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RANDOM MODELS FOR THE JOINT TREATMENT OF RISK AND TIME PREFERENCES

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Abstract. We develop a simple, tractable and sound stochastic framework for the joint treatment of risk and time preferences, in order to facilitate the estimation of risk and time attitudes. In so doing we: (i) study deterministic models of risk and time preferences paying special attention to their comparative statics, (ii) embed the deterministic models and their comparative statics within the random utility framework, and (iii) show how to estimate them, illustrating this exercise on several experimental datasets.

Keywords: Risk preferences; Time Preferences; Comparative Statics; Stochastic Choice; Random Utility Models; Discrete Choice.
JEL classification numbers: C01; D01.

1. Introduction

Economic situations jointly involving risk and time pervade most spheres of everyday live. The main aim of this paper is to develop a simple, tractable and sound stochastic framework for the treatment of risk and time preferences that will enable the proper estimation of risk and time attitudes. This involves three broad steps. In step 1, we analyze several deterministic models of risk and time preferences, establish their formal relationships, and characterize their comparative statics. In step 2, we embed these deterministic models within the random utility framework, show that the comparative statics of the deterministic models extend immediately to this stochastic setting, and discuss how to estimate them. In step 3, we implement our stochastic framework

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to estimate risk and time preferences using datasets from a diverse set of influential experimental papers.

We use two general settings that have had a widespread impact. In the first setting, the individual expresses preferences over *dated lotteries*, where a dated lottery is one in which the player is awarded the prize at a given period of time.[1] This setting includes the particular case in which risk and time are treated separately, in the sense that individuals are faced with menus involving only lotteries awarded in the present, and menus comprising only delayed but certain payouts.[2] The second setting involves *convex budgets*, where the individual decides the proportions for distributing an endowment between an earlier stock and a later stock, in a situation that can ultimately be defined as a pair of lotteries that will be paid out at two different times.[3] Together, these two settings cover the vast majority of the experimental literature on risk and time preferences.

We start the study of the deterministic models of risk and time preferences with the so-called discounted expected utility (DEU), which is overwhelmingly used in practice. This model directly combines the expected utility treatment of risk with the exponentially discounted utility treatment of time, and hence is a natural starting point for the standard treatment of risk and time preferences.[4] We devote Section 3 to the study of DEU, and in particular to deriving its risk and time comparative statics. Our reasons for this are twofold. Firstly, risk and time comparative statics are not immediate in a context where both risk and time considerations are at stake. Secondly, one of the fundamental aims of this paper is to develop a setting that stochastically respects the deterministic comparative statics. Ultimately, this enables a proper interpretation of the model in a stochastic setting, and thus provides a sound framework for econometric purposes. We first show that DEU captures the idea of more risk aversion, i.e. a greater

[1]For papers using this setting see Ahlbrecht and Weber (1997), Coble and Lusk (2010), Baucells and Heukamp (2012) and Cheung (2015).

[2]See Andersen et al. (2008), Burks et al. (2009), Dohmen et al. (2010), Tanaka et al. (2010), Abdellaoui et al. (2013), Benjamin et al. (2013), or Falk et al. (2018).

[3]This setting was proposed by Andreoni and Sprenger (2012b). See also Cheung (2015), Miao and Zhong (2015), Epper and Fehr-Duda (2015), and Kim et al. (2018).

[4]See Phelps (1962) for an early application of the model, and Fishburn (1970) for an axiomatic treatment of DEU in the context of lotteries over sequences of monetary payoffs. In Appendix A we provide an axiomatization of DEU and other models, in the context of dated lotteries, as a way of establishing the formal relationships between the models. We believe that this result may be of independent interest to some readers.

preference for present degenerate lotteries over other present lotteries through the curvature of the monetary utility function, exactly as in expected utility. Similarly, DEU captures the idea of more delay aversion, i.e. a greater preference for present degenerate lotteries over future degenerate lotteries, through the curvature of a normalized monetary utility function, exactly as in exponentially discounted utility.[5] These results are not surprising, since they basically involve canceling one of the two preference dimensions in order to study the other. We then extend the simple classical comparisons just described to more general pairs of objects simultaneously involving risk and time considerations. We argue that the best approach, in this case, is to control for the time (alternatively, risk) attitudes, since this allows us to offer more general comparative statics on the risk (alternatively, time) dimension.

The simplicity of DEU comes with some well-known limitations. In particular, the model uses the same monetary utility function to evaluate both risky payoffs and dated certain payoffs. In Section 4, we introduce two classes of models that allow for different evaluations of money in risk and time preferences, while keeping to the standard treatment of risk preferences by means of expected utility, and that of time preferences by means of exponential discounting. The models are non-recursive, and directly applicable to a wide range of static settings.[6] The first model, which we call the present value of the certainty equivalent (PVCE), reduces, first, the risk dimension by computing the certainty equivalent of the lotteries, and then the time dimension by computing the present value of the (sequence of) certainty equivalents. The second model, which we call the certainty equivalent of the present value (CEPV), reverses the order by first reducing the time dimension by computing the present values of the (sequences of) monetary payoffs involved in the lotteries, and then the risk dimension

[5]This later result is often overlooked in the literature. We emphasize that impatience is not characterized by the discount parameter, but by its joint consideration with the curvature of the Bernouilli function.

[6]Kreps and Porteus (1978) and Selden (1978) introduced recursive expected utility, allowing for the separation of risk and time preferences. Later, Epstein and Zin (1989) and Chew and Epstein (1990) further developed the recursive setting, introduced parametric versions and allowed for behavioral considerations. Our models are non-recursive versions of these. Halevy (2008), Baucells and Heukamp (2012), Cheung (2015), Miao and Zhong (2015), Andreoni et al. (2017), Epper and Fehr-Duda (2019), Lanier et al. (2019) and DeJarnette et al. (2019) also study novel extensions separating risk and time preferences in various non-recursive contexts. Remarkably, DeJarnette et al. (2019) characterize, in a different setting involving monetary prizes awarded at uncertain future dates, a generalization of DEU that is essentially equivalent to our second model.

by computing the certainty equivalent of the lottery formed by these present values. In both models, the certainty equivalents are computed using expected utility with a Bernouilli function, and the present values are obtained using exponential discounting with a discount parameter and another monetary utility function.[7] We derive the comparative statics for risk and time, which follow the logic of the DEU model. Clearly, DEU belongs to the intersection of PVCE and CEPV which are, generally speaking, two different models.

In the second part of the paper, Section 5, we embed the above-mentioned deterministic models into the stochastic framework of random utility models. Given a class of utility functions, e.g. the DEU class, a random utility model is the simplex over the class of utilities, and a particular instance of the random utility model corresponds to a particular probability distribution over the class of utilities. Crucially, we show that all the comparative statics of the deterministic models extend immediately to the random utility model built upon them. This means that we obtain sound risk and time stochastic comparative statics, enabling a proper understanding of risk and time attitudes with stochastic data. To place this result in perspective, let us recall that Apesteguia and Ballester (2018) show that most of the standard stochastic frameworks used in the independent study of risk and time preferences, based on additive iid random utility models, have counterintuitive properties.[8] They show that the additive iid random utility models may generically predict higher choice probabilities for a risky lottery over a degenerate one in individuals with more risk aversion in the baseline, deterministic, utility. This makes it impossible to interpret the additive iid random utility model in terms of stochastic risk aversion. Apesteguia and Ballester (2018) also show that the random utility model respects the notion of more risk aversion in expected utility and of more delay aversion in exponential discounting, while leaving unanswered the question of how to treat risk and time preferences jointly. To the best of our knowledge, this is the first paper to lay down the theoretical properties of a stochastic model of risk and time preferences in terms of their stochastic comparative statics.

Section 5 continues with some relevant results regarding the estimation of random utility models. In actual practice, the estimation of stochastic models is typically

[7]Appendix B extends the analysis to account for non-standard behavioral considerations.

[8]Additive iid random utility models add an error term to the utility valuation, thereby adopting a cardinal approach.

facilitated by the use of particular probability distributions or assumptions over the relevant parameters. We show that these simplifications, if not carefully designed, can bring undesirable consequences. We illustrate with the case of DEU and a homogeneous Bernoulli function, under the assumption of independent probability distributions for the discount parameter and the curvature of the Bernoulli function. In this case, we show that, for sufficiently risk-averse individuals, the prediction is that any earlier lottery is almost sure to be chosen over any later one. This is of course nonsense since the earlier lottery may involve very low payoffs, while the later one may involve arbitrarily large payoffs. We show that the correct approach requires us to account for the dependence of these two parameters, as shown in our study of deterministic more risk and more delay aversion of Section 3.

In the third part of the paper, we illustrate our approach using three different, well-known experimental datasets, which represent the diversity of experimental elicitation methods in common use. We structurally estimate risk and time preferences using the stochastic models studied in Section 5, under the assumption that the monetary functions are homogeneous. We start with the dataset of Andersen et al. (2008), which exemplifies the often used approach in which risk and time attitudes are elicited separately. That is, their experiment involves choices from menus composed of present lotteries, and menus composed of dated certain payoffs. A practical lesson to be learnt from this exercise is that, with this type of data, the separate estimation of risk and time preferences is equivalent to their joint estimation. Furthermore, as we show in Section 4, since the homogeneous versions of DEU, PVCE and CEPV are equivalent in this setting, we simply use DEU in our estimation. We start by analyzing the dataset at the aggregate level, pooling all the individual choices, thereby adopting a representative agent approach. Then, given the richness of the dataset, we estimate preferences for each of the 253 individuals in the sample.

We then use the dataset of Coble and Lusk (2010) to study a general dated lottery setting, where some of the objects of choice simultaneously involve risk and time considerations. The experimental design of Coble and Lusk (2010) involves choices from pairs of equally-dated lotteries, pairs of dated degenerate lotteries, and pairs of dated lotteries where both the risk and the time dimensions are active. This allows us to empirically test the external validity of experimental designs, which, like the previous one, use separate elicitations of risk and time, as opposed to those that incorporate both dimensions at once. Our results suggest that the separate elicitation of risk and time preferences is a valid methodology since the overall estimates are remarkably similar in both scenarios. Another exercise enabled by the dataset of Coble and Lusk is to study the internal correlation of risk and time preferences. We use the pooled dataset and allow dependence between risk and time preferences using a Gaussian copula to model and estimate the correlation between the marginal probability distribution of each behavioral trait. We obtain analogous estimates to those obtained when assuming independence between risk and time preferences, and a positive, albeit statistically insignificant, correlation coefficient between risk and delay aversion.

Finally, we draw on Andreoni and Sprenger (2012b) to illustrate the applicability of our stochastic framework to settings involving convex budgets. An interesting feature of this type of setting is that it allows the separate identification of DEU, PVCE and CEPV, and accordingly, we empirically estimate the three models. We first, obtain that the simple DEU model already captures the idiosyncratic choice patterns remarkably well. That is, we observe that DEU accounts for the large fraction of corner choices, while at the same time rationalizing the interior choices. This is a remarkable result from our stochastic framework, especially in light of the discussion in the literature on the nature of the data generated by convex budgets, and the difficulty of explaining them. Our results show that the use of a valid stochastic methodology allows us to smoothly account for the heterogeneity of observed behavior in convex budget settings. We then show that the more flexible PVCE and CEPV models naturally improve on DEU, with CEPV getting closer to the data.

To conclude, this paper provides a stochastic framework for the estimation of risk and time preferences, contributing to the latest active methodological literature on preference estimations (see, e.g., DellaVigna, 2018; Barseghyan et al. 2018). In so doing, it contributes to the also very active literature on deterministic models of risk and time preferences (see the papers cited in footnote 6). Finally, this paper provides a novel analysis of risk and time preferences, using a diversity of existing experimental datasets, and thus contributes to the empirical estimation of risk and time preferences (see the papers cited in footnotes 1, 2, and 3).

2. Framework

The set of monetary payoffs is $X = \mathbb{R}_+$. A lottery is a finite collection of payoffs and the probabilities with which they are awarded, i.e., a vector $l = [p_1, \ldots, p_N; x_1, \ldots, x_N]$ with $p_n \geq 0$, $\sum_n p_n = 1$ and $x_n \in X$. A degenerate lottery is a lottery composed by a unique payoff received with certainty, i.e., a lottery of the form $[1; x]$. A basic lottery is a lottery containing at most one strictly positive payoff, i.e., a lottery of the form $[p, 1-p; x, 0]$ with $x > 0$. We denote by $\mathcal{L}, \mathcal{D}$ and $\mathcal{B}$ the space of all lotteries, all degenerate lotteries and all basic lotteries, respectively. Time can take any positive real value, i.e. $T = \mathbb{R}_+$. The literature has primarily used two different settings in the study of risk and time preferences.

2.1. **Dated Lotteries.** A setting that has been intensively studied in the joint treatment of risk and time preferences involves individuals facing menus made up of alternatives $(l, t) \in \mathcal{L} \times T$, that represent the situation in which lottery $l \in \mathcal{L}$ is awarded at time $t \in T$.[9] We call these alternatives dated lotteries. The general case is analyzed in Coble and Lusk (2010) and Cheung (2015). Andersen et al. (2008), Burks et al. (2009), Dohmen et al. (2010), Tanaka et al. (2010), Benjamin et al. (2013), or Falk et al. (2018) elicit risk and time attitudes separately, such that individuals face menus made up exclusively either of present lotteries, i.e., elements in $\mathcal{L} \times \{0\}$, or, alternatively, dated degenerate lotteries, i.e., elements in $\mathcal{D} \times T$. Ahlbrecht and Weber (1997) and Baucells and Heukamp (2012) study the case of dated basic lotteries, i.e., elements in $\mathcal{B} \times T$.

2.2. **Convex Budgets.** An alternative setting, in increasing use since it was pioneered by Andreoni and Sprenger (2012b), involves individuals facing convex budget menus. Here, two independent, dated, basic lotteries $([p, 1-p; x, 0], t)$ and $([q, 1-q; y, 0], s)$, with $t < s$, are presented to the individual, who chooses a budget share $\alpha \in [0, 1]$ to be invested in the first dated lottery, leaving $1 - \alpha$ to be invested in the second lottery. Accordingly, if α is chosen, the individual receives the sequence of dated basic lotteries, $([p, 1-p; \alpha x, 0], t)$ and $([q, 1-q; (1-\alpha)y, 0], s)$.[10]

3. Discounted Expected Utility

The most commonly used representation for the joint analysis of risk and time preferences is the discounted expected utility (DEU) model. Formally, denote by $\mathcal{U}$ the set of all continuous and strictly increasing functions $u : \mathbb{R}_+ \to \mathbb{R}_+$ such that $u(0) = 0$. DEU requires us to consider a monetary utility function $u \in \mathcal{U}$ and a discount factor

[9]These menus are typically binary, thereby allowing preferences and choices to be treated as equivalents.

[10]It is typically understood that, if both prizes are awarded, individuals perceive them as being consumed at the times they are awarded.

$\delta \in (0,1)$, to arrive at the following evaluation of the dated lottery (l,t) or the share α:[11]

$$DEU_{\delta,u}(l,t) = \delta^t \sum_{n=1}^{N} p_n u(x_n)$$

$$DEU_{\delta,u}(\alpha) = \delta^t p u(\alpha x) + \delta^s q u((1-\alpha)y).$$

In many applications, monetary utility functions are parameterized. The most common family of monetary utility functions assumes homogeneity, adopting the well-known homegeneous functional $u_h(x) = \frac{x^{1-h}}{1-h}$, with $h < 1$.[12] When homogeneous monetary utility functions are being considered, we refer to DEU as DEU-H. This particular model will be extensively used throughout the paper.

3.1. **More Risk Aversion and More Delay Aversion.** We start by briefly recalling the standard notions of more risk aversion and more delay aversion within the contexts of expected utility and exponentially discounted utility, respectively, and continue by formalizing these notions in the context of DEU.

In expected utility, individual 1 is said to be more risk averse than individual 2 if, whenever individual 2 prefers a degenerate lottery to a non-degenerate lottery, so does individual 1; which is simply equivalent to saying that the utility function of individual 1 is more concave than that of individual 2. Importantly, the notion of more risk aversion allows for comparative static exercises far beyond the simple comparisons used in the definition.[13]

In exponentially discounted utility, more delay aversion is defined by evaluating the preference for present payoffs over delayed payoffs, or, equivalently, based on the combined consideration of the discount parameter and the curvature of the monetary

[11] It is obvious that we could equivalently present DEU in the form of expected discounted utility, i.e., $\sum_n p_n[\delta^t u(x_n)]$ or $p[\delta^t u(\alpha x)] + q[\delta^s u((1-\alpha)y)]$.

[12] In the context of risk preferences, this family is typically called CRRA. Notice how the assumption $h < 1$ is fundamental to guarantee that $u_h \in \mathcal{U}$.

[13] For example, when lotteries are related by mean preserving spreads, whenever the lottery with the least spread is preferred by an individual, it is also preferred by a more risk-averse individual. Similarly, in the familiar case of Holt and Laury (2002) with pairs of lotteries $[p, 1-p; x_1, x_4]$ and $[p, 1-p; x_2, x_3]$ such that $x_1 < x_2 < x_3 < x_4$, whenever the latter lottery is preferred by an individual, it is also preferred by a more risk-averse individual. Notably, an implicit definition of the notion of riskier lotteries emerges from this analysis. We can say that one lottery is riskier than another if, whenever the latter is preferred by one individual, it is also preferred by every more risk-averse individual.

utility function. Crucially, it is very important to stress that the discount parameter alone is uninformative about delay aversion; a fact that is often overlooked in empirical applications. Let us illustrate this with a simple example. Consider two exponentially discounted utilities built upon discount parameters .97 and .95, and monetary utility functions $\sqrt{x}$ and x, respectively. From the comparison of the discount parameters, one may be tempted to claim that the second individual is more delay averse than the first. This would be incorrect. To see this, notice that the present values of \$1 paid at $t = 1$ are .94 and .95 for the first and second individuals, respectively. This shows that there are present payoffs, e.g. \$.945, that the first individual prefers over the \$1 paid at $t = 1$, while the second prefers to wait, suggesting that it is in fact the first individual who is the more delay averse. Correct analysis requires to build a normalized utility function combining the discount factor and the monetary utility function, as formalized below for the case of DEU. Individual 1 is said to be more delay averse than individual 2 if, whenever individual 2 prefers a present payoff over a delayed one, so does individual 1; which is equivalent to saying that the normalized utility function of individual 1 is more concave than that of individual 2. As in the case of more risk aversion, the notion of more delay aversion allows for comparative statics exercises in many other scenarios beyond the simple one used in the definition.[14]

Based on the above discussion, we can now formalize the basic notions of more risk aversion and more delay aversion in the context of DEU.[15]

Proposition 1.

(1) *More risk aversion:* u_1 *is a concave transformation of* u_2 *if and only if, for every* $l \in \mathcal{L}$ *and every* $x \in X$, $DEU_{\delta_2,u_2}([1;x],0) \geq DEU_{\delta_2,u_2}(l,0)$ *implies* $DEU_{\delta_1,u_1}([1;x],0) \geq DEU_{\delta_1,u_1}(l,0)$.

(2) *More delay aversion: Fix* $\theta \in (0,1)$. $u_1^{\frac{\log\theta}{\log\delta_1}}$ *is a concave transformation of* $u_2^{\frac{\log\theta}{\log\delta_2}}$ *if and only if, for every* $x, y \in X$ *and every* $s \in T$, $DEU_{\delta_2,u_2}([1;x],0) \geq DEU_{\delta_2,u_2}([1;y],s)$ *implies* $DEU_{\delta_1,u_1}([1;x],0) \geq DEU_{\delta_1,u_1}([1;y],s)$.

The first part of Proposition 1 closes down the time component by considering only present lotteries, and then compares a pair formed by a riskless degenerate lottery and another lottery, possibly involving risk. Since this is merely a reflection of expected

[14]For example, settings where payoffs or streams of payoffs can be clearly ordered in terms of delay. See Benoît and Ok (2007) for a general treatment of the notion of more delay aversion.

[15]All the proofs are contained in Appendix A.

utility, the classical notion of more risk aversion is immediately reproduced and, naturally, it will also be informative about many other comparisons involving lotteries that can be ordered in terms of riskiness.

The second part of Proposition 1 considers only degenerate lotteries, thereby closing down the risk component, and compares a present payoff with another payoff, possibly in the future. Since DEU reduces to exponentially discounted utility when degenerate lotteries are at stake, the result uses the normalization discussed in Fishburn and Rubinstein (1982). For subsequent analysis, let us briefly stress the nature of this normalization. When DEU evaluates a unique payoff at a given moment in time, it can be shown that $DEU_{\delta,u}$ is equivalent to $DEU_{\theta,\bar{u}}$, where $\theta \in (0,1)$ can be freely chosen, and $\bar{u} = u^{\frac{\log\theta}{\log\delta}}$. We can then use a common discount factor θ for both individuals and state that $([1;x],t)$ is preferred to $([1;y],s)$ by individual i if, and only if, $\theta^t\bar{u}_i(x) \geq \theta^s\bar{u}_i(y)$ or, equivalently, if, and only if, $\frac{\bar{u}_i(x)}{\bar{u}_i(y)} \geq \theta^{s-t}$. Since more concavity of $\bar{u}_i$ is indeed equivalent to larger ratios $\frac{\bar{u}_i(x)}{\bar{u}_i(y)}$, the result follows immediately.[16] Again, since this is merely a reflection of exponentially discounted utility, the notion of more delay aversion in DEU is informative about many other comparisons beyond the simple one used in the definition.

Obviously, the analysis of the parametric family DEU-H is more direct. More risk aversion simply requires us to compare the curvature of power functions $\frac{x^{1-h_1}}{1-h_1}$ and $\frac{x^{1-h_2}}{1-h_2}$, which reduces to the comparison of the parameters h_1 and h_2. Hence, individual 1 is more risk averse than individual 2 if, and only if, $1-h_1 \leq 1-h_2$, i.e., if, and only if, $h_1 \geq h_2$. More delay aversion requires us to compare the curvature of the normalized (power) functions $(\frac{x^{1-h_1}}{1-h_1})^{\frac{\log\theta}{\log\delta_1}}$ and $(\frac{x^{1-h_2}}{1-h_2})^{\frac{\log\theta}{\log\delta_2}}$. That is, we need to evaluate whether $(1-h_1)\frac{\log\theta}{\log\delta_1} \leq (1-h_2)\frac{\log\theta}{\log\delta_2}$, which holds if, and only if, $\hat{\delta}_1 \equiv \delta_1^{\frac{1}{1-h_1}} \leq \delta_2^{\frac{1}{1-h_2}} \equiv \hat{\delta}_2$. The following argument may help in the interpretation of this comparison. Since every monetary utility in the homogeneous family is a power transformation of the linear utility function, we can represent the choice behavior of individual i over dated degenerate lotteries by using the alternative DEU-H composed of the corrected discount factor $\hat{\delta}_i$ and the linear monetary utility function. It is then evident that individual 1 is more delay averse than individual 2 if, and only if, $\hat{\delta}_1 \leq \hat{\delta}_2$.

[16] We can illustrate this analysis using our previous example. Normalize both discount factors by using, for example, that of the second individual. That is, set $\theta = .95$. Then, the first normalized utility function becomes $x^{\frac{\log .95}{2\log .97}} = x^{.84}$, which is a concave transformation of the second (normalized) utility function, x. Thus, the first individual is more delay averse than the second.

3.2. **Comparative Statics with Dated Lotteries.** When risk and time features are considered separately, as is the case in a variety of influential papers (see Section 2.1), the separate consideration of the notions of more risk aversion and more delay aversion allows us to perform comparative statics exercises, as shown above. When dated lotteries have both risk and time features, however, this same logic can be exploited only in very restrictive scenarios. The following result, which is an immediate corollary to Proposition 1, shows that, whenever the dated lotteries are awarded at the same period of time the choice is uniquely governed by the notion of more risk aversion.[17] Similarly, whenever dated basic lotteries with the same probability of winning are being considered choice is uniquely governed by the notion of more delay aversion. Formally,

Corollary 1.

(1) *Consider two DEU individuals such that the first is more risk averse than the second. If l' is riskier than l and the second individual prefers (l, t) to (l', t), so does the first.*
(2) *Consider two DEU individuals such that the first is more delay averse than the second. If $t < s$ and the second individual prefers $([p, 1-p, x, 0], t)$ to $([p, 1-p; y, 0], s)$, so does the first.*

The first part of Corollary 1 follows directly from Proposition 1 by using the stationarity of DEU. The second part of Corollary 1 uses the normalization on $u_i(0)$ which makes that basic lotteries with the same probability of occurrence are compared independently of that probability, and hence we can directly apply Proposition 1.

Naturally, due to the interplay between the risk and time considerations, one should not expect to obtain unambiguous comparative statics using either more risk aversion or more delay aversion separately. The correct approach requires us to control for one of the behavioral components, and then establish comparative statics for the other. The next result illustrates this.

Proposition 2.

(1) *Consider two DEU individuals such that both are equally delay averse but the first is more risk averse than the second. Then, for every $l \in \mathcal{L}$, every $x \in X$*

[17] In Coble and Lusk (2010), for instance, the subjects face a number of choice problems in which they have to compare two lotteries that award prizes at the same time, but not in the present. Corollary 1 essentially states that these problems uniquely concern risk preferences.

and every pair $t_1, t_2 \in T$, if the second individual prefers $([1;x], t_1)$ to (l, t_2), so does the first.

(2) *Consider two DEU individuals such that both are equally risk averse but the first is more delay averse than the second. Then, for every $l, l' \in \mathcal{L}$, and every $t, s \in T$ with $t < s$, if the second individual prefers (l, t) to (l', s), so does the first.*

The first part of Proposition 2 analyzes the case in which two DEU individuals share the same level of delay aversion but one has a higher level of risk aversion. The analysis relies on the fact that equality of delay aversion between the two individuals requires that $(\delta_2, u_2) = (\delta_1^k, u_1^k)$ for some $k > 0$. Under these conditions, if individual 1 is more risk averse then it must be that $k \geq 1$, and hence u_2 must be a convex transformation of u_1 and $\delta_2 \leq \delta_1$. We show in the proof that these properties lead to the claimed comparative statics. The second part of Proposition 2 analyzes the case in which two DEU individuals share the same level of risk aversion but one has a higher level of delay aversion. Equality of risk aversion requires that the monetary utilities must be equal, and more delay aversion then comes with a smaller discount factor. It then follows directly that more delay aversion unequivocally generates a higher preference for earlier lotteries.

For the case of DEU-H, the relations between (δ_1, h_1) and (δ_2, h_2) that must be considered are straightforward. Part 1 analyzes the case of $\hat{\delta}_1 = \delta_1^{\frac{1}{1-h_1}} = \delta_2^{\frac{1}{1-h_2}} = \hat{\delta}_2$ and $h_1 \geq h_2$, while part 2 analyzes the case of $h_1 = h_2$ and $\delta_1 \leq \delta_2$.

3.3. **Comparative Statics in Convex Budgets.** We now discuss the convex budget problem.[18] Denote by $\overline{\alpha}_i$ the share that equates the discounted expected utilities of individual i in both periods, i.e., $\delta_i^t p u_i(\overline{\alpha}_i x) = \delta_i^s q u_i((1 - \overline{\alpha}_i) y)$, and by α_i^* the share that maximizes her discounted expected utility.

Proposition 3.

(1) *If individual i is a risk lover, then $\alpha_i^* \in \{0, 1\}$. Moreover, if individual 2 is a risk lover and $\alpha_2^* = 1$, then if individual 1 is a risk lover and is more delay averse than individual 2 then $\alpha_1^* = 1$.*

(2) *If individual i is (strictly) risk averse, then $\alpha_i^* \in (0, 1)$. Moreover:*

[18] To simplify the exposition, we assume that u is differentiable and that $\lim_{x \to 0} u'(x) = +\infty$, as is typically the case in standard parameterizations.

(a) *If both individuals have the same level of risk aversion, and individual 1 is more delay averse than individual 2 then* $\alpha_1^* \geq \alpha_2^*$.

(b) *If both individuals have the same level of delay aversion, and individual 1 is more risk averse than individual 2 then* $\alpha_1^* \leq \alpha_2^*$ *(resp.* $\alpha_1^* \geq \alpha_2^*$*) whenever* $\alpha_1^* \geq \overline{\alpha}_1$ *(resp.* $\alpha_1^* \leq \overline{\alpha}_1$*).*

The first part of Proposition 3 shows that risk lovers, i.e. those with convex monetary utility functions, allocate their whole share either to the earlier lottery or to the later one.[19] For illustrative purposes, consider the case of a risk neutral individual. One unit of money invested in the earlier period brings a marginal utility return of $\delta^t p$ units, while one invested in the later period brings a marginal utility return of $\delta^s q$ units. Thus, the choice problem of a risk lover reduces to the evaluation of two dated basic lotteries, $([p, 1-p; x, 0], t)$ and $([q, 1-q; y, 0], s)$; and then we can directly apply Part 2 of Corollary 1, with delay aversion governing this choice. In other words, if any risk lover selects the earlier basic lottery $([p, 1-p; x, 0], t)$, so does any other risk lover with more delay aversion.

With risk-averse individuals, the solution will be interior and the interplay between risk and time becomes relevant. The first-order condition of the optimization problem is $\frac{u'(\alpha x)}{u'((1-\alpha)y)} = \frac{\delta^s qy}{\delta^t px}$.[20] With fixed risk aversion, more delay aversion is equivalent to a smaller value of δ, which reduces the right hand side of the first-order condition and leads to a higher share α^*. That is, the same level of risk aversion together with a higher level of delay aversion unequivocally generates the choice of a higher share α^*. If, instead, we fix delay aversion and consider higher levels of risk aversion, the function on the left hand side, which is decreasing in α, can be seen to rotate clockwise, with rotation point at $\overline{\alpha}$. This produces less extreme solutions, depending on whether the solution happens to be to the right or to the left of $\overline{\alpha}$. If the solution of an individual is to the right (respectively, left) of $\overline{\alpha}$, the solution of a more risk-averse individual will be smaller (respectively, larger) than that of the first individual. That is, the same level of delay aversion together with a higher level of risk aversion unequivocally generates the choice of a share α^* closer to $\overline{\alpha}$.

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[19]Convexity is only required in the relevant range $[0, y]$.

[20]Notice that the solution depends on the ratio $\frac{p}{q}$, an observation made and tested in Andreoni and Sprenger (2012b).

Now consider DEU-H. Proposition 3 implies that, whenever $h \leq 0$, the solution is corner and, otherwise, it is interior. With a fixed level of risk aversion, more delay aversion implies a stronger preference for the earlier payoffs. To study the effects of more risk aversion, we can rewrite the first-order condition as $(\frac{(1-\alpha)y}{\alpha x})^h = \frac{\delta^s qy}{\delta^t px} = \frac{(\delta^{\frac{1}{1-h}})^{s(1-h)} qy}{(\delta^{\frac{1}{1-h}})^{t(1-h)} px}$, or equivalently, $\frac{(1-\alpha)y}{\alpha x} = [\frac{\hat{\delta}^{s(1-h)} qy}{\hat{\delta}^{t(1-h)} px}]^{\frac{1}{h}}$, where $\hat{\delta} = \delta^{\frac{1}{1-h}}$ is the normalized discounting factor, which we keep constant in order to fix delay aversion. Under these conditions, the derivative of the right hand side with respect to h is positive (respectively, negative) whenever $\hat{\delta}^t px \leq \hat{\delta}^s qy$ (respectively, whenever $\hat{\delta}^t px \geq \hat{\delta}^s qy$). That is, when a risk-neutral individual with exactly the same level of delay aversion prefers the later prize (and would thus invest nothing in the earlier period), more risk aversion increases investment in the earlier period. Similarly, when the risk-neutral individual prefers the earlier prize (and would thus invest everything on it), more risk aversion reduces investment in the earlier period. In both cases, more risk aversion results in a more balanced allocation of money across time.

4. Other Deterministic Utility Models

It is well-known that DEU does not permit a different treatment of money for the risk-evaluation of uncertain payoffs and the time-evaluation of certain payoffs. To visualize this, consider the case of dated lotteries in $\mathcal{L} \times \{0\}$, involving only risk considerations. These are evaluated as $DEU_{\delta,u}(l, 0) = \sum_{n=1}^{N} p_n u(x_n)$, using an expected utility functional based on u. Now, consider the case of dated degenerate lotteries in $\mathcal{D} \times T$, involving only time considerations. These are evaluated as $DEU_{\delta,u}([1; x], t) = \delta^t u(x)$, which is but an exponentially discounted utility using δ and the *same* utility function u. We now introduce two utility representations that allow money to be treated in a different way for risk than for time. Importantly, in so doing, we maintain the standard expected utility treatment of lotteries and exponentially discounted utility of intertemporal payoffs.[21]

The first of these representations, which we call *present value of the certainty equivalent (PVCE)*, starts by eliminating the risk component by means of a certainty equivalent using expected utility. Once the problem has been reduced to the analysis of degenerate certainty equivalents at some future time, the time component is eliminated by transforming that future hypothetical payoff into its equivalent present value,

[21] In Appendix B, we discuss how to extend these concepts to incorporate behavioral considerations.

under exponentially discounted utility. Formally,

$$PVCE_{\delta,w,v}(l,t) = w^{-1}[\delta^t w(v^{-1}[\sum_{n=1}^{N} p_n v(x_n)])]$$

$$PVCE_{\delta,w,v}(\alpha) = w^{-1}[\delta^t w(v^{-1}[pv(\alpha x)]) + \delta^s w(v^{-1}[qv((1-\alpha)y)])].$$

Notice that $v^{-1}[\cdot]$ represents a certainty-equivalent mapping, obtained throughout the use of expected utility with Bernoulli utility function $v \in \mathcal{U}$. Similarly, $w^{-1}[\cdot]$ represents a present equivalent value, obtained under exponential discounting with discount parameter δ and monetary utility $w \in \mathcal{U}$.[22] Given the assumptions on $\mathcal{U}$, every lottery has a unique certainty equivalent and every dated payoff has a unique present equivalent value.

The second representation, which we call *certainty equivalent of the present values (CEPV)*, reverses the order of analysis. It first eliminates the time component by transforming each of the possible sequences of payoffs into its equivalent present value, using exponentially discounted utility. Once the problem has been reduced to the evaluation of a hypothetical present lottery, the risk component is eliminated by transforming this lottery into its certainty equivalent, throughout expected utility. Formally,[23]

$$CEPV_{\delta,w,v}(l,t) = v^{-1}[\sum_{n=1}^{N} p_n v(w^{-1}[\delta^t w(x_n)])]$$

$$CEPV_{\delta,w,v}(\alpha) = v^{-1}[pqv(w^{-1}[\delta^t w(\alpha x) + \delta^s w((1-\alpha)y)]) + p(1-q)v(w^{-1}[\delta^t w(\alpha x)]) +$$
$$+ (1-p)qv(w^{-1}[\delta^s w((1-\alpha)y)])].$$

The DEU representation is a proper restriction of both the PVCE and the CEPV representations in which the individual uses the same monetary utility to evaluate both intertemporal and risk trade-offs, i.e., $v = w$. Indeed, we can show that, in environments involving dated basic lotteries, the intersection of PVCE and CEPV is exactly DEU.[24] That is, DEU is the only model in which the order of the individual's risk and temporal decision-making processes is inconsequential.

[22] Since $w \in \mathcal{U}$, w^{-1} is strictly increasing and hence, PVCE can be equivalently represented dispensing with w^{-1}.

[23] Analogously to PVCE, since v^{-1} is strictly increasing, CEPV can be equivalently represented dispensing with v^{-1}. Note in addition that by writing $\phi = v \circ w^{-1}$, the CEPV representation is basically the generalized DEU model of DeJarnette et al. (2018) in the context of dated lotteries.

[24] Notice that both domains studied in the paper involve the evaluation of dated basic lotteries.

Proposition 4. *Let $\mathcal{B} \times T$. The set of preferences admitting a DEU representation coincides with the set of preferences admitting both a PVCE and a CEPV representation.*

The intuition for Proposition 4 goes as follows. Notice that PVCE can be written as $\delta^t g(pv(x))$, where $g = w \circ v^{-1}$. Consider one present and one future degenerate lottery, $([1; x], 0)$ and $([1; y], t)$, such that the individual is indifferent between the two. If the preferences admit a CEPV representation, the dated basic lotteries $([p, 1-p; x, 0], 0)$ and $([p, 1-p; y, 0], t)$ must also be indifferent. Since this holds for every p, one can select the exact value which, whenever multiplied by the utility of payoff y, reduces utility by as much as the discount δ^t does, thereby showing that g must be homogeneous. Homogeneity of g allows us to write PVCE as $g(\hat{\delta}^t pv(x))$, which is but a monotone transformation of DEU.

We now briefly turn to the study of more risk aversion and more delay aversion in the two utility representations presented here. By reasoning similar to that applied in Proposition 1 we can argue that, in these models, risk aversion is connected to the curvature of the utility function v, while delay aversion is naturally connected to the curvature of the normalized utility function formed by the intertemporal substitutability function w and the discount parameter δ. We omit the proof of this immediate corollary.

Corollary 2.

(1) *More risk aversion: v_1 is a concave transformation of v_2 if, and only if, for every $l \in \mathcal{L}$ and every $x \in X$, $PVCE_{\delta_2,w_2,v_2}([1; x], 0) \geq PVCE_{\delta_2,w_2,v_2}(l, 0)$ (resp. $CEPV_{\delta_2,w_2,v_2}([1; x], 0) \geq CEPV_{\delta_2,w_2,v_2}(l, 0)$) implies $PVCE_{\delta_1,w_1,v_1}([1; x], 0) \geq PVCE_{\delta_1,w_1,v_1}(l, 0)$ (resp. $CEPV_{\delta_1,w_1,v_1}([1; x], 0) \geq CEPV_{\delta_1,w_1,v_1}(l, 0)$).*

(2) *More delay aversion: Fix $\theta \in (0, 1)$. $w_1^{\frac{\log\theta}{\log\delta_1}}$ is a concave transformation of $w_2^{\frac{\log\theta}{\log\delta_2}}$ if, and only if, for every $x, y \in X$ and every $s \in T$, $PVCE_{\delta_2,w_2,v_2}([1; x], 0) \geq PVCE_{\delta_2,w_2,v_2}([1; y], s)$ (resp. $CEPV_{\delta_2,w_2,v_2}([1; x], 0) \geq CEPV_{\delta_2,w_2,v_2}([1; y], s)$) implies $PVCE_{\delta_1,w_1,v_1}([1; x], 0) \geq PVCE_{\delta_1,w_1,v_1}([1; y], s)$ (resp. $CEPV_{\delta_1,w_1,v_1}([1; x], 0) \geq CEPV_{\delta_1,w_1,v_1}([1; y], s)$).*

Since the comparative statics results obtained under DEU can be analogously extended to the case of PVCE and CEPV, we omit the basic details here. Given their relevance for the empirical section, we briefly comment on some of the implications in the case of convex budgets. Firstly, as in Proposition 3, the convexity of the monetary utility functions is related to corner solutions. In the case of PVCE, for instance, it can be shown that corner solutions require the convexity of w.[25] Secondly, since the first-order condition for PVCE can be written as $\frac{v'(\alpha x)g'(pv(\alpha x))}{v'((1-\alpha)y)g'(qv((1-\alpha)y))} = \frac{\delta^s qy}{\delta^t px}$, with $g = w \circ v^{-1}$, more balanced interior solutions are the result of either more risk aversion or more intertemporal substitutability. Importantly, we can see the relative effect of each of these components through the changes in p and q, which affect only g. To illustrate, notice that the first-order condition of the homogeneous version of PVCE, PVCE-H is $(\frac{(1-\alpha)y}{\alpha x})^{\eta} = \frac{\delta^s y}{\delta^t x}(\frac{q}{p})^{\frac{1-\eta}{1-r}}$ and thus, variation in prize probabilities enables identification of the parameters.[26]

Thirdly, notice also that PVCE-H is in agreement with DEU-H that the solution depends entirely on the ratio $\frac{p}{q}$. Interestingly, the alternative representation CEPV-H is sensitive to the probabilities beyond the ratio, as the first order condition is simply $(\frac{(1-\alpha)y}{\alpha x})^{\eta} = \frac{\delta^s y}{\delta^t x}\frac{pqA_\alpha+(p(1-q))}{pqA_\alpha+(q(1-p))}$, where $A_\alpha = [\delta^t(\alpha x)^{1-\eta} + \delta^s((1-\alpha)y)^{1-\eta}]^{\frac{\eta-r}{1-\eta}}$. Notice how the right hand side is now dependent on α, except for the case in which $\eta = r$, i.e., for DEU-H.

4.1. **An Equivalence Result with Dated Lotteries.** Under the assumption of homogeneity, it is immediate to see that, in the setting of convex budgets, DEU-H is a strict subset of both PVCE-H and CEPV-H. That is, in convex budget settings, there are preferences that can be represented by PVCE-H (or alternatively by CEPV-H) but cannot be represented by DEU-H. Thus, convex budget settings allow us to evaluate the empirical content of these models. This is not the case in the setting of dated lotteries, however. Here, the assumption of homogeneity implies that PVCE-H is equivalent to CEPV-H, which in turn is equivalent to DEU-H.

Proposition 5. *Let $\mathcal{L} \times T$. The set of preferences admitting a DEU-H representation coincides with the set of preferences admitting a PVCE-H representation, which in turn coincides with the set of preferences admitting a CEPV-H representation.*

[25]The case of CEPV is more complex, as it requires analyzing the convexity of both v and w.

[26]The homogeneous versions PVCE-H and CEPV-H use the monetary functions $v_r(x) = \frac{x^{1-r}}{1-r}$ and $w_\eta(x) = \frac{x^{1-\eta}}{1-\eta}$, with $r, \eta < 1$, to capture risk aversion and intertemporal substitutability, respectively. Note that we denote the curvature of the Bernoulli function in DEU-H by h and in PVCE-H (and CEPV-H) by r, to emphasize that they represent different attitudes. Ultimately, the objective functions can be simplified to $PVCE_{\delta,\eta,r}(\alpha) = \delta^t p^{\frac{1-\eta}{1-r}}(\alpha x)^{1-\eta} + \delta^s q^{\frac{1-\eta}{1-r}}((1-\alpha)y)^{1-\eta}$ and $CEPV_{\delta,\eta,r}(\alpha) = pq[\delta^t(\alpha x)^{1-\eta} + \delta^s((1-\alpha)y)^{1-\eta}]^{\frac{1-r}{1-\eta}} + p(1-q)\delta^{\frac{t(1-r)}{1-\eta}}(\alpha x)^{1-r} + (1-p)q\delta^{\frac{s(1-r)}{1-\eta}}((1-\alpha)y)^{1-r}$.

Our proof of Proposition 5 proceeds in a number of steps axiomatically characterizing the utility representations discussed in the paper, within the framework of preferences over dated lotteries. We believe that these results are of independent interest. The properties used in the characterizations are versions of the classical properties used in the independent treatments of risk and time preferences. The equivalence of all the models when using homogeneous utility functions relies on an additional property, which we call Payoff-Scale Invariance (PSI). PSI is an adaptation of a property of commodity bundles due to Lancaster (1963), which implies that the indifference of two dated degenerate lotteries is preserved when the payoffs are multiplied by the same constant. We show that PSI induces the homogeneity of the monetary utilities involved, and forces the PVCE-H, CEPV-H and DEU-H representations to coincide.

5. Random utility models

In this section we discuss the structure of random utility models, and their implementation for the treatment of risk and time preferences. A random utility model can be defined as the simplex over a set of considered utilities Ψ. An instance of the random utility model corresponds to a particular probability distribution f over Ψ, capturing the prevalence of each of the considered utilities.[27] At the choice stage, one of the utilities is realized according to this probability distribution, and maximized, thereby generating random choices.

We now discuss in length two important properties of random utility models: their stochastic comparative statics and the potential implications of restricting the set of allowable probability distributions.

5.1. **Stochastic Comparative Statics.** Crucially for our purposes, one virtue of random utility models is that the comparative statics exercises performed on a deterministic model Ψ extend immediately to the random utility model built upon Ψ. This is so because random utility models are probability distributions over the space of considered utility functions. Hence, it is direct to extend the results based on degenerate distributions over the set of utilities (the deterministic case) to probability distributions over the set of utilities (the random utility model). We now use this feature to establish the stochastic counterparts of the notions of more risk aversion and more delay aversion for DEU.

[27]In the results that follow, we assume that f is measurable in the corresponding sets. This assumption is easily met in the parametric versions used in our data analysis, as discussed later.

In the deterministic setting, we say that DEU_{δ_1,u_1} is more risk averse than DEU_{δ_2,u_2} whenever: (i) the former has a greater preference than the latter for degenerate lotteries, (ii) which is equivalent to u_1 being a concave transformation of u_2. Part (i) of this statement can be easily rewritten in stochastic terms based on the mass of preferences for which a degenerate lottery is better than another lottery or, alternatively, on the probability of choice of a degenerate lottery against another lottery. For expositional purposes, we adopt the second approach and denote by $\rho_f((l,t),(l',s))$ the probability of choice of (l,t) against (l',s), when the probability distribution f over DEU is considered. In order to write Part (ii) in stochastic terms, we only need to consider the following equivalent formulation: for every utility function $u \in \mathcal{U}$ such that u_2 is more concave than u, u_1 is also more concave than u. Then, we can simply denote by $\overline{MCT}_f(u)$ (respectively, $\widetilde{MCT}_f(u)$) the mass, according to f, of DEU utilities with a monetary utility (respectively, a normalized utility) more concave than u, and the next result can be read as an immediate extension of Proposition 1. Given its simplicity, we omit the proof of Proposition 6. We also omit the corresponding results for PVCE and CEPV, which merely reproduce Corollary 2 analogously to Proposition 6.

Proposition 6. *Consider two instances, f_1 and f_2, of the random utility model built upon DEU.*

(1) *Stochastic more risk aversion: $\overline{MCT}_{f_1}(u) \geq \overline{MCT}_{f_2}(u)$ for every $u \in \mathcal{U}$ if, and only if, $\rho_{f_1}(([1;x],0),(l,0)) \geq \rho_{f_2}(([1;x],0),(l,0))$ for every $l \in \mathcal{L}$ and every $x \in X$.*

(2) *Stochastic more delay aversion: Fix $\theta \in (0,1)$. $\widetilde{MCT}_{f_1}(u) \geq \widetilde{MCT}_{f_2}(u)$ for every $u \in \mathcal{U}$ if, and only if, $\rho_{f_1}(([1;x],0),([1;y],s)) \geq \rho_{f_2}(([1;x],0),([1;y],s))$ for every $x,y \in X$ and $0 < s \in T$.*

In words: the probability of choosing a present payoff against a present lottery is larger for the distribution which has stochastically more concave monetary utilities. Accordingly, we call this a stochastically more risk-averse distribution. By now, of course, the intuition is immediate. More concave utility functions lead to greater preference for present payoffs over present lotteries, and hence a distribution f_1 with more weight on the more concave utility functions than another distribution f_2 leads to a larger choice probability for present payoffs. The reverse implication is also immediate. Also, the probability of choosing a present payoff over a future payoff is larger for the distribution that has stochastically more concave normalized utilities. Hence, we call this a stochastically more delay-averse distribution. The intuition for the result is analogous to the one above. Notice that the stochastic notions of more risk and more delay aversion reduce to the corresponding deterministic notions when the probability distributions on DEU are degenerate.

Parametric families, as in the DEU-H model, enable even simpler analysis. Denote by F the bivariate CDF corresponding to the stochastic instance f on DEU-H. Denote by $\bar{F}$ the marginal CDF for the monetary utility curvature. Following Proposition 1, delay aversion can be written in terms of the normalized curvature $\tilde{h} = (1-h)\frac{\log\theta}{\log\delta}$. Accordingly, denote by $\tilde{F}$ the induced marginal CDF over this normalized curvature. Alternatively, as discussed in Section 3.1, we can also write delay aversion in terms of the corrected discount factor $\hat{\delta} = \delta^{\frac{1}{1-h}}$. Denote by $\hat{F}$ the induced marginal CDF over this corrected discount factor.[28] The following result is an immediate corollary of Proposition 1.

Corollary 3. *Consider two instances, f_1 and f_2, of the random utility model built upon DEU-H.*

(1) *f_1 is stochastically more risk averse than f_2 if, and only if, $\bar{F}_1$ first-order stochastically dominates $\bar{F}_2$.*
(2) *f_1 is stochastically more delay averse than f_2 if, and only if, $\tilde{F}_1$ first-order stochastically dominates $\tilde{F}_2$, if, and only if, $\hat{F}_1$ is first-order stochastically dominated by $\hat{F}_2$.*

Corollary 3 establishes simple and intuitive results for stochastic more risk aversion and more delay aversion under DEU-H, based on standard first-order stochastic dominance relations. That is, stochastically more risk-averse individuals will have probability distributions over h biased towards higher values of risk aversion. Similarly, stochastically more delay-averse individuals will have CDFs over normalized curvatures biased towards higher values or, equivalently, CDFs over corrected discount factors biased towards lower values.

Proposition 1 and Corollary 3 are crucial positive results. They enable reliable estimations and interpretations of the preference parameters of interest.

5.2. **Distributional Assumptions.** In applications, the analyst typically simplifies the treatment of random models by restricting the set of probability distributions governing the preference parameters. For instance, the distribution f over DEU-H

[28] Technically, we assume the existence and continuity of all relevant distributions.

may be assumed to belong to a well-known family; or the marginal distributions of the parameters involved could be assumed to be independent. Clearly, this practice reduces the set of admissible instances of the model and, consequently, the set of behaviors it is able to explain. Importantly, this restriction has no relevant implications for stochastic comparative statics. However, as we are about to discuss, distributional assumptions may have important undesirable consequences.

We illustrate using DEU-H, and by showing that the assumption of independence of the parameters h and δ leads to problematic conclusions. In particular, we show that, whenever risk aversion is sufficiently high, an earlier dated lottery is preferred almost surely to a later dated lottery, *no matter how low the earlier payoffs or how high the later payoffs may be.* Similarly, in a convex budget setting, the shares chosen will in no way depend on the magnitude of payoffs x and y. That is, whenever risk aversion is sufficiently high, when it comes to the choice of endowment share, it is irrelevant whether the later payoff y is similar or markedly higher than the earlier payoff x. Formally, denoting by $\rho_f(\alpha)$ the distribution over shares induced by f, we have:

Proposition 7. *Consider an instance f of the random utility model built upon DEU-H satisfying distributional independence for h and δ. Then:*

(1) *For any $(l,t),(l',s)$ with $l \neq [1;0]$ and $t<s$, $\lim_{h\to 1}\rho_f((l,t),(l',s))=1$.*
(2) *$\lim_{h\to 1}\rho_f(\alpha)$ is independent of x and y.*

The intuition of Proposition 7 is as follows. An earlier lottery is chosen under DEU-H if, and only if, the ratio of expected utilities between the later and the earlier lotteries is not high enough to compensate the discounting δ^{s-t}. Under DEU-H, when h approaches 1, the ratio of expected utilities for every pair of lotteries such that $l \neq [1;0]$ converges to 1. Under distributional independence, the discounting δ^{s-t} is independent of h and hence, the mass of utilities for which the earlier lottery is chosen goes to 1. Similarly, in the risk aversion region, the choice of endowment share depends on the term $(\frac{y}{x})^{\frac{1-h}{h}}(\delta^{s-t}\frac{q}{p})^{\frac{1}{h}}$. When h approaches 1, this converges to a constant depending neither on x nor on y.

The predictions described in Proposition 7 are of course nonsense. In practical terms, suppose that an individual is highly risk averse and moderately delay averse. An estimation exercise using DEU-H with distributional independence would severely compromise the correct estimation of risk aversion, because, in this case, high levels of risk aversion would predict an extreme preference for the present, which would contradict the behavior described by the data. A similar problem will appear in practice in convex budget settings. Therefore, the results of the estimation would not be representative of the actual behavior of the individual.

The correct approach for avoiding problems of this nature is suggested in our previous discussion in Section 3.1 on more risk aversion and more delay aversion for DEU-H. To properly account for delay aversion, we need to pay attention to the normalized curvature of the monetary utility function $\bar{u}$, or, equivalently, to the corrected discount factor $\hat{\hat{\delta}}$. Hence, one may assume distributional independence of the parameters governing the notions of more risk aversion and more delay aversion, but clearly not of the parameters h and δ. We illustrate this methodology in our empirical section.

6. Estimation of Risk and Time Preferences

In this section we implement the framework developed in the previous sections, using the experimental datasets of Andersen et al. (2008), Coble and Lusk (2010) and Andreoni and Sprenger (2012b). Andersen et al. (2008) allows us to study the case where risk and time preferences are independently elicited using dated lotteries. We study the general case of dated lotteries, which includes non-degenerate lotteries awarded at different time periods, using the dataset of Coble and Lusk (2010). Finally, Andreoni and Sprenger (2012b) enables the analysis of the convex budget setting.[29]

6.1. **Dated Lotteries: Risk and Time Preferences Independently Elicited.**

The literature has often designed experiments in which individuals must choose, separately, from sets of present lotteries and from sets of dated degenerate lotteries. The separate elicitation of risk and time preferences allows us to illustrate an important advantage of random utility models; namely, that the analyst can jointly estimate risk and delay aversion under DEU using the entire dataset or, alternatively, can separately estimate risk and time attitudes, under expected utility and discounted utility respectively, using the relevant sub-samples of the dataset.[30]

To illustrate this type of estimation exercise, we use the influential dataset of Andersen et al. (2008). They designed 100 menus, which we index by m, each involving either a pair of present lotteries or a pair of degenerate dated lotteries. A group of

[29] Other influential datasets are Tanaka et al. (2010), Dohmen et al. (2010), Cheung (2015) and Miao and Zhong (2015). We can provide the corresponding estimation results upon request.

[30] Notice that the inference of the associated distribution of discount rates δ obviously requires us to use the results of both estimations.

253 individuals, which we index by i, made choices from these menus. This provided a collection of 23,108 observations, i.e. pairs of menus and corresponding choices, which we denote by $\mathcal{O}$.[31]

Our estimations use homogeneous monetary functions and, given Proposition 5, DEU-H. The first estimation adopts a representative agent approach, assuming that the stochasticity of every individual in the population is governed by the same distribution over DEU-H, denoted by f. In accordance with the discussion in Section 5.2, we assume distributional independence between risk aversion and delay aversion, i.e., f will be the product of two independent probability distributions, $\bar{f}$ and $\hat{f}$, defined, respectively, on the risk aversion parameter and the corrected discount factor.[32] For the risk aversion parameter, we assume that $\bar{f}$ is a truncated normal distribution in the interval $(-\infty, 1)$ with parameters μ_h and σ_h^2.[33] For the delay aversion parameter, which we measure in months, we assume that $\hat{f}$ is a beta distribution with parameters $a_{\hat{\delta}}$ and $b_{\hat{\delta}}$.

Now, let an individual i confront menu $m = \{1, 2, \ldots, \mathcal{T}_m\}$. The probability of choosing alternative τ, denoted by $\rho_{im\tau}(f)$, corresponds to the measure of all parameters for which the associated DEU-H utility ranks τ as the best alternative within menu m. Denoting by $\mathbf{1}$ the usual indicator function and by j a generic alternative in the menu, this is

$$\rho_{im\tau}(f) = \int_h \int_{\hat{\delta}} \mathbf{1}\left(\tau = \max_{j\in\{1,2,\ldots,\mathcal{T}_m\}} DEU_{\delta,h}(j)\right) \bar{f}(h)\hat{f}(\hat{\delta})\mathrm{d}h\mathrm{d}\hat{\delta}.$$

Denote by $y_{im\tau}$ the indicator variable, which takes the value 1 when individual i chooses alternative τ from menu m. The log-likelihood function is

$$\log \mathcal{L}\left(f|\mathcal{O}\right) = \frac{1}{|\mathcal{O}|}\sum_{i=1}^{I}\sum_{m=1}^{M}\sum_{\tau=1}^{\mathcal{T}_m} y_{im\tau} \log\left(\rho_{im\tau}\left(f\right)\right).$$ [34]

[31] Not every individual faced all menus, but each was required to make between 84 and 100 choices. The Online Appendix contains further details of all of the three experimental datasets used in this section.

[32] Notice that, for any realization of the risk-aversion coefficient h and the corrected discount factor $\hat{\delta}$, we can back out the implied discount factor as $\delta = \hat{\delta}^{1-h}$, and use it for the relevant computations.

[33] In practice, we simplify the computational analysis by considering the subinterval $[-\underline{h}, 1)$ instead of $(-\infty, 1)$, where $\underline{h}$ is chosen small enough not to bound the estimation. See Appendix C for details.

[34] In order to allow for positive choice probabilities of dominated lotteries, we introduce a fixed small tremble, such that, with very large probability $1-\nu$, the individual chooses according to $\rho_{im\tau}(f)$ and with a very small probability ν, the individual uniformly randomizes. In the Online Appendix, we report the results of a version of the baseline estimation of each model, using all three datasets studied in this section, where we estimate ν as an additional parameter. In general, we find that estimating

Consistent estimation of $(\mu_h, \sigma_h^2, a_{\hat{\delta}}, b_{\hat{\delta}})$ can be achieved via maximization of the log-likelihood, and this estimator summarizes all the information about the estimated distributions of both risk and time attitudes. Robust standard errors for these estimates are computed using the delta method and clustered at the individual level. The computation of integrals is facilitated by means of a Quasi-Monte Carlo method, which can be easily implemented in most statistical packages, and which delivers log-likelihood functions with smooth parameters that can be quickly maximized using gradient-based methods.[35]

Table 1 shows the estimated risk and time preferences, including medians, standard deviations and the corresponding standard errors. Columns 2 and 3 show the results when we estimate risk and delay aversion separately, while column 4 shows the results from the joint estimation of the distributions. As expected, the results are identical in both cases. Figure 1 shows the estimated PDFs of the risk and delay aversion parameters, $\bar{f}$ and $\hat{f}$, and the one implied for δ.[36] We observe high levels of risk aversion, with the mode of the distribution at the upper bound of 1, suggesting that a sizable portion of the generated data may show risk-aversion levels above 1.[37]

The dataset is rich enough to perform individual estimations. Consequently, we now assume that the governing distributions, denoted by f_i, are individual-specific and we use the sub-sample of the corresponding individual observations.[38] Column 5 of Table 1 reports the median value of the median individual estimations of the parameters. Figure 2 shows a scatter-plot with the estimated individual median of risk and corrected discount factor for each of the 253 subjects in the sample, and Figure 3 plots the ordered individual estimates against the CDFs of the pooled estimations. This

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the tremble probability improves the fit of the models by allowing them to explain the observed positive probability of making dominated choices. However, the estimated distributions of risk and time preferences do not change substantially from that obtained by fixing ν as we do here.

[35] In Appendix C, we discuss the numerical evaluation of the log-likelihood function. The Matlab code for implementing these methods is provided via the authors' websites.

[36] The figure plots the normal kernel estimates of the PDF of $\delta = \hat{\delta}^{1-h}$ using the draws from the distributions of $\hat{\delta}$ and h. The results are consistent with the theoretical discussion, in that the high levels of risk aversion push up the discount factors considerably.

[37] In the Online Appendix we show how to extend the model to allow for higher levels of risk aversion, and report the resulting estimates under this extension. We obtain a similar median risk aversion, but capture the upper part of the distribution more closely.

[38] In this case, the asymptotic properties of the maximum likelihood estimator hold as the number of menus grows large.

exercise illustrates a series of points. In the first place, notice that there is substantial heterogeneity of preference across the population.[39] The distributions of median-traits in the population are far from degenerate and, indeed, closely reproduce those of the representative agent. In particular, and in consonance with our previous observation, it is apparent that there is a number of individuals for which the bound of 1 on the risk aversion level is binding. Next, notice that not all heterogeneity is due to the existence of different individual preferences, since, as reported in column 5 of Table 1, the median of the estimated individual standard deviations is clearly non-null. Thirdly, the correlation between the risk-aversion coefficients and the corrected discount factors is slightly positive (0.050), i.e. there is slightly negative correlation between risk and delay aversion, but it is not significant at conventional levels (p-value $= 0.425$).[40]

6.2. **Dated Lotteries: General Case.** A different strand of the literature elicits risk and time preferences over general menus of dated lotteries. Using the techniques discussed in the previous section, we illustrate with the study by Coble and Lusk (2010), which reports on an experiment involving 47 subjects each choosing from 94 menus involving either: (i) pairs of same-dated lotteries, (ii) pairs of dated degenerate lotteries, or (iii) pairs of non-degenerate lotteries awarded at different time periods.

For the sake of comparison, we first run an estimation exercise equivalent to that in the previous section. That is, we use only the subset of the data basically involving only risk or only time considerations, parts (i) and (ii) above.[41] Columns 2 and 3 of Table 2 report that, in this substantially different population of subjects, we find a relatively small decrease in the median levels of risk aversion and of the corrected discount factor. We can now use part (iii) of the dataset to evaluate whether behavior is substantially affected when both risk and time considerations are active. This joint estimation of risk

[39]The variability across estimates may also reflect sampling/estimation variability. Accordingly, the observed heterogeneity can be interpreted as an upper bound of the underlying preference heterogeneity.

[40]This dataset also contains information on individual characteristics. In the Online Appendix, we show how to incorporate this sort of information in the estimations by modeling the parameters of the distribution as a linear function of the observable characteristics. We can assume, for example, that $\mu_h = \gamma_0 + \gamma_l x_l$, where x_l is either a dummy or a real variable and then estimate parameters γ_0 and γ_l. In a recent paper, Jagelka (2019), using a version of the DEU-H model, implements a similar methodology to study the influence of personality traits on risk and time preferences.

[41]As discussed in Corollary 1, the time component does not play a role in the comparison of same-dated lotteries.

and time preferences is reported in column 4. Interestingly, the conclusions reached using (i) and (ii) vary little with respect to those obtained using (iii). This supports the view that the large body of literature using independent elicitations of risk and time preferences is obtaining a picture that is close to the one with dated lotteries involving the dimensions of both risk and time. Column 5 reports the estimation results with the pooled data, and Figure 4 plots the PDFs of the parameters using this pooled dataset.

We next perform individual estimations. Column 6 of Table 2 reports the median and standard deviation of the individual estimates; Figure 5 shows a scatter-plot with the estimated individual medians of risk and corrected discount factors for each of the 47 sample subjects, and Figure 3 plots the ordered individual estimates against the CDFs of the pooled estimations. Again, this analysis suggests great interpersonal heterogeneity, with a slightly negative correlation between individual risk parameters and corrected discount factors, and the presence of some intra-personal variability of preferences.

Interestingly, the richness of this dataset allows us to explore further the idea of correlation between risk and time, since we can now capture both intra-personal and interpersonal correlation. Since risk and time parameters are jointly responsible for choices in part (iii), we can now run a version of the pooled estimation without the independence assumption. In essence, we now express the joint distribution of h and $\hat{\delta}$ in terms of their marginal distributions (which follow the same parametric forms used in the independent case) and a Gaussian copula allowing for correlation between the two.[42] Column 7 in Table 2 shows the results of this exercise. We observe that the estimated correlation coefficient is negative but, due to the high variation, not statistically different from zero. Furthermore, the estimated moments of the marginal distributions are close to those obtained assuming independent distributions, thereby showing that allowing for the correlation of risk and time preferences has very little effect on the estimates in this dataset.

6.3. **Convex Budgets.** We now analyze the setting of convex budgets. We use the original dataset of Andreoni and Sprenger (2012b), which involves 80 subjects, each making 84 decisions from convex budget menus, for a total of 6,720 observations.

Let us start by commenting on some features of the empirical strategy. One is that the experimental implementation uses discretized versions of the continuous share

[42]See Fan and Patton (2014) for an introduction to the use of Copulas in econometrics.

problem, with $\alpha \in \{\frac{0}{100}, \frac{1}{100}, \ldots, \frac{100}{100}\}$. Moreover, since the vast majority of participants tend to choose multiples of 10, for practical reasons we discretize the choice of α to 11 equidistant possible shares. The DEU-H specification of $\rho_{im\tau}(f)$ described in Section 6.1 and its associated log-likelihood immediately extend to this setting. A second important property of convex budgets is that DEU-H, PVCE-H and CEPV-H are not equivalent in general, and the high variability of menus in the experiment allows us to estimate all three models. For PVCE-H and CEPV-H, and, in analogy with the estimation procedure for DEU-H, we assume that the joint probability distribution f is characterized by three independent probability distributions $\bar{f}$, $\tilde{f}$ and $\hat{f}$, defined, respectively, on the curvature of the Bernoulli function r, the curvature of the intertemporal utility function η, and the corrected discount factor $\hat{\delta} = \delta^{\frac{1}{1-\eta}}$. The computation of $\rho_{im\tau}(f)$ now involves a triple integral but is conceptually equivalent. We then use truncated normal distributions for $\bar{f}$ and $\tilde{f}$, and a beta distribution for $\hat{f}$, and therefore, in the baseline model we estimate $(\mu_r, \sigma_r^2, \mu_\eta, \sigma_\eta^2, a_{\hat{\delta}}, b_{\hat{\delta}})$.

Table 3 reports the results for the three models. The baseline estimations are reported in columns (i), (iv) and (vii); while, in columns (ii), (v) and (viii), we report the results when allowing for correlation between the distributions of the parameters using a Gaussian copula. Finally, columns (iii), (vi) and (ix) report on the estimations at the individual level, providing the median and standard deviation of the medians estimated for each individual. Figure 7 plots the estimated PDFs of the preference parameters for all three models. Figure 8 plots the observed and predicted choice probabilities across the different experimental parameters; and Figure 9 shows scatterplots with the estimated individual median parameters for each of the 80 subjects and for all three models. All the figures use the baseline estimated parameters reported in Table 3.

Here, we would like to stress the following findings. The first is that the simple model DEU-H already appears to perform remarkably well. Figure 8 shows that the estimated DEU-H distribution is able to capture the two main empirical regularities in the dataset; namely, a large task-dependent fraction of corner choices, followed by a task-dependent distribution of interior choices. This is due to the fact that DEU-H allows for preference heterogeneity, and, as we know from Proposition 3, interior choices are predicted for risk-averse attitudes while corner choices are predicted for risk-seeking attitudes. The estimation of a positive but close-to-zero median risk-aversion coefficient is the reason why approximately half of the predicted choices are made using negative risk-aversion coefficients leading to corner choices.

Second, a simple inspection of Figure 8 suggests that PVCE-H and CEPV-H models, by being more flexible, bring us even closer to the idiosyncratic nature of the data, being this confirmed by the log-likelihood values.[43] In particular, given the separation of the risk and intertemporal substitution parameters, corner and non-corner choices are now less influenced by risk aversion, and the models may potentially provide different estimates of risk aversion. It proves to be the case that the PVCE-H estimates low levels of risk aversion while CEPV-H yields higher levels, more in line with the experimental results on dated lotteries.

Third, as already mentioned in Sections 3 and 4, DEU-H and PVCE-H impose the same predictions across tasks with the same ratio of probabilities. However, Figure 8 clearly shows that this is not observed in the data. For instance, people seem to choose corner solutions more often when both lotteries are degenerate than they do when both outcomes are realized with equally low probability. Since these models must predict the same choices in both scenarios, an intermediate prediction is observed. CEPV-H has no such restriction and fits the data better across these tasks .

The fourth finding worth noting is that the corrected discount factor estimates are rather stable across both models and estimations, indicating a level of patience somewhere in between those observed in the previous two datasets that involved dated lotteries.

Finally, Figure 9 shows scatter plots of the individually-estimated parameters. Across individuals, there seems to be a negative correlation between risk and the corrected discount factor. The relationship between the intertemporal substitution and corrected discount factor appears to be negative in both models, while that between risk aversion and intertemporal substitution appears to be model dependent. Interestingly, the correlations obtained in the copula estimates of columns (ii), (v) and (viii) do not necessarily coincide with the individual correlations at the qualitative level. This is a further indication of the important role of heterogeneity both between and within subjects. Notice, however, the small magnitude of the correlations, and the fact that

[43]The CEPV-H model outperforms the PVCE-H model in this dataset based on different measures of in-sample fit. In the Online Appendix, we show that CEPV-H also outperforms PVCE-H, based K-fold cross-validation exercise. PVCE-H in turn markedly outperforms DEU-H. These results support the use of CEPV-H over PVCE-H based on in-sample fit and out-sample forecasting performance.

the majority are not statistically significant. Among the individual estimates, the only exception is the correlation between risk aversion and the corrected discount factor in DEU-H, which is approximately -0.345 (p-value=0.002). At a 5% confidence level, we cannot reject the null of zero correlation across individual estimates for the PVCE-H and CEPV-H models.

7. Final Remarks

In this paper, we have developed a sound framework for the analysis of risk and time preferences. We have studied several deterministic models of risk and time that can be used as a basis for estimation exercises; established their risk and time comparative statics; brought them into the framework of random utility models; and empirically illustrated their potential using several experimental datasets. Our framework offers a unique tractable tool for gaining a deeper understanding of risk and time preferences, a cornerstone of economics.

As examples of practical lessons to be learnt with our framework, consider the results obtained with the experimental dataset of Andreoni and Sprenger (2012b). The choice data appears challenging at first; as it has a significant fraction of corner choices, located at both ends of the choice range, and a significant fraction of interior choices. Previous attempts in the literature have had difficulty in accounting for such heterogeneous choice data. However, our stochastic implementation of what can be considered the simplest possible model of risk and time preferences, the discounted expected utility model, already allows us to account for the observed choice patterns remarkably well. The reason for this is that our framework is built upon the consideration of heterogeneous preferences, some of which predict corner choices and others interior ones. As can naturally be expected, the more flexible models studied in the paper capture some other features of the data even better.

Appendix A. Proofs of the Results in the Main Text

Proof of Proposition 1: When $t = 0$, DEU reduces to expected utility and hence, the first part follows immediately from standard results. When lotteries are degenerate, DEU reduces to exponentially discounted utility and we can use the normalization of Fishburn and Rubinstein (1982) to prove the result. Since this normalization plays a key role in this paper, we now discuss its details. When the space $\mathcal{D} \times T$ of dated degenerate lotteries is considered, $DEU_{\delta,u}$ is equivalent to $DEU_{\theta,\bar{u}}$, where θ is any value in $(0,1)$ and $\bar{u} = u^{\frac{\log\theta}{\log\delta}}$. To see this, notice that $DEU_{\delta,u}([1;x],t) \geq DEU_{\delta,u}([1;y],s)$ if and only if $\delta^t u(x) \geq \delta^s u(y)$. For any $\theta \in (0,1)$, since $\frac{\log\theta}{\log\delta} > 0$, the above inequality is equivalent to $(\delta^t u(x))^{\frac{\log\theta}{\log\delta}} \geq (\delta^s u(y))^{\frac{\log\theta}{\log\delta}}$ or, alternatively, $\theta^t \bar{u}(x) \geq \theta^s \bar{u}(y)$. This shows that the normalized model represents the same preferences. Now consider the pair of degenerate lotteries $([1;x];0)$ and $([1;y],s)$ with $0 < s$. We start with the 'only if' part. Clearly, if $x \geq y$, both individuals prefer the present payoff, and the claim follows. Let, then, $x < y$, and suppose that the second individual expresses a preference for the present payoff, i.e., $\bar{u}_2(x) \geq \theta^s \bar{u}_2(y)$, or equivalently, $\frac{\bar{u}_2(x)}{\bar{u}_2(y)} \geq \theta^s$. Then, it must be that $0 < x$. Suppose that $\bar{u}_1$ is more concave than $\bar{u}_2$. Without loss of generality, we can re-scale one of the two normalized utility functions to set $\bar{u}_1(x) = \bar{u}_2(x)$ and then, more concavity of $\bar{u}_1$ implies $\bar{u}_1(y) \leq \bar{u}_2(y)$, or equivalently, $\frac{\bar{u}_1(x)}{\bar{u}_1(y)} \geq \frac{\bar{u}_2(x)}{\bar{u}_2(y)} \geq \theta^s$, leading the first individual also to prefer the present payoff, as desired. We prove the converse by way of contradiction. Assume the existence of two payoffs $x^* < y^*$ and $\gamma \in (0,1)$ such that $\frac{\bar{u}_1(x^*)}{\bar{u}_1(y^*)} > \gamma > \frac{\bar{u}_2(x^*)}{\bar{u}_2(y^*)}$. Trivially, we can find $t^* \in T$ such that $\gamma = \theta^{t^*}$. Thus, selecting the dated lotteries $([1;x^*],0)$ and $([1;y^*],t^*)$, we obtain a contradiction. ■

Proof of Corollary 1: Under exponential discounting the DEU comparison of two lotteries awarded at the same time t is simply equivalent to the DEU comparison of the same two lotteries awarded in the present. We can then apply the first part of Proposition 1 and the definition of riskier lotteries in footnote 13 to conclude the proof of the first statement. For the second part, simply notice that basic lotteries $([p,1-p;x,0],t)$ and $([p,1-p;y,0],s)$ are evaluated by DEU in the same way as the degenerate lotteries $([1;x],t)$ and $([1;y],s)$, and we can then apply the same argument of the second part of Proposition 1 to conclude the proof. ■

Proof of Proposition 2: Denote by $CE_u(l)$ the certainty equivalent of lottery l using expected utility with monetary utility u. Given any pair of dated lotteries $(l_1 \equiv [p_1,\ldots,p_N;x_1,\ldots,x_N],t_1)$ and $(l_2 \equiv [q_1,\ldots,q_M;y_1,\ldots,y_M],t_2)$, it is evident that $\delta^{t_1}\sum_n p_n u(x_n) \geq \delta^{t_2}\sum_m q_m u(y_m)$ is equivalent to $\frac{\sum_n p_n u(x_n)}{\sum_m q_m u(y_m)} \geq \delta^{t_2-t_1}$ and thus, equivalent to $\frac{u(CE_u(l_1))}{u(CE_u(l_2))} \geq \delta^{t_2-t_1}$. We now prove the first part. Suppose that DEU_{δ_1,u_1} is equally delay averse and more risk averse than DEU_{δ_2,u_2}. From Proposition 1, this means that, for every x, $\bar{u}_1(x) = [u_1(x)]^{\frac{\log\theta}{\log\delta_1}} = [u_2(x)]^{\frac{\log\theta}{\log\delta_2}} = \bar{u}_2(x)$.[44] Taking logarithms, this is equivalent to $\frac{\log u_1(x)}{\log\delta_1} = \frac{\log u_2(x)}{\log\delta_2}$ for every x. That is, the ratio $\frac{\log u_2(x)}{\log u_1(x)}$ is equal to the constant $\frac{\log\delta_2}{\log\delta_1} = k > 0$, and hence: (i) $\delta_2 = \delta_1^k$ and (ii) $u_2 = u_1^k$. Since the first individual is more risk averse than the second it must be that $k \geq 1$. We can then rewrite the preference of the second individual for (l_1, t_1) as $\frac{[u_1(CE_{u_2}(l_1))]^k}{[u_1(CE_{u_2}(l_2))]^k} \geq \delta_1^{k(t_2-t_1)}$, which is equivalent to $\frac{u_1(CE_{u_2}(l_1))}{u_1(CE_{u_2}(l_2))} \geq \delta_1^{t_2-t_1}$. Whenever $l_1 = [1; x]$, we have $CE_{u_2}(l_1) = CE_{u_1}(l_1)$. As u_1 is more concave than u_2, we also have $CE_{u_2}(l_2) \geq CE_{u_1}(l_2)$. Hence, $\frac{u_1(CE_{u_1}(l_1))}{u_1(CE_{u_1}(l_2))} \geq \delta_1^{t_2-t_1}$ and the first individual also prefers the (degenerate) dated lottery (l_1, t_1).

For the second part, notice that, if the two individuals are equally risk averse, their monetary utility functions must have the same curvature. If the first individual is more delay averse than the second, the normalized utility function of the first individual must have greater curvature. With fixed risk aversion, it is immediate to see that the first individual must have a lower discount factor. Hence, if $t_1 < t_2$ and the second individual prefers (l_1, t_1) over (l_2, t_2), we have $\frac{\sum_n p_n u_2(x_n)}{\sum_m q_m u_2(y_m)} = \frac{\sum_n p_n u_1(x_n)}{\sum_m q_m u_1(y_m)} \geq \delta_2^{t_2-t_1} \geq \delta_1^{t_2-t_1}$, and the first individual also prefers the earlier lottery, as desired. ■

Proof of Proposition 3: For the first part, let u_i be convex. Then, $\delta_i^t p u_i(\alpha x) + \delta_i^s q u_i((1-\alpha)y) \leq \alpha\delta_i^t p u_i(x) + (1-\alpha)\delta_i^t p u_i(0) + (1-\alpha)\delta_i^s q u_i(y) + \alpha\delta_i^s q u_i(0) = \alpha\delta_i^t p u_i(x) + (1-\alpha)\delta_i^s q u_i(y) \leq \max\{\delta_i^t p u_i(x), \delta_i^s q u_i(y)\}$ and hence, the solution must be corner. As the problem has been reduced to the DEU comparison of the two basic dated lotteries $([p, 1-p; x, 0], t)$ and $([q, 1-q; y, 0], s)$, we can directly use Part 2 of Corollary 1, and the result follows immediately.

For the second part, let u_i be strictly concave. Given the differentiability assumption, the first-order condition of the optimization problem is $\frac{u_i'(\alpha x)}{u_i'((1-\alpha)y)} = \frac{\delta_i^s qy}{\delta_i^t px}$, with the left hand side strictly decreasing in α. The strict concavity of u_i guarantees that the solution is interior. We now start by fixing risk aversion, i.e. $u_1 = u_2$. In this case, we know that individual 1 is more delay averse than individual 2 if and only if $\delta_1 \leq \delta_2$. This implies that $\frac{\delta_1^s qy}{\delta_1^t px} \leq \frac{\delta_2^s qy}{\delta_2^t px}$, and hence, given the strict concavity of u_i and $\frac{u_1'(\alpha x)}{u_1'((1-\alpha)y)} = \frac{u_2'(\alpha x)}{u_2'((1-\alpha)y)}$, it must be that $\alpha_1^* \geq \alpha_2^*$.

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[44] More formally, $\bar{u}_1(x) = a\bar{u}_2(x)$ with $a > 0$. The constant a is inessential to our arguments, and hence we normalize to $a = 1$ without loss of generality. Similar conventions will be adopted in the following proofs.

We now fix delay aversion. As discussed in Proposition 2, this implies that $\delta_2 = \delta_1^k$ and $u_2 = u_1^k$. Next, suppose that the first individual is more risk averse than the second, i.e., $k \geq 1$. The first order condition of the second individual can be written as $(\frac{u_1(\alpha x)}{u_1((1-\alpha)y)})^{k-1} \frac{u_1'(\alpha x)}{u_1'((1-\alpha)y)} = \frac{\delta_1^{ks} qy}{\delta_1^{kt} px} = \frac{\delta_1^s qy}{\delta_1^t px} \frac{\delta_1^{(k-1)s}}{\delta_1^{(k-1)t}}$, or equivalently $\big(\frac{\delta_1^t u_1(\alpha x)}{\delta_1^s u_1((1-\alpha)y)}\big)^{k-1} \frac{u_1'(\alpha x)}{u_1'((1-\alpha)y)} = \frac{\delta_1^s qy}{\delta_1^t px}$. Clearly, considering the constant $\overline{\alpha}_1$ defined in the text, $g(\alpha) = \big(\frac{\delta_1^t u_1(\alpha x)}{\delta_1^s u_1((1-\alpha)y)}\big)^{k-1} \leq 1$ if and only if $\alpha \leq \overline{\alpha}_1$. Let $f(\alpha) = \frac{u_1'(\alpha x)}{u_1'((1-\alpha)y)}$, which we know is strictly decreasing in α. Then, for values of α below (respectively, above) $\overline{\alpha}_1$ the function $h(\alpha) = f(\alpha)g(\alpha)$ on the left hand side of the first-order condition falls below (respectively, is above) $f(\alpha)$. Since the right hand side is a constant, the result follows immediately. ■

Proof of Proposition 4: Since PVCE and CEPV are extensions of DEU, it is evident that DEU belongs to the intersection of both classes. Now suppose that some behavior over basic lotteries belongs to the intersection of both PVCE and CEPV. Consider the PVCE representation of this behavior, and rewrite it as $\delta^t g(pv(x))$, where $g = w \circ v^{-1}$. We now prove that the function g must be homogeneous.

First, consider any $v_0 \in \mathbb{R}_{++}$ in the range of possible utility values associated to v, i.e., there exists a payoff x_0 such that $v(x_0) = v_0$. Consider $0 < p < 1$ and, whenever it exists, the monetary outcome x_1 such that $v(x_1) = \frac{v_0}{p}$. Then, take the value $t_1 \in T$ such that $w(x_0) = g(v_0) = \delta^{t_1} g(\frac{v_0}{p}) = \delta^{t_1} w(x_1)$. That is, t_1 is the value that makes the dated degenerate lotteries $([1; x_0], 0)$ and $([1; x_1], t_1)$ indifferent. Hence, the present value of x_1 awarded at t_1 must be x_0. Clearly, the present value of 0 (awarded at t_1) is 0 and since the choice behavior admits a CEPV representation, it must also be that $([p, 1-p; x_0, 0], 0)$ and $([p, 1-p; x_1, 0], t_1)$ provide the same utility. Hence, it must be that $w(v^{-1}[pv(x_0)]) = g(pv_0) = \delta^{t_1} g(p\frac{v_0}{p}) = \delta^{t_1} w(v^{-1}[pv(x_1)])$. This is simply $g(pv_0) = \delta^{t_1} g(v_0)$ or $\delta^{-t_1} g(pv_0) = g(\frac{1}{p}[pv_0])$. By repeated use of this reasoning, we can obtain, for every positive integer ι, $\delta^{-\iota t} g(pv_0) = g(\frac{1}{p^\iota}[pv_0])$, which proves that the function g is homogeneous of degree $\frac{t \log \delta}{\log p}$ on the sequence of utility points $(pv_0, v_0, \frac{v_0}{p}, \dots)$. By taking v_0 as close to zero as desired and p as close to 1 as desired, the continuity of the functions involved guarantees that g must be homogeneous on the positive orthant. Now, the homogeneity of g, with homogeneity of degree ξ, allows us to rewrite the PVCE representation as $\delta^t g(pv(x)) = g((\delta^t)^{\frac{1}{\xi}} pv(x)) = g(\overline{\delta}^t pv(x))$. Hence, preferences can be represented by a monotone transformation of DEU and the claim follows.■

Proof of Proposition 5: In a series of claims, the proof characterizes the models described in the main text, ultimately showing the stated result. Consider a preference $\succsim$ over $\mathcal{L} \times T$. Here is a list of possible properties for such a preference.

Regularity (REG). $\succsim$ satisfies the following conditions:

(1) Rationality (RAT). $\succsim$ is complete and transitive.
(2) Continuity (CON). $\succsim$ is continuous.
(3) Risk-Monotonicity (R-MON). If l strictly first-order stochastically dominates (FOSD) l', then $(l, 0) \succ (l', 0)$.
(4) Time-Monotonicity (T-MON). If $t < s$, then $([1; 0], t) \sim ([1; 0], s)$ and for every $l \neq [1; 0]$, $(l, t) \succ (l, s)$.

Separability (SEP). $(l, t) \succsim (l', t)$ if and only if $(l, s) \succsim (l', s)$.

First-Time-Then-Risk (FTTR). If $([1; x_n], t) \sim ([1; y_n], 0)$ for every $n \in \{1, \ldots, N\}$, then for every $\{p_n\}_{n=1}^N$ with $p_n \geq 0$ and $\sum_{n=1}^N p_n = 1$, $([p_1, \ldots, p_N; x_1 \ldots, x_N], t) \sim ([p_1, \ldots, p_N; y_1 \ldots, y_N], 0)$.

Stationarity (STAT). If $([1; x], t) \sim ([1; y], s)$ then, for every γ such that $t+\gamma, s+\gamma \geq 0$, $([1; x], t + \gamma) \sim ([1; y], s + \gamma)$.

Independence (IND). $(l, 0) \succsim (l', 0)$ if and only if $(\lambda l + (1 - \lambda)l'', 0) \succsim (\lambda l' + (1 - \lambda)l'', 0)$ for every l'', $\lambda \in (0, 1)$.

Payoff-Scale Invariance (PSI). For every $\kappa > 0$, $([1; x], t) \sim ([1; y], s)$ implies that $[1; \kappa x], t) \sim ([1; \kappa y], s)$.

Claim 1. $\succsim$ satisfies REG and SEP if and only if there exists a continuous mapping $CE : \mathcal{L} \to X$, with $CE([1; x]) = x$ and $CE(l) > CE(l')$ whenever l strictly FOSDs l', and a continuous mapping $PV : X \times T \to X$, with $PV(x, 0) = x$, strictly increasing in X and strictly decreasing in T when $x > 0$, such that $\succsim$ is represented by $(l, t) \succsim (l', s) \Leftrightarrow PV(CE(l), t) \geq PV(CE(l'), s)$.

Proof of Claim 1: Since the proof of necessity is immediate, we only prove sufficiency. We first show that every lottery admits a unique certainty equivalent when evaluated in the present. That is, there exists a mapping $CE : \mathcal{L} \to X$ with the properties stated in the claim, such that $(l, 0) \sim ([1; CE(l)], 0)$. We set $CE([1; x]) = x$, which defines a certainty equivalent for every degenerate lottery. If l is not degenerate, R-MON guarantees that $(l, 0)$ is strictly better (respectively, strictly worse) than the dated lottery giving the worst (respectively, the best) payoff of lottery l, with probability one, in the present. Hence, RAT and CON guarantee the existence of the certainty equivalent $CE(l)$ in the present. Furthermore, RAT, CON and R-MON guarantee that the constructed mapping satisfies all the properties defining a certainty equivalent mapping. Next, we construct a present equivalent mapping $PV : X \times T \to X$ as follows. For a given amount of money x and a given time t, consider the induced dated degenerate lottery $([1; x], t)$. We claim that we can find a degenerate lottery awarded at time 0 that is indifferent to it, and hence, the corresponding payoff is the required present value. That is, $PV(x, t)$ is such that $([1; x], t) \sim ([1; PV(x, t)], 0)$. Whenever $t = 0$ or $x = 0$, the claim can be proved by direct application of T-MON. Whenever both x and t are strictly positive, notice that $([1; x], 0) \succ ([1; x], t) \succ ([1; 0], t) \sim ([1; 0], 0)$. That is, the degenerate lotteries giving x and 0 in the present are strictly better and strictly worse, respectively, than the dated lottery giving the degenerate payoff x at t. RAT and CON guarantee the existence of a monetary value $PV(x, t)$ awarded at time 0 and indifferent to $([1; x], t)$. Again, it is evident that RAT, CON and T-MON guarantee that the mapping PV satisfies all the properties required for a present value mapping.

Now consider two dated lotteries (l, t) and (l', s). Since $(l, 0) \sim ([1; CE(l)], 0)$ and $(l', 0) \sim ([1; CE(l')], 0)$, SEP guarantees that $(l, t) \sim ([1; CE(l)], t)$ and $(l', s) \sim ([1; CE(l')], s)$. Hence, $(l, t) \succsim (l', s)$ if and only if $([1; CE(l)], t) \succsim ([1; CE(l')], s)$ if and only if $([1; PV(CE(l), t)], 0) \succsim ([1; PV(CE(l'), s)], 0)$ if and only if $PV(CE(l), t) \geq PV(CE(l'), s)$, as desired. $\diamond$

Claim 2. $\succsim$ satisfies REG and FTTR if and only if there exists a continuous mapping $CE : \mathcal{L} \to X$, with $CE([1; x]) = x$ and $CE(l) > CE(l')$ whenever l strictly FOSDs l', and a continuous mapping $PV : X \times T \to X$, with $PV(x, 0) = x$, strictly increasing in X and strictly decreasing in T when $x > 0$, such that $\succsim$ can be represented by $(l, t) \succsim (l', s) \Leftrightarrow CE([p_1, \ldots, p_N; PV(x_1, t), \ldots, PV(x_N, t)]) \geq CE([q_1, \ldots, q_M; PV(y_1, t), \ldots, PV(y_M, t)])$.

Proof of Claim 2: Since the proof of necessity is immediate, we only prove sufficiency. That REG implies the existence of certainty equivalents and present values has been proved in Claim 1. Assume FTTR. Consider two dated lotteries (l, t) and (l', s), with $l = [p_1, \ldots, p_N; x_1, \ldots x_N]$ and $l' = [q_1, \ldots, q_M; y_1, \ldots y_M]$. Since, for every n we have that $([1; x_n], t) \sim ([1; PV(x_n, t)], 0)$, the direct application of FTTR leads to $(l, t) \sim ([p_1, \ldots, p_N; PV(x_1, t), \ldots, PV(x_N, t)], 0)$. Similarly, it must also be that $(l', s) \sim ([q_1, \ldots, q_M; PV(y_1, s), \ldots, PV(y_M, s)], 0)$. Now, the lottery constructed by bringing payments of l to the present must be indifferent to one awarding its certainty equivalent. That is, we must have $([p_1,\ldots,p_N;PV(x_1,t),\ldots,PV(x_N,t)],0) \sim ([1;CE(p_1,\ldots,p_N;PV(x_1,t),\ldots,PV(x_N,t))],0)$. A similar reasoning can be applied to the dated lottery (l',s), leading to $([q_1,\ldots,q_M;PV(y_1,s),\ldots,PV(y_M,s)],0) \sim ([1;CE(q_1,\ldots,q_M;PV(y_1,s),\ldots,PV(y_M,s))],0)$. Using REG, we can link $(l,t) \succsim (l',s)$ to the comparison of certainty equivalents $CE(p_1,\ldots,p_N;PV(x_1,t),\ldots,PV(x_N,t)) \geq CE(q_1,\ldots,q_M;PV(y_1,s),\ldots,PV(y_M,s))$, as desired. $\diamond$

Claim 3. $\succsim$ satisfies REG, SEP, STAT and IND if and only if $\succsim$ can be represented by PVCE.

Proof of Claim 3: Since the proof of necessity is immediate, we only prove sufficiency. We start with the representation described in Claim 1. Consider the set $\mathcal{L}\times\{0\}$. Here, our axioms imply those used in the standard treatment of expected utility and, hence, it is immediate to see that there exists a continuous and strictly increasing mapping $v: X\to\mathbb{R}_+$ with $v(0)=0$ such that a certainty equivalent function is constructed from expected utility with Bernoulli utility function v. For the time dimension, we use the results of Fishburn and Rubinstein (1982). Thus, let $\succsim'$ be the preference on $X\times T$ induced by the restriction of $\succsim$ to the set of all dated degenerate lotteries, i.e. $(x,t)\succsim'(y,s)$ if and only if $([1;x],t)\succsim([1;y],s)$. Our axioms imply the axioms of Fishburn and Rubinstein's Theorem 2, and hence there exists $\delta\in(0,1)$ and a continuous and strictly increasing mapping $w:X\to\mathbb{R}_+$ with $w(0)=0$ such that $(x,t)\succsim'(y,s)$ if and only if $\delta^t w(x)\geq\delta^s w(y)$. Since w^{-1} is a strictly monotone transformation of w, it is evident that $(x,t)\succsim'(y,s)$ if and only if $w^{-1}[\delta^t w(x)]\geq w^{-1}[\delta^s w(y)]$. It then follows that $\succsim$ admits a PVCE representation. $\diamond$

Claim 4. $\succsim$ satisfies REG, FTTR, STAT and IND if and only if $\succsim$ can be represented by CEPV.

Proof of Claim 4: It follows from Claim 2, using the same analysis as in the proof of Claim 3. $\diamond$

Claim 5. $\succsim$ satisfies REG, SEP, FTTR, STAT and IND if and only if $\succsim$ can be represented by DEU.

Proof of Claim 5: The proof of necessity is immediate. For sufficiency, it is enough to see that Proposition 4 extends immediately to $\mathcal{L}\times T$ and hence, the result follows from Claims 3 and 4. $\diamond$

Claim 6. $\succsim$ satisfies REG, SEP, FTTR, STAT, IND and PSI if and only if $\succsim$ can be represented by DEU-H or equivalently, by PVCE-H and by CEPV-H.

Proof of Claim 6: The proof of necessity is immediate. For sufficiency, consider any DEU representation of $\succsim$. We now show that, whenever preferences satisfy PSI, the function u in this representation must be homogeneous, and hence we have in fact DEU-H. First, consider any $x \in X$ and real number $\kappa > 1$. From the strict monotonicity of u, we know that there exists $t_\kappa \in \mathbb{R}_{++}$ such that $u(x) = \delta^{t_\kappa} u(\kappa x)$. PSI allows us to use this argument repeatedly to obtain $([1; \kappa^{\iota-1}x], 0) \sim ([1; \kappa^{\iota}x], t_\kappa)$ for every positive integer $\iota \geq 1$. That is, $u(\kappa^{\iota-1}x) = \delta^{t_\kappa} u(\kappa^{\iota}x)$, or equivalently $u(\kappa^{\iota}x) = \delta^{-t_\kappa} u(\kappa^{\iota-1}x)$ for every $\iota \geq 1$, which means that u is a homogeneous function of degree $\frac{-t_\kappa \log \delta}{\log \kappa}$ on the sequence of points $\{x, \kappa x, \kappa^2 x, \dots\}$. By making x as close to 0 as desired and κ as close to 1 as desired and using continuity, homogeneity must hold for the entire positive orthant, as desired. The equivalence with PVCE-H and CEPV-H follows directly from the defining properties and the fact that PVCE and CEPV contain DEU. $\diamond$

The characterization result in Claim 6 concludes the proof. ■

Proof of Proposition 7: For the first part, consider an instance f of the random utility model built upon DEU-H, where h and δ are independently distributed, and a pair of dated lotteries $(l \equiv [p_1, \dots, p_N; x_1, \dots, x_N], t)$ and $(l' \equiv [q_1, \dots, q_M; y_1, \dots, y_M], s)$, with $l \neq [1; 0]$ and $t < s$. We know that, for DEU-H, (l, t) is preferred to (l', s) if and only if $\delta^t \sum_n p_n \frac{x_n^{1-h}}{1-h} > \delta^s \sum_m q_m \frac{y_m^{1-h}}{1-h}$. Since $l \neq [1; 0]$, this is equivalent to $1 > \delta^{s-t} \frac{\sum_m q_m y_m^{1-h}}{\sum_n p_n {x_n}^{1-h}}$. Since the result is trivial for $l' = [1; 0]$, assume that $l' \neq [1; 0]$. Having fixed δ, the right hand side converges to δ^{s-t} whenever h converges to 1. Hence, the independence assumption guarantees that the proportion of choices for which the inequality holds must converge to 1 whenever h approaches 1, thus proving the result.

For the second part, notice that the first order condition of DEU-H can be rewritten as $\frac{(1-\alpha)}{\alpha} = (\frac{\delta^s q}{\delta^t p})^{\frac{1}{h}} (\frac{y}{x})^{\frac{1-h}{h}}$. Having fixed δ, the right hand side converges to $\delta^{s-t} \frac{q}{p}$ whenever h converges to 1. $\alpha^* = \frac{\delta^t p}{\delta^t p + \delta^s q}$ solves the first-order condition corresponding to the limit value $\delta^{s-t} \frac{q}{p}$. The independence assumption then guarantees that as h converges to 1, the mass of choices belonging to a neighborhood of α^* approaches 1. Since this result does not depend on x and y, the result has been proved. ■

Appendix B. Behavioral Models.

Here we outline how to introduce behavioral considerations in the treatment of risk and time preferences, using the dated lottery setting for illustrative purposes. In essence, Claims 1 and 2 in the proof of Proposition 5 axiomatically characterize generalized versions of PVCE and CEPV based on generalized time-invariant certainty equivalent mappings, not necessarily based on expected utility, and generalized present value equivalent mappings, not necessarily based on exponential discounting. The comparative statics in these generalized representations are straightforward, with more risk aversion captured directly by lower values of the certainty equivalent function, and more delay aversion captured by lower values of the present value function. Behavioral considerations can thus be incorporated into this framework through the use of non-standard mappings, as we now illustrate with a simple parametric specification.

In order to save on notation, we assume two-payoff lotteries. In the treatment of risk, we adopt the influential disappointment aversion model of Gul (1991) which, in this setting with two-payoff lotteries, is a special case of the rank-dependent utility of Quiggin (1982). For the treatment of time, we adopt the well-known $\beta - \delta$ model of Laibson (1997), exemplifying with the generalized PVCE representation, and adopting a homogeneous monetary function. Let $([p, 1-p; x_1, x_2], t)$ denote a dated binary lottery with $x_1 \geq x_2$. The certainty equivalent of Gul's model requires us to evaluate the binary lottery as $[\gamma(p)x_1^{1-r} + (1-\gamma(p))x_2^{1-r}]^{\frac{1}{1-r}}$, with weighting function $\gamma(p) = \frac{p}{1+(1-p)\zeta}$, $\zeta \in (-1, \infty)$, where $\zeta = 0$ reduces the model to expected utility, $\zeta > 0$ reflects disappointment aversion and $\zeta < 0$ elation seeking. In the $\beta - \delta$ model, future payoffs are discounted by means of the standard exponential formula multiplied by a parameter $\beta \in (0, 1]$, representing present-bias. Hence, the present value equivalent of a monetary payoff x awarded at time $t > 0$ is $[\beta\delta^t x^{1-\eta}]^{\frac{1}{1-\eta}}$.[45] Then the behavioral version of the PVCE model reduces to $U_{\beta,\delta,\eta,\zeta,r}([p, 1-p; x_1, x_2], t) = \beta\delta^t[\gamma(p)x_1^{1-r} + (1-\gamma(p))x_2^{1-r}]^{\frac{1-\eta}{1-r}}$ whenever $t > 0$, and $U_{\beta,\delta,\eta,\zeta,r}([p, 1-p; x_1, x_2], 0) = [\gamma(p)x_1^{1-r} + (1-\gamma(p))x_2^{1-r}]^{\frac{1-\eta}{1-r}}$ otherwise.

[45] Formally, since the present value function of the standard $\beta - \delta$ model is discontinuous, one can use a continuous decreasing piece-wise linear function $\beta(t)$ taking value 1 when $t = 0$ and value β for any time above a given t_ϵ. The $\beta - \delta$ model is the limit of these models when ϵ goes to zero. Notice also that this assumption is ineffective in practice, since we can always assume that t_ϵ is lower than the time involved in any future lotteries in an experimental dataset. The exponential discounting model simply assumes $\beta = 1$.

The comparative statics in terms of risk follow immediately when noting that the certainty equivalent of a lottery decreases whenever either r or ζ increases. Similarly, the present value of a future payoff decreases whenever either $\hat{\delta}$ or β decreases.

Appendix C. Computational Considerations

In practice, the implementation of the maximum likelihood estimator may be complicated by the computation of probabilities $\rho_{im\tau}$, since these are given by multiple integrals with no closed-form solution. Numerical evaluation of these integrals using quadrature methods can be very slow and fall prey to the curse of dimensionality. Monte-Carlo integration, by directly drawing from the distributions of the parameters, avoids the curse of dimensionality but can still be slow for the problem at hand. Ultimately, this method may lead to log-likelihood functions that are not smooth in the estimated parameters, preventing the use of traditional gradient-based methods to maximize the log-likelihood and compute standard errors. As an alternative, we use Quasi Monte-Carlo methods to evaluate $\rho_{im\tau}$.[46]

Formally, and for the case of DEU-H, we generate K Halton draws $\{h_k,\, \hat{\delta}_k\}_{k=1}^{K}$ on the domain of the parameters h and $\hat{\delta}$. Notice that these are not draws from the distributions of h and $\hat{\delta}$, but quasi-random low-discrepancy sequences dependent upon these distributions. To simplify computation, we assume sufficiently large compact supports, characterized by the intervals $[\underline{h}, \overline{h}]$ and $[\underline{\hat{\delta}}, \overline{\hat{\delta}}]$, and formally work with the associated truncated distributions over these intervals. Using these draws, we can approximate $\rho_{im\tau}$ as follows

$$\rho_{im\tau}(f) \approx \frac{V}{K} \sum_{k=1}^{K} \sum_{i=1}^{I} \sum_{m=1}^{M} \mathbf{1}\left(\tau = \max_{j \in \{1,2,\ldots,\mathcal{T}_m\}} DEU_{\delta_k, h_k}(j)\right) \bar{f}(h_k)\hat{f}(\hat{\delta}_k),$$

where h_k and $\hat{\delta}_k$ are the k-th draw of the parameters, δ_k is derived from the former as usual, and $V = \int_h \int_{\hat{\delta}} \mathrm{d}h \mathrm{d}\hat{\delta} = (\overline{h} - \underline{h})(\overline{\hat{\delta}} - \underline{\hat{\delta}})$ is a normalization constant.

The advantages of this approach are based on the fact that, once the domain of the parameters is specified and the points on the domain of the parameters are drawn, the indicator function is independent of $\bar{f}$ and $\hat{f}$. The indicator function can thus be computed at first, stored, and then used in every step of the maximization of the log-likelihood function, thereby dramatically speeding up the estimation. This is

[46] See Chapter 9.3 in Train (2003) for a textbook introduction.

especially useful when additional parameters are included, as in the cases of PVCE-H and CEPV-H.[47]

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[47] For instance, an estimation of PVCE-H for the pooled dataset of Andreoni and Sprenger, using a Dell Precision Tower with an Intel Core i7-6700 processor of 3.40GHz, 32 GB RAM and running Matlab 2018b takes two minutes with our method and more than 8 hours with quadrature methods.

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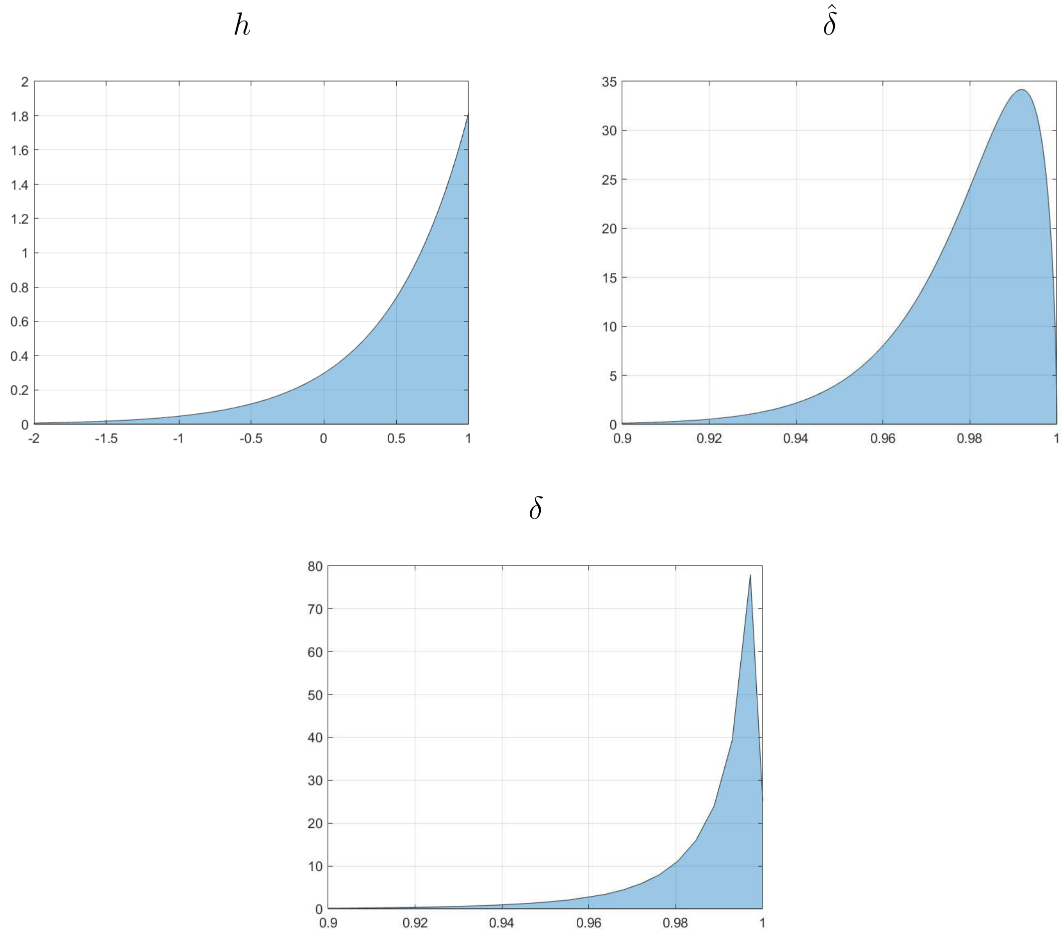
Table 1. Estimated Risk and Time Preferences: Andersen et al. (2008)

Dataset	**Risk Only**	**Time Only**	**Joint**	**by Individual**
Median h	0.620		0.620	0.718
	[0.023]		[0.023]	(0.377)
Std. Dev. h	0.512		0.512	0.517
	[0.023]		[0.023]	(0.284)
Median $\hat{\delta}$		0.983	0.983	0.980
		[0.001]	[0.001]	(0.007)
Std. Dev. $\hat{\delta}$		0.016	0.016	0.085
		[0.001]	[0.001]	(0.051)
# Obs.	7928	15180	23108	23108
Log-Likelihood	-2.128	-0.543	-1.087	-0.991

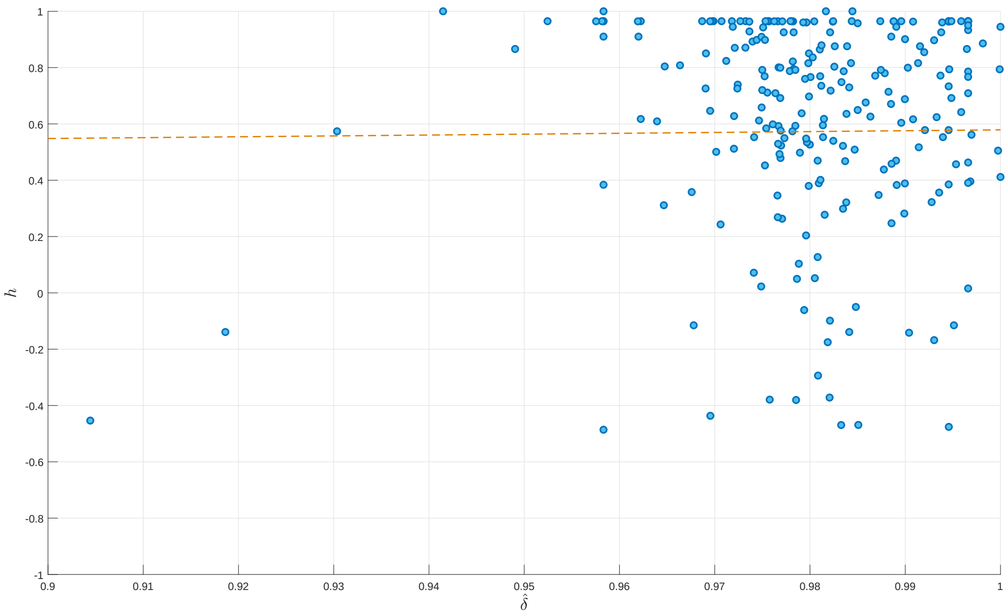
NOTES.- The above table reports the maximum-likelihood estimates of the median and the standard deviation of the distributions of risk and time preferences under the DEU-H representation, using data from Andersen et al. (2008). The second column shows the results obtained using the subsample of menus eliciting risk aversion only. The third column shows the results obtained using the subsample of menus eliciting delay aversion only. The fourth column shows the results of the joint estimation of risk aversion and delay aversion using the pooled menu sample. Standard errors, shown in brackets, are computed using the delta method and clustered at the individual level. The last column shows the median and standard deviation (in parentheses) of the distribution of individual estimates of the respective parameter. In all cases, the coefficient of risk aversion h is assumed to follow a normal distribution truncated at 1, while the corrected discount factor $\hat{\delta}$ follows a beta distribution.

FIGURE 1. PDFs of Estimated Risk and Time Preferences: Andersen et al. (2008)



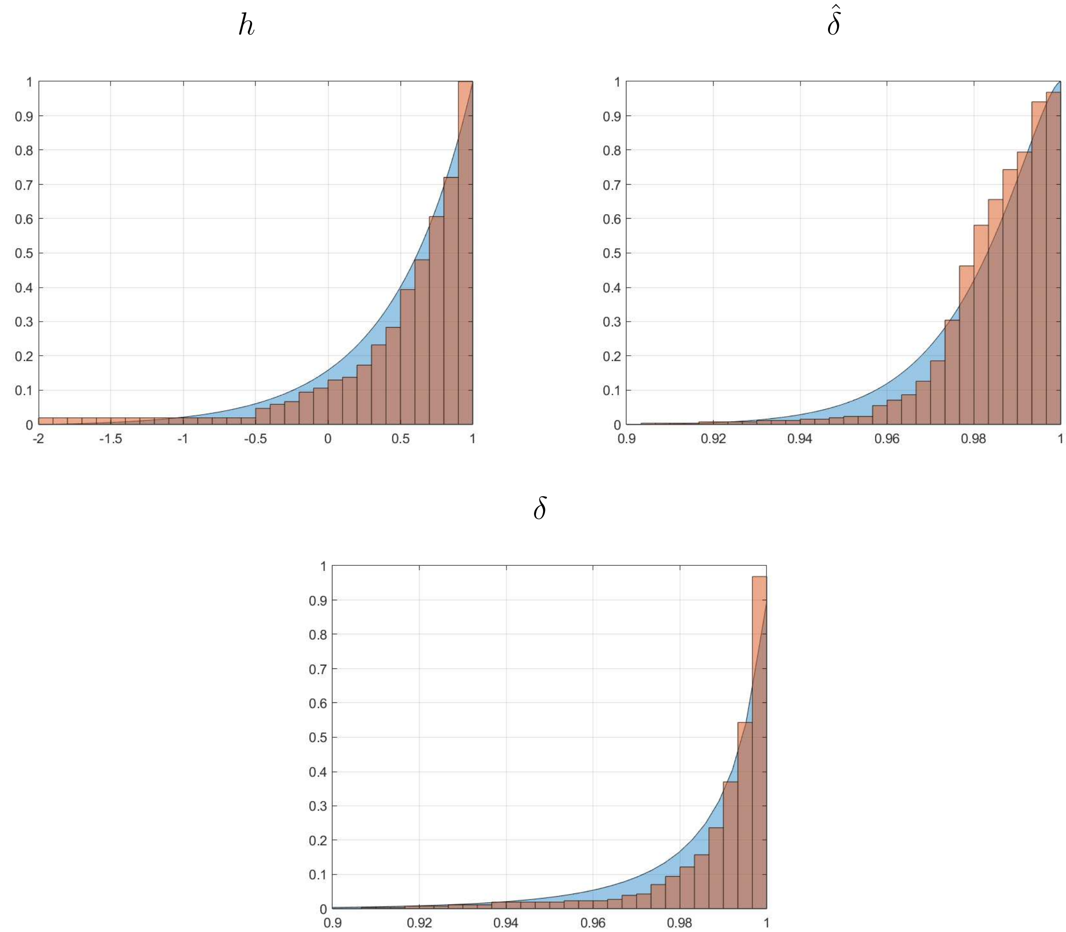
NOTES.- PDFs of the estimated distributions reported in Table 1. The PDF of the discount factor $\delta = \hat{\delta}^{1-h}$ is estimated from the distributions of risk and delay aversion using a normal kernel.

FIGURE 2. Individual Estimates: Andersen et al. (2008)



NOTES.- Each point represents the median of the estimated distributions of the coefficient of risk aversion h and the corrected discount factor $\hat{\delta}$ for the subsample of choice data for a particular individual, following the estimation procedure reported in Table 1.

Figure 3. CDFs of Estimated Risk and Time Preferences and Histograms of Individual Estimates: Andersen et al. (2008)



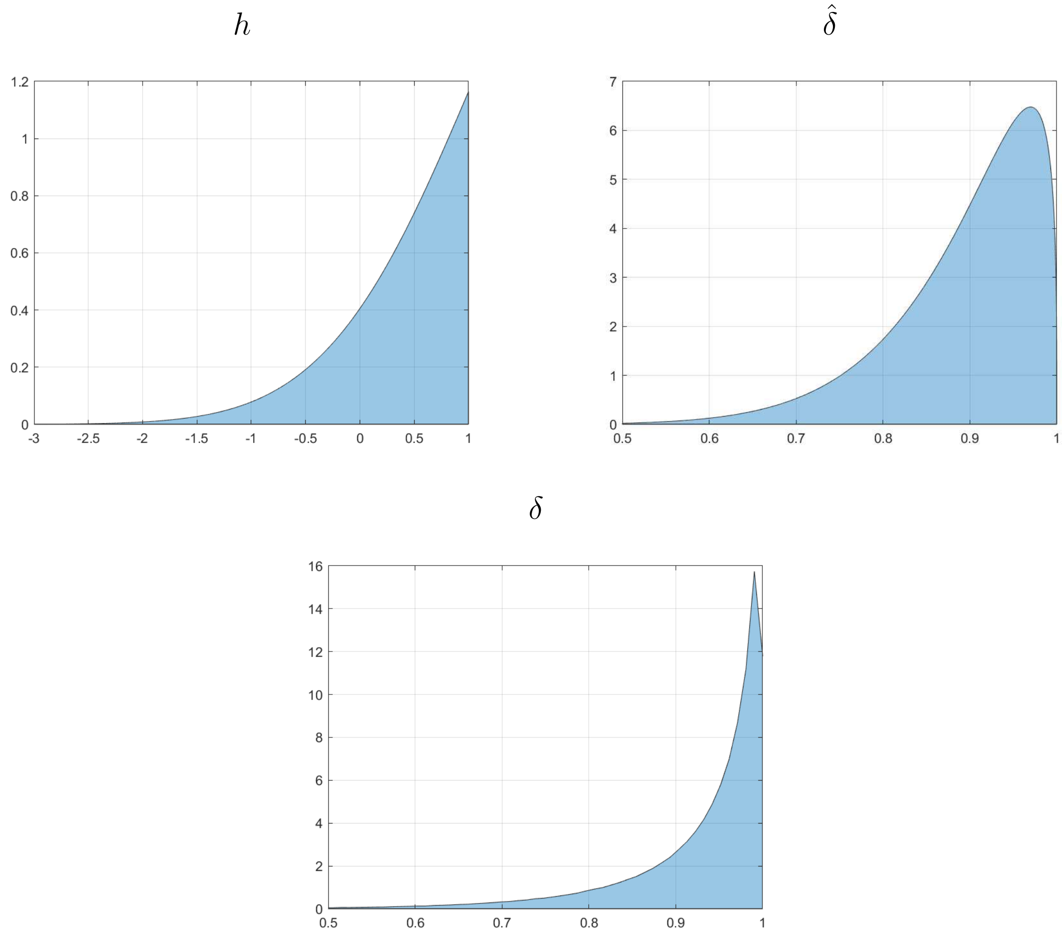
NOTES.- CDFs of the pooled estimation and histograms of the empirical distributions of the individual estimates.

Table 2. Estimated Risk and Time Preferences: Coble and Lusk (2010)

Dataset	**Using Risk Tasks Only**	**Using Discount Tasks Only**	**Using Joint Tasks Only**	**All Tasks**	**All Tasks - Correlated Preferences**	**Pooled Individual Estimates**
Median h	0.503 [0.072]	–	0.485 [0.109]	0.464 [0.076]	0.490 [0.074]	0.238 (0.012)
Std. Dev. h	0.569 [0.084]	–	0.413 [0.061]	0.582 [0.085]	0.554 [0.079]	0.530 (0.278)
Median $\hat{\delta}$	–	0.903 [0.012]	0.939 [0.017]	0.915 [0.010]	0.913 [0.010]	0.918 (0.054)
Std. Dev. $\hat{\delta}$	–	0.089 [0.017]	0.130 [0.034]	0.085 [0.013]	0.082 [0.013]	0.236 (0.031)
Corr(h,$\hat{\delta}$)	–	–	–	–	−0.453 [0.254]	0.023
# Obs.	1880	1128	1410	4418	4418	47
Log-Likelihood	−0.878	−0.436	−0.362	−0.606	−0.604	−0.406

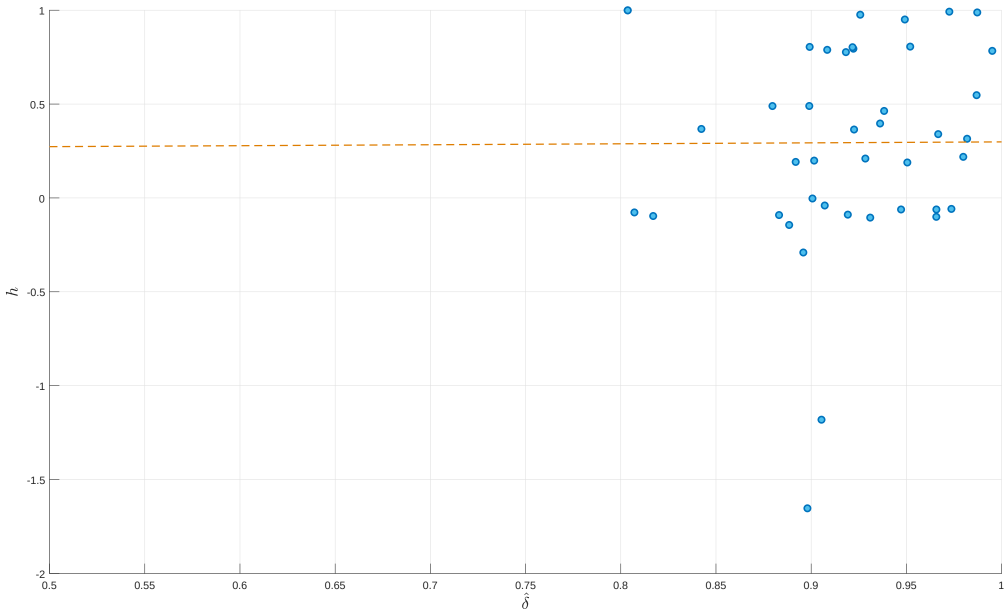
NOTES.- The above table reports the maximum-likelihood estimates of the median and the standard deviation of the distributions of risk and time preferences under the DEU-H representation, using data from Coble and Lusk (2010). The second column shows the results obtained using the subsample of menus eliciting risk aversion only. The third column shows the results obtained using the subsample of menus eliciting delay aversion only. The fourth column shows the results using menus with pairs of non-degenerate lotteries awarded at different time periods. The fifth column shows the results of the joint estimation of risk aversion and delay aversion, using the pooled menu sample. The sixth column shows the estimates obtained when allowing correlation between parameters using a Gaussian copula. Standard errors, shown in brackets, are computed using the delta method and clustered at the individual level. The last column shows the median and standard deviation (in parentheses) of the distribution of individual estimates of the respective parameter. In all cases, the coefficient of risk aversion h is assumed to follow a normal distribution truncated at 1, while the corrected discount factor $\hat{\delta}$ follows a beta distribution.

Figure 4. PDFs of Estimated Risk and Time Preferences: Coble and Lusk (2010)



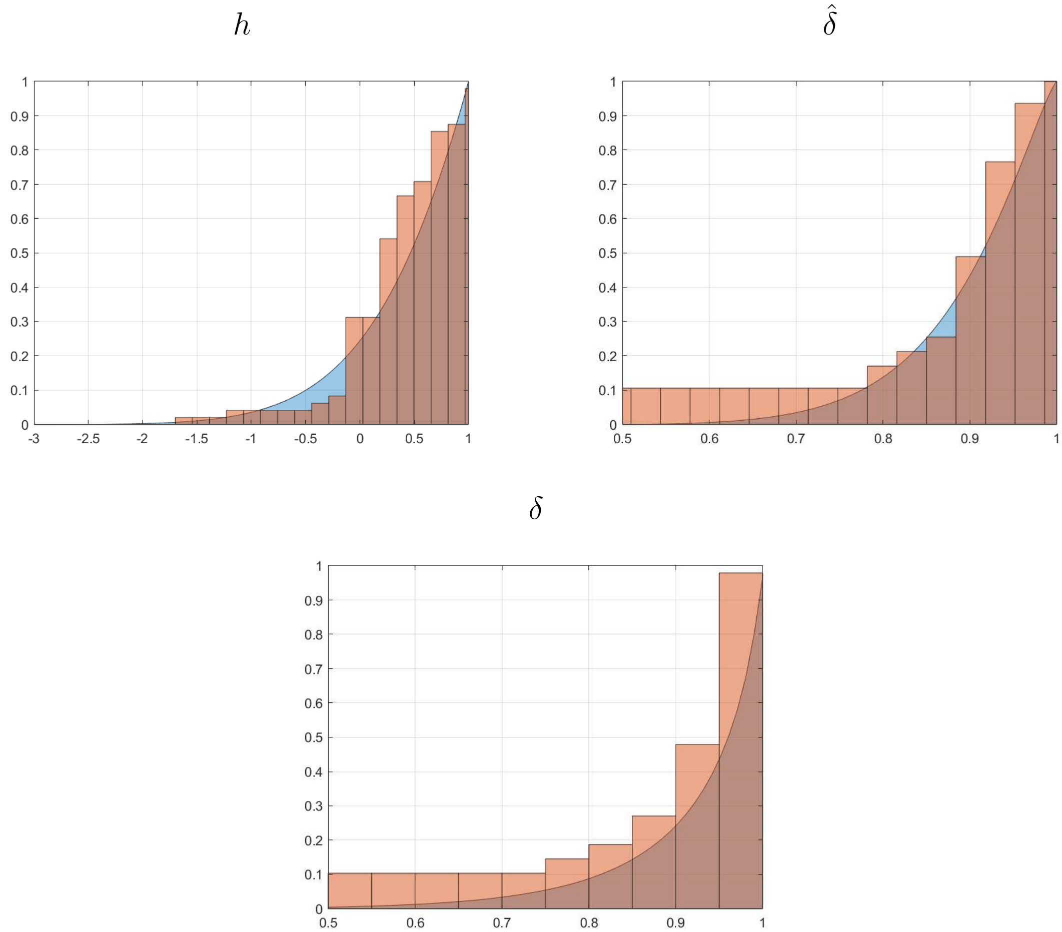
NOTES.- PDFs of the estimated distributions reported in Table 2. The PDF of the discount factor $\delta = \hat{\delta}^{1-h}$ is estimated from the distributions of risk and delay aversion using a normal kernel.

FIGURE 5. Individual Estimates: Coble and Lusk (2010)



NOTES.- Each point represents the median of the estimated distributions of the coefficient of risk aversion h and the corrected discount factor $\hat{\delta}$ for the subsample of choice data for a particular individual, following the estimation procedure reported in Table 2.

Figure 6. CDFs of Estimated Risk and Time Preferences and Histograms of Individual Estimates: Coble and Lusk (2010)



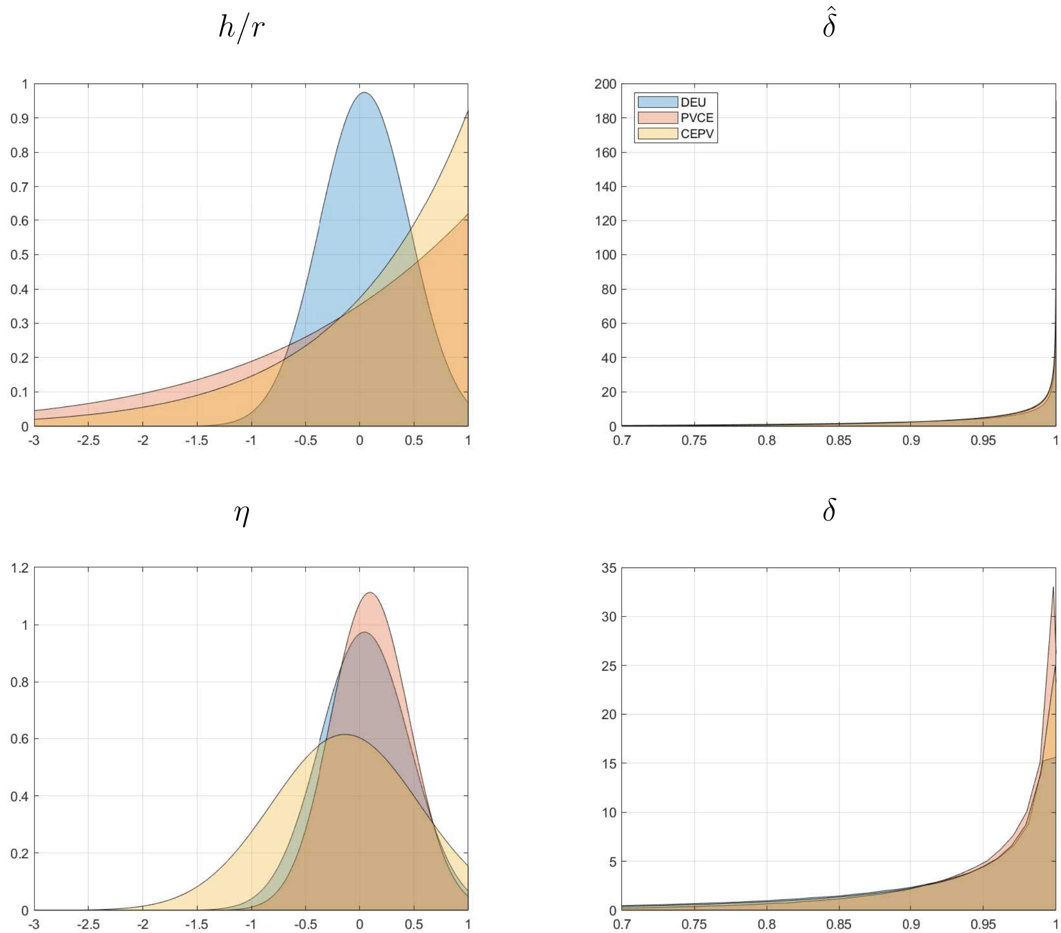
NOTES.- CDFs of the pooled estimation and histograms of the empirical distributions of the individual estimates.

Table 3. Estimated Risk and Time Preferences: Andreoni and Sprenger (2012b)

	DEU-H			PVCE-H			CEPV-H		
Dataset	**(i)**	**(ii)**	**(iii)**	**(iv)**	**(v)**	**(vi)**	**(vii)**	**(viii)**	**(ix)**
Median h/r	0.039	0.048	0.100	−0.064	−0.057	0.220	0.261	0.221	0.270
	[0.035]	[0.035]	(0.238)	[0.137]	[0.140]	(0.649)	[0.095]	[0.096]	(0.967)
Std. Dev. h/r	0.400	0.420	0.443	1.265	1.260	1.238	0.976	1.018	0.964
	[0.032]	[0.020]	(0.215)	[0.126]	[0.193]	(0.787)	[0.101]	[0.096]	(0.577)
Median $\hat{\delta}$	0.953	0.958	0.957	0.968	0.962	0.985	0.966	0.966	0.978
	[0.005]	[0.005]	(0.079)	[0.004]	[0.007]	(0.043)	[0.004]	[0.005]	(0.065)
Std. Dev. $\hat{\delta}$	0.113	0.135	0.055	0.078	0.122	0.094	0.085	0.102	0.142
	[0.008]	[0.012]	(0.060)	[0.009]	[0.019]	(0.083)	[0.010]	[0.014]	(0.091)
Median η	–	–	–	0.094	0.097	0.114	−0.175	−0.070	−0.076
				[0.030]	[0.033]	(0.223)	[0.062]	[0.034]	(0.216)
Std. Dev. η	–	–	–	0.351	0.375	0.315	0.615	0.431	0.796
				[0.025]	[0.035]	(0.191)	[0.059]	[0.026]	(0.453)
Corr(r,$\hat{\delta}$)	–	−0.601	−0.345	–	−0.333	−0.182	–	0.183	−0.177
		[0.046]			[0.143]			[0.160]	
Corr(r,s)	–	–	–	–	0.134	−0.006	–	−0.488	0.137
					[0.272]			[0.070]	
Corr(η,$\hat{\delta}$)	–	–	–	–	−0.539	−0.201	–	−0.574	−0.160
					[0.103]			[0.202]	
Log-Likelihood	−3.650	−3.602	−3.283	−1.858	−1.839	−1.382	−1.774	−1.751	−1.277

NOTES.- The above table reports the maximum-likelihood estimates of the median and the standard deviation of the distributions of risk aversion, corrected discount factor, and intertemporal substitution under different representations, using data from Andreoni and Sprenger (2012b). For each model, the first column shows the results from estimating the model using the pooled sample of observations. The second column shows the estimates obtained when allowing correlation between parameters using a Gaussian copula. Standard errors, shown in brackets, are computed using the delta method and clustered at the individual level. The third column shows the median and standard deviation (in parentheses) of the distribution of individual estimates of the respective parameter. In all cases, the coefficient of risk aversion, denoted h in DEU-H and r in PVCE-H and CEPV-H, is assumed to follow a normal distribution truncated at 1, the corrected discount factor $\hat{\delta}$ to follow a beta distribution, and the curvature of intertemporal substitution η to follow a normal distribution truncated at 1.

Figure 7. PDFs of Estimated Risk and Time Preferences: Andreoni and Sprenger (2012b)

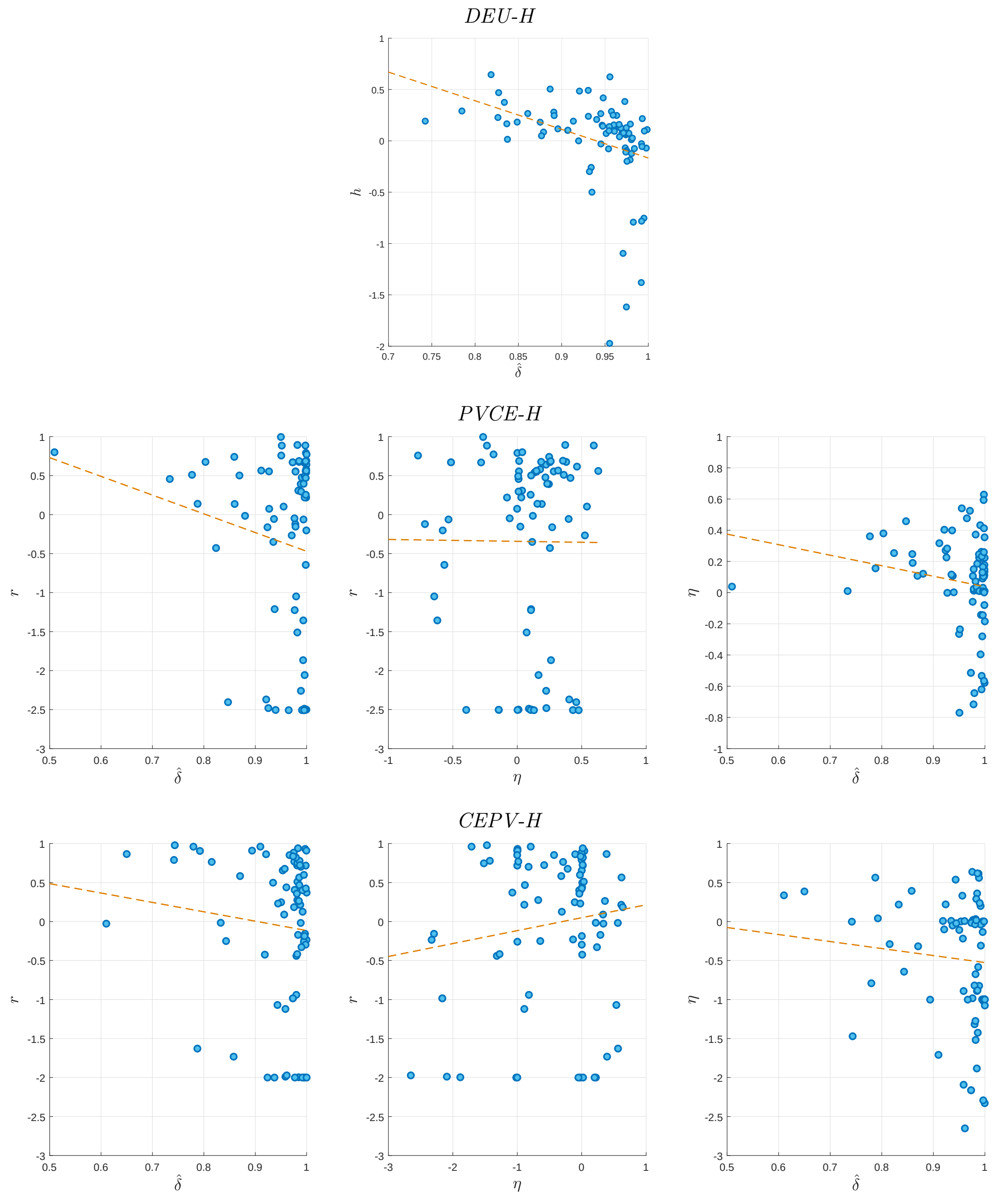


NOTES.- PDFs of the estimated distributions reported in Table 3. The PDF of the discount factor $\delta = \hat{\delta}^{1-\eta}$ is estimated non-parametrically from the distributions of intertemporal substitution and the corrected discount factor using a normal kernel.

Figure 8. Observed and Predicted Distributions of Choices in Andreoni and Sprenger (2012b)

NOTES.- Observed frequencies and predicted probabilities of choosing share α ($\times 100$) under each risk condition considered in Andreoni and Sprenger (2012b). The observed distributions show the relative frequency of each allocation in the data, grouped to the closest multiple of 10. The predicted distributions are computed based on the estimated parameters of the respective representation, as shown in Table 3.

FIGURE 9. Individual Estimates: Andreoni and Sprenger (2012b)



NOTES.- Each point represents a combination of the medians of the estimated distributions of the coefficients of risk aversion h/r, the corrected discount factor $\hat{\delta}$, and the curvature of intertemporal substitution η for the subsample of choice data for a particular individual, following the estimation procedure reported in Table 3.