A Model of Market−making [*]

Nicolaas J. Vriend, *Universitat Pompeu Fabra*, Barcelona, Spain

October 1996

Abstract:

The two essential features of a decentralized economy taken into account are, first, that individual agents need some information about other agents in order to meet potential trading partners, which requires some communication or interaction between these agents, and second, that in general agents will face trading uncertainty. We consider trade in a homogeneous commodity. Firms decide upon their effective supplies, and may create their own markets by sending information signals communicating their willingness to sell. Meeting of potential trading partners is arranged in the form of shopping by consumers. The questions to be considered are: How do firms compete in such markets? And what are the properties of an equilibrium? We establish existence conditions for a symmetric Nash equilibrium in the firms' strategies, and analyze its characteristics. The developed framework appears to lend itself well to study many typical phenomena of decentralized economies, such as the emergence of central markets, the role of middlemen, and price-making.

J.E.L. classification codes: C7, D8, L1, M3

Keywords: Decentralized Trade, Market-making, Communication, Trading Uncertainty

---

N.J. Vriend, Universitat Pompeu Fabra, Dept. Economics, Ramon Trias Fargas 25-27, 08005 Barcelona, Spain, <vriend@upf.es>, http://www.santafe.edu/~vriend/.

[*] I wish to thank Pierre Dehez, Peter Hammond, Esther Hauk, Maarten Janssen, Alan Kirman, Axel Leijonhufvud, PierCarlo Nicola, Martin Shubik, and especially Art De Vany for valuable comments and discussions. All errors and responsibilities are mine.

I. Introduction

Consider the following quote:

> *"Markets rarely emerge in a vacuum, and potential traders soon discover that they may spend more time, energy, and other resources discovering or "making" a market than on the trade itself. This predicament is shared equally by currency traders, do-it-yourself realtors, and streetwalkers! Their dilemma, however, seems to have gone largely unnoticed by economists, who simply assume that somehow traders will eventually be apprised of each other's existence - to their mutual benefit or subsequent regret".* (Blin [1980], p. S193)

Since this quote perfectly captures the motivation behind our paper, and the essence of what this paper is all about, we move directly to the substance of our paper. In order to deal with the indicated lacuna, we will analyze a model of competition with market-making for a decentralized economy characterized by the following basic properties: *(i)* There is a large number of agents and a large number of commodities. The agents and their physical environment are characterized by preferences, endowments, and technologies. *(ii)* Each agent is interested in only a limited number of commodities, while the fraction of agents interested in a given commodity is small for each commodity (cf., Fisher [1983]). *(iii)* The economy is not organized by an auctioneer, intermediary, specialized trader, central distributor, or anonymous random matching mechanism (e.g., Gale [1985]), but is instead one with decentralized trade that depends upon the decisions of individual agents who act on a strictly do-it-yourself basis. *(iv)* Agents, although knowing about the existence of other agents, have no further pre-communication knowledge about each other, such as, for example, their effective demands; not to mention characteristics as endowments and preferences. *(v)* An agent who does not possess any information about the characteristics of other agents is not in a position to find a trading partner. *(vi)* Individual agents may communicate with each other. *(vii)* Communication is costly. *(viii)* There is no exogenous, *'state-of-nature'* uncertainty. *(viii)* Individual agents know the aggregate state of the economy, e.g., aggregate demands, total numbers of sellers/buyers. *(ix)* All commodities are known by all agents (cf. Gary-Bobo & Lesne [1988]), and there is no quality uncertainty (cf. Spence [1974]). *(x)* All transaction costs are information costs, and there are no real transaction costs (see Shubik [1975]).

The assumed properties imply two essential features of a decentralized economy. First, the market interactions that take place depend in an essential way upon the knowledge of the identity of some other agents in the economy.[1] The key property here is *(v)*, which may be explained by the fact that the perceived costs of uninformed search are greater than its perceived gains. This may be due to the fact that the *'psychological'* costs (disutility) of accosting a randomly chosen, unidentified agent to bother him with a question like *'Could you please sell me a refrigerator?'* are high, while the probability that such an agent will in fact be interested in such a transaction is low.[2] Through communication with other agents, an individual agent creates the possibility of meeting potential trading partners. In general, when there are possibilities of trading in a certain commodity, it is said that a *market* for that commodity exists. Hence, by establishing communication with other agents, individual agents create their own

[1] See also Stigler [1961], who points out that the issue of the identity of sellers logically precedes the information problem concerning prices.

[2] Although this is related to the other properties, it is convenient to assume it directly. To give one example of the costs (disutility) of uninformed search: A smoker, after having troubled in vain two non-smokers for a light, might hesitate before asking the next passer-by, and will probably wait until he sees someone actually smoking.

markets. In other words, a market is not a central place where a certain good is exchanged, nor is it simply the aggregate supply and demand of a good. A market is constituted by communication between individual agents. The second essential feature implied by the above assumed properties is that in general the agents will face trading uncertainty.[3] Notice that in the sketched non-Walrasian trading structure[4] individually relevant information would be complete *only* when each agent knows not only the current price vector, but also the complete vector of effective demands of all other agents, and where and when to meet these agents.

In section II we will make these general assumptions concrete, and discuss the specific questions to be considered. In section III we will derive an equilibrium market structure from the optimizing behavior of individual agents, showing necessary and sufficient conditions for its existence, and analyze its characteristics, while section IV will conclude.

II. The Model

Agents and Commodities

Assuming time to be divided into an infinite sequence of discrete periods indexed τ, $\tau \in \{1, 2, ...\}$, we will consider an economy with a single homogeneous, perishable commodity.[5] A set A of N agents, each characterized by preferences, technologies and endowments, is divided into two disjoint classes: a set B of firms and a set D of consumers, with $|B| = m$, $|D| = N-m$, $B \cap D = \emptyset$, and $B \cup D = A$. We think of N as *'large'*. Given the agents' preferences, technologies and endowments, any given consumer i can be characterized by a threshold price $\overline{p}_i$. This threshold price $\overline{p}_i$ corresponds to the utility U_i the agent would derive from the consumption of one unit of the commodity, and is the price above which this agent would certainly not purchase a unit of the commodity (see e.g., Gale [1985] or Kormendi [1979]). Formally $\overline{p}_i$ is defined by $U_i(0, \omega_i) = U_i(1, \omega_i - \overline{p}_i)$, where the first argument of the utility function concerns the commodity considered, and the second represents a *'basket'* of other goods or *'income'*. We assume that the threshold price is 0 for all consumers with respect to any additional unit of the commodity. Formally: $U_i(1, \omega_i - \overline{p}_i) = U_i(a, \omega_i - \overline{p}_i)\ \forall\ a \geq 1\ \forall\ i$. Thus, the number n of interested consumers depends upon the price p, and the aggregate demand may be written n(p), which we assume to be objectively known by all sellers. In other words, given the characteristics of the individual agents, for each price p the set D of consumers consists of n(p) potential buyers and N−m−n(p) agents who are totally uninterested in the commodity. The m firms produce and sell the commodity. They are assumed to be identical in that they use the same technology. The cost C of producing z units of output is given by the function C(z), where $z \in \mathbb{N}$. We assume $C(0) \geq 0$ and $dC(z)/dz > 0\ \forall\ z \geq 0$.[6] The production decided upon at the beginning of the period is immediately available for sale, while unsold stocks perish

[3] Hahn [1980] stresses the distinction between trading uncertainty, which means being uncertain as to whether or not an agent will be able to trade what he wants *at the going price,* and price uncertainty; soliciting an analysis of the first.

[4] See Vriend [1994] for an extensive discussion of Walrasian trading structures.

[5] See appendix B for the notation used throughout this paper.

[6] For reasons of expositional convenience, the notation used in this paper will usually obscure the fact that some variables, in particular z and s, are discrete. We will write, for example, $dg(x, y, ..)/dx$ instead of $g(x+1, y, ..) - g(x, y, ..)$ for any function $g(x, y, ..)$.

at the end. In order for trade to take place, firms and consumers must meet, and they must agree upon the terms of trade. We assume that the price p of the commodity is given and equal for all firms and consumers, and that it is known to all agents. We assume that agents know that other agents exist, but they do not know any of the characteristics of these agents. In particular, they do not know which agent belongs to which class. As some information in this respect is an essential pre-requisite for meeting potential trading partners, we have to specify the way in which agents communicate with each other, i.e., how they create their own markets.

Communication and Trade

Each seller may send information signals to some other agents at the beginning of the period, each signal being directed to one agent. A signal contains, first, the *'name and address'* of the sending agent, and second, the fact that he belongs to the class of firms B. Thus, a signal reveals the type of a given agent.[7] Agents who neither perceive nor send signals cannot find a trading partner. We assume that signaling is costly, the cost being an increasing function of the amount of signals sent. The cost K of sending s signals is given by the function $K(s) = k \cdot s$, where $s \in \mathbb{N}$ and $k > 0$. Thus, we assume constant marginal signaling costs, $dK(s)/ds = k$, while $K(0) = 0$. Receiving signals, on the other hand, is costless.

Agents make their decisions concerning communication and effective demand at the beginning of each period. During each period they try to buy or sell in their markets. We assume that the trading possibilities for each agent are dependent only upon the communication and demands in the given period. Thus, sellers have no reputation, and there are no customer relations. Moreover, the demand above the firm's available supply is simply foregone, and cannot be backlogged. We assume that each consumer who has received one or more signals may visit one firm (*'shopping'*). The order in which buyers make their visit is random.[8] When a firm is sold out, customers will return home dissatisfied; if not, they may buy one unit.[9]

[7] For the moment we suppose that only the sellers may send information. In section III we will show that this is not as restrictive as it may seem. Moreover, it conforms to what we usually observe in reality. Note that the signals give no information about the size of the effective demands, and that this corresponds to what we usually observe in reality.

[8] To assume that consumers may make only one visit and that the order in which they make their visit is random, is only for convenience. It does not restrict the nature of the problem of the firm; it simply changes the value of some of its parameters. It is a simplified version of the more general scenario in which consumers can make more visits during each period, also on more markets to buy various commodities, while the shopping behavior is not synchronized between the consumers. Let us assume that time flows continuously, that the visits and exchanges are discrete events of zero duration, like the arrivals in a Poisson process (see Foley [1975] or Diamond [1982]), associate with each agent a *'random clock'* that rings independently for each agent at the instances of a Poisson process, and let each consumer make a visit when his clock rings (see, e.g., Griffeath [1979]). If the length of a period τ is finite, each consumer will be able to make only a limited number of visits in each period.

[9] Thus, trade in our model is bilateral. In this sense we differ from Ioannides [1990], who also analyzes communication by individual agents in order to make markets, and then considers multilateral trade between all agents who are directly or indirectly informationally linked.

Discussion of the Model

The model is set up in order to focus on the role of communication about the identity of agents and the resulting trading uncertainty. We believe the most convincing arguments to follow this theoretical approach, are *empirical*. First, communication to identify potential trading partners is important in real markets. Readers may wonder why we bother to develop this model with its explicit signaling structure, since in the real world *'a consumer simply knows where to buy something'*? Well-considered, such an observation gives strong empirical support for our approach. For it asserts that transactions do not take place in Walrasian central markets, or through anonymous random matching devices, but that, instead, market interactions depend in a crucial way on local knowledge of the identity of potential trading partners. Such information has to be communicated in one way or another. Most advertising in reality seems indeed to draw the buyers' attention to the fact that someone is selling something somewhere sometime. Advertising is not always in the form of signals sent directly to individual potential buyers. Agents may also hear from radio and tv, or from friends about the newest shops in town, buyers may use the Yellow Pages to find a seller, or they may visit shops randomly, etc. While these possibilities would require slight modifications of the signaling framework used, they would seem to fit rather well into it. The essential characteristic of all these forms of signaling is that agents make information about their own type known to some other agents. And as argued already in the opening quote by Blin, the resources spent in these forms of market-making are enormous in a modern market economy.

Second, markets with trading uncertainty are empirically relevant. There *are* many markets in which goods are sold at fixed prices (whether as a result of legislation, of vertically imposed restrictive practices, or of optimizing behavior of the sellers), and over the period for which prices are fixed, trading opportunities are usually uncertain. Doctors fees are regulated in many countries, the prices of books are fixed by publishers' cartels in a lot of countries, bread prices in Italy are determined centrally, newspaper usually sell for the same price at all stands, etc. Nearly any retail operation has a posted price, invariant through time, orders for stock are placed in advance of knowing what demand will be, and stock outs are commonly faced. Levis jeans are sold that way. So are Seiko watches. McDonald's rarely runs out of food, but it does happen. A more exotic market where this is true is the motion picture industry (see De Vany & Walls [1994]). Admission price is set and does not vary over the run, films are booked at theaters before demand is known, customers cannot always get a seat at their preferred showing and must queue up or wait for another day. Also custom items, e.g., in the finished steel or medical industry, usually are produced to order, at previously posted prices. Clearly, a complete economic analysis would explain such legislation, restrictive practices, or strategies, by which the prices are fixed, as well.[10] But that is not the aim of this paper. Instead of explaining the posted prices, and applying equally to all the possible ways in which these prices may have been determined, our analysis focuses on the directly ensuing not yet fully understood theoretical issue: the short run consequences

[10] For example, Riley & Zeckhauser [1985], who consider the question whether to haggle or not when buyers come in with bids in sequence, show that in a wide variety of markets a fixed, posted price is an optimal strategy for a seller. See also Ross [1993], who finds a similar property solving the so-called streetwalker problem.

for the competitive process within the industry itself.[11] Therefore, focusing on the issue concerning the identity of traders, and the resulting trading uncertainty, the first questions to be considered are: How do firms compete in such markets? And what are the properties of an equilibrium?

III. Strategies and Equilibrium

Objectives and Strategies

We first consider the agents' objectives and strategies. A consumer's utility would increase if he could buy one unit of the commodity at a price below his threshold price $\bar{p}$. If a consumer receives one or more signals at the beginning of the period, and if the price is indeed below his threshold price $\bar{p}$, then he chooses among these signals a firm to visit. Since all firms have the same price, and the signals as such convey no further information about the firms' service reliability, we can think of the consumers picking a firm at random.[12] If this firm is sold out, the consumer will be left dissatisfied in this period, otherwise he will buy one unit.

Firm i's objective is to maximize its expected current profit V_i , which is equal to its expected gross revenue R_i minus its production cost $C(z_i)$ minus its market-making cost $K(s_i)$, by deciding upon its effective supply z_i and signaling s_i. Moreover, it has to decide to which agents it will send these signals. We assume that the destination of each signal is chosen at random. Thus a strategy t_i of firm i is a pair (z_i, s_i). The costs are dependent only upon firm i's own strategy, but the revenue of its market-making and production activity, i.e., firm i's actual sales ($p{\cdot}x_i$, to be specified below), are a function of the vector **t** of the strategies of all firms (including firm i itself).[13] Thus, firm i's objective function is: $V_i(\mathbf{t}) = R_i(\mathbf{t}) - C(z_i) - K(s_i)$, where $R_i(\mathbf{t}) = p \cdot E\big(x_i(\mathbf{t})\big)$.

In this model of a decentralized economy, in which the firms act simultaneously, taking for given the actions of the other firms, we restrict our attention to the existence and characterization of a Symmetric Nash Equilibrium (SNE) in the firms' strategies; neglecting the possibility of asymmetric and multiple equilibria. The reason is that when players do not know *which* equilibrium will be played, and each individual agent simply selects some equilibrium, the resulting combination will in general *not* be an equilibrium. In order to coordinate the actions of the agents, extensive communication would be required. However, the main idea of this paper is to make the communication structure *explicit*. Hence, implicitly assuming that there does exist a communication structure through which the actions of the agents are coordinated in order to choose an equilibrium, would seem paradoxical and counter-productive.

Stochastic Trading Opportunities

Firm i's gross revenue is equal to its sales, $p{\cdot}x_i(\mathbf{t})$. Given the assumptions made in the previous sections about the nature of the commodity and the trading structure, it is clear that firm i cannot sell

[11] This is also in the spirit of the temporary equilibrium literature (see Grandmont [1988a] for an overview), in which prices are supposed to have been quoted at the outset of the period, as the result of an unspecified process of imperfect competition, and to remain momentarily fixed during the period analyzed.

[12] What we mean by *'randomly'*, in a strictly technical sense, is explained in appendix A.

[13] Vectors are denoted by **bold-face** letters.

more than is demanded by its customers, $q_i(\mathbf{s})$, or than it has produced at the beginning of the period, z_i. That is, $x_i(\mathbf{t}) = \min\{q_i(\mathbf{s}), z_i\}$. Therefore, we must specify the demand directed towards firm i, $q_i(\mathbf{s})$.

Proposition 1: The demand directed towards firm i, $q_i(\mathbf{s})$, is a random variable given by a Poisson distribution with parameter $\mu_i = \mu(s_i, S_{-i}) = (s_i/S) \cdot n(p) \cdot (1 - e^{-S/N})$, where S_{-i} denotes the aggregate number of signals sent by the other firms, and $S = s_i + S_{-i}$.

Proof: See appendix A. Here we give an outline only. Firm i sends s_i signals at random into the population. A given signal sent has success when it induces a consumer wanting to buy one unit to visit firm i. This depends upon the probability that the receiver of such a signal is an interested consumer and the probability that he will choose the signal from firm i among the signals he receives. The latter, clearly, also depends upon the aggregate signaling activity of the other firms. It turns out that the probability that any given signal sent by firm i will lead to a consumer visiting firm i is given by $\Pr(S) = \big(n(p)/S\big) \cdot (1 - e^{-S/N})$. Firm i sends s_i signals, and the number of buyers visiting firm i may be approximated by a Poisson distribution with parameter $\mu(s_i, S_{-i}) = s_i \cdot \Pr(S) = (s_i/S) \cdot n(p) \cdot (1 - e^{-S/N})$. □

This result can be read as follows. The potential aggregate demand in the economy, given the price p, is n.[14] The probability that a given potential consumer will not receive any signal at all, and thus will not find his way to a market, is $e^{-S/N}$. Hence, aggregate market demand is $n \cdot (1 - e^{-S/N})$. Finally, each firm’s expected market share turns out to be equal to his share in the aggregate market-making activity, s_i/S. Note that the probability of any single signal from firm i having success is a function only of the aggregate number of signals S sent by all firms. Hence, $\Pr_i(S) = \Pr(S)\ \forall$ i. This is because each interested consumer handles all his received signals identically, putting them all in an urn and drawing just one signal. Notice also that μ_i turns out to be a function only of the number of signals sent by firm i itself, s_i, and the aggregate number of signals sent by all other firms, S_{-i}. Thus, the vector of strategies chosen by the other firms $\mathbf{t}_{-i}$ enters firm i’s decision problem only through the aggregate market-making signaling activity.[15] The resulting transaction possibilities for any given agent are stochastic. Thus agents are uncertain as to whether they will be able to trade as much as they want. There are, as we have seen, two direct causes for this trading uncertainty. First, communication is stochastic, i.e., signals are randomly distributed as agents do not know each other’s characteristics. Second, given that an agent has found or established a market, either he or his potential trading partners may have already fulfilled their demand before they happen to meet, i.e., shopping is a stochastic process. Note that the trading possibilities for firm i are derived explicitly from assumptions with regard to the underlying communication and trading structure of the economy, instead of assuming directly a functional form of each agent’s trading possibilities. The stochastic demand for firm i’s output depends upon one of the (non-price) decision variables of the firm itself. This stochastic demand is not generated by sending an

[14] To lighten notational burden somewhat, we will usually write n instead of n(p).
[15] Allowing for more visits per buyer would simply give a higher value for the parameter μ.

effective demand (i.e., supply) to the market, but by creating the market itself.[16] As a result, firm i's expected gross revenue may be written as follows, where f[|μ] denotes the p.d.f. with parameter μ:

$$R_i(z_i, s_i, S_{-i}) = p \cdot \Big\{ \sum_{q_i=0}^{z_i} q_i \cdot f[q_i |\ \mu(s_i, S_{-i})] + z_i \cdot \sum_{q_i=z_i}^{\infty} f[q_i |\ \mu(s_i, S_{-i})] \Big\} \qquad [1]$$

Observe that the stochastic trading mechanism has an anonymity property. That is, agents who have the same effective demand and have sent out the same number of signals can expect the same realizations. This is due to the fact that trading possibilities depend only upon current period variables, that all signals are for each firm independently distributed, each agent being equally likely to receive such signals, that the firms to visit are chosen independently by all buyers, each firm being equally likely to be chosen among the firms in the buyer's market, and that the order in which buyers make their visits is determined randomly and does not depend upon the agents themselves. We further characterize the stochastic demand directed to firm i through the following claims.

Claim 1 a: For given S_{-i} , $\mu(s_i, S_{-i})$ is a one-to-one function of s_i , which satisfies: $\mu(0, S_{-i}) = 0$, $0 \le d\mu(s_i, S_{-i})/ds_i \approx \Pr(S) \le 1$, $d^2\mu(s_i, S_{-i})/ds_i^2 < 0$, and $\lim_{s_i \to \infty} \mu(s_i, S_{-i}) = n$.

b: For given s_i , $\mu(s_i, S_{-i})$ is a one-to-one function of S_{-i} , which satisfies: $d\mu(s_i, S_{-i})/dS_{-i} < 0$, and $\lim_{S_{-i} \to \infty} \mu(s_i, S_{-i}) = 0$.

Proof: See appendix A.

Thus, a firm that does not signal does not get any demand. The expected change in the demand directed to firm i as a result of sending one additional signal is positive but less than 1, and it depends only upon the aggregate signaling activity in the economy. Notice that for given S this is equal for all firms, and that it is not important which firms send these signals, and in particular it does not matter how many of the S signals are sent by firm i itself. A firm may eventually capture the whole aggregate demand by signaling more and more, given the strategies of the other firms. However, the more the other firms signal, the less will firm i's expected demand be. Suppose all m firms send the same number of signals: $s_i = s \ \forall$ i, $S = m \cdot s$, and $\mu(s_i, S_{-i})$ becomes $\grave{\mu}(s) = \mu\big(s, (m-1)\cdot s\big)$.

Claim 2: $\grave{\mu}(s)$ is a one-to-one function of s, which satisfies: $\grave{\mu}(0) = 0$, $0 \le d\grave{\mu}(s)/ds \le 1$, $d^2\grave{\mu}(s)/ds^2 < 0$, and $\lim_{s \to \infty} \grave{\mu}(s) = n/m$.

Proof: See appendix A.

Thus, if all firms send an infinite number of signals they may expect to share equally the whole aggregate demand. Note that m=1 corresponds to the case of a monopolist. The claims of this section are illustrated in figure 1.

[16] In this sense we differ from the literature on stochastic rationing (e.g., Green [1980]), where it is directly assumed that each agent's trading possibilities are a stochastic function only of his own demand and the aggregate demand and supply in the economy. We also differ from the literature on completely random matching models (e.g., Gale [1985]), where an agent's trading opportunities are independent from his own decisions. And we differ from the fixed price literature in general, where the sending of effective demands is the only means of communication (cf., Drazen's [1980] criticism).

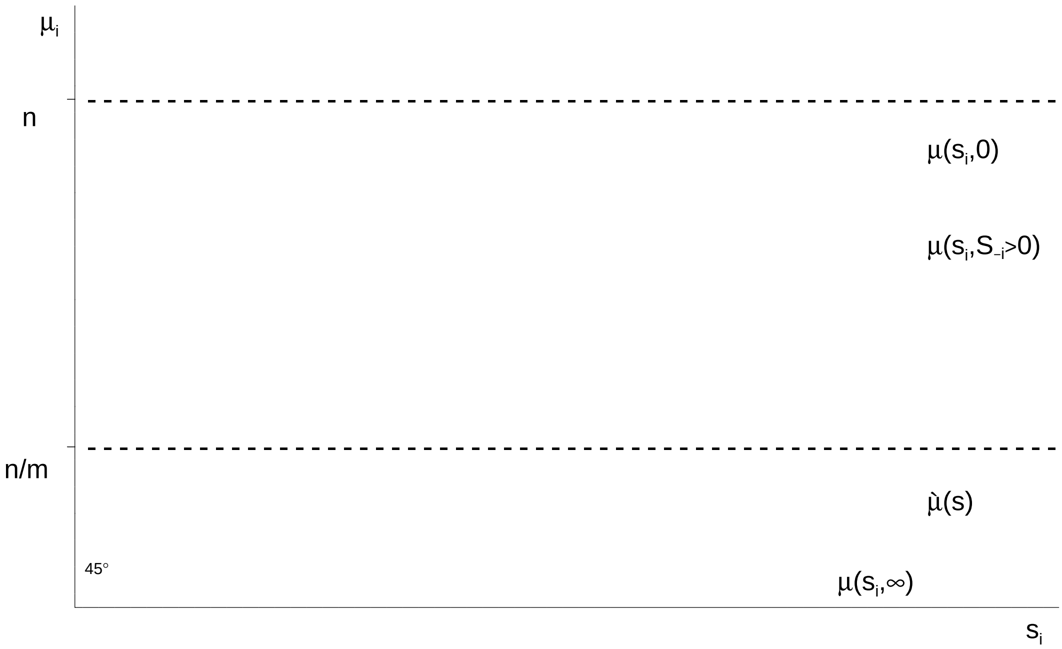


Figure 1 expected demand

Optimization and Equilibrium

We are now in a position to consider firm i's optimization problem. As a strategy t_i for firm i is a pair (z_i, s_i), the first-order conditions (FOCs) for maximization of firm i's payoff are a system of two equations:[17]

$$dV_i(z_i, s_i, S_{-i})/ds_i = dR_i(z_i, s_i, S_{-i})/ds_i - dK(s_i)/ds_i = 0 \qquad [2]$$
$$dV_i(z_i, s_i, S_{-i})/dz_i = dR_i(z_i, s_i, S_{-i})/dz_i - dC(z_i)/dz_i = 0$$

Claim 3 a: $dR_i/ds_i = p \cdot F[z_i-1] \cdot Pr(S)$, where F[z] denotes $\sum_{q=0}^{z} f[q]$

b: $dR_i/dz_i = p \cdot (1 - F[z_i])$

Proof: See appendix A.

In other words, the gross revenue for firm i of sending one additional signal, given the strategies of the other firms, is the price p multiplied by the probability that firm i would have had still at least one unit of the commodity available, multiplied by the probability that this additional signal will lead to a consumer visiting firm i. And the gross revenue for firm i of supplying one additional unit of the commodity, given the strategies of the other firms, is the price p multiplied by the probability that it would have sold out otherwise. It is advantageous for firm i to increase its signaling s_i with one unit, as long as $dR_i/ds_i > dK/ds_i = k$. Similarly, it is advantageous for firm i to increase its supply z_i with one unit, as long as $dR_i/dz_i > dC/dz_i$. As we consider a SNE, having derived the FOCs for maximization of firm i's payoff, we evaluate these conditions only for those cases in which each firm chooses the same

[17] Remember that, in fact, the variables z and s are discrete. Hence, considering unit increments of these variables, the *'true'* FOCs are:
$V_i(z_i, s_i, S_{-i}) - V_i(z_i-1, s_i, S_{-i}) > 0$ while $V_i(z_i+1, s_i, S_{-i}) - V_i(z_i, s_i, S_{-i}) \le 0$
$V_i(z_i, s_i, S_{-i}) - V_i(z_i, s_i-1, S_{-i}) > 0$ while $V_i(z_i, s_i+1, S_{-i}) - V_i(z_i, s_i, S_{-i}) \le 0$

strategy. Hence, $z_i = z$ and $s_i = s \;\forall\; i$, $S = m{\cdot}s$, and by FOC$^+$ we denote a first-order-plus-symmetry condition.

Claim 4 a: For every value of z there exists exactly one value of s, denoted by s(z), for which the first FOC$^+$ is satisfied. This function is characterized by $s(0) = 0$, $ds(z)/dz \geq 0$, $\lim_{z\to\infty} s(z) = s^{max} = \{s: dR_i(z{=}\infty, s)/ds_i = k\}$, and $s(z) \geq z \;\forall\; z$ as long as $s(z) < s^{max}$. Moreover, $s^{max} > 0$ if and only if $n/N > k/p$.

b: For every value of s there exists exactly one value of z, denoted by z(s), for which the second FOC$^+$ is satisfied. This function is characterized by $z(0) = 0$, $dz(s)/ds \geq 0$, $\lim_{s\to\infty} z(s) = z^{max} = \{z: dR_i(z, s{=}\infty)/dz = dC/dz\}$, and $z(s) \leq s \;\forall\; s$. Moreover, if $dC/dz^2 \geq 0 \;\forall\; z$ then $z^{max} > 0$ if and only if $n/m > -\ln(1 - \{dC(0)/dz\}/p)$.[18]

Proof: See appendix A.

Both curves are drawn in figure 2. Clearly, if a firm does not produce, it does not signal either, and vice versa. Moreover, there is a maximum level of signaling, which is related to the fact that beyond that level it is very unlikely that the receiver of an additional signal will respond to that signal. Thus, whatever the level of production the expected gains from an additional signal are below its costs. Similarly, there is a maximum level of production, which is related to the fact that it is very improbable that a customer will ever come to buy it, whatever the level of signaling. At a point of intersection of the two curves, both FOCs for maximization of firm i's payoff are fulfilled, while each firm chooses the same strategy $\hat{t}$.

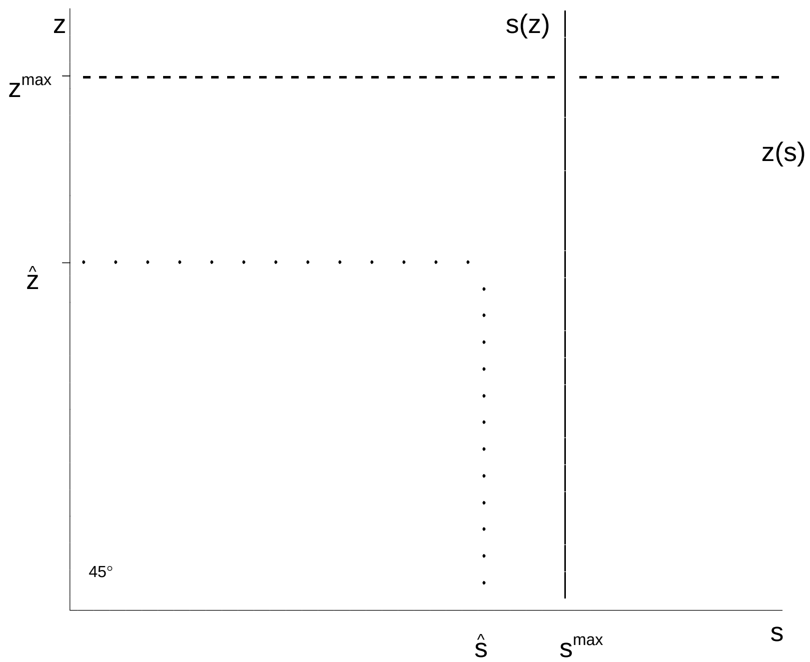


Figure 2 first-order-plus-symmetry condition

[18] Anticipating a result of the analysis, we here avoid giving a rather cumbersome analogous expression for the case in which $d^2C(z)/dz^2 < 0$.

Now, we turn to the second-order condition (SOC) for $\hat{t}$ to be a SNE strategy: $dV_i(z_i, s_i, S_{-i})/dz_i^2 \cdot dV_i(z_i, s_i, S_{-i})/ds_i^2 - \left(d\{dV_i(z_i, s_i, S_{-i})/ds_i\}/dz_i\right)^2 > 0$ and $dV_i(z_i, s_i, S_{-i})/ds_i^2 < 0$. The following claim gives a necessary and sufficient condition for this to be satisfied, given a strategy $\hat{t}$ at which the FOC's are fulfilled.

Claim 5: $\{dC(\hat{z})/d\hat{z}^2\}/p > f[\hat{z}+1]/\hat{z} \iff$ the SOC is fulfilled.

Proof: See appendix A.

Corollary 1: If the SOC is fulfilled then necessarily $dC(\hat{z})/d\hat{z}^2 > 0$.

Proof: This follows directly from claim 5 and the fact that $p > 0$, $f[\hat{z}+1] > 0$ and $\hat{z} > 0$. □

Thus, a necessary condition concerning the production technology is that there are decreasing returns to scale, at least locally. Note that whether the SOC will actually be fulfilled depends also upon the strategy $\hat{t}$ for which the FOC's are fulfilled. The economic meaning of the sign of the second derivative of the cost function is clear, but, a priori, it does not seem to make much sense to make further assumptions concerning the shape of the C(z) function, i.e., with respect to the third derivative. One can only observe that $f[\hat{z}+1]$ is bounded below 1, and that therefore the right-hand side of the equation of claim 5 approaches zero if $\hat{z}$ goes to infinity, implying that it might be more likely that the SOC is fulfilled when $\hat{z}$ is larger. Suppose that there were no trading uncertainty. That is, when firm i supplied $\hat{z}$ units it would know that it would sell $\hat{z}$ units: $f[\hat{z}] = 1$. Then, $f[\hat{z}+1] = 0$, and the condition of claim 5 would be $dC(\hat{z})/d\hat{z}^2 \geq 0$, which is a rather familiar expression for models without trading uncertainty. Finally, one has to consider the payoff V_i to firm i. Clearly, if the expected profit when all firms choose strategy t is negative, firm i will prefer to stay inactive, and no strictly positive SNE will exist.

Proposition 2: Necessary conditions for a SNE to exist are: $n/N > k/p$, $n/m > -\ln\left(1 - \{dC(0)/dz\}/p\right)$, and $dC(z)/dz^2 > 0$ for some z.

Proof: See claim 4 and corollary 1. □

These conditions imply that n, p and $dC(z)/dz^2$ must be large enough, while N, m, k and $dC(0)/dz$ must be small enough. Sufficient conditions for existence of a SNE depend in a complicated way upon the parameter values. As argued by Judd [1995], this is a typical case in which a numerical approach can help the theorist to discern and express patterns that an analytical approach would have difficulty with. Therefore, based on the analytical results presented above, we have done a numerical analysis in order to get more insight into the values of k, C(.), p, n(.), m and N for which the derived existence conditions for a SNE are satisfied. The details can be found in appendix A. The findings are intuitively clear, and can be summarized as follows. A SNE will exist unless the costs of making a market and/or producing for the market are too high relative to the price p for each possible extent of the market at all levels of production. This may be due not only to the level of prices (p) and costs (k and dC/dz) as such, but also to too low a number of potential buyers (n) in the economy or to too high a number of competing firms (m).

Some Characteristics of the SNE

In a SNE, the economy splits up into a number of possibly overlapping markets: each firm creates its own market of size s^*, and produces z^*. We will now analyze its characteristics. We first consider the effects of changes in the parameter values. Without giving all the straightforward analytical details, and observing that each firm's profitability depends upon the net gains per transaction and upon its trading opportunities, our findings can be summarized as follows. Higher net gains per transaction, i.e., a higher price and/or lower production and signaling cost parameters, will, ceteris paribus, lead to a SNE with increased supply, market-making activity and expected profits. A lower number of competing firms or a higher number of potential customers will lead to higher expected profits, but the effect on the level of signaling and production activity is ambiguous. The point is that there are two opposing effects upon the s(z) curve in figure 2. On the one hand, if the probability of success of any given signal, Pr(s), increases, then the probability of success of an additional signal increases. However, on the other hand, the probability of success of all other signals also increases, implying that the firm may expect more visitors, and the probability that the firm would have at least one unit of the commodity left, $F[z-1]$, decreases. As a result, the probability that an additional visitor would be fruitful decreases.[19] However, we at least know what happens when n/m approaches infinity. Each signal sent will lead to a consumer visiting its sender, and hence a situation of certainty is approached. In general, however, a decrease of trading uncertainty will lead to higher expected profits, but not necessarily to increased signaling and production activity.

Claim 6: If $n/m \rightarrow \infty$ then the FOC's will be fulfilled for $\hat{t} \equiv \left(\hat{z} = \{z: dC/dz = p-k\}, \hat{s} = \hat{z}\right)$.
Proof: See appendix A.

The next proposition concerns the size of the economy, asserting that the SNE and individual market outcomes are independent of the size of the economy as long as the proportions of types of agents, i.e., firms and (interested) consumers, remain constant.[20] Hence, we could read the parameters of the model such that the number of agents N is countably infinite, while m and n(.) are the fractions of firms and interested consumers in the population.

Proposition 3: If the *'set of parameters'* $\{k, C(.), p, N, n(.), m\}$ leads to a SNE (z^*, s^*, V^*) then the *'set of parameters'* $\{k, C(.), p, \alpha{\cdot}N, \alpha{\cdot}n(.), \alpha{\cdot}m\}$ leads to exactly the same SNE (z^*, s^*, V^*) for any $\alpha > 0$.
Proof: See appendix A.

In an analysis of market-making, it seems obvious to consider the issue of the extent of the market. Two phenomena that are considered to be related to each other in the literature, are *the division of labor* and *the extent of the market* (see e.g., Smith [1776]). A measure of the division of labor could be the relative number of agents in the economy producing a given commodity. In our model, this would be the number of firms, m, for given N. A measure of the extent of the market could be the number of agents in the economy that is informed about the fact that the commodity is on the market. In our model, this

[19] In appendix A we present some numerical examples of SNE, illustrating these findings.
[20] An additional condition is that the parameters remain such so as to allow for the Poisson approximations.

would be measured by the aggregate number of signals sent, S. The proposition implies, for example, that it cannot be excluded that the aggregate market for a commodity shrinks when the number of firms specialized in producing the commodity increases.

Proposition 4: The extent of the market (S) is a function of the division of labor (m), but the sign of dS/dm is not determined a priori.

Proof: See appendix A, where we show that the number of signals sent by an individual firm, s, is a function of the parameter m, and that an increase in m may lead to a leftward or rightward shift of the s(z) curve in figure 2. This is caused by the two opposing effects mentioned above. On the one hand, the probability of success of an additional signal increases, but on the other hand, the probability of having at least one unit of the commodity left, F[z−1], decreases. □

Up to this point we have assumed that only firms may send signals. The following proposition makes clear why consumers do not generally create buyers' markets, even when the same market-making technology is available to them. Consumers are interested in buying only a very limited number of units, in our model only 1, while firms generally want to sell many more units. Related to this is the fact that, in general, there are many more consumers, n, of a certain commodity than firms, m, selling that commodity, making it more difficult for consumers to find firms than vice versa.

Proposition 5: To get as a result that consumers do not signal, it is sufficient to assume: $m/N < k/\{U_i(1, \omega_i - p) - U_i(0, \omega_i)\}$ for each consumer i.

Proof: Analogous to part of the proof of claim 4. □

Now we consider the efficiency of the SNE. Clearly, the allocation mechanism as such is informationally inefficient. We focus upon efficiency given the trading and communication structure of the model. One of the attractive features of Walrasian models is that *'The Market'* is operated efficiently. A market is efficient if all mutually advantageous trades are carried out, which implies that one will not find rationed demanders and rationed suppliers at the same time (see e.g., Benassy [1982]). In this sense the market outcome of a SNE is inefficient. In a SNE the overall economy will not be orderly for each price, as there may be some buyers as well as some firms rationed at the same time.[21] [22] This would seem to be a rather prominent characteristic of a decentralized economy.

Proposition 6: Prob $[(x_i - z_i) \cdot (x_j - z_j) < 0] > 0$ for each pair i, j where $i \in B$, $j \in D$ and $z > 0$.

Proof: See appendix A. Here we give an outline. Firm i's supply is z^*, while the stochastic demand directed to it is represented by f[q]. Thus, the probability that firm i is rationed is equal to the probability that it will receive less than z^* buyers: Prob $[(x_i - z_i) > 0] = F[z^* - 1] > 0$. Next, an interested consumer j has a unit demand and may visit only one firm. Buyers will be rationed when they do not

[21] Allowing for more visits per buyer would not change the picture. Each buyer dissatisfied in his first round might be more successful in his second or third round. As a result, given the level of signaling s, the probability of rationing will be lower for both firms and buyers.

[22] In this respect the model differs from some other models applying stochastic rationing (e.g., Weinrich [1984]), where it is assumed that markets are orderly, thereby implicitly assuming some kind of '*central lottery*'.

receive any signal or when they visit a firm that has already sold out. In appendix A we show that both possibilities may occur with positive probabilities: Prob $[(x_j - z_j) < 0] > 0$. ◻

As rationing with respect to the consumers is all-or-nothing, from their point of view the probability to be rationed is a good measure of the performance of the economy. For firms, however, rationing is a quite *'natural'* affair. Therefore, we now consider another measure of efficiency concerning the firms. We compare the SNE with an equilibrium that would be in the joint interest of all firms. A Symmetric Cooperative Equilibrium (SCE) is a vector of strategies such that all firms choose the same strategy, and the sum of the payoffs of all firms is maximized. According to the following proposition, a SNE is not efficient from the firms' point of view in the sense that a better, i.e., preferred by all firms, vector of strategies exists. However, each individual firm will have an incentive to deviate from the SCE strategy t^c. Moreover, as proposition 8 asserts, consumers are worse off in a SCE.

Proposition 7: The equilibrium strategy t^* of a SNE involves more communication and production, but lower expected profits, than the equilibrium strategy t^c of a SCE, i.e., $z^* > z^c$ and $s^* > s^c$, while $V^* < V^c$.
Proof: See appendix A. Considering the appropriate FOCs, we show that the z(s) curve in figure 2 remains the same, since the production by the other firms, $\mathbf{z}_{-i}$, does not enter firm i's payoff. But now, the revenue for firm i of sending one additional signal will be lower when all other firms also send one additional signal simultaneously. As a result, the new s(z) curve will be at the left of the s(z) curve in figure 2, and the intersection of the s(z) and z(s) curves will occur at values z^c and s^c that are lower than z^* and s^*. ◻

Proposition 8: Prob [cons. rationed$|$ SNE] $<$ Prob [cons. rationed$|$ SCE]
Proof: See appendix A, where we show that this follows directly from the fact that the values for both z and s are lower in a SCE. ◻

IV. Concluding Remarks

The two essential features of a decentralized economy taken into account in our model are, first, that individual agents need some information about other agents in order to meet potential trading partners, which requires some communication or interaction between these agents, and second, that in general agents will face trading uncertainty. A number of problems concerning decentralized trade is related to these features, and may therefore be studied within this framework very well.

For example, one could analyze the existence of *central markets*. Let us suppose that firms may decide to *'cooperate'* physically in the market-making process. Instead of each firm selling its production in its own market, there may be one or more common, central markets or distribution points. Suppose that the technology of market-making is still the same (the central distributors sending signals giving the address of the distributor and the message that he sells the commodity), and that there are no additional costs of running a central market. When one assumes that each firm may sell in a common market in proportion to its contribution in the market-making costs, one can, for example, consider the non-cooperative solution concept of a Nash equilibrium. Each firm chooses a market to join and decides how much to contribute to the signaling activity, taking the choices of the other firms as given. This seems to capture the essential function of a central distributor.

The functioning of *middlemen* may also be analyzed within the framework of our model. Middlemen are not intrinsically interested in the commodity itself, i.e., they belong to neither the firms nor the consumers. They buy from sellers, and sell to buyers. This description still allows for a number of middlemen functions (e.g., reducing real transaction costs, reducing storing costs, forming a buffer between fluctuating demand and supply, speculation, etc.), but the distinguishing characteristic of middlemen is that they make a profit by taking into account the *'matching'* problem of the economy. Thus, they create markets by sending signals to establish contact with both firms and consumers.

The model would seem to be an interesting starting-point also for the study of *price-making* in a decentralized economy. The resulting market structure in a SNE is imperfectly competitive, although the commodity traded is homogeneous. Each firm signals to s agents, and in so doing it creates its own market. Thus, buyers might know that a firm finds as a maximum s alternative buyers in the market, and therefore buyers have some monopsony power. Each buyer on the other hand, can trade only with those firms from which he has received a signal. Thus, each firm may know that a buyer visiting him will know of only a limited number of alternative firms to visit, and therefore they will have some monopoly power. Finally, the model of decentralized trade proposed seems useful for a study of the phenomenon of *liquidity*: an asset being more liquid if it may be sold more cheaply and surely (see Hahn [1988]), or problems like *effective demand failures*: some agents being unwilling to demand/supply more of one commodity because of uncertainty about their trading possibilities concerning another commodity (see Grandmont [1988b]).

References

Benassy, J.P. (1982). *The Economics of Market Disequilibrium*. New York: Academic Press.

Blin, J.M. (1980). Comments on "The Economics of Markets: A Simple Model of the Market-making Process". *Journal of Business*, *53, No.3, Pt.2,* S193-S197.

Calsamiglia, X. (1987). Informational Requirements of Parametric Resource Allocation Processes. In T. Grooves, R. Radner & S. Reiter (Eds.), *Information, Incentives, and Economic Mechanisms*. University of Minnesota Press.

DeGroot, M.H. (1986). *Probability and Statistics* (2nd ed.). Reading, MA: Addison-Wesley.

De Vany, A., & Walls, W.D. (1994). *Information, Adaptive Contracting, and Distributional Dynamics: Bose-Einstein Statistics and the Movies* (mimeo).

Diamond, P.A. (1982). Aggregate Demand Management in Search Equilibrium. *Journal of Political Economy*, *90*, 881-894.

Drazen, A. (1980). Recent Developments in Macroeconomic Disequilibrium Theory. *Econometrica*, *48*, 283-306.

Fisher, F.M. (1983). *Disequilibrium Foundations of Equilibrium Economics*. Cambridge: Cambridge University Press.

Foley, D.K. (1975). On Two Specifications of Asset equilibrium in Macroeconomic Models. *Journal of Political Economy*, *83*, 303-324.

Gale, D. (1985). *Limit Theorems for Markets with Sequential Bargaining* (CARESS Working Paper 85-15) University of Pennsylvania.

Gary-Bobo, R.J., & Lesne, J.P. (1988). *Advertising Strategies and Competition in a Chamberlinian Market Model* (mimeo) C.O.R.E.

Grandmont, J.M. (Ed.). (1988a). *Temporary Equilibrium, Selected Readings*. San Diego, CA: Academic Press.

Grandmont, J.M. (1988b). Temporary Equilibrium Theory: An Addendum. In J.M. Grandmont (Ed.), *Temporary Equilibrium, Selected Readings* (pp. 41-84). San Diego, CA: Academic Press.

Green, J.R. (1980). On the Theory of Effective Demand. *Economic Journal*, *90*, 341-353.

Griffeath, D. (1979). *Additive and Cancellative Interacting Particle Systems. (Lecture Notes in Mathematics 724)*. Berlin: Springer.

Hahn, F.H. (1980). Discussion. In J. Kareken & N. Wallace (Eds.), *Models of Monetary Economics* (pp. 161-165) Fed. Res. Bank of Minneapolis.

Hahn, F.H. (1988). *Liquidity* (Discussion Paper No.129) University of Cambridge.

Ioannides, Y.M. (1990). Trading Uncertainty and Market Form. *International Economic Review*, *31*, 619-638.

Judd, K.L. (1995). *Computational Economics and Economic Theory: Substitutes or Complements?* (mimeo). Talk given at the First International Conference of the Society for Computational Economics, Austin, TX, May 1995.

Kormendi, R.C. (1979). Dispersed Transaction Prices in a Model of Decentralized Pure Exchange. In S.A. Lippman & J.J. McCall (Eds.), *Studies in the Economics of Search*. Amsterdam: North-Holland.

Riley, J., & Zeckhauser, R. (1983). Optimal Selling Strategies: When to Haggle, When to Hold Firm. *Quarterly Journal of Economics*, *98*, 267-289.

Ross, S. (1993). *Introduction to Probability Models* (5th ed.). San Diego, CA: Academic Press.

Shubik, M. (1975). The General Equilibrium Model Is Incomplete and Not Adequate for the Reconciliation of Micro and Macroeconomic Theory. *Kyklos*, *28*, 545-573.

Smith, A. (1776). *An Inquiry into the Nature and Causes of the Wealth of Nations*. London: Strahan and Cadell.

Spence, A.M. (1974). *Market Signalling*. Cambridge, MA: Harvard University Press.

Stigler, G.J. (1961). The Economics of Information. *Journal of Political Economy*, *69*, 213-225.

Vriend, N.J. (1994). A New Perspective on Decentralized Trade. *Economie Appliquée*, *47, No. 4*, 5-22.

Appendix A. Proofs

Proof proposition 1: Suppose that firm i has sent s_i signals, and consider one of those signals only. Firm i sends this signal at random into the population. (Technically, suppose firm i has put all agents in an urn, draws just one agent to determine the destination of the signal, and replaces that agent). The probability that this signal from firm i is received by an interested consumer is n(p)/N. Each interested consumer puts all received signals in an urn and draws just one signal out of his urn. Supposing that the signal from firm i is received by an agent who has received x signals in total from various firms, the probability that he would draw firm i's signal is 1/x. Thus, one has to determine x, the number of signals received by any given agent.

From the point of view of such an agent, the destination of each signal sent in the economy is the outcome of a Bernoulli trial with two possible outcomes: the signal will reach him or another agent. Thus, the number of signals received by any given agent has a binomial distribution with as parameters the total number of signals sent, and the probability of reaching this given agent. If the number of Bernoulli trials in the sequence is large and the probability of reaching the given agent is close to 0, the binomial distribution may be approximated by a Poisson distribution (see DeGroot [1986]). A glance at the appropriate probability tables suggests that such an approximation is reasonable when the number of trials is greater than 25, while the probability of success is smaller than 0.1. We assume that the sets *B* and *D* are such that both conditions will be fulfilled. Thus, the probability that this signal from firm i is received by an agent who has got (x−1) other signals is:

$$\text{Prob [x−1 other signals]} = e^{-\lambda} \cdot \frac{\lambda^{x-1}}{(x-1)!} \ ,$$

where λ = S/N is the expected number of signals received by any given agent,
with 1/N = Prob [*'hitting'* any given agent]
S = aggregate number of signals sent by all firms

Hence, the probability that any given signal from firm i will lead to an interested consumer visiting firm i is:

$$\begin{aligned} \Pr(S) &= \{n(p)/N\} \cdot \textstyle\sum_{x=1}^{\infty} (1/x) \cdot e^{-\lambda} \cdot \dfrac{\lambda^{x-1}}{(x-1)!} \\ &= \{n(p)/N\} \cdot (1/\lambda) \cdot \textstyle\sum_{x=1}^{\infty} e^{-\lambda} \cdot \dfrac{\lambda^{x}}{x!} \\ &= \{n(p)/N\} \cdot (1/\lambda) \cdot (1 - e^{-\lambda}) \\ &= \{n(p)/S\} \cdot (1 - e^{-S/N}) \quad \forall\, i \end{aligned}$$

This probability Pr(S) refers to one single signal sent. From the point of view of firm i, each signal it has sent is a Bernoulli trial with two possible outcomes: the receiver will or will not visit firm i. The sum of a sequence of s_i of such Bernoulli trials is a random variable that has a binomial distribution with parameters s_i and Pr(S). (Here we make a small error. That is, even when a buyer has received more than one signal from firm i, he will perform only one Bernoulli trial.) If the number of signals is large while the probability that the receiver will visit firm i is close to 0, the number of buyers visiting firm i may be approximated by a Poisson distribution with parameter $\mu_i = \mu(s_i, S_{-i}) = s_i \cdot \Pr(S) = (s_i/S) \cdot n(p) \cdot (1 - e^{-S/N})$. As each visiting agent demands exactly one unit, the demand q_i facing firm i has the same Poisson distribution: $f[q_i \mid \mu(s_i, S_{-i})]$. □

Proof claim 1 a: $\mu(s_i, S_{-i}) = s_i \cdot \Pr(S)$, with $\Pr(S) = (n/S) \cdot (1 - e^{-S/N})$ and $S = s_i + S_{-i} \Rightarrow d\mu(s_i, S_{-i})/ds_i = \Pr(S) + s_i \cdot d\Pr(S)/ds_i$. The last term of the right-hand side of this equation is the indirect effect of sending one additional signal by firm i. This indirect effect is negative as $d\Pr(S)/ds_i = -(n/S^2) \cdot \{1 - e^{-S/N} \cdot (1 + S/N)\} < 0$. That is, each signal sent becomes slightly less likely to be successful when an additional signal competes with it. However, as long as S is relatively large, this indirect effect is negligible from the point of view of firm i: $\lim_{S\to\infty} d\Pr(S)/ds_i = 0$. The direct effect is the probability that any given signal sent will lead to its receiver visiting the sender of the signal. As all probabilities: $0 \le \Pr(S) \le 1$. Hence, $d^2\mu(s_i, S_{-i})/ds_i^2 = d\Pr(S)/ds_i < 0$ (see above).

$$\lim_{s_i\to\infty} \mu(s_i, S_{-i}) = \lim_{s_i\to\infty} \{s_i \cdot n \cdot (1 - e^{-(s_i + S_{-i})/N})\}/(s_i + S_{-i}) = \infty/\infty$$

Applying L'Hôpital's Rule gives:

$$\lim_{s_i\to\infty} n \cdot \{(1 - e^{-(s_i + S_{-i})/N}) + s_i/(N \cdot e^{-(s_i + S_{-i})/N})\} = n \cdot (1 + \infty/\infty)$$

Applying L'Hôpital's Rule again for the last quotient leads to:

$$\lim_{s_i\to\infty} \mu(s_i, S_{-i}) = n \cdot (1 + \lim_{s_i\to\infty} e^{-(s_i + S_{-i})/N}) = n$$ □

b: $d\mu(s_i, S_{-i})/dS_{-i} = d\Pr(S)/dS_{-i}$. As $S = s_i + S_{-i}$, $d\Pr(S)/dS_{-i} = d\Pr(S)/ds_i < 0$ (see above).

$\lim_{S_{-i}\to\infty} \mu(s_i, S_{-i}) = \lim_{S_{-i}\to\infty} s_i/(s_i + {}_{-i}) \cdot n \cdot (1 - e^{-(s_i + S_{-i})/N}) = 0$ □

Proof claim 2: $\grave{\mu}(s) = n/m \cdot (1 - e^{-m \cdot s/N})$. To get $\grave{\mu}(0)$ and $\lim_{s\to\infty} \grave{\mu}(s)$ just substitute s.

$d\grave{\mu}(s)/ds = (n/N) \cdot e^{-m \cdot s/N} \Rightarrow 0 \le d\grave{\mu}(s)/ds \le 1$

$d^2\grave{\mu}(s)/ds^2 = -(n \cdot m/N^2) \cdot e^{-m \cdot s/N} < 0$ □

Proof claim 3 a: We rewrite $R_i(z_i, s_i, S_{-i})$ by omitting subscripts and arguments as much as possible for notational convenience, and observing that

$$\sum_{q=0}^{z} q \cdot f[q] = \sum_{q=1}^{z} q \cdot e^{-\mu} \cdot \frac{\mu^q}{q!} = \mu \cdot \sum_{q=1}^{z} e^{-\mu} \cdot \frac{\mu^{q-1}}{(q-1)!}$$

$$= \mu \cdot \sum_{y=0}^{z-1} e^{-\mu} \cdot \frac{\mu^y}{y!} = \mu \cdot F[z-1]$$

Hence, $R_i = p \cdot \{\mu_i \cdot F[z_i-1] + z_i \cdot (1 - F[z_i])\}$

$dR_i/ds_i = p \cdot \{d\mu_i/ds_i \cdot F[z_i-1] + \mu_i \cdot dF[z_i-1]/d\mu_i \cdot d\mu_i/ds_i - z_i \cdot dF[z_i]/d\mu_i \cdot d\mu_i/ds_i\}$

Now, $dF[z]/d\mu = F[z-1] - F[z] = -f[z]$ and $d\mu/ds = Pr$ (see claim 1).

Hence, $dR_i/ds_i = p \cdot \{F[z_i-1] - \mu_i \cdot f[z_i-1] + z_i \cdot f[z_i]\} \cdot Pr$. As $f[z-1] = (z/\mu) \cdot f[z]$, we get $dR_i/ds_i = p \cdot F[z_i-1] \cdot Pr$. □

b: Again, we first rewrite $R_i(z_i, s_i, S_{-i})$. $R = p \cdot \{ \sum_{q=0}^{z} q \cdot f[q] + z \cdot (1 - F[z])\}$. Summing up the first term by parts and rewriting the second term gives:

$R = p \cdot \{z \cdot F[z] - \sum_{q=0}^{z-1} F[q] + z - z \cdot F[z]\} = p \cdot \{z - \sum_{q=0}^{z-1} F[q]\}$. Then, $dR_i/dz_i = p \cdot \{1 - F[z_i]\}$ □

Proof claim 4 a: The FOC is: $dR_i(z_i, s_i, S_{-i})/ds_i = dK(s_i)/ds_i$. In case of symmetry, $z_i = z$, $s_i = s$ and $S_{-i} = (m-1) \cdot s$ $\forall$ i. Hence, we get $dR_i(z, s)/ds_i = dK(s)/ds_i \Rightarrow p \cdot F[z-1] \cdot Pr(s) = k$. First, we keep constant z. If $s = 0$ then $\grave{\mu}(s) = 0$ and hence $F[z-1] = 1 \ \forall\ z \ge 1$.

$\lim_{s\downarrow 0} Pr(s) = \lim_{s\downarrow 0} n/(m \cdot s) \cdot (1 - e^{-m \cdot s/N}) = 0/0$. Applying L'Hôpital's Rule gives:

$\lim_{s\downarrow 0} (n/N) \cdot e^{-m \cdot s/N} = n/N$. Hence, $dR_i(z, s{=}0)/ds_i = p \cdot n/N\ \forall\ z$.

$d\{dR_i(z, s)/ds_i\}/ds = p \cdot \{dF[z-1]/d\mu \cdot d\mu/ds \cdot Pr(s) + F[z-1] \cdot dPr(s)/ds\} < 0$ as the only negative terms are $dF[z-1]/d\mu$ and $dPr(s)/ds$ (see claims 1 and 3).

$\lim_{s\to\infty} Pr(s) = \lim_{s\to\infty} n/(m \cdot s) \cdot (1 - e^{-m \cdot s/N}) = 0$. Hence, $\lim_{s\to\infty} dR_i(z, s)/ds_i = \lim_{s\to\infty} p \cdot F[z-1] \cdot Pr(s) = 0$.

Now, we consider the variable z. $dR_i(z{=}0, s)/ds_i = 0$, $d\{dR_i(z, s)/ds_i\}/dz = p \cdot f[z] \cdot Pr(s) > 0$ and $\lim_{z\to\infty} dR_i(z, s)/ds_i = p \cdot Pr(s)$. We can draw this in figure A.1. We see that for given z there is an optimal value of s, s(z), with $s(0) = 0$, $ds(z)/dz > 0$ and $\lim_{z\to\infty} s(z) = s^{max} = \{s: p \cdot Pr(s) = k\}$. Observe that if $p \cdot n/N < k$ then $s(z) = 0\ \forall$ z.

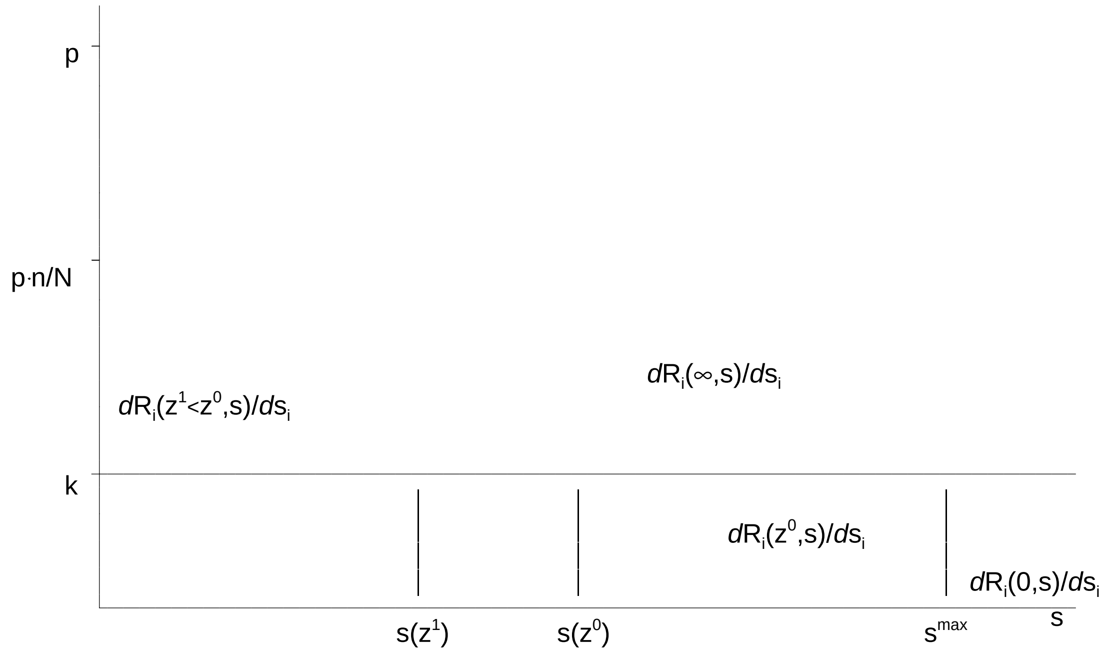


Figure A.1 FOC⁺ with respect to signaling

Finally, we have to prove that $s(z) \ge z\ \forall\ z$ if $s(z) < s^{max}$. Suppose $s < s^{max}$ and $z > s$. If $z > s$ then $F[z-1] = 1$, as no firm can get more customers than the number of signals it has sent. Hence, $dR_i(z, s)/ds_i = p \cdot Pr(s)$. We know that if $s < s^{max}$ then $p \cdot Pr(s) > k \Rightarrow$ if $z > s$ then $dR_i(z, s)/ds_i > k$. Hence, for each given value of z, it will be profitable for each firm i to increase s_i with one unit as long as $s < s^{max}$ and $s < z$. Therefore, $s(z) \ge z\ \forall\ z$ □

b: The FOC is: $dR_i(z_i, s_i, S_{-i})/dz_i = dC(z_i)/dz_i$. With symmetry we get $dR_i(z, s)/dz_i = dC(z)/dz_i \Rightarrow p \cdot (1 - F[z]) = dC(z)/dz_i$. First, we keep constant s. If z = 0 then $F[z] = e^{-\grave{\mu}(s)}$ and hence $dR_i(z, s)/dz_i = p \cdot (1 - e^{-\grave{\mu}(s)})$.

$d\{dR_i(z, s)/dz_i\}/dz = -p \cdot f[z+1] < 0$

$\lim_{z\to\infty} dR_i(z, s)/dz_i = \lim_{z\to\infty} p \cdot (1 - F[z]) = 0$.

Now, we consider the variable s. $dR_i(z, s=0)/dz_i = 0$ as $\grave{\mu}(0) = 0$, and hence $F[z] = 1$. $d\{dR_i(z, s)/dz_i\}/ds = p \cdot f[z] \cdot Pr(s) > 0$ and $\lim_{s\to\infty} dR_i(z, s)/dz_i = p \cdot (1 - F[z|\ \mu=n/m])$ as $\grave{\mu}(\infty) = n/m$ (see claim 2).

The only assumption made with respect to $dC(z)/dz$ is that it is strictly positive. As, at this point, there is no reason to impose a particular shape of the $dC(z)/dz$ curve, in figure A.2, we draw just one possibility, chosen for expositional convenience. We see that there is a function z(s), with $z(0) = 0$, $dz(s)/ds > 0$ and $\lim_{s\downarrow\infty} z(s) = z^{max} = \{z: dR_i(z, s=\infty)/dz_i = dC(z)/dz_i\}$. To prove that $z(s) \le s$, suppose $z = s$. Clearly, $F[z=s] = 1$. Hence, $dR_i(z=s, s)/dz_i = 0 < dC(z)/dz_i$. z^{max} is given by the intersection of the $dR_i(z, s=\infty)/dz_i$ and the $dC(z)/dz_i$ curve. Hence, if $d^2C/dz^2 \ge 0$ then one should have $dR(z=0, s=\infty)/dz_i > dC(0)/dz_i$, which gives $p \cdot (1 - e^{-n/m}) > dC(0)/dz_i$ or $n/m > -\ln(1 - \{dC(0)/dz_i\}/p)$. □

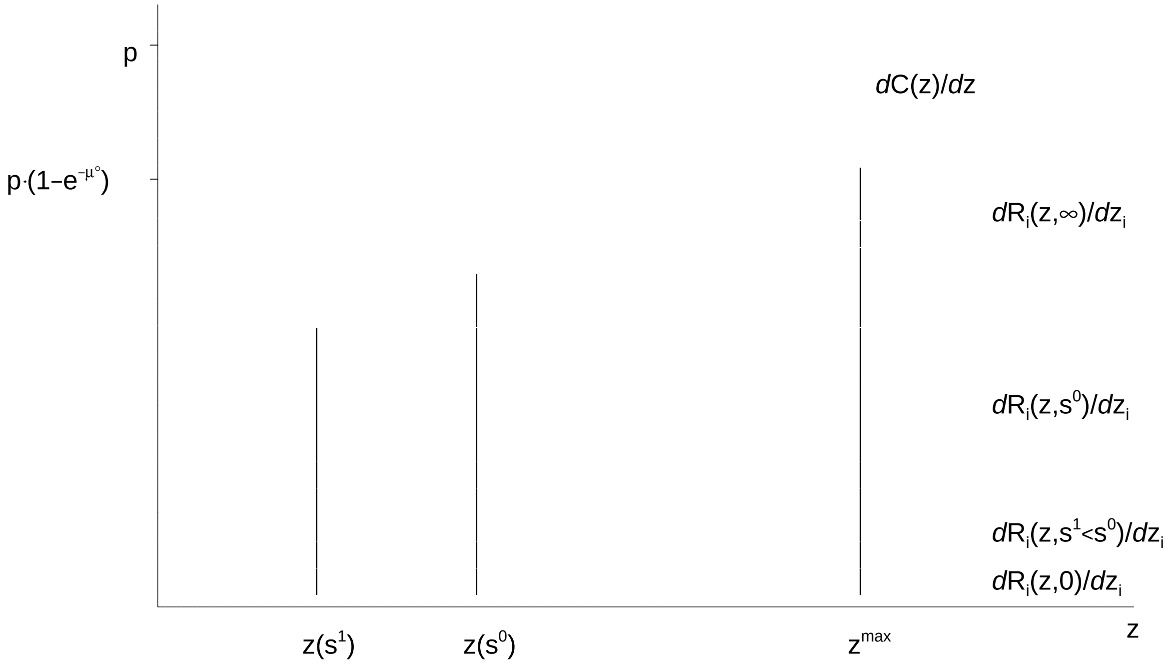


Figure A.2 FOC$^+$ with respect to production

Proof claim 5: The SOC for $\hat{t}$ being the optimal strategy for firm i is: $d^2V_i(z_i, s_i, S_{-i})/dz_i^2 \cdot d^2V_i(z_i, s_i, S_{-i})/ds_i^2 - (d\{dV_i(z_i, s_i, S_{-i})/ds_i\}/dz_i)^2 > 0$ and $d^2V_i(z_i, s_i, S_{-i})/ds_i^2 < 0$. For notational convenience we omit subscripts and arguments as much as possible.

$d^2V/dz^2 = p \cdot \{(1 - F[z+1]) - (1 - F[z])\} - d^2C/dz^2 = -p \cdot f[z+1] - d^2C/dz^2$

$d^2V/ds^2 = p \cdot Pr \cdot dF[z-1]/d\mu \cdot d\mu/ds - d^2K/ds^2 = -p \cdot Pr \cdot f[z-1] < 0$

$d(dV/ds)/dz = p \cdot Pr \cdot f[z]$

Thus, the remaining condition to check is:

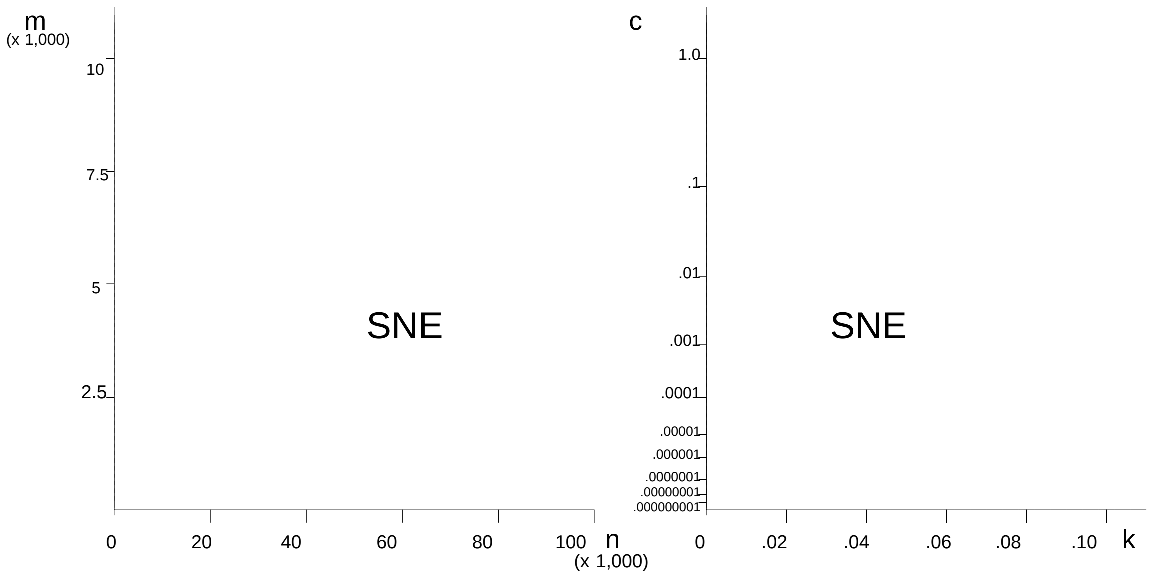
$\{-p \cdot f[z+1] - d^2C/dz^2\} \cdot \{-p \cdot Pr \cdot f[z-1]\} - \{p \cdot Pr \cdot f[z]\} > 0$

$\Rightarrow \; (d^2C/dz^2)/p > -f[z+1] + (f[z])^2/f[z-1]$

$\Rightarrow \; (d^2C/dz^2)/p > -f[z+1] + \{f[z+1] \cdot (z+1)/\mu\}^2/\{f[z+1] \cdot (z/\mu) \cdot (z+1)/\mu\}$

$\Rightarrow \; (d^2C/dz^2)/p > f[z+1]/z$ □

Numerical Analysis: We analyze numerically for which parameter values the FOC$^+$s and SOC for maximization of firm i's payoff V_i are fulfilled, with $V_i > 0$, when firm i chooses a strategy t^* given that all other firms choose the same strategy t^*. The parameters are k, p, N, m and those concerning the functions n(.) and C(.). For matters of convenience of the presentation, in the numerical analysis we restrict the production cost function C(.) to be such that $dC(z)/dz$ is linear through the origin, and hence $d^2C(z)/dz^2 = c$, with $c > 0$ (see corollary 1). We normalize p=1 and fix N = 100,000. So, the parameters to consider are k, c, n and m. In figure A.3.a we consider the importance of the numbers of firms (m) and interested consumers (n), fixing k and c. It seems reasonable to assume that the marginal costs of signaling k are a relatively small fraction of the price p. For example, sending a letter will not cost much more than a stamp, and other means of signaling might be even cheaper. We fix k = 0.01 and c = 0.001. As the marginal cost of production is c·z, the value of c chosen implies that a firm will never produce more than 1,000 units. We see that a SNE with strictly positive values of z and s exists approximately if $n/m \geq 8$. That is, given the other parameters, on average there should be at least 8 interested consumers per firm in the population, which does not necessarily mean that firms should actually get this number of clients. We have drawn no boundary of the shaded area as there are also various combinations of m and n outside this area for which a SNE exists. In any case, the number of interested consumers in the economy should be at least 1000 as k·N/p = 1000 (see proposition 2). In figure A.3.b the role of the values of the cost parameters k and c is considered, fixing the parameters m = 100 and n = 10,000. We put m low in comparison to n in order to reflect the idea of a '*division of labor*'. In figure A.3.b we use a somewhat unconventional scale on the vertical axis for expositional reasons. Clearly, the maximum value of k to consider is .10 as p·n/N = .10 (see proposition 2), while the maximum value for c is 1.0. For those values above the shaded areas for which a SNE does not exist, the FOC's or the condition V>0 are not satisfied. The SOC does not give problems.



a ***b***

Figure A.3 existence of SNE

In table A.1 we give some numerical examples. In the first six rows of table A.1 only the *'parameters'* m and n vary. If we compare, for example, row 3 with row 6, we see that when the relative number of firms and consumers (n/m) does not change the SNE remains the same. A comparison of row 4 with 5, where the number of firms (m) increases enormously, is also interesting. Nevertheless, in this example, each firm's market-making signaling activity does not decrease, but surprisingly increases, while its supply and expected profit fall dramatically. As the final three columns show, the economic situation changes from highly favorable for the firms to highly advantageous for the consumers. In the final six rows the parameters c and k vary. Notice that the probability of a firm being rationed is on average quite high, while the probability of negative profits is much more moderate.

m	n	c	k	z^*	s^*	V^*	Prob [firm rat.]	Prob [cons. rat.]	Prob [neg. prof.]
5	5,000	.001	.010	657	22,053	220.0	.33	.35	.00
100	5,000	.001	.010	61	4,641	1.1	.94	.01	.45
100	10,000	.001	.010	112	8,742	5.8	.87	.01	.26
10	37,500	.001	.010	967	3,196	470.0	.03	.74	.00
750	37,500	.001	.010	61	4,640	1.6	.93	.00	.42
750	75,000	.001	.010	112	8,740	5.8	.87	.01	.26
100	10,000	.010	.050	34	389	5.7	.60	.69	.12
100	10,000	.001	.050	85	1,323	3.3	.90	.27	.33
100	10,000	.010	.010	76	1,762	29.0	.21	.25	.00
100	10,000	.001	.010	112	8,742	5.8	.87	.01	.26
100	10,000	.010	.005	84	2,754	35.0	.15	.17	.00
100	10,000	.001	.005	112	17,480	5.8	.87	.01	.26

Table A.1 some numerical examples of Symmetric Nash Equilibria

Proof claim 6: $\lim_{n/m\to\infty} \Pr(s) = \lim_{n/m\to\infty} n/(m \cdot s) \cdot (1-e^{-m \cdot s/N}) = 1$

That is, each signal sent will surely lead to a consumer visiting its sender $\Rightarrow$ q = s. We know x = min{q, z} $\Rightarrow$ x = min{s, z}. Cost minimization requires s = z. Hence the firm's payoff $V = p \cdot z - C(z) - k \cdot z \Rightarrow$ FOC: $dV/dz = 0 \Rightarrow z(s) = \{z: p - k = dC(z)/dz\}$ ☐

Proof proposition 3: The numbers of agents in the economy influence the economic environment through the stochastic distribution of signals and the stochastic demand directed to any firm. The first can be characterized by the parameter of a Poisson distribution λ, where $\lambda = S/N$ is the expected number of signals received by any given agent. In the case of symmetry we get $\lambda = (m/N) \cdot s$. The stochastic demand directed to any firm is characterized by a Poisson distribution with parameter $\mu = (s/S) \cdot n \cdot (1 - e^{-S/N})$, which gives in the case of symmetry $\mu = (n/m) \cdot (1 - e^{-(m/N) \cdot s})$. Substitute $m = \alpha \cdot m$, $n = \alpha \cdot n$ and $N = \alpha \cdot N$, and observe that both λ and μ remain the same. Hence, for both consumers and firms nothing changes. ☐

Proof proposition 4: Given the set of parameters, individual firms choose z and s. S is simply the aggregate market-making activity: $S = m \cdot s$. Hence, $dS/dm = s + m \cdot ds/dm$. Now, $d(dV_i/ds_i)/d(n/m) = p \cdot \{d\Pr/d(n/m) \cdot F[z-1] + \Pr \cdot dF[z-1]/d\mu \cdot d\mu/d\Pr \cdot d\Pr/d(n/m)\} = p \cdot d\Pr/d(n/m) \cdot \{F[z-1] - \mu \cdot f[z-1]\}$. The term between brackets may be positive or negative depending upon z and μ, while the rest is positive. Hence the s(z) curve in figure 2 may move leftwards or rightwards, depending upon z and μ. Hence, the value of dS/dm is not yet determined. ☐

Proof proposition 6: The case of a firm i is already considered in the text: Prob $[(x_i - z_i) > 0] > 0$. Here we derive the probability that an interested consumer j (i.e., $z_j > 0$) is rationed. In a SNE, the probability for a buyer to be rationed because of lack of communication is $e^{-\lambda}$, where $\lambda = m \cdot s/N$. $0 < e^{-\lambda} < 1$ for $\lambda > 0$. If, instead, an interested consumer j receives one or more signals, he randomly chooses one firm to visit. This firm's supply is z. As customers are served on a first-come first-served basis, the probability to obtain its demand, then, is the probability to be among the first z customers in this firm's '*queue*', every place being equally probable. The number of visitors for a firm is given by a Poisson distribution with parameter μ: f[q]. When we approximate the number of '*rival*' customers visiting this firm by the same Poisson distribution, the probability that a buyer j, having received at least one signal, will be in the position to buy one unit is:

Prob [early enough] $= F[z-1] + \sum_{q=z}^{\infty} \{f[q] \cdot z/(q+1)\} = F[z-1] + z/\mu \cdot \sum_{q=z+1}^{\infty} f[q] = F[z-1] + z/\mu \cdot (1 - F[z])$

Thus, the probability that any given buyer j will not succeed in finding one unit in the period under consideration and will be rationed is: Prob $[(x_j - z_j) < 0] = 1 - (1 - e^{-\lambda}) \cdot \{F[z-1] + z/\mu \cdot (1 - F[z])\} > 0$

(This equation can be made more transparent:

$$\begin{aligned}\text{Prob } [(x_j - z_j) < 0] &= 1 - (1 - e^{-\lambda}) \cdot 1/\mu \cdot \{\mu \cdot F[z-1] + z \cdot (1 - F[z])\} \\ &= 1 - (1 - e^{-\lambda}) \cdot 1/\{(n/m) \cdot (1 - e^{-\lambda})\} \cdot Ex \\ &= 1 - (m \cdot Ex)/n\ ,\end{aligned}$$

where $m \cdot Ex$ = expected aggregate sales

n = aggregate demand)

Drawing i from *B* and j from *D* independently, we get Prob $[(x_i - z_i) \cdot (x_j - z_j) < 0] > 0$ for each pair $i \in B$, $j \in D$ and $z_j > 0$. ☐

Proof proposition 7: Just as in the non-cooperative case, the optimal strategy is the solution of a system of two equations. Now, however, the FOCs must be taken not only with respect to firm i's own strategy t_i , but also with respect to the strategies of the other firms $\mathbf{t}_{-i}$, because a change in t_i implies a simultaneous, equivalent change in $\mathbf{t}_{-i}$. Thus, a SCE is a solution to the following system of two equations:

$$dV_i/ds_i + dV_i/d\mathbf{s}_{-i} = 0$$
$$dV_i/dz_i + dV_i/d\mathbf{z}_{-i} = 0$$

From equation [1] we see that $\mathbf{z}_{-i}$ does not enter firm i's payoff, i.e., $dV_i/d\mathbf{z}_{-i} = 0$. Hence, the second of the FOCs does not change and the z(s) curve in figure 2 remains the same. Turning to the FOC with respect to signaling, we see that

$$\begin{aligned}dR_i/d\mathbf{s}_{-i} &= p \cdot d\mu_i/d\mathbf{s}_{-i} \cdot F[z-1] - z \cdot dF[z]/d\mu_i \cdot d\mu_i/d\mathbf{s}_{-i} \\ &= p \cdot d\mu_i/d\mathbf{s}_{-i} \cdot \{F[z-1] + z \cdot f[z]\} < 0 \quad \text{because } d\mu_i/d\mathbf{s}_{-i} < 0 \text{ (see claim 1).}\end{aligned}$$

That is, an increase in the signaling activity by each of the other firms, $\mathbf{s}_{-i}$, implies a decrease in the expected number of visitors for firm i, $\mu(s_i, S_{-i})$. Hence, the revenue for firm i of sending one additional signal will be lower when all other firms also send one additional signal simultaneously, than in the case where firm i had to take the strategies of the other firms as given. Thus, the dR_i/ds curves will be below the dR_i/ds_i curves in figure A.1, and for every value of z the value of s for which this FOC is satisfied will be lower, i.e., the new s(z) curve will be at the left of the s(z) curve in figure 2. As a result, the intersection of the s(z) and z(s) curves will occur at values z^c and s^c that are lower than z^* and s^*. That $V^c > V^*$ follows from definition 3 and the fact that $t^c \neq t^*$. ☐

Proof proposition 8: Prob [cons. rationed] = $1 - (m \cdot Ex)/n$, where $Ex = \mu \cdot F[z-1] + z \cdot (1 - F[z])$
$dEx/ds = d\mu/ds \cdot F[z-1] + \mu \cdot dF[z-1]/d\mu \cdot d\mu/ds - z \cdot dF[z]/d\mu \cdot d\mu/ds = d\mu/ds \cdot F[z-1]$
$d\mu/ds > 0$ (see claim 2) $\Rightarrow dEx/ds > 0 \Rightarrow d$Prob [cons. rationed]$/ds < 0 \Rightarrow$ as s decreases the Prob [cons. rationed] increases. Next, we consider the effect of the change in z. $Ex = z - \sum_{q=0}^{z-1} F[q]$ (see claim 3). $\Rightarrow$ $dEx/dz = 1 - F[z] > 0 \Rightarrow$ as z decreases the Prob [cons. rationed] increases. ☐

Appendix B. Notation

α	some constant
A	set of all agents
B	set of firms
C(z)	production function
D	set of consumers
$f[\ \mid \mu]$	p.d.f. with parameter μ
F[z]	$\sum_{q=0}^{z} f[q]$
k	*'marginal'* cost of signaling
K(s)	signaling function
λ	expected number of signals received by any given agent
m	number of firms
μ_i	expected demand directed to firm i
$\grave{\mu}$	expected demand directed to any firm in case of symmetry
n	number of interested consumers
N	number of agents in the economy
ω_i	endowments agent i
p	price of the commodity
$\overline{p}_i$	threshold price of consumer i
Pr	probability that any given signal leads to a consumer making a visit
q_i	demand directed towards firm i
$R_i(.)$	expected gross revenue firm i
s_i	number of signals sent by firm i
$\hat{s}_i$	s_i with the first-order-plus-symmetry conditions satisfied
S_{-i}	aggregate number of signals sent by all other firms
S	aggregate number of signals sent by all firms
t^*	SNE strategy t
t^c	SCE strategy t
$\hat{t}$	strategy t with the first-order-plus-symmetry conditions satisfied
$t_i \equiv (z_i, s_i)$	strategy of firm i
$\mathbf{t} \equiv (\mathbf{z}, \mathbf{s})$	complete vector of strategies of all firms
$\mathbf{t}_{-i} \equiv (\mathbf{z}_{-i}, \mathbf{s}_{-i})$	vector of strategies of all other firms
τ	time index
$U_i(.)$	utility function consumer i
$V_i(.)$	payoff or expected profit of firm i
x_i	actual transactions by firm i
z_i	output or effective supply of firm i
$\hat{z}_i$	z_i with the first-order-plus-symmetry conditions satisfied