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Dual decision processes and noise trading

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Abstract

Evidence from financial markets suggests that asset prices can be consistently far from their fundamental value. Prices seem to underreact to news in the short-run and overreact in the long-run. In this paper, we use Dual Process Theory to describe traders behavior. In particular, a part of traders holds *wrong* beliefs anytime the market environment does not change sufficiently. The proportion of traders with wrong beliefs will depend on how similar past market environments are with the present one. We show that such model not only can be seen as a way of endogenizing noise trading, but also provides a justification for noise traders' beliefs and it shows that underreaction and overreaction naturally arise in such framework. Finally, we discuss how the model might help understanding the emergence of the equity-premium puzzle and its variation through time.
(JEL G02, G11, G12)

Keywords: Asset Pricing, Dual Processes, Noise Trading, Underreaction, Overreaction, Equity-Premium Puzzle

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1 Introduction

Price movements depend on traders beliefs and how they use the information they have regarding different assets. Since Grossman and Stiglitz (1980), it has been debated in the literature whether markets can be fully informationally efficient, that is, are agents fully informed? If this is the case, their demand functions should include all the available information and thus, any variation in prices should be the consequence of unexpected noise as summarized in Fama (1970). That is, prices in a fully informationally efficient economy should follow a random walk. Here we propose a behavioral model that presents a channel through which markets might fail to be informationally efficient.

The hypothesis that economic agents in financial markets can be described by the rational model of decision making is in contrast with some of the evidence that has been documented in the last decades in the literature.[1] Asset prices move through time in ways that cannot be fully explained by movements in their fundamental values.

In this paper we model an economy in which markets are not fully efficient because of the presence of some traders, noise traders, that use *bad information*.[2] This idea is not new in the literature and it is present at least since the seminal work by De Long et al. (1990). The main and crucial difference is that we consider a particular cognitive process that makes the presence of noise traders to emerge endogenously due to the changes in information that agents face hence, in contrast with De Long et al. (1990), we explicitly model how noise traders form their beliefs. Having a clear model of beliefs formation allows us to show that the endogenous formation of noise traders helps explain the emergence of underreaction and momentum and also gives some interesting insights regarding overreaction and the equity premium puzzle. These phenomena are at odds with the efficient markets hypothesis because they imply that price movements can be partially predicted.

In section 2 we develop a model that follows Dual Process Theory, a prominent theory in cognitive sciences. Following such theory, human behavior can be seen as the outcome of the interaction between two systems. System 1 is associative and unconscious. Using analogies, it draws from past behavior to influence decisions. System 2 is analytic, conscious and demanding of cognitive capacity. Given it is costly, it is activated to solve the decision problem at hand only when past problems cannot be quickly used by System 1.[3] Traders here are described by these two systems. System 1 compares different market environments through a similarity function while System 2 can be seen as the standard rational model of decision making. The idea behind the model is quite straightforward. Traders receive information regarding present and future dividends

[1] See Shiller (1990, 2003, 2014).

[2] See DeLong (2005) for a discussion regarding agents that trade on bad information.

[3] See Kahneman (2011) for a general description of the theory.

generated by a risky asset and they have to decide how much of their income to invest in such asset. Whenever the information they receive is indistinguishable for their System 1, that is, present and future dividends are similar enough for them, where enough depends on a threshold, they do not update their beliefs. Thus, they trade using old information. Otherwise, System 2 is activated and they fully consider all the information present in the market and decide how much of the asset to buy.[4]

Noise traders arise because for some individuals in the population, the new information is *indistinguishable* from past one, and so they do not update their beliefs. By doing so they use outdated information that is not relevant in the new market situation. On the other hand, individuals that are able to perceive the change in information behave rationally. Thus, in any moment in time, there are two types of traders, noise traders and fully rational ones. Their proportions will depend on the magnitude of the variation in information and on how the similarity threshold, that determines what is *indistinguishable*, is distributed in the population. In general, if for an individual past and new information are similar enough, all traders with an even smaller threshold will consider the two pieces of information indistinguishable too while all others will behave rationally.

In section 3, in an overlapping generations model in which traders live two periods and that behave following the process we explained earlier, we show that in equilibrium prices of risky assets are not at their fundamental value and that, thanks to the endogenous formation of noise traders, their movements qualitatively reproduce empirical facts that have been documented in the literature. First, prices are more volatile because the presence of noise traders increases the overall risk of the economy. Prices vary due to changes in fundamentals in a direct way, as in the standard rational model, but also in an indirect way due to the change in the fraction of noise traders that such changes in fundamentals imply. These changes in noise trading are subject to uncertainty making rational traders unwilling to fully take advantage of arbitrage opportunities and thus prices vary more than what would happen in a rational framework. Second, prices underreact to changes in information because of the fact that at any moment in time there is always a fraction of traders, noise traders, that form their demand functions using *old* information, and thus do not react to the *new* information. Finally, prices can overreact to changes in information in the lung-run. Information gets gradually incorporated into prices due to underreaction, thus, under some circumstances, the effect of old information being incorporated in prices and new information becoming available can sum up and amplify the movement of prices thus causing overreaction.

Finally, in section 4.1 we discuss how analogical thinking can shed new light on the equity-premium puzzle. Since Mehra and Prescott (1985), there is evidence that investment in risky assets is too low given their returns with respect to riskless bonds and such behavior can be

[4] The model follows closely the one presented in Cerigioni (2015). See Glaser and Walther (2014) for experimental evidence of traders using the two described systems.

accommodated in the standard rational framework only by assuming that traders are extremely risk averse. As we previously explained, the kind of noise trading we model here has two effects on financial markets. First, it increases the risk and second it might cause prices to overreact in the long-run. These two effects imply that risky returns can often be higher than normal due to overreaction while on the other hand the increase in market risk implies that the risky asset is less demanded than in a rational economy, thus creating an intuitive channel for the puzzle to arise. Moreover, our model has some further implications that align with some of the empirical evidence presented in Mehra and Prescott (2003). In particular we show how the equity premium should move through time and we show that, if analogical thinking plays the role we describe, the equity premium should be expected to be counter-cyclical.

The remainder of the paper is organized as follows. In section 2 we present the model. Section 3 defines the equilibrium of the economy and provides the pricing function that describes the pricing of the risky asset for any moment in time. Finally, section 4 studies some implications of the model and 5 concludes. All proofs are in the appendix.

2 The Economy: OLG and Dual Processes

The basic structure of our model is taken from De Long et al. (1990). As in their formalization, we consider an overlapping generations model with two-period lived traders with no first period consumption, no labor supply decision, no bequests and resources to invest are exogenous. Similarly the economy contains only two assets. Asset s is riskless, it pays a dividend r in every t and its supply is perfectly elastic: a unit can be created out of, and a unit can be turned back into, a unit of consumption good in every period. The price of this asset, given consumption is the numeraire, is fixed at 1. Asset u is risky, it pays a dividend θ_t in every t, with θ_t defined as a random walk, that is:

$$\theta_t = \theta_{t-1} + \epsilon_t \text{ with } \epsilon_t \sim N(0, \sigma^2).$$

Its supply is inelastic, i.e. it is in fixed and unchangeable quantity normalized at one. The price of u in every t is denoted by p_t. Notice that this is a first difference with respect to De Long et al. (1990). In their model in fact there is no uncertainty concerning the dividends of the risky asset, riskiness of the asset is due to the fact that an *exogenous* proportion of traders holds wrong beliefs regarding the dividend that such asset can produce. As it will be clear from the description of the timing of our economy, also in our framework traders will have no uncertainty regarding dividends that are realized during their lives but uncertainty plays a role for two reasons. First, dividends that will be faced by future generations are uncertain. Second, and more important, the *endogenous* proportion of noise traders is uncertain.

At every moment in time t a continuum of traders of mass 1 is born. Every generation lives two periods. In the first period the traders perceive an exogenous labor income and they decide how many assets to buy in order to maximize their utility in the final period they live. Traders maximize the following mean-variance utility function:

$$\mathbb{E}_t[\omega] - \rho \mathbb{E}_t[\sigma_\omega^2]$$

Where $\mathbb{E}_t$ is the expectation operator at time t, ω is the wealth in the final period, ρ is the parameter measuring the absolute risk aversion and σ_ω^2 is the variance of wealth in the final period.[5] As we intuitively explained in the introduction, traders are described by Dual Process Theory. Each trader i's System 1 is endowed with a similarity function $\sigma : \mathbb{R} \times \mathbb{R} \rightarrow [0, 1]$ and a similarity threshold $\alpha_i \in [0, 1]$. Every time traders face the problem of deciding how much to invest in the risky asset, they can either use old information because the market environment is similar enough for their System 1, or they rationally adapt their behavior to the new environment using System 2. Finally, we assume that the distribution of the different similarity thresholds in every generation follows a continuous and time invariant distribution $F()$ with support $[0, 1]$ and density function $f()$.[6]

The timing of the economy is as follows. For every t:

- A generation of traders is born.
- The dividend θ_t is realized and publicly observed.
- A public and perfect signal of the dividend of asset u in period $t + 1$ is drawn, θ_{t+1}. This implies that there is no uncertainty regarding dividends in period t and $t + 1$.
- Every trader i maximizes his expected utility in period $t + 1$. The only heterogeneity among traders comes from the fact that System 1 determines which information they use to *forecast* the dividend of asset u in period $t + 1$.

  In fact:

  - If $\sigma(\theta_{t+1}, \theta_t) > \alpha_i$ trader i does not perceive the signal and θ_t is used as if it was the true one.

[5] With normally distributed returns to holding a unit of the risky asset, maximizing the previous function is equivalent to maximizing expected utility with a CARA utility function:

$$U = -e^{-(2\rho)\omega}$$

or a quadratic utility function.

[6] Notice that the analysis that follows would not be heavily affected by assuming that ρ is distributed in the population and that such distribution is time invariant and independent form the distribution of the similarity threshold α. We do not analyze this more general case because it would only make the exposition less clear.

  - If $\sigma(\theta_{t+1}, \theta_t) \leq \alpha_i$ then trader i perceives the signal and the utility is maximized with the correct information.

- Asset u is bought at a price p_t that clears the market.[7]

- The old generation of traders consumes all its wealth and dies.

Thus, traders are boundedly rational. Whenever the realized dividend and the signal are *similar enough* a fraction of traders do not update their beliefs and do not take new information into account.[8] This is a way of describing a process in which traders tend to think as if past trends will continue in the future. This is in line with the evidence shown in Greenwood and Shleifer (2014) and Barberis et al. (2015) where traders seem to rely heavily on past performances of assets to predict their future profitability. Moreover, this mechanism as we were saying in the introduction allows for the endogenization of the fraction of *noise traders*. In fact, in every period t, given the realized dividend and signal, a fraction μ_t of traders will use past information to decide how much of the risky asset to buy. Such fraction will depend on how the similarity threshold is distributed in the population. In fact:

$$\mu_t = \int_0^{\alpha^*} f(\alpha) d\alpha = F(\alpha^*)$$

where α^* is such that:

$$\sigma(\theta_{t+1}, \theta_t) = \alpha^*$$

That is, in every t, a proportion μ_t of traders finds the realized dividend and the signal to be similar enough, i.e. indistinguishable, and thus, they do not update their beliefs.

As previously said, we call traders that use the old information, i.e. those traders which decision is driven by System 1, *noise traders*, while the remaining part of the population will be composed by *rational traders*, i.e. those traders which decision is driven by System 2. Then the individual maximization problems are as follows. A noise trader in t maximizes the following:[9]

$$\max_{\lambda_t^N} \mathbb{E}_t^N[\omega] - \rho \mathbb{E}_t[\sigma_\omega^2]$$

That is:

$$\max_{\lambda_t^N} c_0 + \lambda_t^N[\theta_t + \mathbb{E}_t^N(p_{t+1}) - p_t(1+r)] - \rho(\lambda_t^N)^2 \mathbb{E}_t^N(\sigma_{p_{t+1}}^2)$$

Where c_0 is a function of the exogenous labor income, $\mathbb{E}_t^N$ stands for the expectation operator at time t given the information noise traders use and λ_t^N is the quantity of asset u demanded in t by

[7] Following De Long et al. (1990) we allow quantities and prices to be negative.

[8] This is simplification of a much more general model of behavior. The reader should see this model as a way of formalizing the whole market situation where traders receive many and different pieces of information the understanding and usage of which might be cognitively overwhelming.

[9] Note that N stand for noise traders while R for rational traders.

noise traders. This is a concave problem with the following necessary and sufficient FOC:

$$\theta_t + \mathbb{E}_t^N(p_{t+1}) - p_t(1+r) - 2\rho\lambda_t^N \mathbb{E}_t^N(\sigma^2_{p_{t+1}}) = 0$$

That gives the following demand function:

$$\lambda_t^N = \frac{\theta_t + \mathbb{E}_t^N(p_{t+1}) - p_t(1+r)}{2\rho\mathbb{E}_t^N(\sigma^2_{p_{t+1}})}$$

On the other hand, a rational trader in t maximizes the following:

$$\max_{\lambda_t^R} \mathbb{E}_t^R[\omega] - \rho\mathbb{E}_t[\sigma^2_\omega]$$

That is:

$$\max_{\lambda_t^R} c_0 + \lambda_t^R[\theta_{t+1} + \mathbb{E}_t^R(p_{t+1}) - p_t(1+r)] - \rho(\lambda_t^R)^2\mathbb{E}_t^R(\sigma^2_{p_{t+1}})$$

The necessary and sufficient FOC is the following:

$$\theta_{t+1} + \mathbb{E}_t^R(p_{t+1}) - p_t(1+r) - 2\rho\lambda_t^R \mathbb{E}_t^R(\sigma^2_{p_{t+1}}) = 0$$

That leads to:

$$\lambda_t^R = \frac{\theta_{t+1} + \mathbb{E}_t^R(p_{t+1}) - p_t(1+r)}{2\rho\mathbb{E}_t^R(\sigma^2_{p_{t+1}})}$$

Notice that the two types of traders face very similar problems except for the fact that they use different information. Noise traders use the realization of the *dividend* today to forecast dividends tomorrow, on the other hand, rational traders use the realization of the *signal* to decide how much of the risky asset to buy.

3 The Equilibrium

In equilibrium the demand of the risky asset has to be equal to its supply. That is, formally, it must be, for every t:

$$\mu_t\lambda_t^N + (1-\mu_t)\lambda_t^R = 1$$

Thus, from market clearing we get:

$$\mu_t\frac{\theta_t + \mathbb{E}_t^N(p_{t+1}) - p_t(1+r)}{2\rho\mathbb{E}_t^N(\sigma^2_{p_{t+1}})} + (1-\mu_t)\frac{\theta_{t+1} + \mathbb{E}_t^R(p_{t+1}) - p_t(1+r)}{2\rho\mathbb{E}_t^R(\sigma^2_{p_{t+1}})} = 1$$

That gives the following pricing function:

$$p_t = \frac{\mu_t(\theta_t + \mathbb{E}_t^N(p_{t+1}) - 2\rho\mathbb{E}_t^N(\sigma^2_{p_{t+1}}))2\rho\mathbb{E}_t^R(\sigma^2_{p_{t+1}}) + (1-\mu_t)(\theta_{t+1} + \mathbb{E}_t^R(p_{t+1}) - 2\rho\mathbb{E}_t^R(\sigma^2_{p_{t+1}}))2\rho\mathbb{E}_t^N(\sigma^2_{p_{t+1}})}{(1+r)[\mu_t 2\rho\mathbb{E}_t^R(\sigma^2_{p_{t+1}}) + (1-\mu_t)2\rho\mathbb{E}_t^N(\sigma^2_{p_{t+1}})]}$$

Now, to ease the reading, we use the following notation that stresses the information the different traders are using:

- $\mathbb{E}_t^N(p_{t+1}) = P_{t+1}(\theta_t)$.
- $\mathbb{E}_t^R(p_{t+1}) = P_{t+1}(\theta_{t+1})$.
- $\mathbb{E}_t^N(\sigma^2_{p_{t+1}}) = V_{t+1}(\theta_t)$.
- $\mathbb{E}_t^R(\sigma^2_{p_{t+1}}) = V_{t+1}(\theta_{t+1})$.

Thus, the previous pricing function becomes as follows:

$$p_t = \frac{\mu_t 2\rho V_{t+1}(\theta_{t+1})(\theta_t + P_{t+1}(\theta_t) - 2\rho V_{t+1}(\theta_t)) + (1-\mu_t)2\rho V_{t+1}(\theta_t)(\theta_{t+1} + P_{t+1}(\theta_{t+1}) - 2\rho V_{t+1}(\theta_{t+1}))}{(1+r)[\mu_t 2\rho V_{t+1}(\theta_{t+1}) + (1-\mu_t)2\rho V_{t+1}(\theta_t)]}$$

If we define the relative variance of the price due to rational traders' expectations as follows:

$$\beta_t = \frac{\mu_t 2\rho V_{t+1}(\theta_{t+1})}{\mu_t 2\rho V_{t+1}(\theta_{t+1}) + (1-\mu_t)2\rho V_{t+1}(\theta_t)}$$

We can write the previous expression as:

$$p_t = \frac{1}{1+r}\left[\beta_t(\theta_t + P_{t+1}(\theta_t) - 2\rho V_{t+1}(\theta_t)) + (1-\beta_t)(\theta_{t+1} + P_{t+1}(\theta_{t+1}) - 2\rho V_{t+1}(\theta_{t+1}))\right] \qquad (1)$$

The price in t is a convex combination of the utilities the two types of traders expect to get in $t+1$ from holding a unit of the risky asset. The utility a trader gets from holding the asset in t is the dividend he believes the asset will pay, plus its selling value, that is the expected price in $t+1$, minus the expected variance of the price that negatively affects traders' utility. Then, such utility is weighted by the relative frequency of the type in the market multiplied by the variance of the price the other type expects for $t+1$. Thus, in a way, a certain trader type's expectation is more important in driving the price today, the higher is the measure of that type in the economy and the less variant his prediction is with respect to the other type.

3.1 A Consistent Pricing Function

Equation 1 highlights the importance of understanding traders' expectations to obtain a closed-form solution of the model. We assume that traders form their expectation rationally except for the fact that they use different informations. That is, they form expectations knowing that there can be two types of traders in the economy, that their proportions depend on the realized dividend and signal, but they are boundedly rational in the sense that they do not use such information to understand from the price in t whether their decisions are driven by System 1 or System 2.

To obtain the pricing function then we first need to define how $P_{t+1}(\theta_t), P_{t+1}(\theta_{t+1}), V_{t+1}(\theta_t)$ and $V_{t+1}(\theta_{t+1})$ are formed. First notice that, in general, given the information in t the two types of traders will have different expectations regarding the price in the next period, that is:

$$P_{t+1}(\theta_t) \neq P_{t+1}(\theta_{t+1})$$

On the other hand, given the process generating the dividends, the expected variance of prices should not depend on the reference the different traders use. In fact the price can vary due to the signal in $t+1$ and the fraction of noise traders in $t+1$. These two factors only depend on the noise term that realizes with the signal in $t+1$ that is independent of the reference. Hence:

$$V_{t+1}(\theta_t) = V_{t+1}(\theta_{t+1}) = V_{t+1}$$

Given this, we can rewrite the pricing function as follows:

$$p_t = \frac{1}{1+r}\left[\mu_t(\theta_t + P_{t+1}(\theta_t)) + (1-\mu_t)(\theta_{t+1} + P_{t+1}(\theta_{t+1})) - 2\rho V_{t+1})\right]$$

Then, given that traders use the actual pricing function to form their expectations as we said in the beginning of this section, we get:

$$P_{t+1}(\theta_t) = \mathbb{E}_t\left[\frac{1}{1+r}\left[\mu_{t+1}(\theta_t + P_{t+2}(\theta_t)) + (1-\mu_{t+1})(\theta_t + \epsilon_{t+2} + P_{t+2}(\theta_t + \epsilon_{t+2})) - 2\rho V_{t+2}\right]\right]$$

$$P_{t+1}(\theta_{t+1}) = \mathbb{E}_t\left[\frac{1}{1+r}\left[\mu_{t+1}(\theta_{t+1} + P_{t+2}(\theta_{t+1})) + (1-\mu_{t+1})(\theta_{t+1} + \epsilon_{t+2} + P_{t+2}(\theta_{t+1} + \epsilon_{t+2})) - 2\rho V_{t+2}\right]\right]$$

Now notice that a particular trader will form in t the same expectation for all future periods prices and variances given that dividends follow a random walk. Thus expectations have a closed form solution and so it is possible to show the following.

Proposition 1 *The price of the risky asset at time t is defined by the following equation:*

$$p_t = \frac{1}{r}\left[\theta_{t+1} - \mu_t\epsilon_{t+1} - \frac{2\rho}{(1+r)^2}(\sigma_\epsilon^2 + \sigma_{\epsilon\mu}^2 - 2\gamma_{\epsilon^2\mu})\right] - \frac{1}{(1+r)r}\gamma_{\epsilon\mu}, \tag{2}$$

where $\sigma_{\epsilon\mu}^2$ is the variance of the product between the noise and the fraction of noise traders in the market, $\gamma_{\epsilon^2\mu}$ is the covariance between the distance between two consecutive dividends and the *proportion of noise traders and, similarly, $\gamma_{\epsilon\mu}$ is the covariance between the noise and the fraction of noise traders in the market.*

Thus, the price today is a function of the signal of future dividends, θ_{t+1}, the fraction of noise traders in the economy that do not perceive the signal, i.e. μ_t, the variance due to the random walk of dividends, i.e. σ_ϵ^2, the variance due to noise trading, i.e. $\sigma_{\epsilon\mu}^2$, and finally the covariance between the noise ϵ and the fraction of noise traders μ, that is $\gamma_{\epsilon\mu}$. Notice that whenever the similarity function is symmetric, e.g. when it is the inverse of a distance function, $\gamma_{\epsilon\mu}$ must be zero. In fact, if the similarity function is symmetric, negative or positive realizations of the noise ϵ will have the same effect on the similarity comparisons, only their absolute value is of importance. Hence, a symmetric similarity implies that positive or negative noises of the same magnitude have the same effect on μ, thus making the covariance equal to zero.

We leave the interpretation and discussion of equation (2) to the next section. We will just consider the case of a symmetric similarity function for two reasons. First, it is the most sensible assumption on the similarity function in our framework and second, it increases the clarity of exposition.

4 Discussion: Underreaction and Overreaction

First, it is useful to stress an evident characteristic of the model.

Remark 1 *Prices are not at their fundamental value.*

This is immediate to see once we consider a fully rational economy. That is, if traders were fully rational we should have the following pricing function:

$$p_t^* = \frac{1}{r}\left[\theta_{t+1} - \frac{2\rho}{(1+r)^2}\sigma_\epsilon^2\right] \tag{3}$$

The interpretation of such equation is straightforward and quite standard. The price today is the present value of future utility gains from holding the asset. That is, it is the present value of the difference between the expected dividends and the variance of the dividends due to the noise in the dividend generating process.

Less trivial is to understand whether prices can be greater than their fundamental value. When we take the difference between equation (2) and equation (3) we get:

$$\frac{1}{r}\left[-\mu_t\epsilon_{t+1} - \frac{2\rho}{(1+r)^2}(\sigma_{\epsilon\mu}^2 - 2\gamma_{\epsilon^2\mu})\right]$$

This difference can be non-negative only if:

$$-\epsilon_{t+1} \geq \frac{1}{\mu_t}(\frac{2\rho}{(1+r)^2}(\sigma^2_{\epsilon\mu} - 2\gamma_{\epsilon^2\mu})$$

First, a necessary condition for the previous inequality to be satisfied is that ϵ_{t+1} is non-positive. This is in line with the intuition of the model. That is, a bubble can only appear whenever noise traders do not realize that intrinsic value of the asset due to future dividends is lower than the one they use to form their demand functions. Bubbles emerge when noise traders are bullish. They think good past dividend realizations will continue in the future and they do not notice that the market environment is worse than what they think. In a way, anytime the noise is negative, we should expect noise traders to be too optimistic about the future. It is a stylized way of representing the idea of *animal spirits* that have been extensively analyzed in the literature, see for example Shiller (2003), Akerlof and Shiller (2010) and Shiller (2015). Second the difference between the dividend and the signal has to be big enough to offset the depressive effect noise traders have on the economy. This is not a trivial trade-off. The farther away are the dividend and the signal, the smaller is the proportion of noise traders in the economy and thus, the bigger is the right-hand side. Which effect is preponderant depends on the distribution function of the similarity threshold in the economy and so, given the generality of the framework we analyze here, we do not study it in further detail. The next corollary highlights why the presence of noise traders can be depressive for prices.

Corollary 1 *Noise traders increase the risk in the economy.*

Imagine our simple economy without noise traders. As previously said, the price would be the discounted utility gain of holding the risky asset and the risk traders would bear would depend only on the variance of such price due to the random walk dividends follow. On the other hand, in the kind of economy we model here the presence of noise traders, in line with the literature, increases the overall risk of the market. Future demand for the risky asset is even more uncertain because the distance between the realizations of dividends and signals make the proportion of noise traders to vary. Such added uncertainty means that prices are more volatile than what the rational benchmark would imply. This is in line with a lot of evidence that have been documented in the literature in particular by Shiller, see for example Shiller (1992). The evidence shows that prices in stock markets are too volatile to be explained by a rational asset pricing model and that the movement in prices can somehow be predicted in contrast with the idea that, if markets are rational, prices should follow a random walk. Here we suggest that such volatility is due to noise trading as in De Long et al. (1990) but with one important difference. In De Long et al. (1990) noise traders increase the risk of the economy because their beliefs are random. Here, we provide a mechanism through which noise trading emerges and beliefs are formed. Noise traders beliefs are not random. Nevertheless, noise traders increase the risk of the economy because their proportion varies with the realization of the signals. Depending on the information available in the economy and how it compares to past one, noise traders can be more or less present in the economy. Thus, their proportion is ex-ante uncertain. Moreover, given the endogenous formation of noise traders, our model has different implications regarding how prices move. As we show in the following results and in section 4.1, the fact that noise traders are present due to similarity between market environments has some testable implications that are novel and worth studying further. The first one concerns underreaction of prices to news, that is, new information is only partially incorporated into prices.

Corollary 2 *Prices underreact to changes in information in the short-run.*[10]

Thus, the signal influences the price but in a *milder* way with respect to what should happen in a fully rational economy. The signal is not fully incorporated in the price. This is in line with evidence that has been shown in the literature, see for example the seminal paper by Cutler et al. (1991) and Bernard (1993). We here propose a novel explanation of underreaction as a consequence of analogical thinking. Due to their bounded rationality traders can misperceive the changes in the market environment thus they might have demand functions based on *old* information, causing the prices to be sticky. Prices do not adapt immediately because a portion of traders use past realizations of the dividends to forecast future ones due to the fact that they do not perceive the change in the market environment. Another way to see this is that prices show *momentum*. In fact, momentum in the literature is often defined as the slow incorporation of information into prices. That is, given the information in t, the price change between $t+1$ and t can be partially predicted beacuse the price in $t+1$ will incorporate also the information that was not considered in t due to underreaction.

Corollary 3 *Prices show momentum.*

Here we are in line with the literature. Prices show momentum because of underreaction. Again, the novelty is that this is due to the presence of analogical thinking. When noise traders buy the asset they use the *wrong* belief that past performance will repeat in the future. Once they have to sell the asset to the new generation, the price of the asset will correct due to the fact that the actual realized dividend is different than the one that was used to form their beliefs. To put it differently, due to underreaction in the short-run, prices do not incorporate all the information present in the market, thus, they readjust once the information realizes. That is, prices incorporate

[10] Throughout the paper we will refer to *short-run* when analyzing an intra period result and to *long-run* when analyzing an inter-period result.

fundamental value slowly through time. This is in line with the evidence in Jegadeesh and Titman (1993) and Chan, Jegadeesh and Lakonishok (1996) among others, that shows that prices have a *momentum pattern*, that is they slowly drift toward the fundamental value. Thus, price changes can be partially predicted. What is of interest here is that the interplay between momentum and underreaction has an additional implication for the predictability of price movements in the long-run. In our economy prices can overreact in the long-run.

Overreaction since De Bondt and Thaler (1985) is defined as the negative covariance between returns in the long-run. That is, prices tend to overreact to the information present in the market in the long-run and then they adjust toward the fundamental value, creating the negative covariance. It turns out that it is indeed the case in our model.

Corollary 4 *Prices overreact to changes in information in the long-run.*

To understand why this is the case, it is useful to study the difference between two prices in successive periods in our model and in the rational benchmark respectively. That is:

$$\Delta_t(p) = p_{t+1} - p_t = \frac{1}{r}[\epsilon_{t+2} + (\mu_t \epsilon_{t+1} - \mu_{t+1}\epsilon_{t+2})]$$

$$\Delta_t(p^*) = p^*_{t+1} - p^*_t = \frac{1}{r}\epsilon_{t+2}$$

The term in parenthesis is the one that can cause overreaction, a phenomenon studied and documented at least since De Bondt and Thaler (1985).[11] It is interesting to first analyze the two components separately. The first term depends on the underreaction of the price in t to news in t. It is highlighting the fact that past news, i.e. ϵ_{t+1}, affect future prices. By Corollary 3, prices incorporate slowly past information. Thus, the change in price from period t to period $t+1$ can be greater than the one we would have in a rational framework because past information can make the shift in price due to new information even more accentuated. Thus, prices can overreact, in the sense that the change in price from period t to period $t+1$ can be higher than what would be justified by the change in information. Obviously, overreaction depends on the second term in parenthesis that describes underreaction in $t+1$ due to news in $t+1$. Notice that underreaction in $t+1$ is key for the understanding of Corollary 4. The less prices underreact to a change in information in $t+1$, the higher the returns, due to the fact that momentum and new information play together amplifying the movement in prices. But then, the less the underreaction in $t+1$, the less the effect of momentum on the successive price change and so the lower the returns. This creates negative covariance between price changes in two successive periods.

To conclude the section, before analyzing the implications of our model for the equity premium

[11] See Lee and Swaminathan (2000) for empirical evidence of prices underreacting in the short-run and overreacting in the long-run.

puzzle, we think it is important to underline how the model generating our results is different than the ones suggested in Barberis, Shleifer and Vishny (1998), Daniel, Hirshleifer and Subrahmanyam (1998) and Hong and Stein (1999). The three papers provide different models that encompass underreaction and overreaction of prices. Barberis, Shleifer and Vishny (1998) assume that a representative investor, not knowing the true process generating the earnings of an asset, has wrong beliefs about the movements in the asset earnings. In particular they assume that the representative trader thinks that either earnings are mean reverting or they trend. Similarly, Daniel, Hirshleifer and Subrahmanyam (1998) assume that traders are overconfident about the precision of private information and their confidence moves asymmetrically because of biased self-attribution of investment outcomes. Finally, Hong and Stein (1999) assume that the economy is composed by two types of traders, news watchers that receive different pieces of information regarding the assets that slowly diffuse in the market, and momentum traders that trade based on past price changes. While all these papers can explain underreaction and overreaction, they all do so by abandoning the rational framework, that is, there is no space for rational traders in their economies. On the other hand in our model both rational and less than rational traders coexist making our model closer to the standard framework while at the same time proposing a new way of thinking about analogical reasoning and its impact on trading behavior.

4.1 Equity-Premium Puzzle and Analogical Thinking

In the previous subsection we have discussed some of the implications of analogical thinking in financial markets. Here we go further, and study the implications of such cognitive process for the equity premium puzzle. Since De Long et al. (1990), noise traders risk has been seen as a possible source of the equity premium puzzle. In our framework, the same argument that has been used in the literature can be applied, we do not try to provide a new explanation for such puzzle. Nonetheless, we think that understanding the process that makes noise trading emerge can help having a better grasp of how and when we should expect the equity-premium puzzle to arise and be stronger. In fact, we here discuss how the mechanism we propose can shed new light on this phenomenon and on its empirical regularities.

If the increase of the overall risk of the economy due to noise traders is big enough, risk averse traders will demand less of the risky asset for any given dividend than what they would demand if the economy was fully rational. This implies that the riskless asset is overdemanded with respect to the rational benchmark. This is a stylized way of understanding the equity premium puzzle. Since Mehra and Prescott (1985), the literature has tried to understand why risky assets are less demanded than what rational models would imply. All the different explanations that have been proposed to solve the puzzle had to depart from the standard rational framework by assuming for example habits in consumption as in Constantinides (1990). What we propose here is that the inherent risk that noise traders represent for the economy might be enough to offset the potential gains rational traders can obtain due to the erroneous beliefs noise traders have. Noise traders make the demand of the risky asset vary *too much* due to the fact that their proportion varies with the economy, thus amplifying the inherent risk of the market and making non-profitable for rational traders to *bet* too much on the risky asset. The *animal spirits* present in financial markets make them too risky, thus depressing the demand. Thus, a higher variability of the price implies two things. First it can make returns higher, as explained in the analysis of result 4, but second it depresses the demand of risk averse traders, thus giving an alternative way of understanding the equity-premium puzzle.

What is of interest here is analyzing the impact that the introduction of analogical reasoning has on the understanding of the equity premium. In order to to this, it is interesting to look again at a price change in t:

$$\Delta_t(p) = p_{t+1} - p_t = \frac{1}{r}[\epsilon_{t+2} + (\mu_t \epsilon_{t+1} - \mu_{t+1}\epsilon_{t+2})]$$

As we said, overreaction depends on the term in parenthesis. Thus, to get overreaction of prices we need two conditions to be met; (1) News in t and in $t+1$ have the same sign, that is they are both positive noises or negative noises, and (2) in absolute terms, the effect of underreaction in t was bigger than the effect of underreaction in $t+1$, taking into account the marginal impact on the fraction of noise traders. The first implication is in line with empirical evidence, see for example Kaestner (2006). In fact, prices seem to overreact more after periods of news of the same sign, that is positive or negative. Thus we study the case of two consecutive signals of the same sign. The second implication is equivalent to the following:

$$|\mu_t \epsilon_{t+1}| \geq |\mu_{t+1}\epsilon_{t+2}|.$$

We analyze the case when both ϵ_{t+1} and ϵ_{t+2} are positive, the other possibility is symmetric. Clearly the satisfaction of the previous condition depends on the interplay between the change in information, i.e. ϵ, and the change in the fraction of noise traders, i.e. μ. Thus, it depends on the particular distribution of the similarity threshold in the population. Nonetheless, we are still able to say something for any general distribution that satisfies our assumptions. Notice in fact that the previous inequality is equivalent to the following:

$$\frac{\epsilon_{t+1}}{\epsilon_{t+2}} \geq \frac{\mu_{t+1}}{\mu_t}$$

Whenever ϵ_{t+2} is small enough, prices overreact. In fact, the limit of the left hand side when ϵ_{t+2} goes to zero is infinite, while the right hand side converges to $\frac{1}{\mu_t}$ given that, due to analogical reasoning, the closer are the dividend and the signal, the more people will trade on noisy information. Thus, we should expect prices to overreact in the lung-run in those situations where past news where relatively more *surprising* than new ones. This can explain even more clearly how bubbles can arise in the long-run when we take analogical thinking into consideration. Periods of high performances of the market should make prices overreact in the long-run creating the possibility of bubbles. This happens because traders do not perceive that the market environment is changing and is less performing than what they believe. On the other hand, when ϵ_{t+2} is big relative to ϵ_{t+1} the implications of the model are less clear and depend on the distribution of the similarity threshold in the population. Nevertheless, in the extreme case when ϵ_{t+2} tends to infinity, both sides of the inequality converge to zero, which means that in the limit there is no overpricing.

Thus prices overreact when market are bullish, so, in those circumstances, the equity premium should be expected to be higher. This happens when the price of the asset is overreacting and a relative downturn should be expected. Even if this intuition is counter-intuitive following the standard economic analysis, it is in line with empirical evidence, as in Mehra and Prescott (2003), where the equity premium seems to be counter-cyclical. In our simplified framework, we can interpret the dividends of the risky asset as the performance of the economy overall. Then, our result regarding overreaction of prices of the asset, is saying that returns should be expected to be high when the economy is expected to slow down and can be low when the opposite happens. That is, whenever ϵ_{t+2} is small enough relative to ϵ_{t+1}, returns should be high and thus, given the depressing effect of noise traders on the demand of the risky asset, the equity premium should be larger. On the other hand, when the opposite happens, that is ϵ_{t+2} is big enough with respect to ϵ_{t+1}, returns should be lower and so the equity premium smaller.

5 Final Remarks

In this paper we use Dual Process Theory to describe trading behavior in an overlapping generations economy. Whenever present and future dividend realizations are similar enough, traders do not update their beliefs and use old information to decide how much of a risky asset to buy. Otherwise, if dividend are sufficiently dissimilar, traders update their beliefs thus behaving rationally. We show that, whenever the similarity threshold that defines what is similar enough for our traders is continuously distributed in the population, the equilibrium pricing function of such economy can accommodate many empirical regularities by allowing prices to be far from their fundamental value. First, prices are more volatile than what would be normal in a rational framework, second they underreact in the short-run to changes in information and third they overreact in the long-run. Finally, we show how such implications can shed new light on the equity-premium puzzle.

Clearly the model is an oversimplified version of the markets it is supposed to study. It would be interesting to expand the model to the possibility of having more than one risky asset and thus simulate the behavior of the economy to see whether it can reproduce some of the regularities that the empirical distributions of asset prices show. We leave such possibilities for future research.

To conclude, even if stylized, the model we propose gives interesting insights on some empirical facts. In particular we have showed that considering analogical thinking can be useful if we want to understand different puzzling phenomena in a coherent theory of human behavior where *rational* and *less than rational* behaviors can coexist.

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A Appendix

A.1 Proofs

Proof of Proposition 1. First notice that, given dividends follow a random walk, the information in t does not make it possible for traders in t to distinguish between the expected prices of future generations, that is:

$$P_{t+2}(\theta_t) = P_{t+2}(\theta_t + \epsilon_{t+2})$$

$$P_{t+2}(\theta_{t+1}) = P_{t+2}(\theta_{t+1} + \epsilon_{t+2})$$

Similarly, we must have:

$$P_{t+1}(\theta_t) = P_{t+2}(\theta_t) = ... = P_{t+n}(\theta_t) = P(\theta_t) \text{ for any } n > 1$$

$$P_{t+1}(\theta_{t+1}) = P_{t+2}(\theta_{t+1}) = ... = P_{t+n}(\theta_{t+1}) = P(\theta_{t+1}) \text{ for any } n > 1$$

$$V_{t+1} = V_{t+2} = ... = V_{t+n} = V \text{ for any } n > 1$$

So, solving recursively the previous equations we get:

$$P(\theta_t) = \frac{1}{r}\left(\theta_t - 2\rho V - \gamma_{\epsilon\mu}\right) \tag{4}$$

$$P(\theta_{t+1}) = \frac{1}{r}\left(\theta_{t+1} - 2\rho V - \gamma_{\epsilon\mu}\right) \tag{5}$$

Given all these results, we can solve for the expected variance of price. That is:

$$V = \mathbb{E}_t\left[\left(\frac{1}{1+r}(\theta_t + P(\theta_t) + \epsilon_{t+2} - \epsilon_{t+2}\mu_{t+1} - 2\rho V - \theta_t - P(\theta_t) + \gamma_{\epsilon\mu} + 2\rho V)\right)^2\right]$$

Thus, we get:

$$V = \mathbb{E}_t\left[\left(\frac{1}{1+r}((\epsilon_{t+2} - 0) + (\sigma_{\epsilon\mu} - \epsilon_{t+2}\mu_{t+1}))\right)^2\right]$$

Which gives the following result:

$$V = \frac{1}{(1+r)^2}(\sigma_\epsilon^2 + \sigma_{\epsilon\mu}^2 - 2\gamma_{\epsilon^2\mu}) \tag{6}$$

We can use all the previous result to get the final pricing function:

$$p_t = \frac{1}{r}\left[\theta_{t+1} + \mu_t(\theta_t - \theta_{t+1}) - \frac{2\rho}{(1+r)^2}(\sigma_\epsilon^2 + \sigma_{\epsilon\mu}^2 - 2\gamma_{\epsilon^2\mu})\right] - \frac{1}{(1+r)r}\gamma_{\epsilon\mu}$$

Notice that equation (2) given the random walk assumption is equivalent to the following:

$$p_t = \frac{1}{r}\left[\theta_{t+1} - \mu_t\epsilon_{t+1} - \frac{2\rho}{(1+r)^2}(\sigma_\epsilon^2 + \sigma_{\epsilon\mu}^2 - 2\gamma_{\epsilon^2\mu})\right] - \frac{1}{(1+r)r}\gamma_{\epsilon\mu}$$

and the result follows. ■

Proof of Corollary 1. Notice that the variance of prices in a rational economy is:

$$V^* = \frac{1}{(1+r)^2}\sigma_\epsilon^2$$

On the other hand, the variance in the economy described in the paper is:

$$V = \frac{1}{(1+r)^2}(\sigma_\epsilon^2 + \sigma_{\epsilon\mu}^2 - 2\gamma_{\epsilon^2\mu})$$

Thus:

$$V - V^* = \frac{1}{(1+r)^2}(\sigma_{\epsilon\mu}^2 - 2\gamma_{\epsilon^2\mu})$$

which is positive given that $\sigma_{\epsilon\mu}^2$ is positive by definition and $\gamma_{\epsilon^2\mu}$ must be negative. In fact, the higher the distance between dividend and signal, the lower the proportion of noise traders in the economy. Thus the result follows. ■

Proof of Corollary 2. This is easy to see when we study the marginal impact of a change in information in t, *ceteris paribus.* Clearly in our simple model a change in information in t is represented by a change in the signal. Any change in the signal is immediately reflected in the price of the asset in the rational benchmark. On the other hand, in our model, given $f()$ is a continuous density function with full support, there will always be noise traders in the market, making the price less responsive to changes in information. This is even clearer if we rewrite equation (2) as follows:

$$p_t = \frac{1}{r}\left[\theta_t + (1-\mu_t)\epsilon_{t+1} - \frac{2\rho}{(1+r)^2}(\sigma_\epsilon^2 + \sigma_{\epsilon\mu}^2 - 2\gamma_{\epsilon^2\mu})\right] - \frac{1}{(1+r)r}\gamma_{\epsilon\mu}$$

■

Proof of Corollary 3. Define the price change from t to $T+1$ as follows:

$$\Delta_t(p) = p_{t+1} - p_t = \epsilon_{t+2} + (\mu_t\epsilon_{t+1} - \mu_{t+1}\epsilon_{t+2})$$

Then, the expected price change at t is:

$$\mathbb{E}_t(\Delta_t(p)) = \mathbb{E}_t(\epsilon_{t+2} + (\mu_t\epsilon_{t+1} - \mu_{t+1}\epsilon_{t+2})) = \mu_t\epsilon_{t+1},$$

and the result follows. ■

Proof of Corollary 4. We show that the corollary is true by showing that the covariance between two successive price changes conditional on the information in t is negative. First notice that:

$$Cov(\Delta_t(p), \Delta_{t+1}(p)) = \mathbb{E}_t(\Delta_t(p)\Delta_{t+1}(p)) - \mathbb{E}_t(\Delta_t(p))\mathbb{E}_t(\Delta_{t+1}(p)),$$

and that:

$$\mathbb{E}_t(\Delta_{t+1}(p)) = \mathbb{E}_t(\frac{1}{r}[\epsilon_{t+3} + (\mu_{t+1}\epsilon_{t+2} - \mu_{t+2}\epsilon_{t+3})] = 0.$$

Thus:

$$Cov(\Delta_t(p), \Delta_{t+1}(p)) = \mathbb{E}_t(\Delta_t(p)\Delta_{t+1}(p)).$$

That is:

$$Cov(\Delta_t(p), \Delta_{t+1}(p)) = \mathbb{E}_t(\frac{1}{r^2}(\mu_{t+1}\epsilon_{t+2}^2 - \mu_{t+1}^2\epsilon_{t+2}^2)),$$

which implies

$$Cov(\Delta_t(p), \Delta_{t+1}(p)) = \frac{1}{r^2}(\gamma_{\epsilon^2\mu} - \gamma_{\epsilon^2\mu^2}).$$

Clearly $\gamma_{\epsilon^2\mu}$ and $\gamma_{\epsilon^2\mu^2}$ are negative, and moreover, given μ is smaller than one, it must be that $|\gamma_{\epsilon^2\mu}| > |\gamma_{\epsilon^2\mu^2}|$ and the result follows. ■