, a fraction x_2 of the anonymous agents receive one token for sure. For $x_2 > 1$, the interpretation is that a fraction $(x_2 - \lfloor x_2 \rfloor)$ of agents get $\lfloor x_2 \rfloor + 1$ tokens, and the rest get $\lfloor x_2 \rfloor$. But this is only done to accommodate the technical requirement that x_2 lives in an unbounded space. For practical elicitation, the x_2 in this example could be taken from the unit interval, and the token could be a fixed amount of dollars (if $x_2 \in [0, 1]$, the requirement that the third parties have a constant marginal utility over the amount of the token is not needed, as PQ would be ensured by the anonymity of the receiving third parties).

If, furthermore, one were willing to take a stance on the 'exchange rate' between the proxy and the yardstick, what we would call a *pegged proxy* (formally, a viable proxy in which the d_2 parameter is known, or normalized to 1 if the proxy itself is taken as unit of account), then Proposition 5 implies that both the f and α would be uniquely identified:

Corollary 2 (Identification via Peg)**.** *Under the maintained assumptions, if commodity 2 is a* pegged proxy*, then f_1 and α are uniquely identifiable.*

Lastly, we use this discussion to illustrate an additional point. Suppose that one were concerned whether, depending on which proxy we use, the function f could be concave with respect to one (say), but convex with respect to another, so that all the conclusions drawn would rely on the proxy used. The discussion here illustrates that this cannot be the case: if both proxies satisfy PN and PQ, they they can only be positive affine transformations of one another, and hence the function can only differ by a positive affine transformation. The shape, then, will be the same irrespective of the proxy used.

4.3 Parametric assumptions

Turning to parametric assumptions on f, suppose that we assume f to be of the CRRA functional form, i.e. such that that $f(m) = \frac{m^{1-\gamma}}{1-\gamma}$. Then, using Corollary 1, we have:

$$\alpha_u(m) = \alpha m^{-\gamma} + \gamma m^{-1}.$$

Then we could identify the α and γ parameters with the following two-step procedure:

1. In a first stage, estimate $\alpha_u(m)$ for a set of m's, say $M = \{m_1, ..., m_K\} \subset \mathbb{R}$. Since $\alpha_u(m)$ is the local risk attitude parameter over money, at any given m, it can be estimated using standard methods (e.g., Baillon and L'Haridon, 2019).

2. Then, using the collection of $\{(m, \alpha_u(m))_{m \in M}\}$ as the independent and dependent variables, respectively, a maximum likelihood estimation (MLE) procedure can be used to estimate α and γ.

Alternatively, if f is CRRA, then Proposition 1 ensures that u is as follows:

$$u(m; \alpha, \gamma) = \begin{cases} \frac{m^{1-\gamma}}{1-\gamma} & \text{if } \alpha = 0 \\ \frac{1-e^{-\alpha\left(\frac{m^{1-\gamma}}{1-\gamma}\right)}}{\alpha} & \text{if } \alpha \neq 0 \end{cases}.$$

Hence, instead of the above two-step procedure, the two parameters α and γ could also be estimated directly, using standard MLE methods. Note that the α parameter here is pinned down due to a parametric normalization of the CRRA function f. The γ parameter, however, is pinned down uniquely, irrespective of this normalization.

Another important parametric setting, particularly for macroeconomics applications, is the one equation (2), with a CRRA v function with parameter γ. In such settings,

γ and β can be identified or calibrated using savings decisions under certainty, adapting commonly used methods. In addition to that (or in conjunction, using an MLE procedure to estimate all parameters simultaneously), choice data over time lotteries can be employed to estimate the α parameter. Intuitively, let c^* be steady state consumption, and take $c^+ \geq c^*$ that can be either obtained in period t for sure, or in period $t-1$ with probability p or $t+1$ with probability $1-p$, and suppose that one identifies the *'time lottery equivalent'*, p_{TL}, that makes the agent indifferent between the two options. Then, given β and γ, the specific α that leads to this indifference can be identified.[24]

5 Cross-domain inference and elicitation

We now return to the case of multiple commodities (domains) $x = (x_1, x_2, .., x_n)$ introduced in Section 2.2. We discuss first how predictions of risk attitude from one domain to another could be conducted. Next we discuss how multiple domains can be used to elicit α, which in turn can also be used to make predictions to additional domains.

5.1 Cross-domain predictions

Suppose now that we are interested, as is common in economics and psychology, in using risk attitudes elicited in one domain to make predictions of risk attitudes in another.

It is common to simply take the risk attitude in one domain and analyze the correlation and predictions with the risk attitude on other domains (see e.g. Frey et al. (2017); Mata et al. (2018); Einav et al. (2012) and Vieider et al., 2015). But note that our results suggest caution in interpreting the meaning of these correlations. To show this formally, suppose that preferences under certainty are linearly separable between different domains, and maintain the same assumptions over the yardstick (quasilinearity and neutrality) as we have done throughout. Then, in the certainty space, preferences are represented by

$$f(x) = \sum_{k=1}^{n} f_k(x_k),$$

for some collection $(f_k)_{k=1,...,n}$ of functions $f_k : \mathbb{R} \to \mathbb{R}$. Hence, for the overall preferences over risk, we have that $u(x) = g(\sum_k f_k(x))$, where g is CARA with parameter α. It follows that the Arrow-Pratt index for good k, at consumption level x_k, is

$$\alpha_{u,k}(x_k) = \alpha f'_k + \alpha_{f_k}(x_k), \tag{6}$$

where $\alpha_{f_i}(x_k) = -\frac{f''_k(x_k)}{f'_k(x_k)}$. In words, for any k, the (overall) risk attitude over that good depends both on the curvature of f_k and on the common parameter α. It follows that, with no further information, eliciting an agent's risk attitude over a good, say k, cannot be

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[24]That is because $e^{-\alpha\beta^t v(c^*+c^+;\gamma)} = p_{TL} \cdot e^{-\alpha\beta^{t-1} v(c^*+c^+;\gamma)} + (1-p_{TL}) \cdot e^{-\alpha\beta^{t+1} v(c^*+c^+;\gamma)}$.

used to make predictions about the agent's preferences over lotteries of some other good, say l, unless f_k and f_l happen to be identical.

This is in fact consistent with well-known findings in the psychology literature, which states that choice-based measures of risk attitudes commonly used in economics, which typically pertain to the money domain, are often not predictive of risk attitudes in other domains (see, e.g., Frey et al., 2017, Mata et al., 2018). In that literature, it is common to argue that this is a weakness of the economics approach. But, as we see here, it is rather to be expected, given the way that such analyses are often conducted.

Example 2 (A Formula 1 driver)**.** An F.1 driver has preferences over two domains: a financial one, and a racing one. For simplicity, say that x_1 represents quantity of money, and x_2 is a measure of his success as a racer (e.g., points in Championship, holding everything else constant). This particular driver is sufficiently wealthy that the marginal utility of money is essentially constant, and small. Nonetheless, his financial behavior suggests that $\alpha_{u,1} > 0$ (e.g., he buys insurance, diversifies his investments and portfolio holdings, etc.). Yet, on the race-track, he seems willing to take a great deal of risk, which would suggest suggest that perhaps $\alpha_{u,2} < 0$.

Now, suppose we ask ourselves: Is this F.1 driver risk-averse or not? One could give different answers, and it is apparent that they would depend on which domain appears most salient in our mind. But surely enough, taking this driver to the laboratory to estimate his coefficient of risk aversion, $\alpha_{u,1} > 0$, would not be of much use to predict the amount of risk he would be willing to take on the racing track, or even to predict one's willingness to race in F.1 to begin with.

Our approach, however, gives a more nuanced view: If this F.1 driver is wealthy enough that we are willing to assume his utility for money, f_1, is essentially linear, we can conclude that the evidence from his investment behavior, which suggests an overall risk-aversion over money, must come from his 'pure risk' parameter $\alpha > 0$. With this, the fact that he is willing to take so much risk while racing suggests that his preferences over the 'racing greatness' domain must be such that f_2 is convex. Then, his risk-seeking behavior over the racing domain, in this case, must come from an *increasing* marginal utility of 'racing greatness', which in fact must be strong enough as to offset the fact that, when looking at 'pure risk' per se (i.e., the $\alpha > 0$ parameter), this F.1 driver is actually risk-averse.

This suggests that, in order to make predictions *across* domains, one should at least be able to elicit (or be willing to assume) enough about α, f_1, and f_2. Mere risk attitudes over one dimension have no direct bearing on risk-attitudes on other dimensions. □

To make predictions *across* domains, we thus propose the following method instead. Suppose that we have conducted one of the identification exercises discussed in the subsections above. Let us say that we have used Identification by Proxy method (Section 4.2), where good n is the proxy. Suppose that we have used it and data on good 1 only. This has allowed us to elicit (f_1, d_n, α), up to transformation $(af_1 + b, ad_n, \alpha/a)$ for $a > 0$.

Now suppose that we are willing to make some assumption on the shape of f_2 (e.g., that it is linear), or that alternatively we have used the proxy n and data on the certain space 2 to also identify f_2 (up to transformation af_2, where a is the same as above). We now have estimates for (f_1, f_2, d_n, α), which is all that is required to make predictions on the risk attitude of good 2, as we can now use our estimates to obtain $\alpha_{u,2} = \alpha f_2' + \alpha_{f_2}(x_2)$.

This exercise also shows that the estimates being up to constant a in no way interferes with the predictions that we wish to make. This is because $\alpha_{u,2} = \alpha f_2' + \alpha_{f_2} = (\alpha/a)(af_2') + \alpha_{af_2}$, using that $\alpha_{af_2} = \alpha_{f_2}$. Intuitively, since the multiplicative constant that defines the non-uniqueness of the identification of the α parameter is the same across dimensions, it crosses out when going from one dimension to another. The estimate of α obtained from one domain is therefore portable to another, once combined with an estimate of the preferences under certainty in the latter.

5.2 Cross-domain elicitation

Within the psychology literature, the typical methodology to elicit risk attitudes is based on stated preferences, and frequently involves asking a number of questions about different domains (see, e.g., Frey et al., 2017). It is also argued in that literature that the measures of risk attitude elicited in this way are often more stable and more predictive than the ones elicited from standard economics methods, despite the latter being choice-based.

Putting aside for a moment the well-known issues of stated preferences, which we will return to at the end of this section, here we discuss how cross-domain elicitation may effectively help in identifying the 'pure risk' parameter α, and the sense in which this may lead to greater stability and predictability.

Specifically, maintain that there are n goods, and now suppose that there is a population of I individuals, and let $x^i = (x_1^i.,,,.,x_n^i)$ denote the bundle of agent i, with x_k^i being i's quantity of good k. Suppose again that i's preferences in the certainty space linearly separable, i.e. $f^i(x) = \sum_{k=1}^n f_k^i(x_k^i)$, for some collection $(f_k^i)_{k=1,...,n}$ of functions $f_k^i : \mathbb{R} \to \mathbb{R}$. Although not necessary for the argument that follows, let us assume for simplicity that such preferences are linear in each good, i.e. that $f_k^i(x_k^i) = d_k^i \cdot x_k^i$ for all $i = 1,...,I$ and $k = 1,...,n$. We further assume that, for each k, the preference parameters in this population are drawn from some distribution with some (unknown) mean $d_k > 0$, with a noise that is independent and identically distributed across individuals and commodities. More precisely, for each i and k, we have $d_k^i = d_k + \epsilon_k^i$, where ϵ_k^i is i's idiosyncratic preference parameter for good k, drawn from a distribution with 0 mean that is i.i.d. across goods and agents. Under these assumptions, i's utility function is

$$u^i(x^i) = g^i \left(\sum_{k=1}^n (d_k + \epsilon_k^i) x_k^i \right),$$

where g^i is CARA with 'pure risk' parameter α^i.

Next we show that, by eliciting individuals' risk-attitudes over each dimension in this setting, we can obtain a ranking of the individuals' pure risk attitudes. To see this, note that applying the formula for the Arrow-Pratt (AP) index with multidimensional goods (cf. eq. 6) to this setting, for each i and good k for each component, we have:

$$\alpha_{u^i,k}(x_k^i) = \alpha^i(d_k + \epsilon_k^i).$$

Hence, if the number n of dimensions is large enough, for each individual i, the expectation of the average AP-index across dimensions is the following:

$$\bar{\alpha}_{u^i} := \frac{\sum_{k=1}^n \alpha_{u^i,k}(x_k)}{n} = \frac{\sum_{k=1}^n \alpha^i d_k}{n} + \frac{\sum_{k=1}^n \epsilon_k^i}{n} \simeq \frac{\alpha^i \sum_{k=1}^n d_k}{n}, \tag{7}$$

Since the term in the summation is the same for all agents, it follows that if the number of elicited dimensions is large enough, then the average AP-indices, $(\bar{\alpha}_{u^i})_{i=1,\ldots,I}$, are fully ranked by the individuals' pure risk parameters, $(\alpha_i)_{i=1,\ldots,n}$. Notice that this argument does not require knowing what the $(d_k)_{k=1,\ldots,n}$'s are. The main identification assumption is that the ϵ_k^i's are i.i.d. across goods and agents.

Of course, this exercise does not map exactly to how risk indices are measured in the psychology literature, which uses a large number of questions on stated preferences in several domains, and creates an index based on the average of the responses. Rather, it serves to show why eliciting risk-attitudes over multiple dimensions can lead to approximate orderings of the pure risk parameter: Since, unlike the k-specific f_k^i-terms, such pure risk parameters affect the risk attitudes across components, identifying a reliable ranking of the α^i across different individuals may yield better predictive power across domains. Moreover, this ranking could be correlated with risk-taking behavior in domains for which data has not been collected.

As discussed above, it is common in the psychology literature to ask subjects to state their willingness to take risks in various domains, rather than to use choice-based measures used in economics. While we believe that such methods are within the general spirit of the choice-based measures we assumed in the above analysis, they would not be as precise as carrying out the exercise discussed here. In fact, the exercise above can be seen as a *hybrid* of the two methodologies, which maintains the multidimensional logic from psychology and combines it with the choice-based economics methodology of eliciting Arrow-Pratt indices from choice over lotteries over each dimension separately.

We take the tradeoff between the single-domain choice-based measures and multiple-domain stated preferences to be as follows: commonly used choice-based measures are useful because they are precise and based on behavior, but it is difficult to make predictions from one domain to another for the reasons stated here and in the previous Section. It is also impractical to conduct them fully across domain, simply because it may be difficult to implement choice task over some of these relevant domains. As for multiple-domain stated preferences, these are useful for the reasons discussed in this subsection, but suffer

from not being based on behavior. Hence, they may not represent true preferences, and it may often be unclear to the agent what the question is precisely asking for. Overall, to the extent that it can be carried out, it seems to us that a choice-based measure over multiple domains, or even a combination of the two approaches, may well be the most promising avenue, if done in a manner consistent with the logic we discussed in this section.

In closing this section, we note that Qualitative Self-Assessments (QSAs) – which involve a single, simple question on stated-preferences – have been increasingly used in recent years (e.g., Dohmen et al., 2011). When it comes to risk attitudes, the discussion above suggest caution in their use, because it is unclear whether they are eliciting the pure risk attitude α, or the composition with the f_k^i functions in any one domain, or some average composition across domains – we simply do not know. In this sense, this observation on the meaning of QSA methods is complementary to the analysis contained in Chapman, Ortoleva, Snowberg, Yariv, and Camerer (2025).

6   Conclusion

The main idea in this paper has been to separate the vNM's utility function, u, into two components, one corresponding to preferences under certainty and the idea of a 'marginal utility' of money, and the other corresponding to 'pure risk' attitude (cf, Section 2). This has allowed us to derive several implications for economic theory, and to show that beyond interpretation, not accounting for these components of the u function has had important implications in the development of economic thought (cf. Section 3). For instance, the natural application of expected utility theory to profit-maximizing firms has typically assumed that firms must be risk-neutral as a consequence. But this conclusion follows only if one abandons the typical 'agnostic' position of taking EU to be only a representation (which captures a common decision theoretic view), and hence by implicitly making stronger assumption on preferences. We have shown, in contrast, that profit-maximizing firms need not be risk neutral, even within expected utility. Likewise, the natural application of EU to temporal settings is not to take the expected discounted utility (EDU) form, but rather the form discussed in Section 3.2. Interestingly, this is precisely the form that the recent literature has examined to resolve behavioral puzzles that the EDU form cannot accommodate (cf. DeJarnette et al., 2020; Dillenberger et al., forth.). As discussed in Section 3.3, the same formulation may also provide novel insights on the celebrated equity premium puzzle (Mehra and Prescott, 1985). These are only some of the theoretical implications of our main results, but exploring further theoretical implications of our main results, both within EU and outside of it, is likely a promising direction for future research (cf. Section 3.4).

Our analysis has remained close to the classical economics approach, in the sense that all the identification methods discussed in Sections 4 and 5 involve ways of identifying some commodity that could serve as a *proxy* for the yardstick, and to verify that this candidate

proxy is appropriate based only on 'standard', fully *choice-based* datasets. But given the extensive research in economics and neuroeconomics that focuses on other, more direct ways to measure subjective value and some forms of satisfaction (see for instance Glimcher and Rustichini (2004); Camerer (2008); Rustichini (2009), and Glimcher and Tymula, 2023), we believe that a promising avenue for research could employ these techniques as well. For instance, suppose that we are willing to use auxiliary data on physical expressions of utility (such as firing rates, dopamine levels, etc.) and that we accept one such (non-choice based) measure as a common 'unit of account' for the utility index. Then, such a measure could directly serve as a yardstick, and the f function be identified directly as in our thought experiment from Section 2, from which the g can be identified using standard choice data over lotteries.

Clearly, identifying which physical expression of utility is best suited to play the role of a yardstick is inherently a neuroeconomics question, and, as such, it is obviously beyond the scope of this paper. Nonetheless, the key properties for our conceptual yardstick still serve as theoretical guidelines for the properties that a physical measurement should have, in order to serve as a useful unit of measure of utility. Once such a 'physical yardstick' is identified, our exercise can easily be enriched to accommodate these domains, as can the practical identification methods discussed in this paper. This, we think, is a promising direction to further develop, through the lens of our approach.

Appendix

Proof of Proposition 1

Under the vNM axioms and monotonocity, preferences $\succsim$ have an EU representation, where the utility function $u(m, y)$ is strictly increasing in both m and y. By the property of yardstick under certainty, $\delta_{m,y} \succsim \delta_{m,y}$ iff $f(m) + y \geq f(m') + y'$, which holds iff $u(m, y) \geq u(m', y') \Leftrightarrow g(f(m) + y) \geq g(f(m') + y')$ for strictly increasing $g : \mathbb{R} \to \mathbb{R}$ (note that if g is not strictly increasing, then there exists an (m, y) and (m', y') for which $u(m, y) > u(m', y')$ but $g(f(m)+y) \leq g(f(m')+y')$). Since u is unique up to positive affine transformation, it must then be that $u(m, y) = g(f(m) + y)$, or a strictly positive affine transformation of g. Next, by yardstick neutrality, for any p^m, q^m, y, y': $\sum p^m(m) g(f(m) + y) \geq \sum q^m(m) g(f(m) + y)$ iff $\sum p^m(m) g(f(m) + y') \geq \sum q^m(m) g(f(m) + y')$, meaning that function g must be CARA in y. Using the standard results on CARA, g must take the form $g(x) = \frac{1 - e^{-\alpha x}}{\alpha}$ for $\alpha \neq 0$, or $g(x) = x$, or a positive affine transformation thereof.

Proof of Proposition 2

Let $u^* : \mathbb{R} \to \mathbb{R}$ denote a utility function in a EU representation of preference system $(\succsim^*, \Delta(\mathbb{R}))$. Now, construct the preference system $(\succsim, \Delta(Z))$ that is represented by $u(m, y) = g(f(m) + y)$ where $g(x) = x$ and $f(m) = u(m)$. Clearly, this utility function satisfies all the maintained axioms, and $u(m, 0) = g(f(m)) = u^*$.

In the other direction, take any preference system $(\succsim, \Delta(Z))$ that satisfies the maintained assumptions, and let $\succsim^m$ be defined as in the text, i.e., $p^m \succsim^m q^m$ iff $(p^m, 0) \succsim (q^m, 0)$. Letting $u(m, y) = g(f(m) + y)$ represent $\succsim$, clearly $u^*(m) = u(m, 0)$ is strictly increasing and represents preferences $\succsim^* = \succsim^m$, and furthermore the preference system $(\succsim^*, \Delta(Z))$ satisfies all the vNM axioms.

Proof of Proposition 3

We prove the following statements: (1) $\Lambda_\alpha(\alpha, \varepsilon) = R^*_\alpha - R^*_\gamma$; (2) $\Lambda_\varepsilon(\alpha, \varepsilon) = R^*_\varepsilon > 0$; (3) $\Lambda_\alpha(0, \varepsilon) > 0$; (4) $\Lambda_\alpha(k, \varepsilon) < 0$.

To prove these results, first we let $R^*(\alpha, \gamma, \varepsilon)$ be implicitly defined as the unique value of R such that the indirect utilities from the risk-free and risky asset problems coincide:

$$F(R, \alpha, \gamma, \varepsilon) := U_2(R, \alpha, \gamma, \varepsilon) - U_1(R_f, \alpha, \gamma) = 0,$$

where U_1 and U_2 are defined by

$$U_1 := g\left(v(y - s^*_1) + \beta v((1 + R_f)s^*_1)\right), \quad U_2 := \frac{\varepsilon}{2}g(A(R+\delta)) + (1-\varepsilon)g(A(R)) + \frac{\varepsilon}{2}g(A(R-\delta)),$$

with s^*_1 and s^*_2 denoting optimal savings under the risk-free and risky problems, respectively, and $A(x) := v(y - s^*_2) + \beta v((1 + x)s^*_2)$. Let $\Lambda(\alpha, \varepsilon) := R^*(\alpha, \gamma, \varepsilon)$ with $\gamma = k - \alpha$ for some fixed $k > 1$.

(1) $\Lambda_\alpha(\alpha, \varepsilon) = R^*_\alpha - R^*_\gamma$. Since $\Lambda(\alpha, \varepsilon) = R^*(\alpha, k - \alpha, \varepsilon)$, the chain rule gives:

$$\frac{\partial \Lambda}{\partial \alpha} = \frac{\partial R^*}{\partial \alpha} + \frac{\partial R^*}{\partial \gamma} \cdot \frac{\partial \gamma}{\partial \alpha} = \frac{\partial R^*}{\partial \alpha} - \frac{\partial R^*}{\partial \gamma}.$$

(2) $\frac{\partial \Lambda(\alpha, \varepsilon)}{\partial \varepsilon} = \frac{\partial R^*}{\partial \varepsilon} > 0$. Since $\gamma = k - \alpha$ is fixed with respect to ε, we have:

$$\frac{\partial \Lambda}{\partial \varepsilon} = \frac{\partial R^*}{\partial \varepsilon} = -\frac{\frac{\partial F}{\partial \varepsilon}}{\frac{\partial F}{\partial R}},$$

by the Implicit Function Theorem, provided $\frac{\partial F}{\partial R} \neq 0$. Now compute:

$$\frac{\partial F}{\partial \varepsilon} = \frac{1}{2}g(A(R + \delta)) + \frac{1}{2}g(A(R - \delta)) - g(A(R)).$$

Since $A(\cdot)$ is strictly increasing in R (as $v' > 0$, $\beta > 0$, and $s^*_2 > 0$), and g is strictly concave, Jensen's inequality implies $\frac{\partial F}{\partial \varepsilon} < 0$. For the denominator,

$$\frac{\partial F}{\partial R} = (1 - \varepsilon)\frac{d}{dR}g(A(R)) + \frac{\varepsilon}{2}\left[\frac{d}{dR}g(A(R + \delta)) + \frac{d}{dR}g(A(R - \delta))\right],$$

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and each $\frac{d}{dR}g(A(R \pm \delta)) = g'(A(R \pm \delta)) \cdot \beta v'\left((1 + R \pm \delta)s_2^*\right) s_2^* > 0$, since $g' > 0$, $v' > 0$, and $s_2^* > 0$. Hence $\frac{\partial F}{\partial R} > 0$, so

$$\frac{\partial R^*}{\partial \varepsilon} = -\frac{\frac{\partial F}{\partial \varepsilon}}{\frac{\partial F}{\partial R}} > 0.$$

(3) $\frac{\partial \Lambda}{\partial \alpha}(0.\varepsilon) > 0$. From part (1), we have

$$\frac{\partial \Lambda}{\partial \alpha} = \frac{\partial R^*}{\partial \alpha} - \frac{\partial R^*}{\partial \gamma} = \frac{\frac{\partial F}{\partial \gamma} - \frac{\partial F}{\partial \alpha}}{\frac{\partial F}{\partial R}}.$$

At $\alpha = 0$, the aggregator becomes linear: $g(z) \to z$ and evaluated at $\alpha = 0$,

$$\frac{\partial g}{\partial \alpha}(z) = -\frac{1}{2}z^2.$$

Therefore:

$$\frac{\partial U_2}{\partial \alpha} = -\frac{1}{2}\mathbb{E}[A(\widetilde{R})^2], \quad \frac{\partial U_1}{\partial \alpha} = -\frac{1}{2}A(R_f)^2,$$

so that

$$\frac{\partial F}{\partial \alpha} = -\frac{1}{2}\left(\mathbb{E}[A^2] - A(R_f)^2\right) < 0,$$

since $A(\widetilde{R})$ is a mean-preserving spread of $A(R_f)$ and $z \mapsto z^2$ is strictly convex.

Now consider $\frac{\partial F}{\partial \gamma} = \frac{\partial U_2}{\partial \gamma} - \frac{\partial U_1}{\partial \gamma}$. At $\alpha = 0$, $g'(z) = 1$, so:

$$\frac{\partial U_i}{\partial \gamma} = \frac{\partial v(y - s_i^*)}{\partial \gamma} + \beta \frac{\partial v((1 + R_i)s_i^*)}{\partial \gamma}.$$

Since $v_\gamma(c) := \frac{\partial v(c)}{\partial \gamma} = \frac{c^{1-\gamma}[1 - (1-\gamma)\log c]}{(1-\gamma)^2}$ is strictly decreasing in c, and c_{t+1}^2 is a mean-preserving spread of c_{t+1}^1, we obtain

$$\frac{\partial F}{\partial \gamma} > 0.$$

Hence:

$$\frac{\partial F}{\partial \gamma} - \frac{\partial F}{\partial \alpha} > 0 \Rightarrow \frac{\partial \Lambda(0, \varepsilon)}{\partial \alpha} > 0.$$

(4) $\frac{\partial \Lambda(k, \varepsilon)}{\partial \alpha} < 0$.

$$U_i = \mathbb{E}[c_t + \beta c_{t+1}] \Rightarrow \frac{\partial U_i}{\partial \gamma} = 0 \Rightarrow \frac{\partial F}{\partial \gamma} = 0.$$

But since g remains concave for $\alpha > 0$, the agent is risk-averse in the aggregator and:

$$\frac{\partial U_2}{\partial \alpha} > \frac{\partial U_1}{\partial \alpha} \Rightarrow \frac{\partial F}{\partial \alpha} > 0.$$

Therefore:

$$\frac{\partial F}{\partial \gamma} - \frac{\partial F}{\partial \alpha} < 0 \Rightarrow \frac{\partial \Lambda(k, \varepsilon)}{\partial \alpha} < 0.$$

Proof of Proposition 4

1. Let $z = (x_2^1, ..., x_2^K) \in \mathbb{R}^K$ be a vector denoting quantities of viable proxies, so that $x = (x_1, z) \in \mathbb{R}^{K+1}$ denote the vector with the good of interest x_1 and the candidate proxies, where $\succsim^c$ are the certain preferences on $\mathbb{R}^{K+1}$. Assume that a suitable proxy exists, and w.l.o.g.let it be proxy K. Furthermore, assume that all candidate proxies are viable. Noting that here, preferences $\succsim^c$ are defined over outcomes $(x, y) \in \mathbb{R}^{K+1} \times \mathbb{R}$, we define $\succsim^{c,0}$ in the following manner: $x \succsim^{c,0} x'$ iff $(x, 0) \succsim^c (x', 0)$. Then $\succsim^{c,0}$ is clearly (strongly) monotonic and continuous. Moreover, PQ of each viable proxy in $\{1, ..., K\}$ implies that each of these goods satisfies quasilinearity in $\succsim^{c,0}$. Hence, by standard quasilinearity results, it must be that $\succsim^{c,0}$ is represented by $f_1(x_1) + \sum_{k \in \{1,...,K\}} d_2^k x_2^k$ for a strictly increasing function $f_1 : \mathbb{R} \to \mathbb{R}$ and constants $d_2^k > 0$ for all $k \in \{1, ..., K\}$ and that any other function that represents it must have the form $h(f_1(x_1) + \sum_{k \in \{1,...,K\}} d_2^k x_2^k)$, where h is strictly increasing. Returning to $\succsim^c$, it must then also be that $(x, 0) \succsim^c (x', 0)$ iff $f_1(x_1) + \sum_{k \in \{1,...,K\}} d_2^k x_2^k \geq f_1(x_1) + \sum_{k \in \{1,...,K\}} d_2^k x_2^k$, and since, by the maintained assumptions on the yardstick, $(x, 0) \succsim^c (x', 0)$ iff $(x, y) \succsim^c (x', y)$ for all y, it also follows that $(x, y) \succsim^c (x', y)$ iff $f_1(x_1) + \sum_{k \in \{1,...,K\}} d_2^k x_2^k \geq f_1(x_1) + \sum_{k \in \{1,...,K\}} d_2^k x_2^k$, and furthermore that a representation of $\succsim^c$ must have the form $h(f_1(x_1) + \sum_{k \in \{1,...,K\}} d_2^k x_2^k) + y$ for strictly increasing h.

Then, under the maintained assumptions on the yardstick (which imply quasilinearity of the yardstick), full quasilinearity of good K, (strong) monotonicity and continuity of $\succsim^c$, we obtain immediately from known results that $(x, y) \succsim^c (x', y')$ iff $f_{-K}(x_1, x_2^1..., x_2^{K-1}) + b_K x_2^K + y \geq f_{-K}(x_1', x_2'^1...., x_2'^{K-1}) + b_K x_2'^K + y'$, where function $f_{-K} : \mathbb{R}^{K-1} \to \mathbb{R}$ is strictly increasing and $b_K > 0$.

Hence, combining this last result with the necessity of the form

$$h\left(f_1(x_1) + \sum_{k \in \{1,...,K\}} d_2^k x_2^k\right) + y,$$

it must be that h is a positive affine function, and hence that $\succsim^c$ is represented by the form $af_1(x_1) + a \sum_{k \in \{1,...,K\}} d_i x_2^k + y + \kappa$, where $a > 0$ and $\kappa \in \mathbb{R}$, which in turn means that every viable proxy satisfies FQ, and is thus a suitable proxy.

While this completes the proof that the existence of a suitable proxy implies that every viable proxy is suitable, we also show formally here why FQ suffices for PN. Considering now the full preferences $\succsim$, for yardstick neutrality to hold we must have the form $u(x, y) = g(af_1(x_1) + a \sum_{k \in \{1,...,K\}} d_k x_2^k + y)$ for CARA g (or a positive affine transformation of g; note that the constant e is thus included in this transformation). It thus follows immediately that PN holds for each viable proxy $k \in \{1, ..., K\}$.

2. Suppose that no candidate proxy is viable, and hence every candidate proxy violates

PQ or PN. Then, by the previous step, FQ cannot hold for any candidate proxy, and hence none can be ideal.

Proof of Proposition 5

Returning to the notation x_2 for the proxy, suppose (f_1, d_2, α) and (f_1', d_2', α') both observationally represent $\succsim$. Then in the certain domain, for all x_1, x_2 for which $(x_1, 0) \sim^c (0, x_2)$, it must be that $f_1(x_1) = f_1(0) + d_2 x_2$ and $f_1'(x_1) = f_1'(0) + d_2' x_2$, and hence that $x_2 = \frac{f_1(x_1) - f_1(0)}{d_2} = \frac{f_1'(x_1) - f_1'(0)}{d_2'}$. Rearranging, we obtain $f_1'(x) = \frac{d_2' f_1(x)}{d_2} + \left(f_1'(0) - \frac{d_2' f(0)}{d_2} \right) = a f_1(x) + b$, where $a = \frac{d_2'}{d_2}$ and $b = f_1'(0) - \frac{d_2' f(0)}{d_2}$. Hence, it must be that $f_1'(x_1) = a f_1(x_1) + b$ and $d_2' = a d_2$, where $a > 0$ and b is a constant.

Now let α_2 denote the Arrow Pratt coefficient of $\succsim$ with respect to good 2, respectively. Note that it is must be the same for any (f_1, d_2, α) and (f_1', d_2', α') that both observationally represent $\succsim$. Hence it must be that $\alpha_2 = \alpha d_2 = \alpha' d_2'$, and hence that $\alpha d_2 = \alpha' a d_2$, so that $\alpha' = \frac{\alpha}{a}$, which completes the proof.

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