Universitat Pompeu Fabra Barcelona
Department of Economics and Business

Economics Working Paper Series

Working Paper No. 1835

Entropy, directionality theory and the evolution o income inequality

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April 2022

Entropy, Directionality Theory and the Evolution of Income Inequality*

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April 7, 2022

Abstract

A macro-evolutionary theory of income inequality is proposed that is based on a society's dynamic income generating process. Two types of processes are distinguished, namely *dispersing* and *concentrating* ones. A basic result shows that dispersing processes provide a selective advantage for more balanced and mutualistic interaction; whereas concentrating ones favor weaker, less balanced and less mutualistic interaction. We also show that societies with more balanced and mutualistic interaction induce more income equality and a non-stratified society, while less balanced and less mutualistic ones induce more inequality and a possibly stratified society. Also, more equal societies are more resilient in the sense of being quicker to recover from shocks and return to steady state than less equal ones. Stylized examples of pre-modern and modern societies are briefly discussed.

Keywords: Income generating process; interaction network; entropy; cooperation; mutualism; income inequality; fragility; pre-modern society. *JEL Classification*: C73; D31, Z13.

*I am deeply indebted to Professor Lloyd Demetrius, who introduced me to directionality theory and with whom this project started. While freeing him from any responsibility over the paper, it is clear that it would not have been written without his encouragement, vital discussions and comments at various stages of the project. I am also grateful to two anonymous referees and the editor, Friederike Mengel, for their excellent feedback, and thank Ramses Abul-Naga, Jonas Alcaina, Isaac Baley, Larry Blume, Samuel Bowles, Amil Camilo, Patrick François, Cecilia Garcia-Peñalosa, Ramon Marimon, Kiminori Matsuyama, Eulalia Nualart, Giacomo Ponzetto, Joel Sobel and Jaume Ventura for insightful comments and conversations, as well as audiences in Abidjan, Aix-en-Provence, Alicante, Barcelona, Barcelona (Bellaterra), Bologna, Cambridge, Copenhagen, Florence (Fiesole), Louvain-la-Neuve, Luxembourg, Marseille, Montreal, Reus, Toulouse and Vienna. Finally, I thank AMSE-IMéRA at Aix-Marseille Université and the European University Institute in Florence for generous hospitality, and acknowledge financial support from Grants PID2020-115044GB-I00//AEI/10.13039/501100011033 and ECO2017-89240-P (AEI/FEDER, UE), and from the Spanish Agencia Estatal de Investigación (AEI) through the Severo Ochoa Programme for Centres of Excellence in R&D (Barcelona School of Economics CEX2019-000915-S). All errors are my own.

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1 Introduction

The question of why some societies are wealthier and/or exhibit stronger growth than others has received much attention within economics, as has the question of possible consequences of income inequality for development and growth (Acemoglu [1], Aghion et al. [6], Alesina and Rodrik [10], Bénabou [15], Galor [47]). Less studied is the question of why societies differ so much in terms of inequality, and why such different levels of inequality persist for so long across societies (some exceptions include Acemoglu and Robinson [4], Atkinson and Bourguignon [13], Galor and Zeira [48], Kuznets [52], Piketty [59, 61]; further exceptions focusing on pre-modern societies include Boix [17], Borgerhoff Mulder [20], Bowles et al. [22], Mayshar et al. [54]). This paper addresses this question by taking a novel entropy-based macro-evolutionary approach that identifies specific aspects of a society's income generating process as critical for the emergence and persistence of disparate levels of income inequality. To focus the analysis, we limit ourselves to pre-modern or very early societies with state that appear to be a more direct fit for the simple types of income generation processes considered here.[1]

The macro-evolutionary analysis introduced, distinguishes income generating processes between what we call *dispersing* and *concentrating* ones. Dispersing processes are characterized by *scarce* and *diverse* resource environments, meaning that how much individuals can generate for their own group or for another group is limited (scarce), and there are multiple or spread out sources of income generation (diverse). Both properties are shown to reinforce each other in promoting stronger and more mutualistic interaction, and eventually in generating and sustaining a more equal income distribution. Within pre-modern societies, some types of societies that correspond to this kind of income generation are hunter-gatherer societies or horticultural societies that are often found to be cooperative and equal.[2] By contrast concentrating processes are characterized by *abundant* and *singular* resource environments, which allow for relatively high rates of income generation for individuals' own group or for another group (abundant), and have few, concentrated sources of relatively high income generation (singular). Again, both properties reinforce each other and promote weaker and less mutualistic interaction and eventually generate a less equal income distribution. Pre-modern society examples for this are fertile agricultural or pastoral societies that are often less cooperative and less equal. Thus certain eatures of income generating processes act as basic forces pushing towards more or less balanced interaction and through that, also towards more or less income equality. To the extent that the basic underlying productive structure and resource environment are fixed for given societies over long periods of time, and through that whether the associated income process is dispersing or concentrating, they can explain both the persistence of different levels of income inequality, as well as the emergence of perpetuating trends.

[1] We do not exclude, however, that the basic forces pushing for more or less equality presented in this paper, that are derived from the underlying income generating processes, may apply to more modern and complex societies.

[2] See Borgerhoff Mulder et al. [20], Bowles et al. [22], Smith et al. [69], Flannery and Marcus [44] for discussions and comparisons of various examples of pre-modern societies. We discuss related examples in Section 5.

More specifically, the paper introduces a simple dynamic macroeconomic framework that combined with an evolutionary analysis shows that:

(I) When the macro-economic income generation process is *dispersing* (that is, *scarce* and *diverse*), stronger and more balanced and mutualistic interaction within and across groups is more effective at generating income. A society in such an environment tends to become more equal and more robust.

By contrast:

(II) When the macro-economic income generation process is *concentrating* (that is, *abundant* and *singular*), weaker and possibly less balanced and mutualistic interaction is more effective. Such a society tends to become more unequal and more fragile.

Thus, starting from a mathematical macro-model of interaction and income generation, the present approach allows us—within a single framework—to address three critical dimensions of the problem alluded to above, namely, the origin and spread of income equality and inequality, the positive correlation between balanced interaction or cooperation and income equality and the social and economic instability of highly unequal societies. To the best of our knowledge this has not been addressed so far in a single framework in the economics literature.

Inequality has traditionally been addressed in economics in terms of supply and demand for different types of factor endowments (Atkinson and Bourguignon [13] contains a survey). These models are typically formulated within the context of (competitive) markets and are sufficiently flexible to provide insight into a variety of empirical aspects of inequality, including the explanation of a widening gap between skilled and unskilled labor due to technological change. An important limitation, is that they take as given not just the socio-economic context, but also the distribution of factor endowments, as well as the mechanisms that determine the compensation of the different factors. Not surprisingly, they cannot explain certain empirical observations that set apart stratified and non-stratified societies, namely, (*i*) the positive correlation between economic inequality and intergenerational immobility; (*ii*) the negative correlation between income inequality and social cohesion: non-stratified societies tend to have high levels of social cohesion and trust, stratified societies tend to have low levels of social cohesion and trust, which in turn can lead to a dysfunctional society; (*iii*) the negative correlation between economic inequality and robustness or resilience, that is, the capacity of the economic institutions to maintain a high level of economic and political stability and functionality in spite of internal and external shocks.[3]

More recently, the literature has addressed questions of persistent inequality and intergenerational mobility (Piketty [59] contains a survey). It has integrated socio-cultural factors in its framework, which include: the transmission of wealth from parents to children through inheritance, the intergenerational transmission of ambition and the recognition of economic success and

[3]For evidence on (*i*)–(*iii*), we refer to, respectively, the 2012 US Economic Report of the President [27], the 2011 UN Human Development Report [78], and Wilkinson and Pickett [82]; see Section 7 below for further discussion.

social prestige, the statistical discrimination or the prevalence of self-fulfilling discriminatory beliefs. These socio-cultural perspectives have helped to elucidate the limitations of the supply and demand model and to resolve some of its anomalies. But they also have shortcomings in that they do not account for the large variation *across* societies with respect to their key features, namely, the transmission of wealth through inheritance, the intergenerational transmission of ambition and work ethic, and the extent of discrimination.

Societies which show significant differences in economic inequality typically differ in terms of their income generating processes. Societies with severe economic inequality often have economies which are largely dependent on a small (singular) set of relatively abundant dominant economic resources; more equal societies tend to have economies based on relatively scarce and diverse sets of resources (Piketty [60, 61]). Our analysis shows that the underlying income generation process confers advantages to different interaction modes, which in turn, whether more or less mutualistic, further feed back into the income generating process. Over the long run, the underlying interdependence thus reinforces different ways of interacting, which in turn can be seen as directly reinforcing basic norms of behavior (Tomasello [77]). In his more recent book, Piketty [61] highlights the importance of having an ideology in place, often reinforced by the elites, that justifies inequality and that is accepted by the members of the society in order to support significant societal stratification; this point is also emphasized by Flannery and Marcus [44] in their study of early societies. Our approach further suggests that the income-generating process itself may contribute towards reinforcing certain behavioral norms, such as more or less mutualistic behavior or sharing between individuals within and across groups.

At a formal level, our analysis revolves around a statistical measure called *evolutionary entropy*, which has its origins in the ergodic theory of dynamical systems (Demetrius [31, 33]). In our macroeconomic context, it is a measure of the number of productive interaction links between individuals within and across groups in the economy. A low entropy network is described by few and typically weakly mutualistic interaction links and is characteristic of unequal societies. A high entropy network is described by a large number of strongly mutualistic interaction links and is characteristic of equal societies. The significance and centrality of the evolutionary entropy concept towards our understanding of the origin of equal and unequal societies derives from its relation to several key socio-economic variables. First of all, as mentioned, it is a measure of the strength and balancedness in the sense of mutualistic or reciprocal interactions and hence of *cooperation* in the underlying economic network. Moreover, it is negatively related with income inequality measured by the Theil index (Proposition 1), positively related with Field's measure of income equalization (Proposition 2), and positively related with the robustness or resilience of the network (Theorem 3). Interestingly, a rich Markovian stochastic structure is derived from an otherwise simple, linear dynamic macro-model. The stochastic structure is best understood through the *genealogy model*, which also shows the centrality of interactions and the entropy measure in understanding income inequality. Establishing the connection between entropy, cooperation and income inequality is novel and constitutes an important contribution of the paper.

Building on this stochastic structure, we study an evolutionary process, whose outcome is characterized by the *entropic selection principle* (stated within our macroeconomic context as Theorem 1). This is the central tenet of *Directionality theory* (see Demetrius [33], Demetrius and Gundlach [34] for recent expositions). It describes the outcome of competition between an incumbent and a variant population, and asserts that, when the income generation process is *dispersing* (in the sense of *scarce* and *diverse*), communities which have a higher evolutionary entropy will have a selective advantage and increase in frequency; whereas when the income generation process is *concentrating* (in the sense of *abundant* and *singular*), communities which have lower entropy will have a selective advantage and increase in frequency. At a basic level, the *entropic selection principle* can be seen as partitioning income generation processes into what we refer to as *dispersing* and *concentrating* ones.[4]

As an example of a *dispersing* process consider rice agriculture. It is well known that it requires hard work but also joint participation of farmers and villagers in irrigating and maintaining the rice fields. A more interactive society performs better in such a situation, whereas a less interactive one would not do as well. By contrast, as an example of a *concentrating* process, consider wheat agriculture in a relatively fertile area. In this case, a less interactive society, but one where corresponding households concentrate on their own productive plots performs better than one that is more interactive but does not sufficiently exploit the highly productive plots. Consistent with this, it has been found that more interdependent culture has developed (and persists today) in regions of China where rice agriculture was prevalent, and more individualistic culture has developed in regions where wheat was prevalent (Talhelm et al. [76]).

Because of the negative relation between changes in evolutionary entropy and changes in income inequality (Proposition 1), the *entropic selection principle* allows us to relate the income generation process to changes in income inequality (Theorem 2). Specifically, in environments that are dispersing, societies with low income inequality will be more successful and will grow faster and steadier; while in environments that are concentrating, societies with high inequality will be more successful. Thus whether a society tends to be more or less equal depends on aspects of its underlying income generating process.

Another important tenet of Directionality theory is the *complexity-stability principle* (stated here as Theorem 3). In the context of our paper, it implies that unequal or highly stratified societies are inherently unstable and sensitive to shocks; they take longer to return to steady state than more equal ones. More equal societies are more stable and more robust to shocks; they are quicker in returning to steady than less equal societies.

Related Literature. The framework introduced seems particularly suited for the study of pre-modern societies. Borgerhoff Mulder [20] and the articles in the special issue of *Current Anthropology* on "Intergenerational Wealth Transmission and Inequality in Premodern Societies" (Bowles et al. [22], Smith et al. [69]) are especially relevant. They provide first comprehensive

[4]Note that Theorem 1 covers all possible cases, including, scarce and singular, and abundant and diverse.

estimates of intergenerational wealth transmission and inequality in pre-modern societies. Besides containing a wealth of empirical data and insights, their theoretical approach is complementary to ours in that it emphasizes the role of capital, particularly material capital as a driver of inequality. To the extent that the introduction and extensive use of material capital leads to what we call an abundant income generation process, then the two theories may be connected. There is also a long-standing literature that identifies the emergence of hierarchical societies from more egalitarian hunter-gatherer or early agrarian societies with the capability to produce surplus from increases in productivity (e.g., Lenski [53]). This literature is discussed and qualified in Mayshnar et al. [54] and Scott [66], who show the importance that some of the key goods produced also be appropriable by the elites in order to maintain and reinforce a hierarchical structure. To the extent that appropriability affects the income generation process and is reflected in the underlying interaction matrix, these results are not inconsistent with our mechanism.

Another literature related to our approach is the multidisciplinary literature on the evolution of cooperation in human societies, which includes theoretical, experimental and empirical work, also covering different types of societies and over different periods of human history (Boix [18], Bowles and Gintis [21], Boyd and Richerson [23], Henrich [49], Mesoudi [57], Tomasello [77] contain surveys and expositions of different aspects of the literature). This literature contains a variety of models which study the evolution and coevolution of cooperation and other related norms and institutions under various assumptions and perspectives, and includes explicit agent-based, multilevel and gene-culture analyses (see especially Bowles and Gintis [21] for a detailed overview). What we contribute to this rich literature is a reduced form model that is embedded in a simple macroeconomic framework that we view as complementary to the models studied in the literature. For us, cooperation is modeled in terms of the evolutionary entropy of the underlying economic network and its income generating process that evolves at the macro-group level.[5]

Also, a vast literature studies how institutional and non-institutional factors may explain long run differences in various economic variables across countries (Acemoglu and Robinson [4], Currie et al. [30], Engerman and Sokoloff [40] and Spolaore and Wacziarg [72] contain surveys). Although most of the focus has been on understanding growth and development, part of the literature has studied institutional and non-institutional effects on inequality, for example, whether a country being politically organized as a democracy may contribute to income being more equally distributed (Alesina and Rodrik [10], Meltzer and Richard [55] and more recently Acemoglu et al. [3]). To this general literature, our approach contributes a novel entropy-based mechanism, that formally puts the countries' income generating processes at the center of the analysis of long-run determinants. The income generating process affects the interaction between individuals which in turn contributes to the coevolution of stronger or weaker behavioral norms of cooperation and sharing. In our approach, whether a society is more or less equal is not necessarily a matter of whether or not it is a democracy, or whether it is capitalistic, or industrially or technologically

[5]Establishing a formal connection to some of the explicit agent-based or multilevel models in the literature goes beyond the scope of this paper and is left for further research.

developed, but rather it depends on specific parameters of the income generating process, which in turn affect the society's underlying social and behavioral norms of cooperative behavior.

Finally, the paper also contributes to the rapidly growing literature on fluctuations and the macroeconomics of networks. Acemoglu et al. [2], Carvalho and Gabaix [25], and Gabaix [45], model aggregate fluctuations through the propagation of individual firm-level shocks across the economy as a function of the network structure and the relative sizes of the firms. The present application of Directionality theory studies networks of dynamic economic interactions between different groups and derives a rich stochastic structure from the average group level interactions. Evolutionary entropy, as an analytic measure of network structure, may prove to be a potentially useful conceptual tool for studying dynamic aggregate behavior.

The remainder of the paper is organized as follows. Section 2 presents the macroeconomic and interaction framework. Section 3 introduces the evolutionary process and shows our key analytical result, the entropic selection principle. Section 4 contains the economic results of the paper and establishes the relation between evolutionary entropy and measures of income inequality and income equalization. Section 5 contains examples of simple economies that include a discussion of pre-modern and very early modern societies from the point of view of the present framework. Section 6 addresses the fragility of unequal societies, and Section 7 concludes. The proofs and some background material are contained in the Appendix.

2 Macroeconomic Framework

Our analysis takes as its point of departure a society's intertemporal income generating process, which depends on the economic interaction of individuals within and across different groups and which we capture by what we call the interaction matrix A, described below. It reflects both allocation and production of goods and services. Moreover, it gives rise to the Markov matrix P, which provides the basic underlying stochastic structure of our model.

Population and Income. Consider a society with a **total population** $N(t)$ of individuals distributed in d **groups**, written as,

$$N(t) = \sum_{i=1}^{d} n_i(t), \tag{1}$$

where $n_i(t)$ denotes the number of individuals (or households) in group i in period t. The groups may be thought of as describing occupational groups or extended families, which, for expositional purposes, we assume throughout to be of equal size $n_i(t) = N(t)/d$.[6] The individuals engage in several activities in order to produce and exchange commodities and services. The **total income**

[6] This allows for easier comparison of income levels across groups. Clearly, the theory can be adapted to account for classes of significantly different sizes.

(or production) $Y(t)$ of the economy is the sum of the income of the different groups,

$$Y(t) = \sum_{i=1}^{d} y_i(t), \tag{2}$$

where $y_i(t)$ denotes the income of group i in period t; we also write $y(t) = (y_1(t), \ldots, y_d(t))^\top$ for the (column) vector of incomes of the different groups.[7] Prices are not modeled explicitly in this set-up. We assume all goods (outputs, inputs, services) in all periods to be measured by a fixed common numéraire commodity.

Interaction Matrix and Income Generation Process. The individuals are jointly involved in a process of income generation, where, besides exchanging and transforming resources or producing goods and services within groups, they also exchange and transform resources *across* groups. We summarize this by what we call the society's **intertemporal income generation process**, that is, a law of motion for income in the different groups,

$$y(t+1) = Ay(t), \tag{3}$$

where

$$A = (a_{ij}), \qquad a_{ij} \geq 0, 1 \leq i, j \leq d, \tag{4}$$

is a $d \times d$ matrix, which we refer to as the society's **interaction matrix**. The elements a_{ij} measure the rate of contribution or transformation of income in group j in period t towards income in group i one period later.

The interaction matrix A can be thought of as representing a directed graph over d nodes (for the d groups), where a_{ij} corresponds to a directed link from node j to node i of intensity a_{ij}. We assume the matrix is *irreducible*, meaning that there exists a finite $n \geq 1$ such that $a_{ij}^{(n)} > 0$ for all i, j, where $a_{ij}^{(n)}$ is the (i, j)-th element of A^n.

We assume the entries of the interaction matrix are initially fixed and that the economy is at steady state. The Perron-Frobenius theorem guarantees that the matrix A has a maximal, positive and real-valued eigenvalue λ with corresponding positive, real-valued right and left eigenvectors, $v = (v_1, \ldots, v_d)^\top$ and $u = (u_1, \ldots, u_d)$, respectively, such that,

$$Av = \lambda v \qquad \text{and} \qquad uA = \lambda u. \tag{5}$$

If we further assume the vectors are normalized such that $\sum_{i=1}^{d} v_i = 1$ and $\sum_{i=1}^{d} u_i = 1$, then v_i represents group i's steady state share of *received* income, and u_i represents group i's steady state share of *contributed* income. We also refer to $g = \log \lambda$ as the **growth rate** at steady state.

Eq. (3) can be viewed as representing a reduced form of the steady state of a process of economic interaction between individuals in different groups in a society with exchange (including

[7] Throughout the paper, we denote the transpose of a vector (or matrix) x by $x^\top$.

on markets) and with production (including with capital). Mathematically speaking it can be viewed as a linear approximation to a more complex, nonlinear and higher dimensional system, for example, of the form $z(t+1) = F(z(t))$, where $z(t) = (x(t), y(t))$ is an $m+d$ vector with $x(t)$ a vector of m further variables, besides $y(t)$ introduced above, with $F : \mathbb{R}^{m+d} \to \mathbb{R}^{m+d}$ a function that is not necessarily linear. Importantly, following Solow and Samuelson [71], if all equations $z_k(t+1) = F_k(z(t))$ are homogenous of degree one, then, at steady state, all variables grow at the same rate, say g, and satisfy $z_k(t) = \psi_{k,\ell} \cdot z_\ell(t)$ with some constant $\psi_{k,\ell} > 0$, for any pair $0 \leq k, \ell \leq m+d$. Thus, the framework allows for underlying production functions of the Cobb-Douglas form with capital variables, as long as the functions (and all equations in the model) are homogeneous of degree one. An example of such a system for early pre-modern societies and with a composite measure of capital, possibly used jointly between the groups, is presented in Appendix A.4. It can be seen as providing a background for almost all the examples discussed throughout the paper.

Interactions, Evolutionary Entropy and Productive Potential. From the interaction matrix A, we can define the **Markov matrix**,

$$P = (p_{ij}) = \left(\frac{a_{ji} u_j}{\lambda u_i} \right), \quad 0 \leq p_{ij} \leq 1, 1 \leq i, j \leq d, \tag{6}$$

where an element p_{ij} is the fraction of income contributed by group i that is contributed from group i to group j. The rows of the matrix P always sum to one. Let $\pi = (\pi_1, \ldots, \pi_d)$ denote the corresponding stationary distribution satisfying $\pi P = \pi$, then π_i (≥ 0) can be interpreted as the fraction of total income that "passes" through group i. It can also be shown that $\pi_i = \frac{u_i v_i}{\sum_i u_i v_i}$. The *genealogies model* presented below at the end of this section provides further intuition for the underlying probabilistic structure.

Next we define the **evolutionary entropy**,

$$H = -\sum_{i=1}^{d} \pi_i \sum_{j=1}^{d} p_{ij} \log p_{ij}, \quad 0 \leq H \leq \log d, \tag{7}$$

where $H = \log d$ and $H = 0$ indicate, respectively, maximal and minimal entropy; H is our measure of *cooperation* and is the central concept of the approach of the present paper.

The evolutionary entropy H is a measure of the strength and balancedness of overall interaction between individuals in the economy, within and across groups. Figure 1 represents the interaction networks associated to high entropy (left-hand panel) and low entropy income processes (right-hand panel). In the former there are strong and balanced interactions between all groups, describing a high entropy (and strongly reciprocal or mutualistic) network; in the latter, the interactions are weaker and less balanced and are directed towards a single group (node 1, since $a \gg \epsilon > 0$ and $\epsilon \approx 0$), describing a low entropy (and weakly reciprocal or mutualistic) network. More technically, the evolutionary entropy describes the effective number of productive

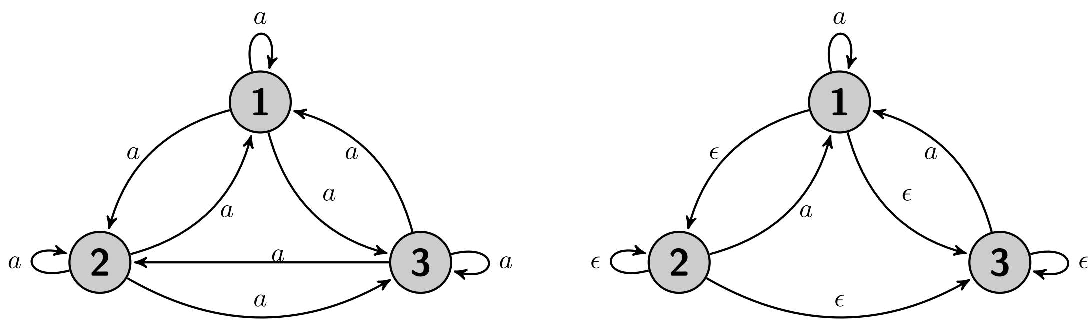

Figure 1: High entropy (left) and low entropy (right) interaction graphs, $a \gg \epsilon > 0$, $\epsilon \approx 0$.

cycles of interaction between individuals in the different groups thereby directly measuring an aspect of mutualism or reciprocity of interaction within and across groups.[8] Besides being a measure of *cooperation* in the sense of strong and mutualistic or reciprocal interaction, H is further related to an economy's robustness or resilience in the sense of the rate at which it returns to steady state after an arbitrary shock (see Theorem 3 in Section 6).

Scarcity and Diversity. The central tenet of our evolutionary analysis, the *entropic selection principle*, distinguishes the income generation process along two dimensions, namely, what we refer to as *scarcity* and *diversity*. As is clear from the proof of the entropic selection principle, the economy's selective advantage (see Eq. (15)) depends on these two measures as well as on the direction of change of the society's level of entropy. We define them formally.

Scarcity. The parameter used to define scarcity is the **productive potential** defined by,

$$\Phi = \sum_{i=1}^{d} \pi_i \sum_{j=1}^{d} p_{ij} \log a_{ji}, \tag{8}$$

[8]A fundamental characterization of *evolutionary entropy* relates it to a measure of the number of cycles of interaction of the associated interaction network (see Demetrius and Gundlach [34], pp. 5461-62): Fix a group $\alpha \in \{1, 2, \ldots, d\}$ which can be thought of as a node of the corresponding interaction network and consider an arbitrary path $\widetilde{\alpha}$ of length $n \geq 1$ that starts and ends at α without otherwise passing through α (but possibly repeatedly passing through other nodes $\beta_k \neq \alpha$), written as

$$\widetilde{\alpha} = [\alpha, \beta_1, \ldots, \beta_{n-1}, \alpha], \text{ where } \beta_k \in \{1, 2, \ldots, d\}, \beta_k \neq \alpha.$$

Let $\widetilde{X}_\alpha$ be the set of all such paths of arbitrary length $n \geq 1$. Then it can be shown that, for any $\alpha \in \{1, 2, \ldots, d\}$,

$$H = \frac{-\sum_{\widetilde{\alpha} \in \widetilde{X}_\alpha} p_{\widetilde{\alpha}} \log p_{\widetilde{\alpha}}}{\sum_{\widetilde{\alpha} \in \widetilde{X}_\alpha} |\,\widetilde{\alpha}\,|\, p_{\widetilde{\alpha}}} \equiv \frac{H_\alpha}{T_\alpha},$$

where, $p_{\widetilde{\alpha}} = p_{\alpha,\beta_1} p_{\beta_1,\beta_2} \cdots p_{\beta_{n-1},\alpha}$ is the probability of the given cycle $\widetilde{\alpha}$, and $|\,\widetilde{\alpha}\,|$ denotes its length (i.e., the number of nodes through which it passes). H_α is a measure of the average uncertainty of the cycles starting at α, and T_α is the average length of such cycles. Thus the characterization shows that the evolutionary entropy is directly related to the number of paths connecting individuals in a group to individuals in the own and other groups in the network. The characterization makes clear that the more *mutualistic* the network is, the higher the H_α's are and the lower the T_α's are, and hence the higher the entropy is, and vice versa. Demetrius [33] and Demetrius and Gundlach [34] contain further discussions, also with respect to other measures of entropy.

which can be interpreted as an average of the "$\log a_{ji}$'s" under the Markov process associated with P. With Eq. (7) it can be shown to satisfy $\Phi = g - H$.

When $\Phi < 0$, we say the income generation process is **scarce** and when $\Phi > 0$ we say it is **abundant**. Given Eq. (8), the condition $\Phi < 0$ implies that the (individual) intertemporal "productivity rates" a_{ii} and a_{ij} are predominantly < 1, and therefore the environment exhibits a certain degree of scarcity and limited abundance; $\Phi > 0$, on the other hand, implies productivity rates that are predominantly > 1, which is typical of environments exhibiting a certain minimum degree of abundance or concentrated sources of high returns and/or individual contributions to income generation.[9]

Diversity. The parameter for diversity is $\gamma = 2\sigma^2 + \kappa$. We say the income generation process is **diverse** if $\gamma > 0$; we say it is **singular** if $\gamma < 0$. As mentioned, σ^2 is a measure of variance and κ a measure of skewness of the $\log a_{ij}$'s and hence of the sources of income generation. The condition $\gamma < 0$ holds when the variance measure σ^2 is small and the skewness measure κ is negative, so that the density over the $\log a_{ij}$'s is concentrated on few links with sufficiently large values. This reflects what we call a *singular* income generating process with few but large and dominating sources of productivity. The condition $\gamma > 0$ holds in the opposite case with many small and spread out sources of productivity, which we refer to as *diverse*.

Dispersing and Concentrating. Finally, we also say that the income process is **dispersing** if it is both scarce and diverse so that $\Phi < 0$ and $\gamma > 0$, and we say it is **concentrating** if it is both abundant and singular so that $\Phi > 0$ and $\gamma < 0$.

Genealogies Model and Macroscopic Variables. The point of departure for our analysis of cooperation is the society's income generation process summarized by the matrix A of Eq. (3). The complexity of its underlying interactions are modeled through what we call the *genealogies model*. It introduces a stochastic structure that is based on the ergodic theory of random dynamical systems, which provides a general formalism for generating a family of *macroscopic variables* from the matrix A (see Arnold et al. [12] or Demetrius [33]). These macroscopic variables capture critical aspects of dynamic aggregate behavior, besides the growth rate g, such as the entropy H and the productive potential, variance and skewness measures Φ, σ^2, and κ mentioned above. Before introducing the formalism, we provide some intuition to motivate the space Ω.

Heuristics for the Genealogies Model. Consider a society of foragers, where individuals collect, share, exchange and ultimately consume, say, apples. This heuristics is inspired by interpreting the interaction matrix A as a generalized Leslie matrix, typically used to model population dynamics by age groups (Arnold et al. [12], Demetrius [33]). While in such a population model, the quantities of interest are the numbers of individuals in each age group, in our model the quantities of interest

[9] The notion of scarcity (abundance) does not perfectly represent the condition $\Phi < 0$ (> 0), since an environment with $\Phi < 0$ may nonetheless have several relatively large rates $a_{ij} < 1$, evenly distributed over several interactions, and, for example, have a greater growth rate and be in some sense "more abundant" than one with $\Phi > 0$ that nonetheless has one or more rates $a_{ij} > 1$. In other words, Φ also distinguishes between dispersed vs. concentrated sources of income generation.

are total apples in each group (household or extended family). Furthermore, in the population model, mothers give birth to one or more daughters, allowing one to associate to each daughter a unique mother. Here, assume individuals collect apples, but before collecting one or more apples, each individual needs to consume an apple so that each apple collected can be associated to a unique apple consumed. In turn the apple consumed was either collected by the individual himself or it was contributed from another individual from the same group or from another one. What is important is that, for each apple consumed or collected, one can trace the sequence of all "ancestor" apples with the corresponding group of the individual who contributed the given "ancestor" apple. Thus, if we fix an apple of an individual in, say, group 1 at time t_0, we can go backwards in time and follow all the successive ancestors of that apple. Doing so for, say, 5 periods, from t_0 to $t_0 - 5$ the corresponding sequence might take the form (the t_0 entries are underlined for expositional purposes):

$$\cdots \longrightarrow 3 \xrightarrow{a_{23}} 2 \xrightarrow{a_{12}} 1 \xrightarrow{a_{31}} 3 \xrightarrow{a_{13}} 1 \xrightarrow{a_{11}} \underline{1} \longrightarrow \cdots$$

with ancestors in the groups:

$$(\ldots, x_{t_0-5}, x_{t_0-4}, x_{t_0-3}, x_{t_0-2}, x_{t_0-1}, \underline{x_{t_0}}, \ldots) = (\ldots, 3, 2, 1, 3, 1, \underline{1}, \ldots).$$

The interpretation is simply that at t_0 the apple consumed by the individual in group 1 ($x_{t_0} = 1$), was contributed by someone in group 1 ($x_{t_0-1} = 1$), which in turn was contributed by someone in group 3 ($x_{t_0-2} = 3$) and so on until in period $t_0 - 5$ the apple consumed in group 2 was contributed from someone in group 3 ($x_{t_0-5} = 3$). These sequences, which are the elements of our space Ω, can be followed backward (direction of ancestor/contributor) and forward (direction of successor/reciever). The different sequences and their distribution in the space Ω can be used to describe the intensity and balancedness of interaction in the network associated to the matrix A. Specifically, in this context, the *evolutionary entropy* H associated to the Markov chain with Markov matrix P and stationary distribution π, measures of the average uncertainty about what the ancestor group of a given apple is (one step backward), given the current group of that apple. The larger the uncertainty the larger the entropy, which therefore is also natural measure of *cooperation* in the sense of strong and balanced interaction, derived from the links in the network graph G associated to the matrix A.

We now describe the formalism which we refer to as the **genealogies model** (see Arnold et al. [12] or Demetrius [33] and also Appendix A.3 for more details), which provides formal definitions of the space Ω and of macroscopic variables such as H, Φ, σ^2 and γ. This formalism embeds a stochastic model of productive interactions in our linear dynamic macro-model of (Eq. (3)). It can be skipped on a first reading.

Genealogies Model. Consider the macro-model of Eq. (3) and let G denote the directed graph associated with the interaction matrix $A = (a_{ij})$ over the nodes $D = \{1, 2, \ldots, d\}$, which in our model coincide with the d groups. Figure 1 above illustrates two examples of graphs over three nodes. To study such systems at steady state, let

$$\Omega = \{x \in X : a_{x_{\nu+1}x_\nu} > 0\}, \qquad \text{where } X = \prod_{\nu=-\infty}^{\infty} D_\nu \text{ and } D_\nu = D.$$

The set Ω, which we refer to as the **phase space**, is the set of all paths (or *genealogies*) of the graph G, associated with the matrix A, of the form (again, the $\nu = 0$ entries are underlined for expositional purposes):

$$x = (\ldots, x_{-1}, \underline{x_0}, x_1, \ldots), x_\nu \in D.$$

In the case of a graph with $d = 3$ nodes (e.g., Figure 1, right panel), the elements in the set Ω are described by sequences (or genealogies) of the form:

$$(\ldots 1\,1\,1\,2\,3\,\underline{1}\,1\,1\,2\,1\,1 \ldots)$$
$$(\ldots 3\,1\,2\,3\,3\,\underline{1}\,1\,1\,3\,1\,2 \ldots)$$
$$(\ldots 2\,1\,3\,1\,2\,\underline{2}\,1\,2\,1\,2\,3 \ldots)\,.$$

Consider now the shift map $\tau : \Omega \to \Omega, (x_\nu) \mapsto (\tilde{x}_\nu)$, where $\tilde{x}_\nu = x_{\nu+1}$. Then the dynamics for τ is related to the dynamics given by Eq. (3) but they are not the same object. While the latter describes the dynamics of income levels of the groups in the society, the shift on Ω is only concerned with the genealogical history of income generation and consequently corresponds to the dynamics defined by P. To establish a connection between the two, define the **potential function** φ by:

$$\varphi : \Omega \to \mathbb{R},\ \varphi(x) = \log a_{x_1 x_0},$$

which represents the intensity of the individual interactions reflected in the interaction matrix A. The steady state of the dynamical system induced by Eq. (3) can be described by an equilibrium **probability measure** μ induced by the function φ, which is invariant under the shift map τ defined above, and which can be written in terms of the Markov matrix $P = (p_{ij})$ and its eigenvector π.[10] Together with Ω, these yield the dynamical system (Ω, μ, φ) that replaces the classical system of difference or differential equations $(\mathbb{R}^d, y, f)$ (see Demetrius [33] and also Appendix A.3 for more discussion).

Macroscopic Variables. The dynamical system (Ω, μ, φ) with its underlying genealogies model is the basis for generating a number of *macroscopic variables* that characterize diverse aspects of the income process $Y(t)$ besides the growth rate g, which has been almost exclusively at the center of economic growth theory. As shown, for example, in Petersen [58] and Walters [81], the associated Markov chain (P, π) has entropy H as defined in Eq. (7). This means that, if we

[10] It satisfies, for any n and and sequence of finite length k, $(i_n, \cdots, i_{n+k}) \in D^{k+1}$,

$$\mu(x_n = i_n, \ldots, x_{n+k} = i_{n+k}) = \pi_{i_n} p_{i_n i_{n+1}} \cdots p_{i_{n+k-1} i_{n+k}},$$

as shown, for example, in Petersen [58] and Walters [81].

read genealogies in direction of successive ancestors, then H can be interpreted as the average uncertainty about the ancestor class $x_{n-1} \in D$ that appears at time $n-1$, given the class $x_n \in D$ seen at time n. As mentioned above, this further motivates our interpretation of the entropy as measure of cooperation (see also footnote 8 above).

Moreover, let

$$S_n\varphi(x) = \sum_{k=0}^{n-1} \varphi(\tau^k x) = \sum_{k=0}^{n-1} \log a_{x_{k+1}x_k},$$

and

$$Z_n(\varphi) = \sum_{(x_0,x_1,\ldots,x_n)} \exp S_n\varphi(x) = \sum_{(x_0,x_1,\ldots,x_n)} a_{x_1x_0}a_{x_2x_1}\cdots a_{x_nx_{n-1}},$$

then, using the Perron-Frobenius Theorem, we can represent the growth rate as the limit,

$$g = \log \lambda = \lim_{n\to\infty} \frac{1}{n} \log Z_n(\varphi). \tag{9}$$

Furthermore, define the following **macroscopic variables** as moments of the system (Ω, μ, φ):

$$\Phi = \lim_{n\to\infty} \frac{1}{n}\mathbb{E}_n\left[S_n\varphi\right] = \int \varphi d\mu, \quad \sigma^2 = \lim_{n\to\infty} \frac{1}{n}\mathbb{V}_n\left[S_n\varphi\right], \quad \kappa = \lim_{n\to\infty} \frac{1}{n}\mathbb{E}_n\left[S_n\varphi - \mathbb{E}_nS_n\varphi\right]^3, \tag{10}$$

where $\mathbb{E}_n$ and $\mathbb{V}_n$ denote the expectation and the variance with respect to the measure μ_n on finite sequences of length n, $(x_0, x_1, \ldots, x_n)$, and which is defined by:

$$\mu_n = \frac{S_n\varphi(x)}{\sum_{(x_0,x_1,\ldots,x_n)} S_n\varphi(x)}.$$

The first macroscopic variable, Φ, defined in Eq. (10) is our measure of **scarcity**. Next, the variables σ^2 and κ represent respectively **variance** and **skewness** measures, from which we define $\gamma = 2\sigma^2 + \kappa$, which is our measure of **diversity** of the income generating process. Together with g and H, these parameters allow us to characterize the long-run outcome of our evolutionary process. In Appendix A.3, we present an alternative way to compute the parameters Φ, σ^2, κ, and γ, by expressing them as moments of the Taylor series expansion of the growth rate g as a function of a certain perturbation parameter δ.

The following example provides more intuition for the macroscopic variables and the overall framework developed so far.

Example 1. To illustrate the framework introduced, we consider a few simple 2×2 interaction matrices that represent different income generation processes between two (similar) groups of individuals. In Appendix A.4 we sketch a structural model of an economy that yields such interaction matrices as special cases and discuss the possible underlying productive context in more detail. Here we limit ourselves to illustrating the connection between the stylized interaction matrices and the macroscopic variables introduced, including the relation between evolutionary entropy and cooperation. The first case is of a fully symmetric income generation process (matrix A), while the remaining two are asymmetric with respect to the distribution of income (matrix A') or with respect to contribution towards income generation (matrix A'').

Consider the interaction matrix between two groups of individuals, $i = 1, 2$, given by:

$$A = \begin{pmatrix} a & b \\ b & a \end{pmatrix},$$

where $a > 0$ measures rate of (intertemporal) resource contribution to individuals in the own group, and $b > 0$ measures rate of resource contribution to individuals in the other group.

We can readily compute $\lambda = a + b, u = \left(\frac{1}{2}, \frac{1}{2}\right), v = \left(\frac{1}{2}, \frac{1}{2}\right)^\top$ and hence also:

$$\begin{aligned}
P &= \begin{pmatrix} \frac{a}{a+b} & \frac{b}{a+b} \\ \frac{b}{a+b} & \frac{a}{a+b} \end{pmatrix} \text{ (Markov matrix)} \\
\pi &= \left(\frac{1}{2}, \frac{1}{2}\right) \text{ (stationary distribution of } P) \\
H &= -\frac{a}{a+b} \log \frac{a}{a+b} - \frac{b}{a+b} \log \frac{b}{a+b} \text{ (entropy)} \\
\Phi &= \frac{a}{a+b} \log a + \frac{b}{a+b} \log b \text{ (scarcity)} \\
g &= \Phi + H = \log (a+b) \text{ (growth rate)} \\
\sigma^2 &= \frac{ab \log \left(\frac{a}{b}\right)^2}{(a+b)^2} \text{ (variance)} \\
\kappa &= -\frac{ab\,(a-b) \log \left(\frac{a}{b}\right)^3}{(a+b)^3} \text{ (skewness)} \\
\gamma &= 2\sigma^2 + \kappa \text{ (diversity)}
\end{aligned}$$

In particular, defining $\rho = \frac{b}{a+b}$ as a normalized rate of contribution to individuals in the other group, we can write:

$$H = -(1-\rho) \log (1-\rho) - \rho \log \rho.$$

This expression is maximized at $\rho = \frac{1}{2}$, where there is maximally mutualistic interaction within and across groups, and is minimized at $\rho = 0$ or $\rho = 1$, where there is minimally mutualistic interaction. To make this connection even clearer, using the characterization in footnote 8, we can further compute:

$$\begin{aligned}
H_1 = H_2 &= -(1-\rho) \log (1-\rho) - \sum_{i=0}^{\infty} \rho^2 (1-\rho)^i \log (\rho^2 (1-\rho)^i) \\
&= -2(1-\rho) \log (1-\rho) - 2\rho \log \rho = 2H \\
T_1 = T_2 &= 1 \cdot (1-\rho) + \sum_{i=0}^{\infty} (2+i) \cdot \rho^2 (1-\rho)^i = (1-\rho) + (1+\rho) = 2,
\end{aligned}$$

for $0 < \rho < 1$. For $\rho = 0$ and $\rho = 1$, clearly, $H_1 = H_2 = 0$, whereas, $T_1 = T_2 = 1$ for $\rho = 0$ and

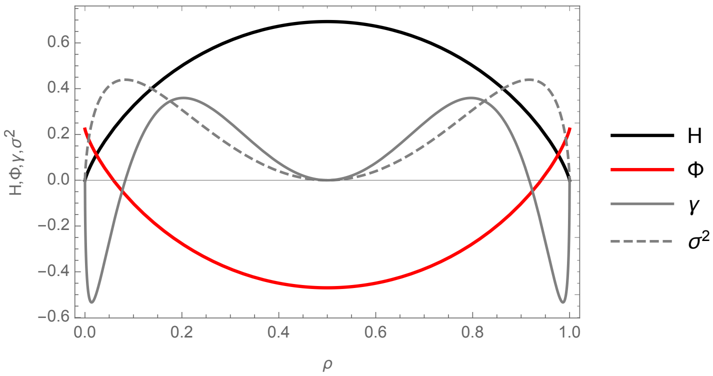

Figure 2: Macroscopic variables of Example 1 as a function of $\rho = \frac{b}{a+b}$ for $\lambda = a + b = \frac{5}{4}$.

$T_1 = T_2 = 2$ for $\rho = 1$. Interestingly, since H is directly related to H_1 and H_2, it is also a measure of the amount of uncertainty over the productive cycles that start and end at node 1 or 2; T_1 and T_2 measure how long it takes on average to return to the initial node in such cycles.

Figure 2 shows the key macroscopic variables as a function of ρ, assuming $\lambda = a + b = \frac{5}{4}$.[11] Notice that for $\rho \approx 0$ and $\rho \approx 1$ the income generation process is what we call concentrating ($\Phi > 0$, $\gamma < 0$) and is associated with a low entropy ($H \approx 0$), while for interior values of ρ it is dispersing ($\Phi < 0$, $\gamma > 0$) and is associated with higher entropy.[12] In the next sections, we study the reasons for distinguishing the income generation processes in terms of (the signs of) Φ and γ, as well as the implications this has for the evolution of cooperation and for income inequality.

Next, consider the following two variations of matrix A:

$$A' = \begin{pmatrix} a & a \\ b & b \end{pmatrix}, \; A'' = \begin{pmatrix} a & b \\ a & b \end{pmatrix}.$$

They differ from A in that they are no longer symmetric. All three matrices have the same eigenvalue $\lambda = \lambda' = \lambda'' = a + b$, yet they differ with respect to the eigenvectors. With the matrix A both groups receive and contribute the same at the steady state: $u = \left(\frac{1}{2}, \frac{1}{2}\right), v = \left(\frac{1}{2}, \frac{1}{2}\right)^\top$; with A' both groups contribute the same but receive different shares: $u' = \left(\frac{1}{2}, \frac{1}{2}\right), v' = \left(\frac{a}{a+b}, \frac{b}{a+b}\right)^\top = (1-\rho, \rho)^\top$; and with A'' both groups receive the same shares but contribute differently: $u'' = \left(\frac{a}{a+b}, \frac{b}{a+b}\right) = (1-\rho, \rho), v'' = \left(\frac{1}{2}, \frac{1}{2}\right)^\top$. Nonetheless, the matrices A' and A'' have the same Markov matrix with the same stationary distribution:

$$P' = P'' = \begin{pmatrix} \frac{a}{a+b} & \frac{b}{a+b} \\ \frac{a}{a+b} & \frac{b}{a+b} \end{pmatrix} = \begin{pmatrix} 1-\rho & \rho \\ 1-\rho & \rho \end{pmatrix}, \pi' = \pi'' = \left(\frac{a}{a+b}, \frac{b}{a+b}\right) = \left(1-\rho, \rho\right).$$

Interestingly, all the remaining macroscopic variables, $H, \Phi, g, \sigma^2, \kappa$, and γ have the same formulas

[11]Assuming $\lambda = a + b = \frac{5}{4}$ implies $a = \frac{5}{4}(1-\rho)$ and $b = \frac{5}{4}\rho$, which allows us to express all macroscopic variables as a function of ρ alone. Otherwise, the number $\frac{5}{4}$ is taken purely for expositional purposes.

[12]It can be calculated that for the matrix A and for the matrices A' and A'' below, Φ changes sign at $\rho \approx 0.06$ and $\rho \approx 0.94$, and γ changes sign at $\rho \approx 0.08$ and $\rho \approx 0.92$.

as functions of a and b for all three matrices. Given that with the matrices A' and A'' the parameter $\rho = \frac{b}{a+b}$ can be seen as a measure of similarity/discrepancy between what is received and contributed by the two groups, it further shows how the formula for the entropy H (expressed as a function of ρ) captures balancedness in the sense of mutualism of the income generation process in terms of what is received and contributed between the two groups. When $\rho = \frac{1}{2}$ there is no discrepancy and hence maximal mutualism, and entropy is maximal, whereas when $\rho = 0$ or $\rho = 1$ there is maximum discrepancy and hence minimal mutualism and minimal entropy. Finally, with respect to the characterization in footnote 8, for A' and A'', we have $H_1' = H_1'' = (1-\rho)H$, $T_1' = T_1'' = (1-\rho)^{-1}$, and $H_2' = H_2'' = \rho H$, $T_2' = T_2'' = \rho^{-1}$, for $0 < \rho < 1$. □

3 Evolutionary Dynamics and Entropic Selection

This section introduces the evolutionary framework and states a key result, the Entropic Selection Theorem (Theorem 1), based on the *entropic selection principle* of Demetrius [32] and Demetrius and Legendre [37]. We assume the economy is in a steady state described by Eq. (3) with an initially given matrix A. This matrix is then allowed to evolve in response to small perturbations, some of which are successful, while others are not. Unsuccessful perturbations are not adopted and do not change the matrix A, while successful ones do. In this sense while initially exogenous, the matrix A is subject to small changes. The perturbations in A are modeled by what we call the perturbed matrix $A^* = A(\delta)$ of the initial (or incumbent) matrix $A = A(0)$, where $\delta \in \mathbb{R}$ is a small perturbation parameter. Whether or not a perturbation is successful is determined by the evolutionary process. Each perturbed matrix A^* has an entropy level $H^* = H(\delta)$ that corresponds to it. The Entropic Selection Theorem (Theorem 1) characterizes the successful perturbations in terms of the entropy of the (successful) perturbed matrix (H^*) compared to the one of the initial matrix (H), given Φ and γ. Importantly, the perturbations do not change the signs of Φ and γ, so that an environment that is dispersing (or concentrating) will remain so even as the matrix A is subject to repeated, successive perturbations. In this sense, the type of income generating process, whether dispersing or concentrating is determined by the initial matrix A and is exogenous throughout (Demetrius [33]).

The evolutionary process has two aspects. The first aspect occurs on a *short term* level. It is characterized by competition between the incumbent population with interaction matrix A and entropy H, and a variant population with perturbed interaction matrix A^* and entropy H^*. The second aspect occurs on a *long term* level. It is characterized by the interaction between the external environment and the composite population (incumbent and variant) as it evolves from one steady state with entropy H to a new steady state with entropy $\widehat{H}$. The short term entropy change $\Delta H = H^* - H$ induced by invasion of the (successful) variant population is determined by the entropic selection principle. The long term entropy change $\widehat{\Delta} H = \widehat{H} - H$, which describes the transition from one steady state to the next, is determined by the external environment and the population dynamics. It is shown in Demetrius [33], Section 9, that the change in H

from one steady state to another ($\widehat{\Delta}H$) is related and in fact positively correlated to the short term change (ΔH), that is, $\Delta H \widehat{\Delta} H > 0$, under regularity conditions. Therefore, the long term changes in entropy can be inferred from the short term changes (ΔH), captured by the entropic selection principle. What the theory contributes then, is not so much endogenously determined matrices, but rather directions or, better, classes of matrices A^* (distinguished by their associated level of entropy H^*) that constitute successful perturbations or variants to the exogenously given matrix A. We now introduce the evolutionary framework, which underlies the short and long term changes in entropy.

Evolutionary Dynamics. We use a model of evolution that is based on the interaction between an incumbent population and a variant (or mutant) population that can potentially increase in frequency and lead to a displacement of the incumbent types. This can be seen as being representative of various models of cultural evolution. To determine the evolutionary outcome, we compute the probability that the variant population completely displaces the types of the incumbent population.

Consider an **incumbent population** N operating with an interaction matrix A and producing Y, and consider a (small) **variant population** N^* operating with an interaction matrix A^* that is modeled as a perturbation of the matrix of the incumbent population; the variant population produces Y^*. Importantly, in order to keep things tractable, while attempting to capture a large class of variants, we consider, throughout the paper, **perturbations** of the form:

$$A^* = A(\delta) = (a_{ij}(\delta)), \text{ where } a_{ij}(\delta) = a_{ij}^{1+\delta}, \text{ for } \delta \in \mathbb{R}. \tag{11}$$

These constitute in a precise sense a canonical class of perturbations that can be modeled with a single one-dimensional parameter, $\delta \in \mathbb{R}$, small in absolute value. They represent changes in the income generating process that are derived as small perturbations of the original potential function φ that preserve the multiplicative structure of the model.[13] As shown in Appendix A.3, they also allow us to easily compute the macroscopic parameters Φ, σ^2 and γ.

The population N^* can be thought of as a small (sub-)population or sub-community (with the same d groups), that operates with $A^* = A(\delta)$ that is similar to A but slightly different. For example, the underlying interaction mode might involve slightly more or less exchange or sharing (as in Example 2 below). The question we ask is whether this variant or perturbation will be successful and hence increase in frequency and displace the traits of the original population, or whether it will be unsuccessful and simply disappear. This is calculated based on the competitive interaction of the corresponding processes Y and Y^*.

The introduction of a variant type via perturbations thus constitutes the first step in describing the evolutionary dynamics. The second step is given by the **invasion dynamics**, which studies

[13] These take the form $\varphi(\delta) = \varphi + \delta\varphi = (1+\delta)\varphi = (1+\delta)\log a_{ij} = \log a_{ij}^{1+\delta}$ for $\delta \in \mathbb{R}$ small in absolute value. More generally, the perturbations can take the form $\varphi(\delta) = \varphi + \delta\psi$, where ψ has the same productive potential and the same directional derivative as φ, see Demetrius et al. [36], Section 6, and Demetrius [33], Ch. 4; we provide further details in Appendix A.2.

the interaction and competition between the incumbent population N and the variant population N^* for the resources which the economic environment makes available for the corresponding productions Y and Y^*. Given that each population is producing according to its own interaction matrix A and A^*, we are essentially studying a situation where the two populations produce side by side with a constraint on total production such that $Y + Y^*$ is locally fixed while the "invasion" of the variant takes place.

When analyzing the invasion dynamics and the evolution of Y and Y^*, we follow Demetrius et al. [36] and work with continuous time representations $Y(t)$ and $Y^*(t)$ derived from techniques of diffusion approximation originally studied by Feller [41] integrated with techniques of ergodic theory from Arnold et al. [12]. Starting from the system (Ω, μ, φ) of the genealogies model for Eq. (3) introduced in Section 2, we can write $Y(t)$ as the solution to the stochastic differential equation,

$$dY(t) = gY(t)dt + \sigma\sqrt{Y(t)}dW(t), \tag{12}$$

where $W(t)$ is Brownian motion, and r and $\sigma\sqrt{Y(t)} \geq 0$ are respectively the growth rate and the standard deviation of the process $Y(t)$, where g and σ^2 coincide with the macroscopic variables defined in Section 2.[14] This can be obtained using a version of the Central Limit Theorem as sketched in Appendix A.1 (see Demetrius et al. [36] for details). The process $Y(t)$ can be viewed as a Feller-type process with parameters, g and σ^2;[15] it belongs to the class of Cox-Ingersoll-Ross (CIR) processes also studied in finance (see Cox et al [29]). The same applies to the process $Y^*(t)$ (with parameters g^* and σ^{*2}). We use these representations of processes $Y(t)$ and $Y^*(t)$ when studying the invasion dynamics.

Entropic Selection. Formally, the (global) selective dynamic is determined by the **entropic selection principle**. As studied in Directionality theory, the changes in evolutionary entropy under the process of variation and selection are contingent on the income process and can be characterized in terms of the following local result:

(I) When the income generating process is *dispersing* (*scarce* and *diverse*), a community with higher entropy will have a selective advantage and will increase in relative size.

[14] This in itself provides a novel yet natural technique of generating macroeconomic fluctuations from disaggregated interactions based on law of motion described by Eq. (3) and the underlying system (Ω, μ, φ) derived from the network structure associated to A. Note that our source of fluctuations are distinct and complementary, to the ones of Acemoglu et al. [2] and Gabaix [45], who model aggregate fluctuations through the propagation of individual firm-level shocks across the economy as a function of the network structure and the relative sizes of the firms.

[15] This process is likely to underestimate the variance of contemporary processes, see Stock [75] who estimates a variance of the order $Y(t)$ rather than $\sqrt{Y(t)}$ for postwar US GNP. To better understand the difference, write the discrete time version of our process as $Y(t) = (1 + g)Y(t-1) + \epsilon_t$, where $\epsilon_t \sim N(0, \sigma^2 Y(t))$, and notice that our process yields an error term that decreases in $Y(t)$, hence vanishes in the limit, when estimated in log-differences ($\Delta \log Y(t)$) rather than a stationary one as in [75]. On the other hand, the present model seems to explain a nontrivial amount of fluctuations. This is unlike macro-models, where stationary idiosyncratic shocks typically do not lead to aggregate fluctuations (e.g., Aiyagari [7], Bewley [16]). In our case, the fluctuations originate from the randomness of the interaction patterns (or sequences) of individual interactions. Gabaix et al. [46] introduce a very flexible class of processes to study the dynamics of income inequality, including transition dynamics.

(II) When the income generating process is *concentrating* (*abundant* and *singular*) , a community with low entropy will have a selective advantage and will increase in relative size.

We exploit *(I)* and *(II)* to address questions about the origin, spread, and persistence of inequality, which have hitherto seemed intractable within classical frameworks of economic growth and socio-cultural evolution. The entropic selection principle can also be interpreted loosely as characterizing when to expect more or less mutualistic or cooperative behavior to spread locally.

Through the entropic selection principle, our main result characterizes whether globally the economy tends towards higher or lower entropy. This indirectly characterizes which perturbations will be successful and which will not. We make this more precise. Consider an incumbent population $N(t)$ operating in steady state with interaction matrix A and producing $Y(t)$. We say that in this economy, **entropy tends to increase (decrease)** if, for any $\delta \in \mathbb{R}$ small and for any variant population $N^*(t)$ of $N(t)$, $t > t_0$, operating with A^*, producing $Y^*(t)$, for which $H^* > H$ ($H^* < H$), we have:

$$\text{Prob}\left[\lim_{t\to\infty} \frac{Y^*(t)}{Y(t)+Y^*(t)} = 1 \,\middle|\, Y^*(t_0) > 0\right] = 1. \tag{13}$$

If $\text{Prob}\Big[\lim_{t\to\infty} \frac{Y^*(t)}{Y(t)+Y^*(t)} = 1 \mid Y^*(t_0) > 0\Big] = p$, we say **entropy tends to increase (decrease) with probability** p. An analogous definition holds for the inequality measure introduced in Section 4, replacing respectively T, T^* for H, H^*.

The next theorem shows that whether higher or lower entropy interactions prevail, depends on characteristics of the underlying income process. Scarce and diverse processes are conducive to higher entropy interactions, while abundant and singular ones are conducive to lower entropy interactions. The following result states how the level of entropy of the underlying interaction will evolve *globally* in all possible cases.

Theorem 1 (Entropic Selection)**.** *The outcome of the selection process in a society evolving according to the income process $Y(t)$ described by Eq. (3) above is characterized by the following four cases:*

(Ia) If the income process is scarce and diverse ($\Phi < 0, \gamma > 0$), entropy tends to increase;

(Ib) If the income process is scarce and singular ($\Phi < 0, \gamma < 0$), entropy tends to increase, provided total income is sufficiently large ($Y > \gamma/\Phi$); otherwise for small total income ($Y < \gamma/\Phi$) entropy increases with a probability that increases in the total level of income;

(IIa) If the income process is abundant and singular ($\Phi > 0, \gamma < 0$), entropy tends to decrease;

(IIb) If the income process is abundant and diverse ($\Phi > 0, \gamma > 0$), entropy tends to decrease, provided that total income is sufficiently large ($Y > \gamma/\Phi$); otherwise for small total income ($Y < \gamma/\Phi$) entropy decreases with a probability that increases in the total level of income.

This suggests that it is essentially in societies with scarce income processes ($\Phi < 0$) that one should expect to find higher entropy societies. In large economies the result is general. When aggregate income is not sufficiently large ($Y < \gamma/\Phi$) then, to guarantee the same result, one needs that the process also be diverse ($\gamma > 0$). Conversely, when the income process is abundant ($\Phi > 0$) one should expect to find low entropy societies. Again, in large economies this is general. When aggregate income is not sufficiently large ($Y < \gamma/\Phi$) then, to guarantee the same result, one needs that the process also be singular ($\gamma < 0$). In particular, it follows that in dispersing environments ($\Phi < 0, \gamma > 0$) a variant with higher entropy ($H^* > H$) will be successful and grow faster and steadier, whereas in concentrating ones ($\Phi > 0, \gamma < 0$) a variant with lower entropy ($H^* < H$) will be successful and grow faster and steadier. This follows from basic relations between the macroscopic variables derived in Appendix A.3 (see statements (f) and (g)).

The analytical basis for the Entropic Selection Theorem involves the integration of the ergodic theory of dynamical systems with the theory of diffusion processes (Demetrius [33], Demetrius and Gundlach [34]). A proof of the theorem can be found in Demetrius et al. [36]. Below, we sketch some of the key steps (with further details also given in the Online Appendix). Before that, we offer the following intuition for the result.

Heuristics for Entropic Selection. Consider two communities of foragers competing for resources. A first community is more egalitarian and shares the proceeds from foraging more equally; the second one is less egalitarian and awards significantly larger shares to the more productive "star-hunters" (and their families). We go through the selection process in the two types of environments.

In the *scarce* environment ($\Phi < 0$, all or almost all a_{ij}'s are less than 1), the first, more egalitarian community, will have all members relatively well provided for. They do well, as all members are effectively contributing and their contributions are reciprocated. In the second, less egalitarian, community the star-hunters receive a larger share of the produce at a cost to the others. Because resources are scarce, this does not significantly increase overall produce by star-hunters, but as it reduces the resources to the *non*-star-hunters, it can seriously threaten their capacity to effectively contribute. This can hinder the community's survival, especially when competing with a more egalitarian one.

In the *abundant* environment ($\Phi > 0$, at least some a_{ij}'s are sufficiently greater than 1), the first, more egalitarian, community will have all members relatively well provided for, but now the star-hunters forgo a more significant quantity of produce. The less egalitarian group, on the other hand, awards a larger share to the star-hunters, thus potentially significantly increasing total produce, while at the same time being able to provide sufficiently for the others. This gives the less egalitarian community an advantage over the more egalitarian one, and overall improves its survival when competing for resources.

A second important aspect of the evolutionary process is the variance or steadiness of the resulting income generation process. A more egalitarian community provides for its members in a more steady way than a less egalitarian one, which gives it a further advantage over the less egalitarian community. An exception occurs with singular income processes, where the underlying resources are very concentrated. In such cases, a less egalitarian community that strongly rewards the most productive "star-hunters" and is more focused on the singular resources may achieve an even steadier income generation process and therefore possibly do better.

The heuristic is consistent with several of the fascinating, historical examples discussed in Flannery and Marcus [44], where the scarcity ($\Phi < 0$) or abundance ($\Phi > 0$) of the resources available for production and consumption play an important role for the appearance and disappearance of more or less intense and reciprocal modes of interaction. In many of the examples, (long-term) scarcity indeed seems to encourage agents to cooperate in order to get by, whereas (long-term) abundance (especially coupled with the possibility to store) generates competition for the "lion's share" or for the surplus production, which encourages and rewards less cooperative or self-regarding behavior. The same heuristics, reinforced by diversity, which naturally encourages cooperation and interaction, also seems relevant for understanding socio-cultural evolution in resource-cursed economies.[16] More recently, Bartos [14] studies sharing and enforcement of sharing norms within poor communities of subsistence farmers in Afghanistan. He finds that, while the overall sharing norms appear to be constant over the periods considered, the reinforcement norms become laxer in periods of seasonal scarcity and stronger in periods of seasonal abundance. An interpretation consistent with our model would be that, from a long-term perspective, enforcement may be more necessary (and effective) in periods of abundance than in periods of scarcity.[17] Buggle and Durante [24] show that areas in Europe with historically higher climatic variability (measured by inter-annual variability in precipitation and temperature) are associated with higher levels of cooperation and trust. To the extent that higher climatic variability is associated with limits on crop yield productivity (Mendelsohn [56], Semenov and Porter [67]) and thus encourages more diverse production modes (Reidsma and Ewert [64]), it may well contribute to what we call scarce and diverse environments, thereby providing a further, complementary explanation for the observed positive association of climatic variability with cooperation and trust, as well as more intense economic exchange. As discussed earlier, Talhelm et al. [76] show that rice agriculture, which is associated with a typically more scarce income process, as compared to wheat, has led to stronger sharing norms in regions of China where it was prevalent, while more individualistic norms developed in regions where wheat was prevalent. Finally, somewhat related, in a comprehensive survey carried out in 67 countries, Elbaek et al. [39] study the relationship between morality, social class and income inequality, and find that relative chronic economic scarcity, in-

[16] A particularly insightful example is that of the Kachin society in Highland Burma, described in Flannery and Marcus [44], Ch. 10, pp. 195-199. Here the primary commodity of the income generating process is rice, and the scarcity or abundance of key resources is determined by the environmental constraints: highlands (low rainfall) and lowlands (high rainfall).

[17] Relatedly, Aksoy and Palma [8] study cheating and sharing behavior as well as in- and out-group favoritism among poor communities of coffee farmers in Guatemala. Among other things, they show that overall levels of cheating and sharing are constant across periods of scarcity and abundance, but also that while in-group favoritism is higher than out-group favoritism under abundance, the bias vanishes during periods of scarcity.

dexed by social-economic status, predicts, among other things, higher morality-as-cooperation and higher pro-social behavior.

We now sketch the main steps of the argument presented in Demetrius et al. [36]. Besides the first-order effect of the scarcity of the environment, it also includes the effect of diversity as the two enter the formula for the variant's selective advantage.

Invasion Dynamics and Sketch of Proof for Entropic Selection Theorem. The essence of the argument lies in the interaction between the aggregate production of a given incumbent population (whose production process is described by A, (Ω, μ, φ), and the process $Y(t)$ satisfying Eq. (12)) and that of the variant population (described by A^*, $(\Omega, \mu^*, \varphi^*)$, and the related process $Y^*(t)$). We thus model variants (mutants or invaders) as producing alongside the incumbent community, according to their own production and allocation technology represented by the matrix A^*, defined in Eq. (11), and which we model as being a perturbation of the incumbent's matrix A. For the matrix A^*, we can compute corresponding macroscopic parameters $g^* = g(\delta)$, $\sigma^{*2} = \sigma^2(\delta)$, and $H^* = H(\delta)$. Given the incomes of the two populations, we study the evolution of the share of the variant population's income,

$$p(t) = \frac{Y^*(t)}{Y(t) + Y^*(t)},$$

when the two are competing for resources, that is, for a locally fixed total production $Z = Y(t) + Y^*(t)$. The idea is that invasion takes place on a significantly faster scale, during which total resources are fixed, which is captured by assuming a fixed total production Z. Computing the probability of a complete displacement of the incumbent population by the variant population ($p(t) \to 1$), we obtain that it depends on the sign of the expression,

$$s = \Delta g - \frac{\Delta\sigma^2}{Z}, \tag{14}$$

where $\Delta g = g^* - g$ and $\Delta\sigma^2 = \sigma^{*2} - \sigma^2$. This expression describes the **selective advantage** of the variant population over the incumbent population which, using the properties of the perturbations and assuming $\Phi \neq 0$ and $\gamma \neq 0$, can be conveniently rewritten as:

$$s = -\left(\Phi - \frac{\gamma}{Z}\right)\Delta H, \tag{15}$$

where $\Delta H = H^* - H$. This highlights the joint role of the parameters Φ and γ of the incumbent's income process ($Y(t)$), coupled with the entropy differential between the two populations' income processes (ΔH). In particular, if $\Phi < 0$ and $\gamma > 0$, then the variant population has a positive selective advantage ($s > 0$) when $H^* > H$; on the other hand, if $\Phi > 0$ and $\gamma < 0$, then the variant has a positive selective advantage ($s > 0$) when $H^* < H$. The studied invasion dynamics covers a wide class of models of cultural evolution.[18]

[18]The present approach does not assume an infinite amount of available resources. The Malthusian case where

4 Entropy, Inequality and Redistributive Selection

In this section, we establish the implications of the Entropic Selection Theorem for the evolution of inequality. In order to state our main result, Theorem 2, we first introduce measures of income inequality, which we relate formally to underlying measures of evolutionary entropy.

Theil Index and Income Inequality. As a measure of income inequality, we use a well-known entropy-based measure, the *Theil index* (Cowell [28], Sen [68]), which can be written as:[19]

$$T(v) = \sum_{i=1}^{d} v_i \log(d \cdot v_i), \quad 0 \leq T(v) \leq \log d, \tag{16}$$

where $v = (v_1, \ldots, v_d)^\top$ is an income distribution written as a vector of income shares. We also use the notation $T(0) \equiv T(v)$, and write $T(\delta) \equiv T(v(\delta))$ for the Theil index of the eigenvector (of income shares) of the perturbed matrix $A(\delta)$, $\delta \in \mathbb{R}$. The more unequal the incomes across groups, the larger the index T is. To see this, let $H_{EQ}(v) = -\sum_{i=1}^{d} v_i \log v_i$ denote an entropy-based measure of *equality* of the income distribution v. When all income shares are equal ($v_i = 1/n$ for all i) the corresponding entropy H_{EQ} is maximal and equal to $\log d$, while the Theil index is minimal and equal to 0. It is easy to see that the Theil index can also be written as $T(v) = \log d - H_{EQ}(v)$.

Our first result shows that inequality as measured by T is negatively correlated with the evolutionary entropy H of the process $y(t)$. This result is novel and is key in making the connection between the results of directionality theory with corresponding statements on income inequality.[20]

Proposition 1 (Evolutionary Entropy and Inequality)**.** *For perturbations of the form $A(\delta) = (a_{ij}(\delta))$, where $a_{ij}(\delta) = a_{ij}^{1+\delta}$, we have that the Theil index and the evolutionary entropy move in opposite directions, $\Delta T \Delta H \leq 0$, where $\Delta T = T(\delta) - T(0)$ and $\Delta H = H(\delta) - H(0), \delta \in \mathbb{R}$ small; with strict inequality if $\Delta T, \Delta H \neq 0$.*

A few comments are in order. First, while it is straightforward to show that T is negatively correlated with the equality measure H_{EQ}, this does not automatically show that it is negatively correlated with the evolutionary entropy measure H. To see the relation, we need to show that H_{EQ} is positively correlated with H. This involves showing that, starting from a matrix A, perturbations of the form $A(\delta) = (a_{ij}^{1+\delta})$ that increase entropy H will generate more interaction and will redistribute income more such that H_{EQ} increases, while T decreases; similarly, perturbations that decrease H will decrease H_{EQ} and increase T. Second, this result confirms the intuition from

the selective advantage s is based exclusively on the growth rate differential (Δg) between the two populations, is a special case and is obtained in the limiting case where total production (and resources) tend to infinity (see Demetrius [33], Sections 2.3 and 6.3, for discussion).

[19] The use of the variable T for the Theil index here and in the rest of the paper is not to be confused with the variable T_α used in Section 2 to denote the length of time of a cycle starting at node α. Notice that the version we use is not normalized to lie between $[0, 1]$.

[20] Knack and Keefer [51], Putnam [62] and Wilkinson and Pickett [82] among others find that measures of trust and indices of social capital (e.g., civic, social, and political participation), which are often taken as measures of "cooperation" are typically negatively correlated with various measures of income inequality.

the genealogy model that a society with a higher evolutionary entropy H, and hence stronger and more balanced interactions within and across groups, implying a greater uncertainty about from what group a given unit of income was contributed, will be associated with a higher degree of income redistribution.

Redistributive Selection. We now turn to our main implication, which establishes a link between the income generating process and the evolution of inequality in the society. Given the last proposition, the notion of evolutionary entropy plays a central role in establishing the link. Again, in our evolutionary model, variants with different modes of redistribution (perturbed matrices A^* with levels of inequality T^*) are introduced; these variants have to compete with the original types (operating with matrix A with level of inequality T) for the existing resources. The outcome of the evolutionary process depends on which of the two types has a selective advantage. This in turn depends on the underlying income generating process.

Theorem 2 (Redistributive Selection)**.** *The outcome of the selection process facing a society evolving according to Eq. (3) is such that, if the income process is dispersing ($\Phi < 0, \gamma > 0$), income inequality tends to decrease; whereas, if it is concentrating ($\Phi > 0, \gamma < 0$), income inequality tends to increase.*

This result follows as a direct application of cases *(Ia)* and *(IIa)* of Theorem 1 together with Proposition 1. It shows that, whether equal or unequal societies tend to prevail depends on the underlying income process, concretely, on the matrix A. More equal societies have a selective advantage in dispersing (scarce and diverse) processes, while less equal ones are favored in concentrating (abundant and singular) ones. From relations (f) and (g) in Appendix A.3, we expect again that more equal variants ($T^* < T$) grow faster and steadier in dispersing processes, while less equal ones ($T^* > T$) grow faster and steadier in concentrating ones.

Income Equalization. To end this section, we briefly consider the issue of persistence of equality or inequality. For this, we use Fields' [42, 43] measure of income equalization, that compares two (successive) income distributions, say $v = (v_1, \ldots, v_d)^\top$ and $v' = (v'_1, \ldots, v'_d)^\top$, and is defined by

$$E(v, v') = 1 - \frac{T(\frac{v+v'}{2})}{T(v)}. \tag{17}$$

As is discussed in Fields [42, 43], this is a measure of the degree of equalization of income distributions v and v', in the sense that positive values indicate a higher income equality at v' since average income $\frac{v+v'}{2}$ is more equally redistributed than base income v with respect to the inequality measure T. We take this as a basic measure of *persistence* of changes in income equality or inequality, and show that it is positively correlated with changes in the evolutionary entropy. As before, we use the notation $E(\delta, 0) \equiv E(v(\delta), v(0))$.

Proposition 2 (Evolutionary Entropy and Income Equalization)**.** *For perturbations of the form $A(\delta) = (a_{ij}(\delta))$, where $a_{ij}(\delta) = a_{ij}^{1+\delta}$, we have that income equalization as measured by E and* *changes in the evolutionary entropy H move in the same direction, $E\Delta H \geq 0$, where $\Delta H = H(\delta) - H(0), \delta \in \mathbb{R}$ small, with strict inequality if $E, \Delta H \neq 0$.*

Small perturbations that increase the entropy H are also associated with higher income equalization, as captured by the measure E. Applying this measure for successive income distributions along an evolutionary process with multiple perturbations, where the signs of the changes in entropy are the same, further shows the persistence of the changes in income equality or inequality.

5 Examples

In this section we provide some examples that illustrate the mechanics of the theory developed so far, while also providing some context for the framework used. Example 2 extends Example 1 of Section 2 to allow for a richer class of interactions. Besides computing the relevant macroscopic variables for the corresponding interaction matrices, it also shows key evolutionary implications of the theory. Next, Example 3 sketches possible interaction matrices for income generation processes of selected types of pre-modern societies such as hunter-gatherer, horticultural or early agricultural societies, and uses these to show how our theory can be used to interpret some of the evidence on cooperation and inequality found in the literature. All these examples can be derived as special cases of a relatively general production economy presented in Appendix A.4. There are two groups that produce goods using labor and a composite measure of capital, according to a production function homogeneous of degree one, and where the two groups may share their production as well as their production activities to varying degrees. Example 4, finally, sketches an exploitative type of society, where a group of workers or farmers contribute a significant multiple of what they produce to a group of elites or clergy.

Example 2. Consider an economy with two groups ($i = 1, 2$) operating with an asymmetric version of the interaction matrix of Example 1. Let $a_i > 0$ measure the rate of contribution to the own group and $b_i > 0$ the rate of contribution to the other group. To better interpret the resulting income generation processes, define, $\rho_i = \frac{b_i}{a_i+b_i}$ as a normalized rate of contribution of individuals in group i to individuals in the other group, and let $\eta_i = a_i + b_i$ be a measure of productivity of class i. We can then write the interaction matrix between the two groups as:

$$A = \begin{pmatrix} a_1 & b_2 \\ b_1 & a_2 \end{pmatrix} = \begin{pmatrix} \eta_1(1-\rho_1) & \eta_2\rho_2 \\ \eta_1\rho_1 & \eta_2(1-\rho_2) \end{pmatrix}. \tag{18}$$

For concreteness, fix $\eta_1 = \frac{5}{4}$, $\eta_2 = 1$, so that group 1 has a productivity advantage over group 2. The resulting matrix takes the form:

$$A = \begin{pmatrix} \frac{5}{4}(1-\rho_1) & \rho_2 \\ \frac{5}{4}\rho_1 & 1-\rho_2 \end{pmatrix}.$$

The associated graph is depicted in Figure 3 and the level curves for the different macroscopic

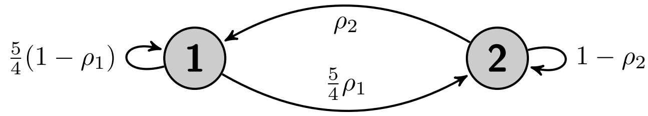

Figure 3: Interaction graph for the economy of Example 2 with $\eta_1 = \frac{5}{4}$, $\eta_2 = 1$.

variables as a function of the parameters ρ_1 and ρ_2 are depicted in Figure 4. Notice that for ρ_1 not too small the income generation process is what we call scarce and diverse ($\Phi < 0$, $\gamma > 0$), while for ρ_1 small it is abundant and singular ($\Phi > 0$, $\gamma < 0$). The red contour line in Figure 5 separates the two regions.

To illustrate our results on entropic selection (Theorems 1 and 2), consider the following three cases:

$$\rho^{(1)} \left(= \left(\rho_1^{(1)}, \rho_2^{(1)}\right)\right) = \left(\tfrac{3}{10}, \tfrac{3}{10}\right), \rho^{(2)} = \left(\tfrac{7}{10}, \tfrac{3}{10}\right), \text{ and } \rho^{(3)} = \left(\tfrac{1}{20}, \tfrac{1}{10}\right).$$

Each case can be seen as representing a different type of interaction and hence a different income generation process. It can be checked that the first two are scarce and diverse (ρ_1 not so small), while the third is abundant and singular (ρ_1 small). Perturbations of the corresponding matrices are successful when they are associated with a positive selective advantage ($-(\Phi - \gamma/Y)\Delta H > 0$).

Consider the perturbed matrices associated with $\rho^{(1)}$ and $\rho^{(2)}$, for δ small in absolute value:

$$A^{(1)}(\delta) = \begin{pmatrix} \left(\frac{7}{8}\right)^{1+\delta} & \left(\frac{3}{10}\right)^{1+\delta} \\ \left(\frac{3}{8}\right)^{1+\delta} & \left(\frac{7}{10}\right)^{1+\delta} \end{pmatrix}, A^{(2)}(\delta) = \begin{pmatrix} \left(\frac{3}{8}\right)^{1+\delta} & \left(\frac{3}{10}\right)^{1+\delta} \\ \left(\frac{7}{8}\right)^{1+\delta} & \left(\frac{7}{10}\right)^{1+\delta} \end{pmatrix}.$$

Since $\Phi - \gamma/Y < 0$ at $\rho^{(1)}$ and $\rho^{(2)}$ (see Figure 5), successful perturbations are ones where $\delta < 0$. Such perturbations have a selective advantage over the original matrices $A^{(1)}(0)$ and $A^{(2)}(0)$, as measured by $-(\Phi - \gamma/Y)\Delta H > 0$, and are associated with higher entropy ($\Delta H > 0$) and lower income inequality ($\Delta T < 0$). The directions of corresponding successful perturbations are plotted (in blue) in Figure 5. Notice that both $\rho^{(1)}$ and $\rho^{(2)}$ are inside the red region where $\Phi - \gamma/Y < 0$ and hence both perturbations point roughly towards ρ^H, where the entropy is maximal.

Consider now the perturbed matrix associated with $\rho^{(3)}$, for δ small in absolute value:

$$A^{(3)}(\delta) = \begin{pmatrix} \left(\frac{19}{16}\right)^{1+\delta} & \left(\frac{1}{10}\right)^{1+\delta} \\ \left(\frac{1}{16}\right)^{1+\delta} & \left(\frac{9}{10}\right)^{1+\delta} \end{pmatrix}.$$

Since $\Phi - \gamma/Y > 0$ at $\rho^{(3)}$ (see Figure 5), successful perturbations are ones where $\delta > 0$. Such perturbations have a selective advantage over the original matrix $A^{(3)}(0)$, as measured by $-(\Phi - \gamma/Y)\Delta H > 0$, and are associated with lower entropy ($\Delta H < 0$) and higher inequality ($\Delta T > 0$). The direction of such a successful perturbation is plotted (in red) in Figure 5.[21]

[21] To make the successful perturbations more tangible, and using as measure of selective advantage $\Delta g - \Delta\sigma^2/Y$, take the cases $\rho^{(1)}$ and $\rho^{(3)}$ and in both cases consider a slight increase in the level of contribution to the other group ($\Delta\rho_1, \Delta\rho_2 > 0$ small). Then it can be checked that for $\rho^{(1)}$, this leads to a positive term for the selective

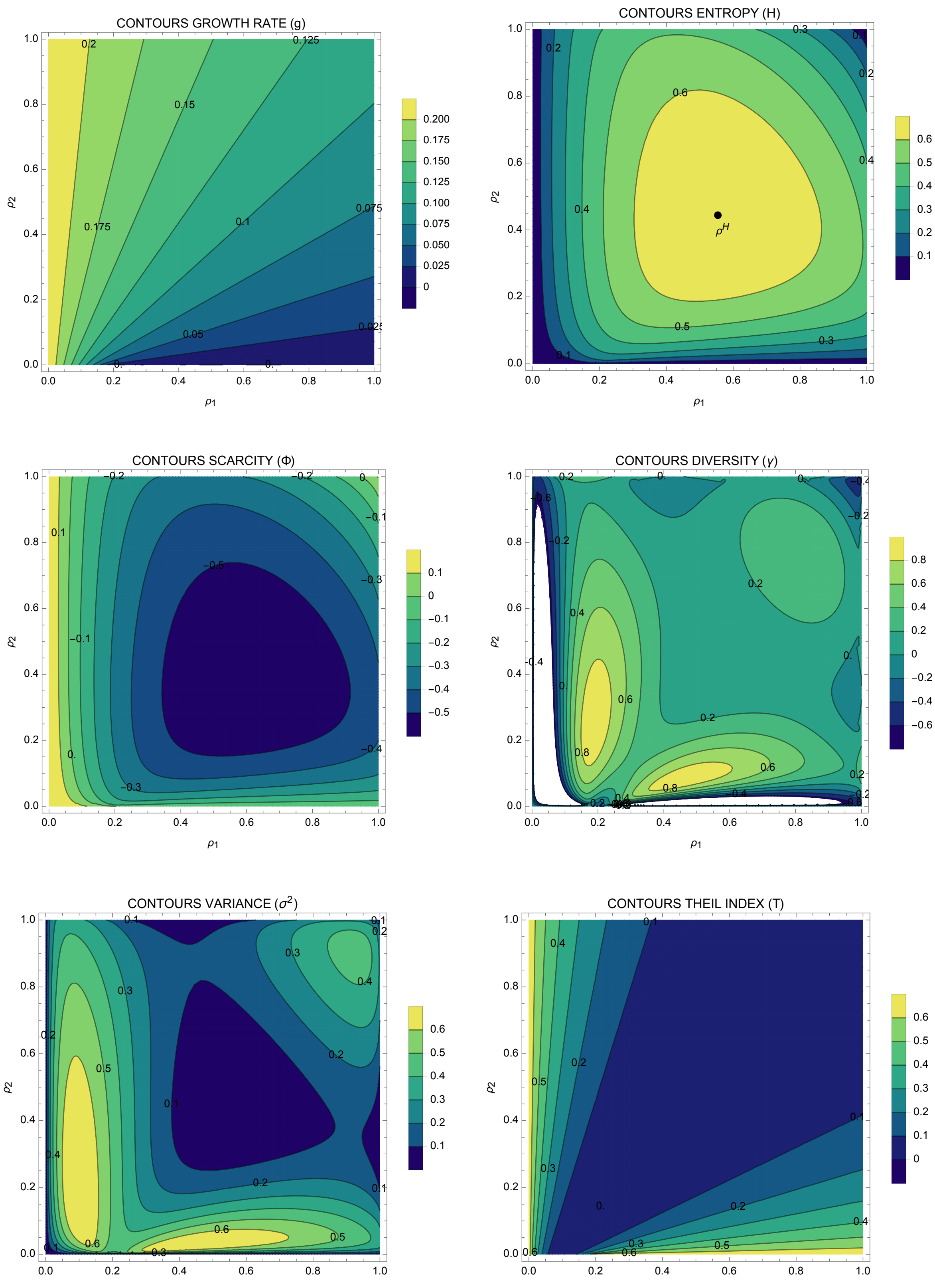

Figure 4: Contour levels for the main macroscopic variables of Example 2

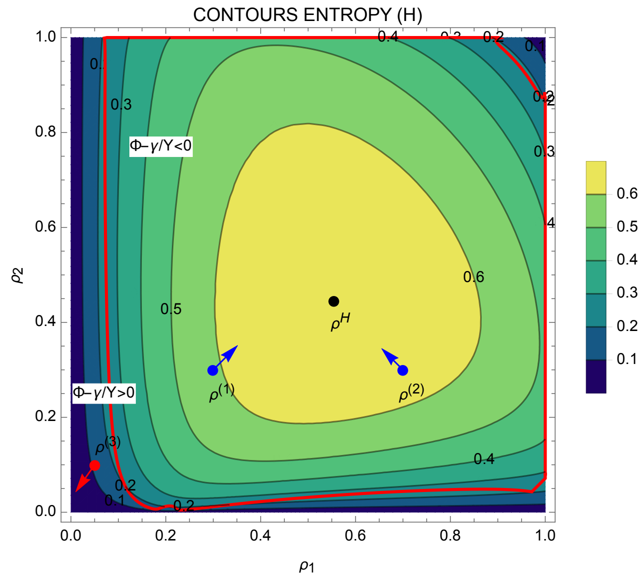

Figure 5: Successful perturbations for $\rho^{(1)}$, $\rho^{(2)}$ and $\rho^{(3)}$.

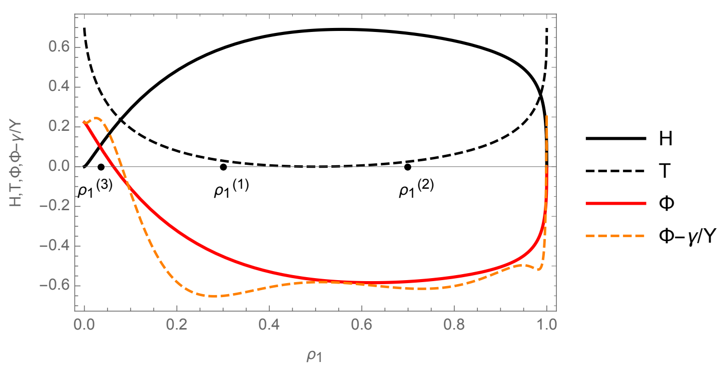

Figure 6: Macroscopic variables of Example 2 as a function of ρ_1 for $\rho_2 = \sqrt{\rho_1(1-\rho_1)}$.

Finally, to further illustrate the relation between the macroscopic variables, in particular between the entropy and the Theil index, we consider a subset of the parameters $(\rho_1, \rho_2) \in [0,1]^2$, namely, parameters along the (inverse U-shaped) curve satisfying $\rho_2 = \sqrt{\rho_1(1-\rho_1)}$, which roughly passes through the points $\rho^{(1)}$, $\rho^{(2)}$ and $\rho^{(3)}$, as well as through $\rho = (0,0), (\frac{1}{2}, \frac{1}{2})$ and $(1,0)$. Figure 6. shows the key macroscopic variables. In particular, we see the regions where Φ and $\Phi - \gamma/Y$ go from being positive to being negative. We also see the negative correlation between the entropy (H) and the Theil index (T). □

Next we apply some of the insights obtained from this example to discuss stylized examples of early pre-modern societies, including hunter-gatherer or early agricultural societies.

advantage $(\Delta g - \Delta\sigma^2/Y)$, whereas it leads to a negative term in the case of $\rho^{(3)}$. It follows that $\Delta\rho_1, \Delta\rho_2 > 0$ is a successful perturbation for $\rho^{(1)}$ but not for $\rho^{(3)}$. A perturbation in the opposite direction $(\Delta\rho_1, \Delta\rho_2 < 0)$ is a successful perturbation for $\rho^{(3)}$, which also can be shown to lead to an increase in the corresponding selective advantage $(\Delta g - \Delta\sigma^2/Y)$.

Example 3. *(Early Pre-Modern Societies)* Consider again the interaction matrix A given in Eq. (18), a version of which is depicted in Figure 3. We saw that, as a consequence of the Entropic Selection Theorem, the space of such interaction matrices is divided into ones at which entropy tends to increase (namely, where $\Phi - \gamma/Y < 0$) and ones at which it tends to decrease (where $\Phi - \gamma/Y > 0$). By Proposition 1 these also correspond to matrices at which inequality tends to decrease and increase respectively. Given various descriptions of some simple pre-modern societies such as hunter-gatherer, horticultural or early agricultural societies (Boix [18], Flannery and Marcus [44], Smith and Winterhalder [70], or the special issue of *Current Anthropology* [22]) it appears that their income generation processes may be approximated by interaction matrices as in Eq. (18). To the extent that we can place those matrices in corresponding regions (essentially by distinguishing whether $\Phi - \gamma/Y < 0$ or $\Phi - \gamma/Y > 0$), our theory can be used to infer whether to expect the economies to tend towards more or less cooperation or more or less inequality.

With the objective of placing the matrices in corresponding regions, we draw from the above mentioned literature to sketch some special cases that provide highly idealized and speculative examples of possible interactions between say two extended families (or clans) within some types of pre-modern societies.[22] In Appendix A.4, we provide a simple structural model involving a form of capital that gives a derivation of a matrix, slightly more general than the one of Eq. (18).[23]

(A) *Hunter-gatherer society.* For simplicity, suppose the hunter-gatherer society consists of two main families who frequently hunt and gather together and share almost all the goods hunted and gathered roughly equally. This would yield $\rho_1 \approx \rho_2 \approx 0.5$. Suppose further that their (slightly asymmetric) productivity is such that they can sustain a small growth rate of $g \approx 0.02$, that is, $\eta_1 \approx 1.04 > 1 \approx \eta_2$. Then a corresponding (highly stylized) interaction matrix could take the form: $A \approx \begin{pmatrix} 0.52 & 0.50 \\ 0.52 & 0.50 \end{pmatrix}$ (with $\Phi \approx -0.67$, $H \approx 0.69$, and $T \approx 0$).

(B) *Agricultural society (wheat).* This type of pre-modern society includes very many types of organizations and hence different types of income generation processes and corresponding interaction matrices. The one we consider here is where, say, two main families are fairly independent of each other, that is, $\rho_1 \approx \rho_2 \approx 0.01$ and can sustain a slightly larger growth rate of $g \approx 0.03$, that is, allowing again for some productivity asymmetry, suppose $\eta_1 \approx 1.04 > 1 \approx \eta_2$. Then a corresponding (highly stylized) interaction matrix could take the form: $A \approx \begin{pmatrix} 1.03 & 0.01 \\ 0.01 & 0.99 \end{pmatrix}$ (with $\Phi \approx 0.01$, $H \approx 0.03$, and $T \approx 0.19$).

(C) *Horticultural society.* This type of society can be seen roughly as a mixture between a

[22] The matrices are clearly overly simplified and are meant to at best capture possible representative features of the interactions for different types of societies. The numbers chosen are based loosely on the analysis and descriptions from Boix [18], Flannery and Marcus [44], Smith and Winterhalder [70], or the special issue of *Current Anthropology* [22]. For example, Figure 1, p. 26, from Flannery and Marcus [44] on Netsilik Eskimo meat sharing partnerships partly motivated the numbers chosen for the matrix for the hunter-gatherer society (Case (A)). Another example is Wood and Marlowe [83], Table 5, p. 300, which provides numerical estimates for Hazda hunter-gatherers.

[23] Clearly, there are many meaningful ways in which a society can be partitioned in different groups, besides extended families or professions, such as by gender or by age. Each one leads to a different matrix A. Studying the corresponding matrices and respective macroscopic variables, may reveal further interesting aspects of the societies.

hunter-gatherer and an agricultural society. Accordingly, we could have $\rho_1 \approx \rho_2 \approx 0.255$ and $\eta_1 \approx 1.04 > 1 \approx \eta_2$, implying a growth rate $g \approx 0.02$, and hence an example of an interaction matrix of the form: $A \approx \begin{pmatrix} 0.77 & 0.26 \\ 0.26 & 0.74 \end{pmatrix}$ (with $\Phi \approx -0.55$, $H \approx 0.57$, and $T \approx 0.001$).

It can be checked that the matrices of cases (A) and (C) satisfy $\Phi - \gamma/Y < 0$ whereas the one of case (B) satisfies $\Phi - \gamma/Y > 0$.[24] Hence, we expect societies with matrices in the neighborhood of (A) and (C) to tend towards increasing levels of cooperation and redistribution, whereas the ones in the neighborhood of (B) we expect to tend towards decreasing levels of cooperation and redistribution. This is not inconsistent with what the literature finds (Borgerhoff Mulder et al. [20], Bowles et al. [22], Smith et al. [69], Flannery and Marcus [44]), namely, more equality in hunter-gatherer and horticultural societies (represented by cases (A) and (C)) as opposed to a stronger tendency towards inequality and stratification in agricultural (and pastoral) societies (represented here by case (B)). Cleary more work is needed to obtain empirically founded interaction matrices for different societies in order to estimate corresponding parameters Φ and γ and match them with observed levels of inequality (T) and cooperation (H). □

We conclude with a stylized example of a more exploitative type of society that has two distinct groups (or classes) of individuals. Group 1 can be seen as representing a well-off elite (warriors, landowners, clergy), while group 2 represents a hard-working group (farmers, artisans, workers) possibly living on subsistence. The example illustrates how our approach can be applied to feudal or ternary societies (Flannery and Marcus [44] or Piketty [61]).

Example 4. *(Exploitative Society)* Consider now an interaction matrix between two groups of individuals, $i = 1, 2$, given by:

$$A = \begin{pmatrix} a & b \\ a & a \end{pmatrix},$$

where $b > a$ is meant to be a possibly large rate of (intertemporal) resource contribution from group 2 to group 1, whereas $0 < a < 1$ is a parameter for the rate of contribution to the own group for both groups 1 and 2 as well as for the rate of contribution from group 1 to group 2. Group 1 is meant to represent the "exploiting" group or the elite (clergy and nobility in ternary societies), whereas group 2 represents the potentially "exploited" group (farmers or workers) when b is very large compared to a. This is a clearly highly stylized example with just two parameters.

If we set $\rho = \frac{\sqrt{b}}{\sqrt{a}+\sqrt{b}}$, then we can readily compute $\lambda = a + \sqrt{ab}$, $u = (1-\rho, \rho)$, $v = (\rho, 1-\rho)^\top$ and hence:

$$P = \begin{pmatrix} 1-\rho & \rho \\ \rho & 1-\rho \end{pmatrix}, \quad \pi = \left(\tfrac{1}{2}, \tfrac{1}{2}\right), \quad g = \log\left(a + \sqrt{ab}\right),$$

[24] If we were to plot the parameters, say $\rho^{(A)}$ and $\rho^{(C)}$ in a figure similar to Figure 5, then the corresponding points would be in a neighborhood of the points $\rho^{(1)}$ and $\rho^{(2)}$ with successful perturbations also pointing in direction of higher entropy. By contrast, $\rho^{(C)}$ would correspond to a point in a neighborhood of $\rho^{(3)}$ and a successful perturbation would point in direction of lower entropy.

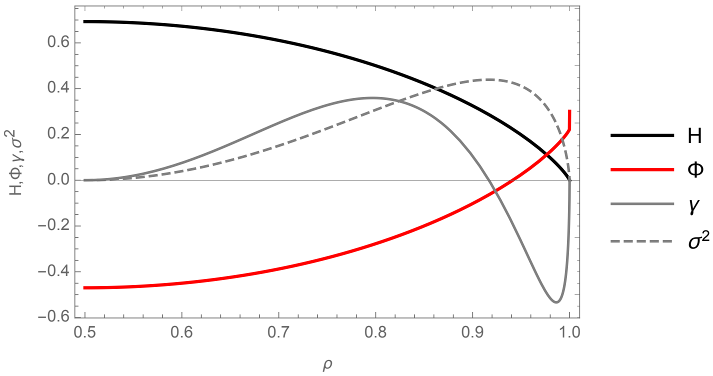

Figure 7: Macroscopic variables of Example 4 as a function of ρ for $\lambda = a + \sqrt{ab} = \frac{5}{4}$.

but also:

$$H = -(1-\rho)\log(1-\rho) - \rho\log\rho\,, \qquad \Phi = (2-\rho)\log\sqrt{a} + \rho\log\sqrt{b}\,.$$

In particular, entropy is maximized at $\rho \to \frac{1}{2}$ (or $b \to a$), where there is maximally mutualistic interaction within and across groups, and it is minimized at $\rho = 1$ (or $b \to \infty$, $a \to 0$), where there is minimally mutualistic interaction and full dependence on group 2. Figure 5 shows these and other key macroscopic variables as a function of ρ, assuming that $\lambda = a + \sqrt{ab} = \frac{5}{4}$.[25]

Notice that for $\rho > 0.94$ (or $b > 18, a < 0.075$), the income generation process is what we call concentrating ($\Phi > 0$, $\gamma < 0$) and is associated with a low entropy, while for smaller values of ρ, it is dispersing ($\Phi < 0$, $\gamma > 0$) and is associated with higher entropy.[26] Accordingly, and perhaps paradoxically, societies with large levels of exploitation may tend to increase the level of exploitation, and hence inequality, over time, whereas ones with lower levels may tend to decrease the level of exploitation. As we show in the next section, the former will tend to become more robust, whereas the latter will tend to become more fragile in response to small perturbations. □

6 Entropy, Inequality and Robust Selection

Do equal societies present advantages over unequal ones? Wilkinson and Pickett [82] suggest that economic inequality is "socially corrosive" in that it is positively correlated with a variety of "undesirable" variables that cover issues ranging from health, life expectancy and literacy rates to fairness, trust, and happiness of individuals in society. Aghion, Banerjee, and Piketty [5], Alesina

[25]Assuming $\lambda = a + \sqrt{ab} = \frac{5}{4}$ implies $a = \frac{5}{4}(1-\rho)$ and $b = \frac{5}{4}\frac{\rho^2}{1-\rho}$, which allows us to express all macroscopic variables as a function of ρ alone. As with Example 1, the number $\frac{5}{4}$ is taken purely for expositional purposes.

[26]To better interpret the parameters, consider a value of $\rho = 0.95$. This implies left and right eigenvectors $u = (0.05, 0.95)$ and $v = (0.95, 0.05)^\top$, and further implies that, at the steady state, households in group 2 contribute 19 (= 0.95/0.05) times more towards income generation than households in group 1, while households in group 1 receive 19 times more income than households in group 2. This stylized example also assumes that the population in the two groups is equal, which is often not the case, see Piketty [61]. Clearly, the framework allows for a more nuanced analysis, that distinguishes more groups and with different sizes of population.

and Perotti [9], and Rodrik [65] study models that associate inequality within a society with political instability. More recently, Stiglitz [73, 74], among others, examine various macro-economic and financial channels for how inequality contributes to economic instability and financial crises; see Van Treeck [80] for a survey.

Using a notion of robustness (or resilience) that measures the capacity or the speed with which a society returns to steady state, we show that more equal societies are more robust than less equal ones. This is important since more robust societies will also be more stable and less sensitive to shocks. Roughly speaking, a society that is slow in recovering from a shock can become dysfunctional in the sense that, after a shock, interaction levels deviate from the expected ones, and if this occurs for prolonged periods, there is a chance that further shocks occur, creating further deviations and so on, making the society overall more dysfunctional and fragile. Being quicker to recover from shocks, reduces the chances that a society is subject to dysfunction. This is the case for societies with high entropy and low inequality.

Following Demetrius, Gundlach and Ochs [35], we first formally define our notion of robustness and show that it is positively related with the evolutionary entropy (Theorem 3). We then invoke Proposition 1 to also show a positive relation between robustness and income equality (Proposition 3). Finally, we invoke Theorem 1, to show that dispersing income processes lead to more robust societies, while concentrating ones lead to more fragile ones (Proposition 4), thus providing a rationale in favor dispersing processes.

Robustness. At a general level, robustness is associated with the invariance of key macroscopic variables to endogenous or exogenous shocks. This is captured here using the formalism of large deviation theory (see Demetrius [33] and Demetrius, Gundlach and Ochs [35]). Formally, the robustness result to be shown focuses on the macroscopic parameter Φ, and introduces, for fixed $\epsilon > 0$, the probability:

$$P_\epsilon(n) = \mu \left\{ x \in \Omega \,\middle|\, \left| \frac{1}{n} S_n \varphi(x) - \Phi \right| > \epsilon \right\},$$

that is based on averages of sample trajectories of length n, $\widehat{\Phi}(n) \equiv \frac{1}{n} S_n \varphi(x)$, that differ by more than ϵ from the steady state average Φ over all trajectories.[27] The ergodic theorem states that $P_\epsilon(n)$ converges to zero for large enough sample lengths; moreover, it can be shown that there exist constants, $c_0, c_1 > 0$, such that, $P_\epsilon(n) \leq c_0 \exp^{-c_1 n}$, (see Appendix A.5.1), so that the convergence rate is at least exponentially fast. Hence, we define **robustness** $\mathcal{R}$ as the following fluctuation decay rate:

$$\mathcal{R} = \lim_{n \to \infty} \left[-\frac{1}{n} \log P_\epsilon(n) \right]. \tag{19}$$

Large values of $\mathcal{R}$ correspond to fast rates of convergence of the sample averages ($\widehat{\Phi}(n)$) to their steady state values; small values of $\mathcal{R}$ correspond to slow rates of convergence. Thus,

[27] Recall the definition, $\Phi = \lim_{n\to\infty} \frac{1}{n} \mathbb{E}_n [S_n \varphi]$ (see Section 2). As average of the potential function φ, the parameter Φ is central in determining all other macroscopic parameters of the process $Y(t)$, including g, σ^2 and γ. It is for this reason that the robustness measure is based on the probability of sample averages approaching Φ.

$\mathcal{R}$ characterizes the adjustment rate of the (fundamental) macroscopic variable ($\widehat{\Phi}(n)$) in the face of general shocks or perturbations in the underlying system. More generally, because all further macroscopic variables are directly determined through the variable Φ, the variable $\mathcal{R}$ also provides a measure of the convergence rate of sample averages of all macroscopic variables to their steady state values.

The next result asserts that changes in robustness are positively correlated with changes in evolutionary entropy.

Theorem 3 (Robustness and Evolutionary Entropy)**.** *For perturbations of the form $A(\delta) = (a_{ij}(\delta))$, where $a_{ij}(\delta) = a_{ij}^{1+\delta}$, we have that robustness and evolutionary entropy move in the same direction, $\Delta\mathcal{R}\Delta H \geq 0$, where $\Delta\mathcal{R} = \mathcal{R}(\delta) - \mathcal{R}(0)$, $\Delta H = H(\delta) - H(0)$, for $\delta \in \mathbb{R}$ small; with strict inequality if $\Delta\mathcal{R}, \Delta H \neq 0$.*

Consider two societies operating according to A and $A^*(= A(\delta))$ with corresponding levels of robustness $\mathcal{R}$ and $\mathcal{R}^*(= \mathcal{R}(\delta))$, satisfying $\mathcal{R} > \mathcal{R}^*$, then from the definition of $\mathcal{R}$, we can say that the first society is more resilient to shocks in the sense of returning faster to steady state. In this sense, the proposition suggests that societies with higher evolutionary entropy are more resilient to shocks than societies with lower evolutionary entropy. In view of Proposition 1, this further implies that changes in robustness are negatively correlated with changes in inequality.

Proposition 3 (Robustness and Inequality)**.** *For perturbations of the form $A(\delta) = (a_{ij}(\delta))$, where $a_{ij}(\delta) = a_{ij}^{1+\delta}$, we have that robustness and the inequality index move in opposite directions, $\Delta\mathcal{R}\Delta T \leq 0$, where $\Delta\mathcal{R} = \mathcal{R}(\delta) - \mathcal{R}(0)$, $\Delta T = T(\delta) - T(0)$, for $\delta \in \mathbb{R}$ small; with strict inequality if $\Delta\mathcal{R}, \Delta T \neq 0$.*

These results point to a significant advantage of equal over unequal societies, namely, the former are more robust and have a higher level of resilience to shocks in the sense that they are quicker in returning to steady state; unequal societies are more fragile and slower in getting back to the steady state. The role of inequality in affecting the resilience of a country to respond to shocks was an important topic in the 2011 and 2014 reports of the United Nations Development Programme (UNDP [78, 79]), which finds that less equal societies are slower in adapting to shocks and also tend to have less stable environments. The ability to quickly respond to shocks is particularly relevant when shocks are adverse. Anbarci et al. [11] and Kahn [50] study respectively earthquakes and natural disasters and show that besides national income, the level of income inequality has an important effect on the death toll and fatalities; Alesina and Perotti [9] show that income inequality fuels discontent and is empirically positively correlated with political instability.

Finally, Theorems 1 and 3 readily imply the following.

Proposition 4 (Robust Selection)**.** *The outcome of the selection process facing a society evolving according to Eq. (3) is such that, if the income process is dispersing, it will tend towards higher levels of robustness; whereas if it is concentrating, it will tend towards lower robustness.*

If the previous results (Theorems 1 and 2) show that dispersing processes tend to favor higher levels of cooperation as well as higher income equality, this last result shows that they also generate more robust societies. By contrast, concentrating processes, while favoring lower levels of cooperation and higher inequality, tend to generate more fragile societies. This seems to provide an important rationale for preferring dispersing processes over concentrating ones.

7 Conclusion

Discussions regarding the phenomenon of economic inequality, its origin and spread, have moved recently from the confines of specialized academic departments to being among the most widely debated topics by the general public worldwide. At least three issues fuel the debate: the empirical reality that the gap between rich and poor has shown a remarkable increase in a number of countries over the last 30 years (Bourguignon [19], Piketty [60, 61]); the empirical observation that countries with large economic inequality have a high degree of dysfunction in a number of canonical indices, including health, corruption, and crime, as well as economic and political instability (Acemoglu and Robinson [4], Case and Deaton [26], Piketty [61], Stiglitz [74], Wilkinson and Pickett [82]); the instability and dysfunctionality of highly unequal societies often generates flows of immigrants that in turn create further problems for hosting countries (UNDP [78]).

The theory developed here contributes to the understanding of the origin and spread of inequality by relating changes in inequality to specific features of the underlying income generation process. The theory distinguishes between *dispersing* and *concentrating* processes. The former favor more reciprocal interactions and cooperation between individuals within and across groups as well as more equal redistribution. The latter favor weaker or less reciprocal interactions and less equal redistribution.

The income process, whether dispersing or concentrating, further impacts the society's resilience. This occurs in a way that reinforces and amplifies the original effect on inequality and cooperation, thus explaining the emergence and persistence of more equal or egalitarian societies on one hand, and unequal or stratified ones on the other. The intrinsic instability of highly unequal societies that we associate with concentrating income processes entails their potential vulnerability to shocks which can seriously compromise their functionality and welfare.

Economic policy may be designed to take such underlying forces into account. While the paper focused on pre-modern or early societies with state, we believe all the main insights and especially the distinction between dispersing and concentrating income processes may well be relevant for contemporary societies. In this sense, our central policy implication wold be that regulation and fiscal policies should be designed to ensure that societies' income processes be dispersing rather than concentrating. This may be obtained through regulation and taxation and other fiscal policies that constrain and redistribute excessive returns or sources of returns, typically associated with concentrating processes; as well as macroeconomic and industrial policies that stabilize and diversify the underlying income generation process. Policies that attempt to address issues of, say, inequality and immobility, without monitoring the underlying type of income generation process may fail to achieve their intended objectives.

Clearly, significant empirical work is needed to estimate the relevant matrices A and the associated parameters and to test the different mechanisms and correlations derived in this paper for simple pre-modern or early societies as well as for contemporary ones.[28] At the same time, more theoretical work is needed to extend the present framework to contemporary societies with financial sectors and social states. This may yield an improved understanding of key aspects of income generating processes as driving forces of equality, cooperation and robustness, and may contribute to a better design of long-run economic policies.

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[28] Some have raised the question of how our model relates to the dramatic increase in inequality in the US and other Western countries in recent decades. To address this, one would need to identify and estimate appropriate interaction matrices and derive the parameters Φ and γ. For the US, we suspect that since the 1960's the parameters have been such that the income generating process was concentrating and has remained such since. To check it, as the US is a large economy, one would need to check whether the empirical value of Φ has been positive over the period. This would be consistent with the observed increase in income inequality (Bourguignon [19], Piketty [60, 61], Stiglitz [74]) and decrease in various measures of social capital (Putnam [62] and Putnam and Garrett [63]). Less clear is the transition from the period prior to the 1960s to the post 1960's. The recent book by Putnam and Garrett [63] studies various measures and shows how over the last 125 years economic, social, political and cultural measures trace roughly a synchronous bell-shaped curve with an *upswing* from the 1900's to the mid 1960's, followed by a *downswing* from the mid 1960's until now. The authors describe this as a transition from an "I"-society to a "we"-society (upswing) and back to an "I"-society (downswing). In our approach, we expect a high entropy society to be associated with a "we"-society and a low entropy society to be associated with an "I"-society; Tomasello [77] also discusses the evolution of moral norms in terms of psychological and cultural notions of "I", "you" and "we". And while the upswing can potentially be explained by a dispersing income generation process (where one would need to check whether Φ was negative for at least the initial part of the period prior to the 1960's, which is not inconsistent, e.g., with the years of the Great Depression), what our theory cannot explain is the *transition* from one type of process to another, or in other words, the necessary change in the underlying parameter Φ. This may have occurred through a change in the productive and organizational structure of the industry, exogenous to the model and its evolutionary forces. The US participation in the second world war, where the US economy emerged as incomparably strong at the end of the war, may represent such an event.

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APPENDIX

A Background Analysis and Proofs

We here build on Arnold et al. [12] and Demetrius and Gundlach [34] to sketch some basic properties of the dynamic system and of its macroscopic variables at steady state. This allows us to better understand the terminology and connections between the different variables in the model. These facts are derived in detail in the cited articles using the formalism of random dynamical systems and statistical mechanics. While we can only limit ourselves to sketching the main steps, we refer to those articles for a complete discussion; see also Demetrius [33] and Demetrius et al. [36] for further discussion.

A.1 Random Dynamical Systems

We assume that the (possibly nonlinear) dynamic system,

$$v(t+1) = A(t)v(t)$$

evolves to a steady state. At steady state we assume the process is represented by means of a constant $d \times d$ matrix $A = (a_{ij})$ with $a_{ij} > 0$. Let $D = \{1, 2, \ldots, d\}$ and define the set of all possible doubly infinite sequences

$$X = \prod_{\nu=-\infty}^{\infty} D_\nu, \qquad \text{where } D_\nu = D,$$

and let

$$\Omega = \{x \in X : a_{x_{\nu+1}x_\nu} > 0\},$$

define the space of all such sequences associated to paths on the graph associated with the matrix A. Let further $\tau : \Omega \to \Omega, (x_k) \mapsto (\tilde{x}_k)$, where $\tilde{x}_k = x_{k+1}$ be the shift map, and let $\mathcal{M}$ denote the set of probability measures that are invariant under the shift map τ. Defining μ as the natural Markov measure on Ω at the steady state, one can show that (see [36], Theorem 4.2) this is the unique probability measure that maximizes $H_\mu(\tau) + \int \varphi d\mu$ such that,

$$\log \lambda = \sup_{\mu \in \mathcal{M}} \left\{ H_\mu(\tau) + \int \varphi d\mu \right\}, \tag{20}$$

where $H_\mu(\tau)$ is the Kolmogorov-Sinai entropy for the system (Ω, μ, φ), and $\varphi : \Omega \to \mathbb{R}$ is given by

$$\varphi(x) = \log a_{x_1 x_0}.$$

Analytically, it can be described explicitly by means of the Markov matrix $P = (p_{ij})$, where as usual $p_{ij} = \frac{a_{ji}u_j}{\lambda u_i}$, and $u = (u_i)$ is the left (row) eigenvector corresponding to the largest eigenvalue λ of A; (see Arnold et al. [12] and also Demetrius et al. [36], Theorem 4.2).

For future reference, for $x \in X$, define the sample path,

$$S_n\varphi(x) = \sum_{k=0}^{n-1} \varphi(\tau^k x) = \sum_{k=0}^{n-1} \log a_{x_{k+1}x_k},$$

and recall that $\lim_{n\to\infty} \frac{1}{n}\mathbb{E}_n \left[S_n\varphi\right] = \int \varphi d\mu = \Phi$.

Diffusion Approximation of the Income Process. Using a generalization of the Central Limit Theorem, (see Demetrius et al. [36], Theorem 7.1), one can show that, in the genealogies model of the dynamical system defined by Eq. (3) and (Ω, μ, φ), for the natural measure μ and any $\widehat{t} \in \mathbb{R}$, we have,

$$\lim_{n\to\infty} \mu \left\{ x \in \Omega \,\middle|\, \frac{S_n\varphi(x) - n\int \varphi d\mu}{\sqrt{n}} \right\} \leq \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\widehat{t}} \exp\left(-\frac{t^2}{\sigma^2}\right) dt.$$

Thus, asymptotically, the deviations of the sample paths $S_n\varphi$ for genealogies $x \in X$ from their mean, as $n \to \infty$, can be approximated by a Brownian motion with variance $\sigma^2 t$ for the process in continuous time. The density of $f(Y,t)$ of the process $Y(t)$ can then be characterized by the Fokker-Planck equation,

$$\frac{\partial f}{\partial t} = -g\frac{\partial(fY)}{\partial Y} + \frac{\sigma^2}{2}\frac{\partial^2(fY)}{\partial Y^2}.$$

Hence, we can characterize $Y(t)$ as the solution to the stochastic differential equation,

$$dY(t) = gY(t)dt + \sigma\sqrt{Y(t)}dW(t), \tag{21}$$

where $W(t)$ is a Brownian motion. In particular, this yields a continuous time process with growth rate g and variance $\sigma^2 Y(t)$. (See Demetrius et al. [36], Section 7, for the details.) This process belongs to the class of so-called Cox-Ingersoll-Ross processes studied in mathematical finance (see Cox et al [29]).

A.2 Perturbations

Throughout the paper we make use of perturbations of the matrix A of the form $A(\delta) = (a_{ij}(\delta)$, where $a_{ij}(\delta) = a_{ij}^{1+\delta})$, for $\delta \in \mathbb{R}$. We here sketch a motivation for the specific one-parameter form we adopt throughout the text.

Consider two dynamical systems at steady state given respectively by (Ω, μ, φ) and $(\Omega, \mu^*, \varphi^*)$. To capture the fact that, the latter is a variation (mutation) of the former, we assume that

$$\varphi^* = \varphi(\delta) = \varphi + \delta\psi,$$

satisfying the conditions

$$\int \varphi d\mu = \int \psi d\mu \quad \text{and} \quad \frac{d}{d\delta}\int \varphi d\mu\,|_{\delta=0} = \frac{d}{d\delta}\int \psi d\mu\,|_{\delta=0}.$$

The first condition says that the deviation ψ of the variant (mutant) population has the same productive potential as that of the incumbent population; the second condition says that the deviation ψ also has the same directional derivative as that of the productive potential of the incumbent population. This is sufficient for our results. However, for ease of representation and to simplify the analysis we consider the special case $\psi = \varphi$, which, if we assume $\varphi = \log a_{ij}$ (where $A = (a_{ij})$ is the interaction matrix), implies

$$\varphi(\delta) = \varphi + \delta\varphi = (1+\delta)\log a_{ij} = \log a_{ij}^{1+\delta},$$

which corresponds to perturbations of the interaction matrix of form $A(\delta)$, with $a_{ij}(\delta) = a_{ij}^{1+\delta}$, considered throughout the paper. See Demetrius et al. [36], Section 6, for details. The perturbations $A(\delta)$ are used both to model the variant populations' interaction matrices and to generate the macroscopic variables.

A.3 Macroscopic Variables and Their Relations

In Section 2, we defined several macroscopic variables using the genealogies model. We here summarize some of the relationships that hold between the macroscopic variables used in the paper. We refer to Demetrius et al. [36] and Demetrius and Gundlach [34] for derivations and further details. As mentioned in the body of the paper, we can also generate the macroscopic variables from the moments of the growth rate, using perturbations. This provides an easier way of computing the macroscopic variables. Let $\lambda(\delta)$ denote the dominant eigenvalue of the perturbed matrix $A(\delta) = (a_{ij}^{1+\delta})$, and $g(\delta) = \log \lambda(\delta)$, for $\delta \in \mathbb{R}$. Then it can be shown that:

$$g(\delta) = g(0) + \delta g'(0) + \frac{\delta^2}{2!} g''(0) + \frac{\delta^3}{3!} g'''(0) + \ldots,$$

and

$$\begin{aligned} g'(0) &= \lim_{n\to\infty} \frac{1}{n} \mathbb{E}_n \left[S_n \varphi \right] = \int \varphi d\mu = \Phi \\ g''(0) &= \lim_{n\to\infty} \frac{1}{n} \mathbb{V}_n \left[S_n \varphi \right] = \sigma^2 \\ g'''(0) &= \lim_{n\to\infty} \frac{1}{n} \mathbb{E}_n \left[S_n \varphi - \mathbb{E}_n S_n \varphi \right]^3 = \kappa. \end{aligned}$$

Here $\mathbb{E}_n$ and $\mathbb{V}_n$ denote the expectation and variance with respect to the measure μ_n on finite sequences of length n of the form $(x_0, x_1, \ldots, x_n)$, which is defined by

$$\mu_n = \frac{S_n \varphi(x)}{\sum_{(x_0, x_1, \ldots, x_n)} S_n \varphi(x)}.$$

Consider now the perturbed variance $\sigma^2(\delta)$ obtained by further perturbing $A(\delta)$ again. Then we can define

$$\gamma = \frac{d\sigma^2(\delta)}{d\delta} |_{\delta=0} \ .$$

The following relations between the macroscopic variables hold:

(a) $\Phi = g - H$

(b) $\gamma = 2\sigma^2 + \kappa$.

In our evolutionary analysis, an incumbent population is in competition with a variant (or invader) population, which we capture in terms of a dynamic interaction between the two populations. The incumbent and the variant (invader) population steady state dynamics are given respectively by (Ω, μ, φ) and $(\Omega, \mu^*, \varphi^*)$, where to capture the fact that the latter is a mutation of the former, as discussed in A.2, we assume that

$$\varphi^* = \varphi(\delta) = \varphi + \delta\varphi,$$

which corresponds to an interaction matrix of the invader population of the form $A(\delta) = (a_{ij}^{1+\delta})$.

We can then determine the macroscopic variables for the variant population as with the incumbent population, so that setting $g^* = g(\delta)$, $\sigma^{*2} = \sigma^2(\delta)$, and $H^* = H(\delta)$, we get:

(c) $\Delta g = g(\delta) - g(0) \approx \Phi\delta$

(d) $\Delta\sigma^2 = \sigma^2(\delta) - \sigma^2(0) \approx \gamma\delta$

(e) $\Delta H = H(\delta) - H(0) \approx -\sigma^2\delta$.

For small $\delta \in \mathbb{R}$, this readily gives $\Delta g \Delta H \approx -\Phi\sigma^2\delta^2$ and $\Delta\sigma^2\Delta H \approx -\gamma\sigma^2\delta^2$, and hence the following relations:

(f) $\Phi < 0 \Rightarrow \Delta g \Delta H > 0$ and $\Phi > 0 \Rightarrow \Delta g \Delta H < 0$

(g) $\gamma > 0 \Rightarrow \Delta\sigma^2\Delta H < 0$ and $\gamma < 0 \Rightarrow \Delta\sigma^2\Delta H > 0$.

These will play an important role in the derivation of the main results.

A.4 A Simple Model for Pre-Modern Economies

To give a more structural derivation of the matrix A that might work for various types of pre-modern societies, we postulate a simple not necessarily linear model, which implies a steady state as in Eq. (3), and under further assumptions, also of the more specific form of Eq. (18). More concretely, we assume production functions that are homogeneous of degree 1.[29]

Assume the society consists of two groups, $i = 1, 2$, that can also be thought of as extended families. Let $x_i(t)$ denote what is actually produced in period t by group i (e.g., amount of food over a year measured with a common numeraire); let $y_i(t)$ denote what is obtained in terms of consumption goods in period t by group i (this is what we refer to as income); and let $h_i(t)$ denote a measure of capital in period t by group i (e.g., Borgerhoff Mulder et al. [20] distinguish three types of capital, embodied, relational and material capital; our measure can be viewed as a compound measure of the three). We assume the following equations:

$$x_1(t) = \alpha_{11} h_1(t)^{\alpha_{12}} h_2(t)^{1-\alpha_{12}} \tag{22}$$
$$x_2(t) = \alpha_{21} h_1(t)^{1-\alpha_{22}} h_2(t)^{\alpha_{22}} \tag{23}$$
$$y_1(t) = \rho_{11} x_1(t) + \rho_{12} x_2(t) \tag{24}$$
$$y_2(t) = \rho_{21} x_1(t) + \rho_{22} x_2(t) \tag{25}$$
$$h_1(t) = \beta_{11} h_1(t-1) + \beta_{12} y_1(t-1) \tag{26}$$
$$h_2(t) = \beta_{21} h_2(t-1) + \beta_{22} y_2(t-1) \tag{27}$$

where all coefficients are nonnegative, and $0 \leq \alpha_{12}, \alpha_{22} \leq 1$.

Eqs. (22) and (23) capture how consumption goods (mainly food) are produced by means of capital and labor from both groups, where labor, being constant in our model, is reflected in the coefficients α_{11} and α_{21}.[30] To the extent that the capital variables incorporate embodied capital, allows them to also reflect technology and skills and hence also labor productivity. Importantly, Eqs. (22) and (23) are homogeneous of degree one, somewhat similar to Borgerhoff Mulder et al. [20]. Eqs. (24) and (25) capture actual income or total consumption goods obtained by groups 1 and 2. Again, group 1 might directly give goods to group 2 and vice versa. Finally, Eqs. (26) and (27) capture how capital is transmitted from one period to the next, in a way that depends on the levels of own capital and consumption of the previous period.

[29] Solow and Samuelson [71] show that allowing for equations that homogeneous of degree 1 (and hence not necessarily all linear) leads to a steady state where all variables grow at the same rate. We follow their approach in deriving such a steady state with a Cobb-Douglas type production functions.

[30] To keep the model simple, labor and population, but also natural resources are not explicitly modeled and are assumed to be indirectly reflected in the equations, and ultimately in the coefficients of the matrix A.

We now sketch how Eqs. (22) to (27) together imply a process for $y_1(t), y_2(t)$ as the one of Eq. (18) and a fortiori of Eq. (3). For simplicity and to keep the notation as close as possible to the one of the examples of Section 5, we make the following assumptions:

$$\rho_{11} = 1 - \rho_1, \rho_{21} \equiv \rho_1 \text{ and } \rho_{12} \equiv \rho_2, \rho_{22} = 1 - \rho_2,$$

as well as:

$$\beta_{11} = \beta_{21} \equiv \beta_1 \text{ and } \beta_{12} = \beta_{22} \equiv 1 - \beta,$$

which implies symmetry of the capital transmission, Eqs. (26) and (27). Furthermore, assume:

$$\alpha_{11} \equiv \eta, \alpha_{21} = 1 \text{ and } \alpha_{12} = \alpha_{22} \equiv \alpha.$$

This yields the following equations:

$$x_1(t) = \eta h_1(t)^{\alpha} h_2(t)^{1-\alpha} \quad (28)$$

$$x_2(t) = h_1(t)^{1-\alpha} h_2(t)^{\alpha} \quad (29)$$

$$y_1(t) = (1 - \rho_1)x_1(t) + \rho_2 x_2(t) \quad (30)$$

$$y_2(t) = \rho_1 x_1(t) + (1 - \rho_2)x_2(t) \quad (31)$$

$$h_1(t) = \beta_1 h_1(t-1) + \beta_2 y_1(t-1) \quad (32)$$

$$h_2(t) = \beta_1 h_2(t-1) + \beta_2 y_2(t-1) \quad (33)$$

Under the above assumptions, we can derive a steady state approximation of the linear form $y(t) = Ay(t-1)$ as in Eqs. (3) and (18). For this, we substitute equations (32), (33) into equations (28), (29) and subsequently equations (28), (29) into equations equations (30), (31) to get a system of four difference equations for the y_i's and h_i's. Next, we can write the steady state relation between $h_i(t)$ and $y_i(t)$ as $h_i(t) = \frac{\beta_2+\lambda}{\beta_1} y_i(t)$. This reduces the system to one of two difference equations, which can be linearized at the steady state to obtain equations of the form $y(t) = Ay(t-1)$. To illustrate this, consider two cases:

Case 1. $\alpha = \frac{1}{2}$:

$$\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(\beta_1 + \beta_2\chi) & \frac{\eta(1-\rho_1)+\rho_2}{2(\eta\rho_1+(1-\rho_2))}(\beta_1 + \beta_2\chi) \\ \frac{\eta\rho_1+(1-\rho_2)}{2(\eta(1-\rho_1)+\rho_2)}(\beta_1 + \beta_2\chi) & \frac{1}{2}(\beta_1 + \beta_2\chi) \end{pmatrix} \begin{pmatrix} y_1(t-1) \\ y_2(t-1) \end{pmatrix}, \quad (34)$$

where here $\chi = \sqrt{(\eta(1-\rho_1)+\rho_2))(\eta\rho_1 + (1-\rho_2))}$ (note that $\chi \approx \sqrt{\eta} \approx 1$ for $\eta \approx 1$ and $\rho_1 \approx \rho_2$; and $\beta_1 + \beta_2\chi \approx 1$ if furthermore $\beta_1 + \beta_2 \approx 1$).

Case 2. $\alpha = 1$:

$$\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} \eta(1-\rho_1)(\beta_1\chi + \beta_2) & \rho_2(\beta_1\chi + \beta_2) \\ \eta\rho_1(\beta_1\chi + \beta_2) & (1-\rho_2)(\beta_1\chi + \beta_2) \end{pmatrix} \begin{pmatrix} y_1(t-1) \\ y_2(t-1) \end{pmatrix}, \quad (35)$$

where $\chi = 2\left(\eta(1-\rho_1) + (1-\rho_2) + \sqrt{(\eta(1-\rho_1) + (1-\rho_2))^2 - 4(1-\rho_1-\rho_2)}\right)^{-1}$ (note that $\chi \approx \frac{2}{1+\eta} \approx 1$ for $\eta \approx 1$ and $\rho_1 \approx \rho_2$; and $\beta_1\chi + \beta_2 \approx 1$ if furthermore $\beta_1 + \beta_2 \approx 1$).

These two cases are examples of a generalization of the matrices postulated in Examples 2 and 3 of Section 5. The forms of Eqs. (34) and (35) allow us to better interpret the elements of the matrix A in two different cases. In Case 1 production depends symmetrically on both groups' capital ($\alpha = \alpha_{12} = \alpha_{22} = \frac{1}{2}$)

and can be seen as a case of joint production. It is usually associated with a dispersing process, especially if the productivity variable η is not too large and the parameters ρ_1, ρ_2 not too small or asymmetric. It can represent cases of rice agriculture or of hunter gatherer societies. Case 2 reflects a case where group i's production depends only on i's own capital (since $\alpha = \alpha_{12} = \alpha_{22} = 1$). This is a situation that for sufficiently large η and sufficiently small ρ_1, ρ_2 can qualify as a concentrating process (as in Example 3-B). However, as, for example, ρ_1, ρ_2 increase, can become dispersing. It can represent cases of wheat agriculture but also of hunter gatherer societies (e.g., for $\rho_1, \rho_2 \approx \frac{1}{2}$ as in Example 3-A) and horticultural societies (e.g., for $\rho_1, \rho_2 \approx \frac{1}{4}$ as in Example 3-C). Of course, horticultural societies (as also rice, wheat and hunter gatherer societies) could also be represented by cases that are intermediate between Cases 1 and 2.

Finally, Example 2 of Section 5, which can be seen as a special case of Case 1 (for $\eta = \frac{5}{4}$ among other restrictions), also shows how to interpret the perturbations of the matrices A in the context of the model above. Namely, they can be interpreted as small changes in the parameters ρ_1 and ρ_2. Figure 5 shows directions for successful perturbations, starting from the original matrices $A^{(1)}, A^{(2)}$ and $A^{(3)}$, and consistent with the postulated perturbations of the form $A^{(1)}(\delta), A^{(2)}(\delta)$ and $A^{(3)}(\delta)$, respectively.

A.5 Proof of Proposition 1

We need to show that $\Delta T \Delta H \leq 0$, with strict inequality if $\Delta T, \Delta H \neq 0$. Consider again the following measure of entropy, which can be viewed as a measure of *equality*:

$$H_{EQ}(v) = -\sum_{i=1}^{d} v_i \log v_i, \qquad 0 \leq H_{EQ}(v) \leq \log d,$$

where in the steady state v is the normalized eigenvector of A so that v_i is the share of income of class i. Then it is easy to see that $H_{EQ}(v)$ and $T(v)$ are related as follows,

$$T(v) = \log d - H_{EQ}(v),$$

so that $\Delta H_{EQ} \Delta T \leq 0$, with strict inequality if $\Delta H_{EQ}, \Delta T \neq 0$, where $\Delta H_{EQ} = H_{EQ}(v(\delta)) - H_{EQ}(v(0))$ and where $v(0) = v$ and $v(\delta)$ are the (normalized) eigenvectors of A and $A(\delta)$, respectively.

Because $\Delta H_{EQ} \Delta T \leq 0$, to show that $\Delta T \Delta H \leq 0$, it suffices to show that $\Delta H_{EQ} \Delta H \geq 0$, with strict inequality if $\Delta H_{EQ}, \Delta H \neq 0$, for $\delta \in \mathbb{R}$ small. Now, it can be shown that, for δ small, the (normalized) eigenvector $v(\delta)$ satisfies:

$$v_i(\delta) \approx \frac{v_i^{1+\delta}}{\sum_{j=1}^{d} v_j^{1+\delta}} \approx \frac{v_i(1 + \delta \log v_i)}{\sum_{j=1}^{d} v_j(1 + \delta \log v_j)}. \tag{36}$$

We work with the latter approximation formula, with which we compute:

$$H_{EQ}(v(\delta)) \approx -\sum_{i=1}^{d} \frac{v_i(1 + \delta \log v_i)}{\sum_{j=1}^{d} v_j(1 + \delta \log v_j)} \log \frac{v_i(1 + \delta \log v_i)}{\sum_{j=1}^{d} v_j(1 + \delta \log v_j)}.$$

Since we know that for δ small we have $\Delta H(\delta) \approx -\sigma^2 \delta$, it suffices to show that $\Delta H_{EQ}(v(\delta))$ can also be written as $\Delta H_{EQ}(v(\delta)) \approx -\psi \delta$ for some $\psi > 0$ and $\delta \in \mathbb{R}$ small. To see this we compute the derivative of

$\Delta H_{EQ}(v(\delta))$ at $\delta = 0$. This gives:

$$\frac{\partial \Delta H_{EQ}(v(\delta))}{\partial \delta}\,|_{\delta=0} = -\sum_{i=1}^{d} v_i(\log v_i)^2 \sum_{\substack{j=1\\j\neq i}}^{d} v_j + \sum_{i=1}^{d} v_i \log v_i \sum_{\substack{j=1\\j\neq i}}^{d} v_j \log v_j,$$

where we use the fact that at $\delta = 0$, we have $\sum_{j=1}^{d} v_j(1+\delta \log v_j) = 1$. We proceed by induction to show that the expression is non-positive. Notice first that the expression can be rewritten as:

$$\frac{\partial \Delta H_{EQ}(v(\delta))}{\partial \delta}\,|_{\delta=0} = -\sum_{i=1}^{d} v_i \log v_i \sum_{\substack{j=1\\j\neq i}}^{d} v_j \left(\log v_i - \log v_j\right).$$

For $d = 2$, it is easy to check that this expression is negative for any values of the v_i's except for $v_i = 0, \frac{1}{2}$ or 1, where it is zero. Next, for general d, the expression can be written in two parts as the sum of the above term with $d = d-1$ and another term, which is clearly negative (except for some boundary cases where it is zero):

$$\frac{\partial \Delta H_{EQ}(v(\delta))}{\partial \delta}\,|_{\delta=0} = -\sum_{i=1}^{d-1} v_i \log v_i \sum_{\substack{j=1\\j\neq i}}^{d-1} v_j \left(\log v_i - \log v_j\right) - v_d \sum_{i=1}^{d-1} v_i \left(\log v_i - \log v_d\right)^2.$$

From here it is clear that, assuming the expression is negative for $d = d-1$, then it is negative for general d (except for some boundary cases where it is zero). This shows that the above derivative is negative locally around $\delta = 0$. Hence we can write $\Delta H_{EQ}(v(\delta)) \approx -\psi\delta$ for some $\psi > 0$. This implies that $\Delta H_{EQ}(v(\delta))$ has the same sign as $\Delta H(\delta)$, or in other words, $\Delta H_{EQ}(v(\delta))\Delta H(\delta) \geq 0$, which in turn shows that $\Delta T \Delta H \leq 0$ given that $\Delta T \Delta H_{EQ} \leq 0$. It remains to show that the approximation for $v(\delta)$ used in Eq. (36) is valid. This can be checked by computing $A(\delta)v(\delta)$ for the approximation in Eq. (36) and showing that it is equal to $\lambda(\delta)v(\delta)$ again for the approximation in Eq. (36), for $\delta \in \mathbb{R}$ small.

A.5.1 Proof of Proposition 2

This follows from Proposition 1 after noticing that for $\delta \in \mathbb{R}$ small, the eigenvector $v(\delta)$ corresponding to the perturbed matrix $A(\delta) = (a_{ij}^{1+\delta})$ satisfies

$$v\left(\delta/2\right) \approx \frac{v(\delta)+v(0)}{2}.$$

This implies that $T(\frac{v(\delta)+v(0)}{2}) - T(0)$ has the same sign as ΔT and the opposite sign as $E(v(\delta), v(0))$. Hence $(T(\frac{v(\delta)+v(0)}{2}) - T(0))\Delta H \leq 0$ with strict inequality if $T(\frac{v(\delta)+v(0)}{2}) - T(0)), \Delta H \neq 0$, and also $E(v(\delta), v(0))\Delta H \geq 0$ with strict inequality if $E(v(\delta), v(0)), \Delta H \neq 0$.

SUPPLEMENTARY APPENDIX (Not for Publication)

Proof of Theorem 1

We show the main steps of the argument. For a detailed proof, we refer the reader to Demetrius et al. [36].

Consider the aggregate production of the incumbent population, described by $Y(t)$ (which is derived from A and (Ω, μ, φ))) and satisfies Eq. (21) above, and that of the variant population Y^* (which is derived from A^* and $(\Omega, \mu^*, \varphi^*)$) and satisfies Eq. (37) below. Let $Z(t) = Y(t) + Y^*(t)$ denote total aggregate production. The share of aggregate production of the variant population can be written as:

$$p(t) = \frac{Y^*(t)}{Z(t)}.$$

We are concerned with the evolution of this ratio.[31]

As mentioned in the text, the matrix A^* is given as the perturbation $A^* = A(\delta) = (a_{ij}^{1+\delta})$ of the original matrix $A = (a_{ij})$, for $\delta \in \mathbb{R}$ small in absolute value. Let $g^* = g(\delta)$, $H^* = H(\delta)$, $\Phi^* = \Phi(\delta)$, ${\sigma^*}^2 = \sigma^2(\delta)$, $\kappa^* = \kappa(\delta)$, and $\gamma^* = \gamma(\delta)$ be corresponding macroscopic parameters. Similarly to the process $Y(t)$, the density $f^*(Y^*, t)$ of the income process $Y^*(t)$ is also characterized by the Fokker-Planck equation,

$$\frac{\partial f^*}{\partial t} = -g^* \frac{\partial (f^* Y^*)}{\partial Y^*} + \frac{{\sigma^*}^2}{2} \frac{\partial^2 (f^* Y^*)}{\partial {Y^*}^2}.$$

Again, we can characterize $Y^*(t)$ respectively as the solution to the stochastic differential equation

$$dY^*(t) = g^* Y^*(t) dt + \sigma^* \sqrt{Y^*(t)} dW^*(t), \tag{37}$$

where we assume the processes $Y(t)$ and $Y^*(t)$ evolve simultaneously and stochastically independently, so that the Brownian motion $W^*(t)$ is independent of $W(t)$.

It can be shown (see Demetrius et al. [36], Theorem 7.2) that equations (21) and (37) are equivalent to the system of stochastic differential equations,

$$dZ(t) = (g + p(t)\Delta g)\, Z(t) dt + \sigma \sqrt{(1 - p(t)) Z(t)} dW(t) + \sigma^* \sqrt{p(t) Z(t)} dW^*(t), \tag{38}$$

and

$$dp(t) = p(t)(1 - p(t)) \left(\Delta g - \frac{\Delta \sigma^2}{Z(t)} \right) dt - \sigma p(t) \sqrt{\frac{(1 - p(t))}{Z(t)}} dW(t) + \sigma^* (1 - p(t)) \sqrt{\frac{p(t)}{Z(t)}} dW^*(t). \tag{39}$$

We need to solve this for the process $p(t)$. Assuming total aggregate production is constant, $Z(t) = Y$,[32]

[31] Initially, the share $p(t)$ is small and the two populations evolve independently of each other. The invader population can be seen as drawing from resources not used or available to the incumbent. Then, as the invader population grows, the two populations compete for resources. We assume the two populations are in steady state assuming indirectly that the convergence to steady state is much faster than the selection process. This also justifies focusing on the case where the overall production is fixed ($Z(t) = Y$); see also Demetrius et al. [36], Section 2.

[32] Strictly speaking, we need only to assume that this holds for $t > t_0$ for some t_0 that represents the instant where the exploitation competitive interaction between incumbent and invader population begins; this is consistent with the case we consider, where resources are finite and limited. See Demetrius et al. [36], Section 2, for further discussion on this point.

then the process $p(t)$ can be shown to be a diffusion process with drift,

$$\alpha(p(t)) = p(t)\,(1-p(t))\left(\Delta g - \frac{\Delta\sigma^2}{Y}\right),$$

and variance,

$$\beta(p(t)) = \frac{p(t)(1-p(t))}{Y}\left(\sigma^2 p(t) + {\sigma^*}^2(1-p(t))\right);$$

and that the process $p(t)$ has density ψ solving the Fokker-Planck equation (see Demetrius et al. [36], Theorem 7.3),

$$\frac{\partial\psi}{\partial t} = -\frac{\partial[\alpha(p)\psi]}{\partial p} + \frac{1}{2}\frac{\partial^2[\beta(p)\psi]}{\partial p^2},$$

with natural boundary conditions, $\psi(0,t) = 0, \psi(1,t) = 1$, that correspond to the cases $p = 0$ (when the variant population becomes extinct) and $p = 1$ (when the incumbent population becomes extinct). Notice that we set $\alpha(p) \equiv \alpha(p, Y)$ and $\beta(p) \equiv \beta(p, Y)$, so that $\alpha(0) = \alpha(1) = 0$ and $\beta(0) = \beta(1) = 0$. This implies a unique solution for any initial value $\psi(p, 0)$.

Letting $p_0 = p(0)$ denote the initial frequency of the mutant and letting $\rho(p_0)$ denote the probability that the diffusion process leads to an absorption in the state $p = 1$ (extinction of the incumbent population), appealing to the backward Kolmogorow equation,

$$\frac{\partial\psi}{\partial t} = \alpha(p)\frac{\partial\psi}{\partial p} + \frac{1}{2}\beta(p)\frac{\partial^2\psi}{\partial p^2}$$

and integrating, one shows that the invasion probability $\rho(p_0)$ can be written as,

$$\rho(p_0) = \frac{1-\left(1-\frac{\Delta\sigma^2}{\sigma^{*2}}p_0\right)^{\frac{2Ys}{\Delta\sigma^2}+1}}{1-\left(1-\frac{\Delta\sigma^2}{\sigma^{*2}}\right)^{\frac{2Ys}{\Delta\sigma^2}+1}},$$

where $s = \Delta g - \frac{\Delta\sigma^2}{Y}$ (again, see Demetrius et al. [36], Section 7). The sign of the expression s thus becomes crucial in determining whether a variant is successful in invading or not. Except for the degenerate case of $\frac{2Ys}{\Delta\sigma^2} + 1 = 0$, we have $\rho'(\cdot) \neq 0$, and it is easy to show that convexity or concavity of $\rho(\cdot)$ is determined by s alone, namely,

$$s > 0 \Rightarrow \rho(\cdot) \text{ is convex,} \qquad s < 0 \Rightarrow \rho(\cdot) \text{ is concave.}$$

The exact curvature of $\rho(\cdot)$ then depends on the magnitude of s and hence on the values of $\Delta g, \Delta\sigma^2$, and Y. The exact relations between these variables in determining the sign of s and their effect on the invasion probability provides the conditions under which an invader's level of entropy should be higher or lower than H in order to be successful.

Now, consider initial values p_0 close to zero, then the solution $p(t)$ is absorbed in state $p = 0$ (extinction of the invader population) for any small perturbation, if

$$\Delta g < 0, \Delta\sigma^2 \geq 0 \qquad \text{or} \qquad \Delta g \leq 0, \Delta\sigma^2 > 0. \tag{40}$$

Under these conditions, one of the following two cases occurs,

(I) $\Phi < 0$, $\gamma \geq 0$, or $\Phi \leq 0$, $\gamma > 0$;

(II) $\Phi > 0$, $\gamma \leq 0$, or $\Phi \geq 0$, $\gamma < 0$.

In case *(I)*, condition (40) for all perturbations is equivalent to $\Delta H < 0$ (the variant population has lower entropy; and the incumbent population with higher entropy takes over; recall that, $\Phi < 0 \Rightarrow \Delta r \Delta H > 0$ and $\gamma > 0 \Rightarrow \Delta\sigma^2 \Delta H < 0$); in case *(II)*, it is equivalent to $\Delta H > 0$ (the variant population has higher entropy; and the incumbent population with lower entropy takes over; recall that, $\Phi > 0 \Rightarrow \Delta g \Delta H < 0$ and $\gamma < 0 \Rightarrow \Delta\sigma^2 \Delta H > 0$); (see [36], Theorem 7.4). This yields the more general formula for the selective advantage

$$s = -\left(\Phi - \frac{\gamma}{Y}\right)\Delta H,$$

where $\Delta H = H^* - H$.

In the limit, as $Y \to \infty$, the diffusion equation for p degenerates to a linear differential equation, and the convexity criterion in terms of s reduces to the growth rate differential Δg. In this case, we have, $\Phi < 0 \iff \Delta H < 0$ and $\Phi > 0 \iff \Delta H > 0$, and the sign of the productive potential alone determines the selective advantage for the entropy (again, see Demetrius et al. [36], Section 7).

Proof of Theorem 3

We here provide a sketch of the main steps, and refer to Demetrius et al. [35], Section 3, for more details. We need to show $\Delta H \Delta \mathcal{R} > 0$. We first recall the definition of our robustness measure $\mathcal{R}$. Fix $\epsilon > 0$ and define the probability that the sample mean differs from the value Φ by more than ϵ,

$$P_\epsilon(n) = \mu\left\{x \in \Omega \;\middle|\; \left|\frac{1}{n} S_n\varphi(x) - \Phi\right| > \epsilon\right\},$$

where, as we saw, the sample mean is given by,

$$\begin{aligned} S_n\varphi(x) &= \sum_{j=0}^{n-1} \varphi(\tau^j x) \\ &= \log a_{x_0x_1} + \log a_{x_1x_2} + \ldots + \log a_{x_{n-1}x_n} \\ &= \log a_{x_0x_1} a_{x_1x_2} \cdots a_{x_{n-1}x_n}, \end{aligned}$$

and is such that $\lim_{n\to\infty} S_n\varphi(x) = \int \varphi d\mu = \Phi$.

By the ergodic theorem, $\lim_{n\to\infty} P_\epsilon(n) = 0$; moreover, the convergence rate is at least exponentially fast, so that there exist constants, $c_0, c_1 > 0$, such that,

$$\mu\left\{x \in \Omega \;\middle|\; \left|\frac{1}{n} S_n\varphi(x) - \Phi\right| > \epsilon\right\} \le c_0 \exp^{-c_1 n}.$$

This motivates the robustness measure given by the fluctuation decay rate,

$$\mathcal{R} \equiv \mathcal{R}_\epsilon = -\lim_{n\to\infty}\left[\frac{1}{n}\log P_\epsilon(n)\right],$$

which characterizes the asymptotic value of the probability of the set of trajectories that deviate from the typical trajectory by ϵ or less.

In order to better characterize $\mathcal{R}$, consider the more general decay measures,

$$\overline{\mathcal{R}}(\varphi, E) = -\frac{1}{n}\limsup_{n\to\infty}\left[\mu\left\{x \in \Omega \;\middle|\; \frac{1}{n}\sum_{j=0}^{n-1}\varphi(\tau^j x) \in E\right\}\right]$$

and

$$\underline{\mathcal{R}}(\varphi, E) = -\frac{1}{n} \liminf_{n\to\infty} \left[\mu \left\{ x \in \Omega \,\middle|\, \frac{1}{n} \sum_{j=0}^{n-1} \varphi(\tau^j x) \in E \right\} \right],$$

where E stands for arbitrary subsets of the real line. (Later we will be interested in sets of the form $E = \{s \in \mathbb{R} : |s - \Phi| > \epsilon\}$.)

Next, one defines the function

$$k_\varphi(s) = g - s - \sup_\nu \left\{ H_\nu(\tau) \,\middle|\, \nu \text{ invariant under } \tau \text{ and } \int \varphi d\nu = s \right\}.$$

Then we have,

$$\overline{\mathcal{R}}(\varphi, E) \geq -\inf\{k_\varphi(s) \,|\, s \in E\} \text{ for every open set } E,$$

and

$$\underline{\mathcal{R}}(\varphi, E) \leq -\inf\{k_\varphi(s) \,|\, s \in E\} \text{ for every closed set } E.$$

Moreover, $k_\varphi(s)$ is continuous and satisfies $k_\varphi(\Phi) = 0$ by the variational principle of Eqn. (20). Hence,

$$\mathcal{R} = \underline{\mathcal{R}}(\varphi, E) = \overline{\mathcal{R}}(\varphi, E) = -\inf\{k_\varphi(s) : s \in E\} = -\min\{k_\varphi(\Phi - \epsilon), k_\varphi(\Phi + \epsilon)\},$$

and actually attains its minimum.

Now consider perturbations of the form $A(\delta) = (a_{ij}^{1+\delta})$ corresponding to $\varphi(\delta) = (1+\delta)\varphi$, where again, $\varphi = \log a_{x_0 x_1}$. One can then define $\mathcal{R}(\delta)$ using $\varphi(\delta)$ instead of φ and show that

$$\mathcal{R}(\delta) = -\min\{k_{\varphi(\delta)}(\Phi(\delta) - (1+\delta)\epsilon), k_{\varphi(\delta)}(\Phi(\delta) + (1+\delta)\epsilon)\},$$

where $\lim_{n\to\infty} S_n\varphi(\delta)(x) = \int \varphi(\delta) d\mu(\delta) \equiv \Phi(\delta)$. Also, $H(\delta) = H_{\mu(\delta)}(\tau)$, where $\mu(\delta)$ is the measure corresponding to $\varphi(\delta)$.

Finally, one shows,

$$\begin{aligned} k_{\varphi(\delta)}(\Phi(\delta) - (1+\delta)\epsilon) &= H(\delta) + (1+\delta)\epsilon - \sup\left\{ H_\nu(\tau) \,\middle|\, \nu \text{ invariant under } \tau \text{ and } \int \varphi d\nu = \Phi - \epsilon \right\} \\ k_{\varphi(\delta)}(\Phi(\delta) + (1+\delta)\epsilon) &= H(\delta) - (1+\delta)\epsilon - \sup\left\{ H_\nu(\tau) \,\middle|\, \nu \text{ invariant under } \tau \text{ and } \int \varphi d\nu = \Phi + \epsilon \right\}. \end{aligned}$$

This readily implies that, from $\Delta\mathcal{R} = \mathcal{R}(\delta) - \mathcal{R}$ and $\Delta H = H(\delta) - H$, we have,

$$\Delta H - \delta\epsilon \leq \Delta\mathcal{R} \leq \Delta H + \delta\epsilon,$$

and hence (for ΔH bounded away from zero) we obtain $\Delta H \Delta\mathcal{R} > 0$.