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Abstract

What is the role of monetary policy in a bubbly world? To address this question, we study an economy in which financial frictions limit the supply of assets. The ensuing scarcity generates a demand for "unbacked" assets, i.e., assets that are backed only by the expectation of their future value. We consider two types of unbacked assets: bubbles, which are created by the private sector, and money, which is created by the central bank. Bubbles and money share many features, but they also differ in two crucial respects. First, while the rents from the creation of bubbles accrue to entrepreneurs and foster investment, the rents from money creation accrue to the central bank. Second, while bubbles are driven by market psychology, and can rise and fall according to the whims of the market, money is under the control of the central bank. We characterize the optimal monetary policy and show that, through its ability to supply assets, monetary policy plays a key role in the bubbly world. The model sheds light on the recent expansion of central bank liabilities in response to the bursting of bubbles.

JEL classification: E32, E44, O40

Keywords: bubbles; financial frictions; optimal monetary policy; liquidity trap.

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1 Introduction

We live in a world of low interest rates and volatile asset values. Over the last three decades, real interest rates have declined continuously throughout the western world, having been in negative territory throughout much of the last decade. Although there are different views as to the ultimate cause of this decline, most agree that it reflects some form of asset scarcity, i.e., an increased demand for assets that raises their price thereby depressing returns. Over these same three decades, Japan, the United States, and parts of the Eurozone have all exhibited large booms and busts in asset prices with significant consequences for economic activity. The long slump that has characterized Japan since the early 1990s is commonly interpreted as the result of the collapse of real estate and equity bubbles. In the United States, the recent recession has also been associated with the development and subsequent burst of a large bubble in the real estate and equity markets. In the same vein, some of the most severe recessions experienced in the Eurozone in the aftermath of the global financial crisis – such as those in Ireland and Spain – have coincided with the bursting of real estate bubbles.

How should monetary policy be conducted in this bubbly world? Most of the debate surrounding this question has focused on the ability of interest rate policy to prevent or control the appearance and growth of bubbles. But there is an alternative aspect of policy that was central to all of these episodes: the collapse of asset prices was accompanied by a fall into a liquidity trap and a substantial growth of central bank balance sheets. As the value of private assets evaporated, market participants turned to central banks for stores of value, which supplied them by expanding the monetary base (mostly in the form of reserves, but also through cash). In the United States, for instance, the monetary base grew fivefold, from approximately $880 billion in 2008 to a peak of $4.1 trillion in 2014. The Eurozone's monetary base experienced a similar expansion, from approximately €800 billion in 2008 to €3.5 trillion in 2018. This suggests that a key aspect of monetary policy in dealing with bubbles and their aftermath has been precisely to supply stores of value. Yet this role is completely absent in the New-Keynesian paradigm that has dominated monetary economics over the last few decades, which emphasizes the role of money as a unit of account and of nominal rigidities as drivers of monetary transmission (e.g., Galí (2009) and Woodford (2011)).

Without denying the usefulness of this paradigm, the events outlined above suggest that a shift in perspective may be called for. In a bubbly world (i.e., a world of low interest rates and large booms and busts in asset prices), we can no longer disregard the role of money as a store of value, and the role of monetary policy as a supplier of stores of value. This raises a number of fundamental questions. When is money valuable as a store of value? How is this value connected to the rise and bursting of bubbles? Can the central bank always supply stores of value? If so, how much should it supply? This paper develops a framework to address these questions.

To be sure, we are not the first to think of money as a store of value, as there is an old tradition in economics of adopting this perspective (e.g., Wallace (1981)). We build on that tradition and its insights, but combine it with the recent literature on asset bubbles and financial frictions (e.g., Martin and Ventura (2018)). The framework that emerges provides a unified view on the connection between low interest rates, booms and busts in asset prices, and the role of money as a store of value. In particular, it enables us to study the similarities and differences between money and bubbles, their interaction, and the conduct of monetary policy in a bubbly world. In order to make the results as stark as possible, we completely neglect the role played by money as a unit of account and by nominal rigidities emphasized in the literature. Nonetheless, as we shall see, monetary policy retains a powerful role.

The main elements of this view are easily explained. Consider an overlapping-generations world, in which some agents (entrepreneurs) want to borrow because they have productive investment opportunities, and other agents (savers) want to save because they do not have them. Normally, entrepreneurs would supply "backed" assets, i.e., assets backed by the fruits of their investment, and savers would demand these assets as stores of value. But what if financial frictions restrict the supply of backed assets? Since savers still demand stores of value, these frictions depress the interest rate and open the door for "unbacked" assets to be issued, i.e., assets that are backed only by expectations of their future value. Unbacked assets can be thought of as pyramid schemes, in which present contributions to the schemes (present purchases of the asset) give the right to receive future contributions (future purchases of the asset): as long as the expected return to these assets or pyramid schemes is sufficiently high, agents will be willing to hold them in equilibrium.

The dynamics of unbacked assets are driven by two forces, with differing effects on economic activity. First, their creation generates a wealth effect. New unbacked assets generate a rent for their creators because they are costless to produce and yet they have positive market value. For example, if an entrepreneur issues debt that is unbacked because the market expects it to be rolled over indefinitely, then she receives a pure rent. Second, the existence of unbacked assets generates an overhang effect. Old unbacked assets must be purchased and this diverts resources that could have been used for productive investment. In our example, the savers that actually finance the roll-over of the entrepreneur's debt must divert their funds from other uses. In short, the wealth effect of unbacked assets depends on their issuer, whereas the overhang effect depends on their buyer.

We consider two types of unbacked assets. The first one is created by private entrepreneurs, and we refer to it as a bubble. The second one is created by a public central bank, and we refer to it as money. Money can be valued as an asset if its expected rate of return (the inverse of the rate of inflation) equals that of the market (the real interest rate): in other words, if the economy is in a liquidity trap. Both bubbles and money have wealth and overhang effects, but they differ in two crucial respects. First, bubbles are superior to money because their wealth effect accrues to entrepreneurs and, as a result, they foster investment. The wealth effect of money (i.e., seigniorage) accrues instead to the central bank, and its effect depends on how it is distributed among economic agents. Second, bubbles are inferior to money because they are driven by market psychology, so that their behavior may be volatile and unpredictable, whereas money is under the control of the central bank. Ultimately, nothing guarantees that the size of bubbles supplied by the market will be optimal.

We analyze the role of monetary policy in this bubbly world. To constrain the role of policy as much as possible, we restrict the central bank's actions along three key dimensions. First, we assume that the central bank lacks fiscal backing, so that it cannot use tax proceeds to back the value of money. Second, we assume that the central bank is constrained in its use of seigniorage, so that it cannot choose how to distribute this revenue between entrepreneurs and savers. Finally, we assume that the central bank cannot affect market psychology, so that it must take the evolution of private bubbles as given. Despite these restrictions, we show that monetary policy has the ability to supply unbacked assets and – through this ability – plays a powerful role.

We find two main results. First, we show that the central bank *can* always intervene in the bubbly world, adjusting the money supply to provide unbacked assets over and above those supplied by private bubbles. Should it choose to do so, moreover, we show that the central bank can fully stabilize the economy's total supply of unbacked assets at a target of its choice! The choice of target is constrained by the market psychology that governs private bubbles, however. In particular, the central bank can only add to – but not substract from – the unbacked assets supplied by private bubbles, and it is limited to implementing policies that guarantee non-explosive paths for these bubbles.

Second, we show that the central bank *should* intervene in the bubbly world. We characterize an optimal monetary policy that – by adjusting the supply of unbacked assets – raises the consumption of all generations along all histories. To derive this result, we identify sequences of unbacked assets that are Pareto optimal, in the sense that – given any such sequence – it is impossible to raise the consumption of any one generation without reducing it for some other generation. The intuition here is the familiar one, i.e., unbacked assets are beneficial insofar as their overhang effect crowds out dynamically inefficient investments. By adjusting its supply of unbacked assets, monetary policy can ensure Pareto optimality.

In a nutshell, the main takeaway of the theory is that central banks have a key role to play in the bubbly world: to supply assets. This resonates well with the conduct of monetary policy in the wake of the global financial crisis, when central banks expanded their balance sheets in response to the bursting of bubbles. But it is also quite different. In a standard balance sheet expansion, the central bank issues some assets and purchase others, leaving the net supply of assets available to the private sector unchanged. In the bubbly world, instead, the key aspect of welfare-enhancing interventions is that they expand the net supply of assets available to the private sector, thereby soaking up inefficient investment.

Finally, the theory also clarifies how the central bank's ability to supply assets is shaped by different aspects of its institutional design. First, although the absence of fiscal backing does not limit the central bank's ability to supply assets, it may lead to volatile inflation. The reason is that the central bank can always expand the supply of real balances when there is demand for unbacked assets, but – without fiscal backing – the only way of reducing real balances when there is no such demand is by inflating their value away. Thus, if inflation volatility is costly, so is asset-supply stabilization in the absence of fiscal backing. Second, the distribution of seigniorage is crucial in shaping the aggregate effects of monetary policy. In our benchmark, bubbles outperform monetary policy because the central bank cannot target the distribution of seigniorage to entrepreneurs. If it could do so, seigniorage could be used to foster investment and monetary policy could potentially replicate the effects of private bubbles. Somewhat paradoxically, we show that this might come at the cost of creating the volatility typically associated with bubbles. Finally, the interaction between monetary policy and market psychology is central. Our benchmark shows that monetary policy is powerful even when the central bank cannot affect market psychology: if it could do so, monetary policy would be even more powerful in the sense that it could prevent "undesirable" bubbles altogether.

1.1 Related literature

We build on the traditional models of rational bubbles (Samuelson, 1958; Tirole, 1985) and on the recent literature connecting bubbles and credit (Caballero and Krishnamurthy, 2006; Farhi and Tirole, 2011; Martin and Ventura, 2012, 2015, 2016; Miao and Wang, 2012, 2018). Although we share many similarities with these frameworks, these papers do not study monetary policy.

A small literature has considered the role of monetary policy in economies with rational bubbles, but it abstracts from credit and financial frictions (Galí, 2014, 2017). We are thus among the first to provide a model of rational bubbles, credit cycles and monetary policy.[1]

Our work is also closely related to the literature on the "financial accelerator," in which borrowers' net worth in general - and asset prices in particular - determine the level of financial intermediation and economic activity (Bernanke and Gertler, 1989; Kiyotaki and Moore, 1997). Building on this literature, some recent work has shown that balance sheet policies by the central bank can help alleviate financial frictions during periods of financial stress (Gertler and Kiyotaki, 2010; Gertler and Karadi, 2011; Del Negro et al., 2017). Our paper differs from this body of work in two key dimensions. First, in our theory net worth and asset prices are not just a transmission channel for fundamental shocks: instead, they are driven by expectations and can therefore be a source of shocks themselves. Second, in these models the central bank expands its balance sheet to increase lending to constrained agents, or to sustain asset prices and thus the value of their collateral. We emphasize, instead, a different role for balance sheet policies. In our theory, in fact, due to financial frictions the economy's supply of assets is scarce, leading to inefficient investment. Against this background, the central bank expands its balance sheet to increase the economy's stock of assets and to reduce inefficient investment. We thus show that a balance sheet expansion by the central bank might increase welfare, even if it does not lead to an increase in borrowers' net worth or in the value of their collateral.

Our paper also contributes to the vast literature on liquidity traps (Krugman, 1998; Eggertsson and Woodford, 2003; Di Tella, 2018). In particular, it is related to the work identifying financial shocks as the source of liquidity traps (Eggertsson and Krugman, 2012; Guerrieri and Lorenzoni, 2012). Different from existing work, we provide a framework where financial shocks arise due to changes in expectations. Our paper is also closely related to the literature on secular stagnation (Hansen, 1939; Summers, 2013; Krugman, 2013), which is the idea that structural factors can depress interest rates and generate long-lasting liquidity traps. A recent literature has formalized the secular stagnation hypothesis in microfounded frameworks (Benhabib et al., 2001; Eggertsson and Mehrotra, 2014; Caballero and Farhi, 2014; Benigno and Fornaro, 2015; Bacchetta et al., 2015). We contribute to this literature by showing that a long-lasting liquidity trap can be the outcome of a bubble crash, which depresses the economy's supply of assets.

[1] In contemporaneous work, Wang et al. (2017) also study the role of monetary policy in a world with rational bubbles. However, they do not consider the central bank's role as a supplier of stores of value.

2 A model of money and bubbles

We develop an overlapping-generations model with bubbles and money. Entrepreneurs invest and raise funds by issuing debts and bubbles. The former are backed by future output, while the latter are not. Savers purchase these assets, and they might also hold money if inflation is low enough. The focus of the model is on the interaction between money and bubbles. Both are unbacked assets, but they differ in two crucial aspects. First, bubbles are issued by the entrepreneurs, while money is issued by the central bank. Second, there is an arbitrarily small but stable demand for money, while there is no such demand for bubbles.

2.1 Setup

We consider an economy populated by overlapping generations of individuals that live for two periods. All generations have the same size, which is normalized to one. Time is discrete and infinite, $t \in \{0, ..., \infty\}$. This economy does not experience technology or preference shocks, but it displays stochastic equilibria with asset price and monetary policy shocks. We define h_t as the realization of these shocks in period t; h^t as a history of shocks until period t, that is, $h^t = \{h_0, h_1, ..., h_t\}$; and H_t as the set of all possible histories until period t.

The technology to produce goods takes the standard Cobb-Douglas form:

$$Y_t = \left(\gamma^t \cdot L_t\right)^{1-\alpha} \cdot K_t^{\alpha}, \tag{1}$$

with $\alpha \in (0, 1)$, where Y_t, L_t and K_t denote the output, the labor force and the capital stock in the economy. Labor productivity grows at rate $\gamma > 1$. Each generation supplies one unit of labor during youth, so that $L_t = 1$. Capital fully depreciates in one period. To produce one unit of capital for period $t + 1$, one unit of goods must be invested in period t. Factor markets are competitive so that all factors are employed and paid their marginal products:

$$W_t = (1 - \alpha) \cdot \gamma^{(1-\alpha)\cdot t} \cdot K_t^{\alpha} \tag{2}$$

$$R_t^K = \alpha \cdot \gamma^{(1-\alpha)\cdot t} \cdot K_t^{\alpha-1} \tag{3}$$

where W_t and R_t^K are the wage and the rental rate, respectively.

Each generation contains three agent types: entrepreneurs, savers and money holders. All agents maximize old-age consumption:

$$U_t^i = E_t C_{t+1}^i, \tag{4}$$

where U_t^i and C_{t+1}^i are the utility and consumption of agent i of generation t.[2] Thus, young agents save their entire income and choose those assets that offer the highest expected return.

All agents are endowed with one unit of labor when young, but they differ in terms of the assets that they can hold and issue. Both entrepreneurs and savers participate in the financial market. However, while entrepreneurs can hold capital and issue private assets in the financial market, savers can do neither. For technical reasons, namely to generate a positive demand for money, we assume that there is an arbitrarily small share of money holders $\upsilon \approx 0$, who can neither hold capital nor participate in the financial market.[3] Throughout, we assume that the share of entrepreneurs ε is smaller than the share of savers, i.e. $\varepsilon < \frac{1}{2} \cdot (1 - \upsilon) \approx \frac{1}{2}$.

Entrepreneurs issue two types of assets. The first one is debt backed by the return to a unit of capital. Creating these debts costs a fraction ϕ of their promised return. Thus, the return to these debts is given by $\frac{R_{t+1}^K}{1+\phi}$. One interpretation of the intermediation cost ϕ is that there is some need to monitor or screen entrepreneurs (see Section 2.4 for a formal microfoundation). The second type of asset is unbacked and we refer to it as a bubble. Let B_t be the value of all bubbles started by earlier generations of entrepreneurs, i.e., "old" bubbles. Let N_t be the value of all bubbles started by the current generation of entrepreneurs, i.e., "new" bubbles. Free-disposal implies that old and new bubbles must be non-negative: $B_t \geq 0$ and $N_t \geq 0$. The return to holding all bubbles from t to $t+1$ is given by $\frac{B_{t+1}}{B_t+N_t}$. To see this, note that the value of all bubbles traded in period t is $B_t + N_t$ in period t, and B_{t+1} in period $t+1$.

There is a central bank that issues unbacked money. Let M_t be the *real* value of money in period t. Let μ_{t+1} be the (gross) growth rate of nominal money from t to $t+1$. Thus, the return to holding money from t to $t+1$ is given by:

$$\pi_{t+1}^{-1} = \mu_{t+1}^{-1} \cdot \frac{M_{t+1}}{M_t} \tag{5}$$

where π_{t+1} is the (gross) inflation rate from period t to period $t+1$. Seigniorage (or the value of "new" money printed in period t) is given by $\frac{\mu_t - 1}{\mu_t} \cdot M_t$. The rest is "old" money printed in earlier periods. New money is distributed lump-sum among the old. Since the central bank does not have the ability to tax, seigniorage must be non-negative, i.e., $\mu_t \geq 1$ for all t and h^t.

[2] All variables are indexed by h^t. We could be more explicit about this dependence by writing $C_{t+1,h^{t+1}}^i$, but we prefer to streamline the notation and omit the history sub-index.

[3] All monetary models introduce a small demand for money. This is often done by including money in the utility function, or by adding a cash-in-advance constraint. One can interpret our money holders in one of these two ways. Or one can interpret them as agents that have a high cost of participating in the financial market.

2.2 The value of money and bubbles: three regimes

Let R_{t+1} be the real interest or the required expected return to assets in the financial market. Then, the value of bubbles and money is determined as follows:

$$B_t + N_t = \frac{1}{R_{t+1}} \cdot E_t B_{t+1}, \tag{6}$$

$$M_t = \max \left\{ \upsilon \cdot W_t, \frac{1}{R_{t+1}} \cdot E_t \left\{ \frac{M_{t+1}}{\mu_{t+1}} \right\} \right\}. \tag{7}$$

Equation (6) says that the value of bubbles today is their expected value tomorrow discounted. Equation (7) says that, if the expected value of money tomorrow discounted is large enough, savers hold money and its value exceeds the savings of money holders: $M_t > \upsilon \cdot W_t$. In this case, $E_t \pi_{t+1}^{-1} = R_{t+1}$, and we say that the economy is inside the liquidity trap. If instead the expected value of money tomorrow discounted is not large enough, savers do not hold money and $M_t = \upsilon \cdot W_t$. In this case, $E_t \pi_{t+1}^{-1} < R_{t+1}$, and the economy is outside the liquidity trap.

This economy can be in one of three regimes depending on the value of unbacked assets. In the first regime, unbacked assets fall short of the combined savings of money holders and savers, i.e., $M_t + B_t + N_t < (1 - \varepsilon) \cdot W_t$. Savers hold backed assets and, possibly, also some unbacked assets. Entrepreneurs invest their wage plus the proceeds from selling assets. Thus, we have that:

$$K_{t+1} = \varepsilon \cdot W_t + N_t + \frac{(1 - \varepsilon) \cdot W_t - M_t - B_t - N_t}{1 + \phi}, \tag{8}$$

$$R_{t+1} = \frac{R_{t+1}^K}{1 + \phi}. \tag{9}$$

The capital stock equals the wage of the entrepreneurs plus the revenue raised by selling assets. Since bubbles are costless to issue, they generate one unit of capital per unit sold. Since debts are costly to issue, they generate only $(1 + \phi)^{-1}$ units of capital per unit sold. Since the marginal buyer of unbacked assets is a saver, the required return to unbacked assets is $\frac{R_{t+1}^K}{1+\phi}$.

In the second regime, unbacked assets equal the combined savings of money holders and savers, i.e., $M_t + B_t + N_t = (1 - \varepsilon) \cdot W_t$. In this regime, savers hold only unbacked assets. Entrepreneurs do not hold unbacked assets and invest their wage plus the proceeds from selling bubbles. Thus, we have that:

$$K_{t+1} = \varepsilon \cdot W_t + N_t, \tag{10}$$

$$R_{t+1} \in \left[\frac{R_{t+1}^K}{1+\phi}, R_{t+1}^K \right]. \tag{11}$$

The capital stock equals the wage of the entrepreneurs plus the revenue from selling bubbly assets. The value of unbacked assets has grown large enough to crowd out all debts. Since the marginal buyer of unbacked assets depends on the direction of the change, the required return to unbacked assets is now higher than $\frac{R_{t+1}^K}{1+\phi}$, but lower than R_{t+1}^K.

In the third regime, unbacked assets exceed the combined savings of money holders and savers, i.e. $M_t + B_t + N_t > (1-\varepsilon) \cdot W_t$. In this regime, both savers and entrepreneurs hold unbacked assets. Thus, we have that:

$$K_{t+1} = W_t - M_t - B_t, \tag{12}$$

$$R_{t+1} = R_{t+1}^K. \tag{13}$$

The capital stock equals the wage of the entrepreneurs, plus the revenue from selling bubbles minus their purchases of money and bubbles. The value of unbacked assets has grown so large that it does not only crowd out all debts, but it also crowds out some investments financed by the entrepreneurs themselves. Since the marginal buyer of unbacked assets is now an entrepreneur, the required return to unbacked assets is R_{t+1}^K.

We summarize this discussion as follows:

$$K_{t+1} = \min \left\{ \varepsilon \cdot W_t + N_t + \frac{(1-\varepsilon) \cdot W_t - M_t - B_t - N_t}{1+\phi}, W_t - M_t - B_t \right\}, \tag{14}$$

$$R_{t+1} \begin{cases} = \frac{R_{t+1}^K}{1+\phi} & \text{if } M_t + B_t + N_t < (1-\varepsilon) \cdot W_t, \\ \in \left[\frac{R_{t+1}^K}{1+\phi}, R_{t+1}^K \right] & \text{if } M_t + B_t + N_t = (1-\varepsilon) \cdot W_t, \\ = R_{t+1}^K & \text{if } M_t + B_t + N_t > (1-\varepsilon) \cdot W_t. \end{cases} \tag{15}$$

Equations (14)-(15) show the evolution of the capital stock and the real interest rate as a function of factor prices and the value of unbacked assets.

We define a competitive equilibrium of this economy as an initial condition K_0 and a non-negative sequence for $\left\{ W_t, R_t^K, B_t, N_t, M_t, \mu_t, K_t, R_t \right\}$ that satisfies Equations (2)-(3), (6)-(7) and (14)-(15) and $K_t > 0$ and $\mu_t \geq 1$ for all t and $h^t \in H_t$. This guarantees that all individuals maximize and all markets clear in all periods and histories.

It is worth here to say a few words about the sources of uncertainty and, therefore, the collection of sets H_t (one for each t) that need to be considered. In period t, all individuals know $\{B_t, N_t, M_t, \mu_t, K_t\}$. It follows from Equations (2)-(3) and (14)-(15) that K_{t+1} must be known also.[4] There are however, two potential sources of uncertainty that individuals could face. The first one is the value of bubbles and money tomorrow, i.e., $\{B_{t+1}, N_{t+1}, M_{t+1}\}$. As we shall see, under some parameter conditions, there are random sequences for these variables that satisfy Equations (6)-(7). When this is the case, we say that there are asset price shocks since individuals cannot perfectly forecast the value of future bubbles and money. The second potential source of uncertainty is future money growth, i.e., $\{\mu_{t+1}\}$. Monetary policy could be uncertain because it responds to asset price shocks. Monetary policy could also add additional sources of uncertainty. When this is the case, we say that there are monetary policy shocks since individuals cannot perfectly forecast future money growth *conditional* on asset price shocks. Thus, the collection of sets H_t must be specified for each competitive equilibrium.

2.3 Constructing competitive equilibria

To study the dynamics of this economy, we define k_t as the capital stock in efficiency units, i.e., $k_t \equiv \gamma^{-t} \cdot K_t$. Also, we define b_t, n_t and m_t as the value of bubbles and money as a share of aggregate wages, i.e., $b_t \equiv \frac{B_t}{W_t}$, $n_t \equiv \frac{N_t}{W_t}$ and $m_t \equiv \frac{M_t}{W_t}$. By using these de-trended variables, we make the system stationary.

A useful property of this model is that we can construct competitive equilibria recursively. Substituting Equations (2)-(3) and (14)-(15) into Equations (6)-(7), we obtain the following system of equations:

$$b_t + n_t \begin{cases} = [1 - m_t - b_t + \phi \cdot (\varepsilon + n_t)] \cdot \frac{1-\alpha}{\alpha} \cdot E_t b_{t+1} & \text{if } \frac{m_t + b_t}{1-\varepsilon-n_t} < 1, \\ \in \left[(1+\phi) \cdot (\varepsilon + n_t) \cdot \frac{1-\alpha}{\alpha} \cdot E_t b_{t+1}, (\varepsilon + n_t) \cdot \frac{1-\alpha}{\alpha} \cdot E_t b_{t+1}\right] & \text{if } \frac{m_t + b_t}{1-\varepsilon-n_t} = 1, \\ = (1 - m_t - b_t) \cdot \frac{1-\alpha}{\alpha} \cdot E_t b_{t+1} & \text{if } \frac{m_t + b_t}{1-\varepsilon-n_t} > 1, \end{cases} \tag{16}$$

$$m_t = \lim_{\upsilon \to 0} \max \{\upsilon, m_t^*\}, \tag{17}$$

[4]Note that the factor prices and the real interest rate are functions of K_t and t only. Thus, it is enough to focus on $\{B_t, N_t, M_t, \mu_t, K_t\}$.

where m_t^* is given as follows:

$$m_t^* \begin{cases} = [1 - m_t^* - b_t + \phi \cdot (\varepsilon + n_t)] \cdot \frac{1-\alpha}{\alpha} \cdot E_t\left\{\frac{m_{t+1}}{\mu_{t+1}}\right\} & \text{if } \frac{m_t^*+b_t}{1-\varepsilon-n_t} < 1, \\ \in \left[(1+\phi)\cdot(\varepsilon+n_t)\cdot\frac{1-\alpha}{\alpha}\cdot E_t\left\{\frac{m_{t+1}}{\mu_{t+1}}\right\}, (\varepsilon+n_t)\cdot\frac{1-\alpha}{\alpha}\cdot E_t\left\{\frac{m_{t+1}}{\mu_{t+1}}\right\}\right] & \text{if } \frac{m_t^*+b_t}{1-\varepsilon-n_t} = 1, \\ = (1 - m_t^* - b_t)\cdot\frac{1-\alpha}{\alpha}\cdot E_t\left\{\frac{m_{t+1}}{\mu_{t+1}}\right\} & \text{if } \frac{m_t^*+b_t}{1-\varepsilon-n_t} > 1. \end{cases} \quad (18)$$

Note that, from now on, we focus on the limiting case in which $\upsilon \to 0$. When money balances are only held by money holders, we will abuse language and say that $m_t = 0$, even though it should be understood that the real value of money in our setting is always positive.

The key observation is that the capital stock is not present in Equations (16)-(17)-(18) and, as a result, the model has a recursive structure. Any non-negative sequence $\{b_t, n_t, m_t, \mu_t\}$ such that $m_t + b_t < 1$ and $\mu_t \geq 1$ is part of a competitive equilibrium if it satisfies Equations (16)-(17)-(18) for all t and h^t.[5] Let $\mathcal{E}$ be the set that contains all such sequences. The first step to construct a competitive equilibrium is to pick a sequence from this set.

The second step is to determine the implications of the chosen sequence for the capital stock and consumption. It follows from Equations (2) and (14) that:

$$k_{t+1} = \max\left\{\frac{1 - m_t - b_t + \phi\cdot(\varepsilon + n_t)}{1+\phi}, 1 - m_t - b_t\right\} \cdot \frac{1-\alpha}{\gamma} \cdot k_t^\alpha, \quad (19)$$

$$c_t = [\alpha + (m_t + b_t)\cdot(1-\alpha)]\cdot k_t^\alpha \quad (20)$$

where $c_t \equiv \gamma^{-t} \cdot C_t$ and C_t is the average consumption at time t.[6] For any sequence $\{b_t, n_t, m_t, \mu_t\}$, Equations (19)-(20) describe the evolution of the capital stock and consumption from any initial condition $k_0 > 0$.

Unbacked assets have two key effects, as shown by Equations (19)-(20). The first one is an *intra*generational transfer equal to the supply of new bubbles that young entrepreneurs sell to young savers, i.e., n_t. This "wealth" effect of bubble creation transfers resources from savers to entrepreneurs, raising the capital stock. Note that money creation also has a wealth effect that transfers resources to the central bank in the form of seigniorage: given our assumption on the distribution of seigniorage to the old, however, it does not affect capital accumulation. The second effect is an *inter*generational transfer equal to the supply of money and old bubbles that the old sell to young savers, i.e., $m_t + b_t$. This "overhang" effect is the same for money and bubbles, and – by transferring resources from savers to the old – it raises consumption by

[5]The condition $m_t + b_t < 1$ is equivalent to $k_{t+1} > 0$.

[6]Formally, $C_t = R_t^K \cdot K_t + M_t + B_t$, since the old finance their consumption with their capital income, plus the proceeds of selling their (old plus new) money, and their (old) bubbles.

reducing the capital stock. Most of the analysis that follows can be understood in terms of these two transfers or effects. Their relative strength determines whether unbacked assets have an expansionary or a contractionary effect on capital accumulation.

2.4 Some remarks on the modeling approach

Before moving on, we make a few remarks on the nature of bubbles for those readers that remain skeptical about this modeling strategy; convinced readers may skip to the next section.

A bubble is nothing but a pyramid scheme. Participants in a pyramid scheme make a voluntary contribution to the scheme that gives them right to the next voluntary contribution. Even though this might seem quite abstract or exotic at first sight, it is easy to find real-world situations that correspond fairly well to this concept. Consider, for instance, the case of a firm with a stock market value that exceeds the value of the capital that it employs. Although such a valuation could be deemed "excessive," it is perfectly rational if agents expect the firm's valuation to be excessive in the future as well. In this case, a firm's excessive valuation can be interpreted as a voluntary contribution to the firm that gives the right to the next voluntary contribution. This leads us to think of real-world firms as portfolios of capital and bubbles. And this is exactly what entrepreneurs in our model stand for: in fact, with minor modifications, our model can be easily recast as a model of stock market bubbles.

To see this, consider first a simple microfoundation for the financial friction that we assumed. In particular, suppose that each entrepreneur has access to an unbounded pool of investment projects or ideas. Each project in this pool allows an entrepreneur to convert a unit of the consumption good into a unit of productive capital. The projects, however, are of heterogeneous quality, denoted by $\theta \in \{L, H\}$. The low (L) quality projects introduce an agency problem in that they allow the entrepreneur to divert all of their output for private consumption, whereas the high (H) quality projects do not.

Before investing, entrepreneurs observe project qualities and choose which projects to undertake. Savers, however, can only find out a project's quality by paying ϕ units of the consumption good, a form of costly screening or monitoring. It is then straightforward to show that:

1. The equilibrium return to debt contracts for savers is $\frac{R^K_{t+1}}{1+\phi}$. The reason is that debt contracts must involve costly screening; otherwise, entrepreneurs would always choose low quality projects and keep all their output. With screening, however, entrepreneurs choose high quality projects and, as a result, savers appropriate all output.

2. The equilibrium return on internal funds of entrepreneurs is R^K_{t+1}. The reason is that

entrepreneurs are willing to self-finance projects of either type, and these have the same return.

These are the two properties that we use throughout the analysis.

Next, we explain how it is possible to re-interpret the two assets issued by entrepreneurs, debts and bubbles, as a single asset, equity. Assume entrepreneurs buy old firms, create new ones and sell equity claims to finance these purchases plus any investment in new capital. Then the value of equity depends on the dividends/production that can be appropriated by the equity-holder plus the re-sale value of the equity. In our setting, savers buy these equity claims from the entrepreneurs. In the first regime of Section 2.2, savers pay the screening cost and entrepreneurs choose the high quality projects. Thus, the return to the claims issued by entrepreneurs has both a fundamental ($\frac{R^K_{t+1}}{1+\phi}$) and a bubbly component ($\frac{B_{t+1}}{B_t+N_t}$). In the second and third regimes of Section 2.2, savers do not pay the screening cost and entrepreneurs choose the low quality projects. Thus, the claims issued by entrepreneurs only have the bubbly component. Notice that we have converted our two assets, debt and bubbles, into a single asset, equity, which has both fundamental and bubbly components. All of the results that follow remain unchanged under this re-interpretation.

3 Monetary policy: general considerations

We know now how to construct competitive equilibria and evaluate them. For each sequence $\{b_t, n_t, m_t, \mu_t\} \in \mathcal{E}$, Equation (19) generates a unique path for k_t from any initial condition $k_0 > 0$. Thus, we refer somewhat loosely to $\mathcal{E}$ as the set of all equilibria. The next steps are to characterize this set, and then choose equilibria within it.

3.1 The bubbly world

The set $\mathcal{E}$ contains all sequences $\{b_t, n_t, m_t, \mu_t\}$ that are part of a competitive equilibrium. Sequences such that $\{b_t, n_t, m_t, \mu_t\} = \{0, 0, 0, \mu_t\}$ for all t and h^t describe equilibria in which there is no demand for unbacked assets. All other sequences describe equilibria in which there are some histories with a positive demand for unbacked assets in some periods. We say that this demand is *extreme* if the economy is in the second or third regimes described in Section 2.2. Recall that in these regimes only bubbles are traded in the financial market. If the demand for unbacked assets is not extreme, the economy is in the first regime of Section 2.2. Here, both backed debts and bubbles are traded simultaneously in the financial market.

The following proposition provides a characterization of the set $\mathcal{E}$:

Proposition 1 *$\mathcal{E}$ always contains sequences with $\{b_t, n_t, m_t, \mu_t\} = \{0, 0, 0, \mu_t\}$ for all t and h^t.*

1. ***Non-bubbly world.*** *If $\frac{\alpha}{1-\alpha} > \max\left\{1 + \phi \cdot \varepsilon,\ \frac{1}{4} \cdot \frac{1+\phi}{1-\varepsilon}\right\}$, then $\mathcal{E}$ contains no additional sequences.*

2. ***Bubbly world.*** *If $\frac{\alpha}{1-\alpha} < \max\left\{1 + \phi \cdot \varepsilon,\ \frac{1}{4} \cdot \frac{1+\phi}{1-\varepsilon}\right\}$, then $\mathcal{E}$ also contains sequences with a positive demand for unbacked assets. Moreover,*

   (i) if $(1 + \phi) \cdot \varepsilon < \frac{\alpha}{1-\alpha}$, then $\mathcal{E}$ contains no sequences with an extreme demand for unbacked assets;

   (ii) if $(1 + \phi) \cdot \varepsilon > \frac{\alpha}{1-\alpha}$, then $\mathcal{E}$ also contains sequences with an extreme demand for unbacked assets.[7]

Our proof of this result proceeds as follows. First, showing that the set $\mathcal{E}$ contains a sequence $\{b_t, n_t, m_t, \mu_t\} = \{0, 0, 0, \mu_t\}$ for some $\{\mu_t\}$ is straightforward. Second, we characterize the "most sustainable" sequence $\{b_t, n_t, m_t, \mu_t\} \in \mathcal{E}$ where $b_t > 0$ and/or $n_t > 0$ and/or $m_t > 0$ for some t and h^t. If this sequence implies an explosive path for either bubbles or money, there cannot be another sequence with the property that $b_t > 0$ and/or $n_t > 0$ and/or $m_t > 0$ for some t and h^t that is non-explosive.[8] We show that the condition $\frac{\alpha}{1-\alpha} > \max\left\{1 + \phi \cdot \varepsilon,\ \frac{1}{4} \cdot \frac{1+\phi}{1-\varepsilon}\right\}$ is necessary and sufficient for this most sustainable sequence to be explosive. Instead, the condition $(1 + \phi) \cdot \varepsilon > \frac{\alpha}{1-\alpha}$ is necessary and sufficient for it to be explosive if the equilibrium is ever outside of the first regime described in Section 2.2.

Figure 1 provides a graphical illustration of Proposition 1. The demand for unbacked assets is positive whenever the return to backed debts is low enough. This depends on the values of α and ϕ.[9] A low value of α reduces the marginal product of capital. A high value of ϕ reduces the share of this marginal product that can be appropriated by the savers. Both depress the return to backed debts and create the conditions for unbacked assets to have positive values. The unshaded region at the top of the figure depicts the non-bubbly world, where the return to backed debts is high, and there is thus no demand for unbacked assets. Instead, the other two regions depict the bubbly world, where the return to backed debts is low, and there is thus a potential demand for unbacked assets. This demand is not extreme in the shaded region of

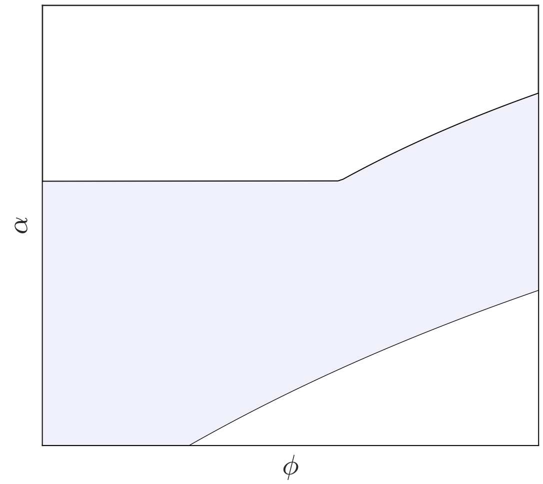

FIGURE 1: Illustrates combinations of α and ϕ such that: there is no demand for unbacked assets (top-unshaded region); there is potential demand for unbacked assets that is not extreme (shaded region); and there is potential demand for unbacked assets that is extreme (bottom-unshaded region).

the figure, but it is extreme in the unshaded region.

In the non-bubbly world, the capital stock and consumption are given by:

$$k_{t+1} = \frac{1+\phi\cdot\varepsilon}{1+\phi}\cdot\frac{1-\alpha}{\gamma}\cdot k_t^{\alpha}, \tag{21}$$

$$c_t = \alpha\cdot k_t^{\alpha}. \tag{22}$$

Thus, from any initial condition, $k_0 > 0$, these variables monotonically and deterministically converge to:

$$k_{\infty} = \left(\frac{1+\phi\cdot\varepsilon}{1+\phi}\cdot\frac{1-\alpha}{\gamma}\right)^{\frac{1}{1-\alpha}}, \tag{23}$$

$$c_{\infty} = \alpha\cdot\left(\frac{1+\phi\cdot\varepsilon}{1+\phi}\cdot\frac{1-\alpha}{\gamma}\right)^{\frac{\alpha}{1-\alpha}} \tag{24}$$

The economy exhibits standard growth dynamics without asset price shocks.

The non-bubbly world is very quiet and monetary policy makes no difference. We shall not analyze it further. Instead, we focus on the bubbly world from now on. To streamline the

[7]We ignore the non-generic parameter cases where $\frac{\alpha}{1-\alpha} = \max\left\{1+\phi\cdot\varepsilon,\ \frac{1}{4}\cdot\frac{1+\phi}{1-\varepsilon}\right\}$ or $(1+\phi)\cdot\varepsilon = \frac{\alpha}{1-\alpha}$.

[8]The process for b_t or m_t is explosive if $m_t + b_t \geq 1$ with positive probability in finite time.

[9]This discussion keeps $\varepsilon < \frac{1}{2}$ fixed in the background. Since $1-\varepsilon$ is the share of savings that need to be intermediated, the effect of an increase in ε is similar to that of an increase in ϕ.

discussion, we also restrict our analysis to the shaded parameter region of Figure 1. That is, we focus on the case in which the demand for unbacked assets is never extreme and the economy always remains in the first regime described in Section 2.2. Though a full analysis of the bubbly world that includes the parameter region with an extreme demand for unbacked assets is also possible, it is cumbersome and adds few additional insights.

3.2 A model of market psychology

The bubbly world is characterized by multiple equilibria, each of which corresponds to a different sequence $\{b_t, n_t, m_t, \mu_t\} \in \mathcal{E}$. To select among them, we introduce now the concept of market psychology. Formally, we define a market psychology $\mathcal{P}$ as a rule to produce a subset $\mathcal{E}^{\mathcal{P}} \subseteq \mathcal{E}$, i.e., a selection rule that discards equilibria that are not in $\mathcal{E}^{\mathcal{P}}$. We say that a market psychology $\mathcal{P}$ is feasible if $\mathcal{E}^{\mathcal{P}}$ is non-empty.

Next, we define $\mathcal{M}$ to denote a monetary policy rule, i.e., a sequence $\{\mu_t\}$ such that $\mu_t \geq 1$ for all t and h^t. Given a market psychology $\mathcal{P}$, we let $\mathcal{E}^{\mathcal{M},\mathcal{P}} \subseteq \mathcal{E}^{\mathcal{P}}$ be the subset of $\mathcal{E}^{\mathcal{P}}$ containing all equilibria that are consistent with $\mathcal{M}$. We say that $\mathcal{M}$ is *feasible* if $\mathcal{E}^{\mathcal{M},\mathcal{P}}$ is not empty. We say that $\mathcal{M}$ is *decisive* if $\mathcal{E}^{\mathcal{M},\mathcal{P}}$ is a singleton. That is, given a market psychology, a decisive monetary policy rule selects one equilibrium only. Instead, a monetary policy that is feasible but not decisive selects two or more equilibria.

This procedure to select equilibria could be interpreted as a sequential game between nature and the central bank. Nature 'moves first' and picks a market psychology $\mathcal{P}$. The central bank 'moves second' and picks a monetary policy $\mathcal{M}$ that is feasible given this market psychology. In this section, we propose a specific class of market psychologies and, in the next section, we search for an optimal monetary policy rule conditional on this class of market psychologies. Thus, we are essentially adopting the role of a central banker who is designing monetary policy with a limited ability to influence market sentiment.

We focus on a class of market psychologies defined in terms of an initial condition b_0 and two types of shocks: (i) bubble return shocks: $u_{t+1} \equiv \frac{b_{t+1}}{E_t b_{t+1}} \geq 0$; and (ii) bubble-creation shocks: $n_t \geq 0$. Let $\mathcal{S} = \{(u_1, n_1), (u_2, n_2), ..., (u_S, n_S)\}$ be a finite state space for these shocks and let $\mathcal{T}$ be an $S \times S$ matrix of constant transition probabilities. To simplify some arguments that follow, we assume that $\mathcal{T}$ has no zeros. We define the following class of market psychologies:

$$\mathcal{P}(\beta, \mathcal{S}, \mathcal{T}) \equiv \{b_0 = \beta \text{ and } \{u_t, n_t\} \text{ is a Markov chain on } \mathcal{S} \text{ with transition matrix } \mathcal{T}\}.$$

That is, we index market psychologies by β, $\mathcal{S}$ and $\mathcal{T}$. This family turns out to be very useful analytically. Moreover, it is quite intuitive from an economic point of view: bubble-return shocks capture the notion that there are random movements in the value of old bubbles, whereas bubble-creation shocks capture the notion that the value of new bubbles is also random.

Naturally, a market psychology has to be feasible. In the non-bubbly world, for instance, the only feasible market psychology within this family is the trivial one with $\beta = 0$, $\mathcal{S} = \{(0,0)\}$. We next characterize the set of feasible market psychologies in the bubbly world.

Proposition 2 *The market psychology $\mathcal{P}(\beta, \mathcal{S}, \mathcal{T})$ is feasible if and only if:*

1. *For all $\{u_s, n_s\} \in \mathcal{S}$:*

$$\Gamma_s \equiv \left[1 + \phi \cdot (\varepsilon + n_s) - \frac{\alpha}{1-\alpha} \cdot u_s\right]^2 - 4 \cdot \frac{\alpha}{1-\alpha} \cdot u_s \cdot n_s \geq 0. \tag{25}$$

2. *$b_H \equiv \min_{s\in\mathcal{S}} b_{H,s} \geq \max_{s\in\mathcal{S}} b_{L,s} \equiv b_L$, where $b_{H,s}$ and $b_{L,s}$ are defined as follows:*

$$b_{H,s} \equiv \frac{1}{2} \cdot \left[1 + \phi \cdot (\varepsilon + n_s) - \frac{\alpha}{1-\alpha} \cdot u_s + \sqrt{\Gamma_s}\right], \tag{26}$$

$$b_{L,s} \equiv \frac{1}{2} \cdot \left[1 + \phi \cdot (\varepsilon + n_s) - \frac{\alpha}{1-\alpha} \cdot u_s - \sqrt{\Gamma_s}\right]. \tag{27}$$

3. *$\beta \leq b_H$.*

Our proof of this result proceeds as follows. First, we show that if $\mathcal{P}$ is feasible and $\mathcal{E}^{\mathcal{P}}$ contains sequences with $m_t > 0$ for some t and h^t, then it must also contain sequences with $m_t = 0$ for all t and h^t. Intuitively, money raises the interest rate and makes the set of feasible bubbles smaller. Second, we compute the set of feasible market psychologies as the set of non-explosive solutions to Equation (16) with $m_t = 0$ for all t and h^t. This places an upper bound on the initial size of the bubble and it also limits the extent to which the bubble can vary across states and, therefore, histories. The first part of the proposition provides the conditions under which the bubble process is non-explosive conditional on remaining in any given state forever. The second part guarantees that there is a bubble process that is non-explosive even if there are transitions across states. Finally, the third part states that the initial bubble must be small enough; otherwise, it will be on an explosive path with positive probability.

Proposition 2 shows us how to construct market psychologies that are feasible. We next use this flexible model of market psychology to study the conduct of monetary policy.

4 What should the central bank do?

Given a feasible market psychology $\mathcal{P}$, a feasible monetary policy is given by a rule $\mathcal{M}$ such that the set $\mathcal{E}^{\mathcal{M},\mathcal{P}}$ is non-empty. Recall that this set consists of all non-explosive sequences $\{b_t, n_t, m_t, \mu_t\}$ satisfying the following equations:

$$b_t + n_t = [1 - m_t - b_t + \phi \cdot (\varepsilon + n_t)] \cdot \frac{1-\alpha}{\alpha} \cdot E_t b_{t+1}, \tag{28}$$

$$m_t = [1 - m_t - b_t + \phi \cdot (\varepsilon + n_t)] \cdot \frac{1-\alpha}{\alpha} \cdot E_t \left\{ \frac{m_{t+1}}{\mu_{t+1}} \right\}, \tag{29}$$

which are the same as Equations (16)-(18), since we have assumed that the economy is in the first regime of Section 2.2 where the demand for unbacked assets is never extreme.

Through its choice of sequence $\{\mu_t\}$, monetary policy affects the evolution of the supply of unbacked assets $\{b_t, m_t\}$; recall that n_t is driven by market psychology. Moreover, it is only through its effect on the total supply of unbacked assets, which we denote by $\{x_t\} \equiv \{b_t + m_t\}$, that monetary policy affects the capital stock and consumption:

$$k_{t+1} = \frac{1 - x_t + \phi \cdot (\varepsilon + n_t)}{1+\phi} \cdot \frac{1-\alpha}{\gamma} \cdot k_t^{\alpha}, \tag{30}$$

$$c_t = [\alpha + x_t \cdot (1-\alpha)] \cdot k_t^{\alpha}, \tag{31}$$

which are the same as Equations (19)-(20) since we are in the first regime.

We adopt a generational perspective of optimality and focus on the implications of monetary policy for the evolution of average consumption (or, equivalently, welfare) c_t. In doing so, we abstract from issues of intra-generational redistribution. Given a feasible market psychology $\mathcal{P}$, we say that a monetary policy rule $\mathcal{M}$ is *optimal* if (i) it is feasible and decisive, and (ii) there does not exist another feasible monetary policy rule $\mathcal{M}'$ that increases the consumption of any one generation without decreasing the consumption of some other generation.

The following proposition states our main result.

Proposition 3 *Given a market psychology $\mathcal{P}$, there exists a feasible and decisive monetary policy rule $\mathcal{M}$, in which the (stationary) equilibrium supply of unbacked assets is:*

$$x^*_{\mathcal{P}} = \begin{cases} b_L & \text{if } \max_{s\in\mathcal{S}} \Omega(n_s) \leq b_L \\ \max_{s\in\mathcal{S}} \Omega(n_s) & \text{if } b_L < \max_{s\in\mathcal{S}} \Omega(n_s) \leq b_H \\ b_H & \text{if } b_H < \max_{s\in\mathcal{S}} \Omega(n_s) \end{cases}, \tag{32}$$

where $\Omega(n_s) \equiv 1 + \phi \cdot (\varepsilon + n_s) \cdot (1-\alpha) - \frac{\alpha}{1-\alpha}$. *Moreover, the monetary policy rule* $\mathcal{M}$ *is optimal.*

Proposition 3 identifies a specific optimal monetary policy rule, which completely stabilizes the supply of unbacked assets even in the face of fluctuations in market psychology. We now demonstrate the macroeconomic effects of this rule and provide an intuition for its optimality and implementation by considering a series of examples.

4.1 Deterministic bubbles

Consider first the case of a "deterministic" market psychology that does not generate fluctuations in the supply of unbacked assets. In particular, we assume that $\mathcal{S}$ is a singleton, and $u_t = 1$ and $n_t = \eta > 0$ for all t and h^t. In the discussion that follows, we ignore transitional dynamics and focus on stationary equilibria, in which the bubble has already converged to its stationary or steady state value.

Figure 2 depicts the law of motion of the bubble (as given by Equation (28)) under a *passive* monetary policy rule, which sets $m_t = 0$ for all t and h^t.[10] It is immediate that, for any initial bubble $\beta < b_H$, the bubble converges monotonically to its steady state value b_L, where the values for b_H and b_L are defined in Proposition 2.[11] The paths starting from any initial bubble above b_H are in turn explosive. Thus, ignoring the non-generic case where the initial bubble is exactly equal to b_H, it follows that under a passive monetary policy the equilibrium supply of unbacked assets equals b_L, and the capital stock and consumption are respectively given by:

$$k^{\text{pass}} = \left(\frac{1 - b_L + \phi \cdot (\varepsilon + \eta)}{1 + \phi} \cdot \frac{1-\alpha}{\gamma} \right)^{\frac{1}{1-\alpha}}, \tag{33}$$

$$c^{\text{pass}} = [\alpha + b_L \cdot (1-\alpha)] \cdot (k^{\text{pass}})^{\alpha}. \tag{34}$$

Although passive monetary policy is always available to the central bank, it may not be optimal. The intuition for this is well-known in overlapping generations economies: absent unbacked assets, investment may be inefficiently high. Optimality requires that the supply of unbacked assets be large enough to eliminate dynamically inefficient investments. In our economy, the extent of dynamic inefficiency depends both on the financial friction, as captured by ϕ, and on the market psychology, as captured by η. A more severe financial friction (higher

[10]For a formal proof that such a monetary policy always exists, see Lemma 1 in the Appendix. Intuitively, any such policy sets the money growth rate μ_t sufficiently high for all t and h^t, ensuring high enough inflation so that money never becomes attractive as a store of value.

[11]It can be readily verified that this market psychology is feasible as long as η is not too large.

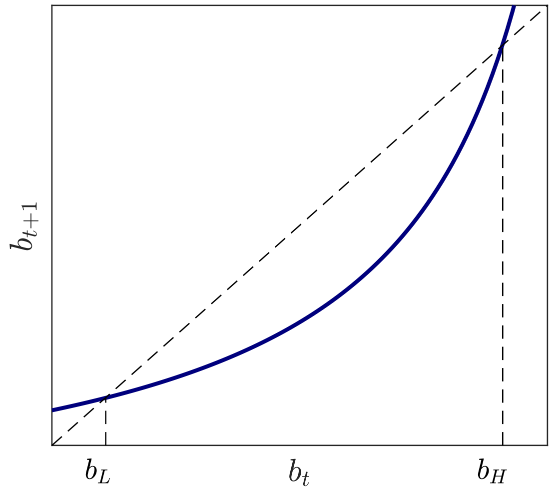

FIGURE 2: Illustrates the law of motion of the bubble under deterministic market psychology and passive monetary policy.

ϕ) reduces the return to intermediated investment and raises the demand for unbacked assets. Instead, a larger wealth effect of bubbles (higher η) raises the capital stock and reduces the return to investment, thereby also increasing the demand for unbacked assets.

Against this backdrop, the target $\Omega(\eta)$ defined in Proposition 3 is the minimum supply of unbacked assets that is required to eliminate inefficient investments. Whether this target is attainable or not depends on market psychology, which determines both b_L and b_H, i.e., the lower- and upper-bound to the supply of unbacked assets that are consistent with equilibrium. This supply can never fall below b_L, as the central bank can only add to – but not subtract from – the unbacked assets supplied by the bubble. But the supply of unbacked assets cannot exceed b_H either, because doing so would put the bubble on an explosive path.

Figure 3 depicts the law of motion of the bubble under the passive monetary policy (dashed curve) and the optimal policy rule (solid curve) of Proposition 3, in the case when $b_L < \Omega(\eta) < b_H$, so that $x^*_{\mathcal{P}} = \Omega(\eta)$. In this scenario, the private supply of unbacked assets is too small and the optimal policy rule requires that the central bank complement it by providing additional stores of value in the form of real balances. By doing so, the central bank raises the equilibrium interest rate, which is captured in the figure by an upward shift in the law of motion of the bubble. Moreover, by raising equilibrium interest rates, the optimal policy accelerates the growth of the bubble and raises its stationary value to $\widetilde{b}_L > b_L$.[12]

[12] Though we ignore transitional dynamics, they are straightforward. For instance, if the bubble starts below $\widetilde{b}_L$, then the central bank temporarily places the economy in a liquidity trap, sets $m_t = x^*_{\mathcal{P}} - b_t$, and gradually withdraws money as the bubble converges to $\widetilde{b}_L$. If the bubble starts above $\widetilde{b}_L$, then the central bank supplies

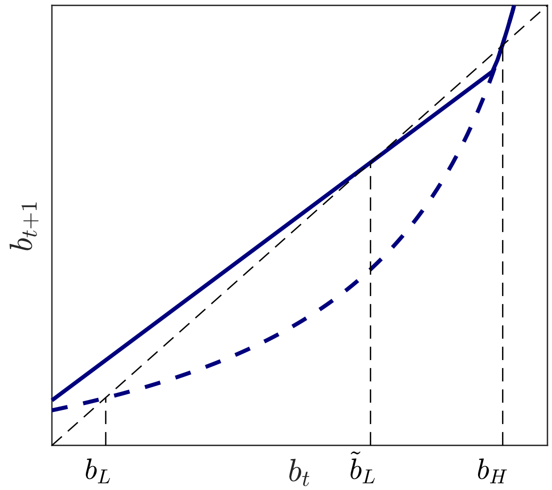

FIGURE 3: Illustrates the law of motion of the bubble under deterministic market psychology and passive monetary policy (dashed curve) vs. optimal monetary policy (solid curve).

By altering the equilibrium supply of unbacked assets, the optimal monetary policy rule affects the capital stock and consumption, which are now:

$$k^{\text{opt}} = \left[\frac{1 - x^*_{\mathcal{P}} + \phi \cdot (\varepsilon + \eta)}{1 + \phi} \cdot \frac{1 - \alpha}{\gamma}\right]^{\frac{1}{1-\alpha}}, \tag{35}$$

$$c^{\text{opt}} = [\alpha + x^*_{\mathcal{P}} \cdot (1 - \alpha)] \cdot \left(k^{\text{opt}}\right)^{\alpha}. \tag{36}$$

As we had already anticipated, an increase in the supply of stores of value reduces the capital stock because it eliminates some investment. To the extent that this displaced investment is inefficient, consumption actually increases. In fact, given our definition of $\Omega(\eta)$, any increase in the supply of stores of value up to $\Omega(\eta)$ is guaranteed to raise steady-state consumption. Even though targeting an asset supply beyond this level is also optimal (because it still eliminates all inefficient investment), doing so may reduce steady-state consumption, however, by eliminating some efficient investment as well! Under a deterministic market psychology, therefore, the monetary policy rule described in Proposition 3 has the additional attractive feature of maximizing steady-state consumption, as depicted in Figure 4.

What changes when $\Omega(\eta) \notin (b_L, b_H)$? In this case, the optimal policy is constrained by the market psychology and it is unable to implement the supply of assets that maximizes consumption. When $\Omega(\eta) \leq b_L$, the market psychology implies that the equilibrium supply of assets is necessarily higher than $\Omega(\eta)$: thus, the optimal policy does not alter the supply of assets

negligible amount of money until the bubble declines below $x^*_{\mathcal{P}}$.

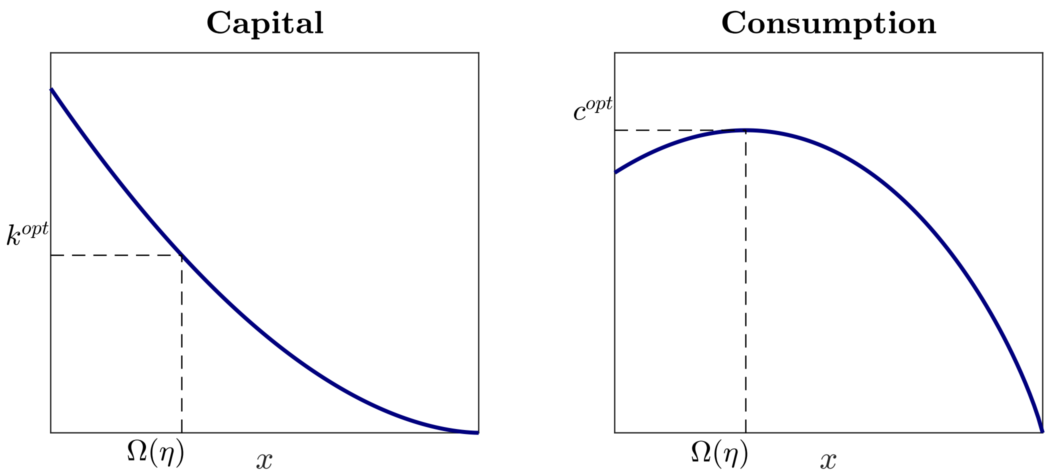

FIGURE 4: Illustrates the effect of the supply of unbacked assets on the steady state values for the capital stock and consumption.

and has no effect on the capital stock or consumption. When $b_H \leq \Omega(\eta)$, the market psychology implies that the equilibrium supply of assets can be no higher than b_H, since otherwise the bubble would be explosive. Thus, the optimal policy increases the equilibrium supply of assets as much as possible, i.e., to b_H, but it is unable to attain the level that would maximize consumption.

Finally, as for implementation, the central bank can guarantee that the equilibrium supply of unbacked assets is $x^*_{\mathcal{P}}$ by simply setting the money growth rate to:

$$\mu^* = \frac{1-\alpha}{\alpha} \cdot \left[1 + \phi \cdot (\varepsilon + \eta) - x^*_{\mathcal{P}}\right], \tag{37}$$

which is shown to pin down the value of money balances uniquely; in this sense, the monetary policy rule is decisive.[13] Intuitively, in order to increase the supply of stores of value, the central bank must decrease money growth and thus inflation, so as to make money more attractive.

[13]Our proof of decisiveness relies on showing that, if the central bank set the money growth rate to μ^* but the supply of unbacked assets exceeded $x^*_{\mathcal{P}}$ in any period, then it would be explosive thereby contradicting the notion of equilibrium. On the other hand, if the supply of unbacked assets fell short of $x^*_{\mathcal{P}}$ in any period, then the value of money would eventually collapse below the arbitrarily small but positive demand of money holders, which cannot be part of competitive equilibrium either. In this sense, the role of money holders in our setting is akin to that of fundamental backing of money in Obstfeld and Rogoff (1983), who show that (even if arbitrarily small) such backing can rule out hyper-inflationary paths in which the value of fiat money goes to zero.

4.2 Stochastic bubbles

We now turn to "stochastic" market psychologies, which generate fluctuations in the supply of unbacked assets. These market psychologies are arguably more interesting as they can generate bubbly business cycles in the spirit of Martin and Ventura (2012). As before, we ignore transitional dynamics and focus on stationary equilibria, in which the bubble has already converged to its stationary or steady state distribution.

To fix ideas, we consider a simple market psychology that gives rise to 'bubbly episodes'. In particular, suppose that the state-space is $\mathcal{S} = \{F, B\}$ and the economy fluctuates between a fundamental state $s = F$, in which $n_F = 0$ and $u_F = 0$, and a bubbly state $s = B$, in which $n_B = \eta > 0$ and $u_B > 1$. Let $\tau_{ss'}$ denote the transition probability from state s to s', then it must be that $u_B \cdot \tau_{BB} = 1$, so that the evolution of the bubble satisfies Equation (28).[14] Thus, whenever the economy is in the bubbly state, there is a probability τ_{BF} that it transitions to the fundamental state, in which case $u_{t+1} = 0$ and the bubble bursts fully. Conversely, there is a probability τ_{BB} that the economy continues in the bubbly state, in which case $u_{t+1} > 1$ and the realized return to the bubble is higher than the market interest rate.

In this example, the law of motion of the bubble fluctuates with the state of the economy. Consider first the *passive* monetary policy rule that sets $m_t = 0$ for all t and h^t. In the fundamental state, the law of motion of the bubble is simply $b_{t+1} = 0$. In the bubbly state, the law of motion is instead qualitatively similar to the one depicted in Figure 2.[15] It is immediate that, for any initial bubble $\beta < b_H$, the bubble gradually converges to the interval $[0, b_L]$ and, once there, it oscillates permanently within this interval. As long as the economy is in the bubbly state, the bubble grows towards b_L; when the economy transitions to the fundamental state, however, the bubble collapses to zero and stays there until the next bubbly episode begins. Therefore, ignoring the non-generic case where the initial bubble is exactly equal to b_H, the supply of unbacked assets in this economy fluctuates in the interval $[0, b_L]$, and the capital stock and consumption are given by Equations (30) and (31), respectively, with $x_t = b_t$.

Figure 5 illustrates the fluctuations in the capital stock and consumption under passive monetary policy by simulating a particular realization of the market psychology. The economy begins in the fundamental state with $b_t = 0$ (top-left panel). After a few periods, it transitions to the bubbly state, during which both bubble creation and high bubble returns fuel the growth of the bubble. In this simulation, the wealth effect of bubble creation dominates the overhang

[14] It can be readily verified that this market psychology is feasible provided that η and u_B are not too large.

[15] The only difference is that, because $u_{t+1} > 1$, the law of motion of the bubble is shifted up and increases faster than the deterministic one; as a result, now b_L is larger and b_H is smaller.

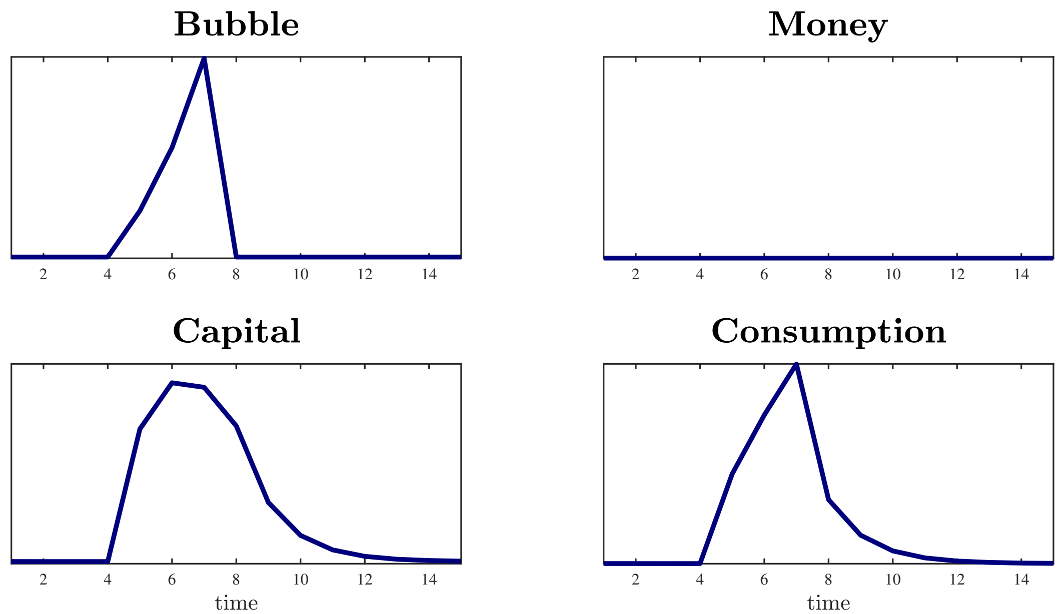

FIGURE 5: Illustrates the equilibrium dynamics of the economy under stochastic market psychology and passive monetary policy.

effect of bubble returns and therefore investment, output and consumption all expand for as long as the economy remains in the bubbly state. Eventually, however, all these effects operate in reverse when the economy transitions back to the fundamental state and the bubble bursts.

This example illustrates that, when monetary policy is passive, the supply of assets x_t follows the whims of market psychology. This need not be desirable. To see this, we now turn to the optimal policy of Proposition 3. Figure 6 depicts the evolution of the same variables as in Figure 5, under both the passive monetary policy (solid lines) and the optimal policy rule (dashed lines). The figure assumes that $b_L < \Omega(\eta) < b_H$, so that the optimal policy sets $x^*_{\mathcal{P}} = \Omega(\eta)$. In this case, the private supply of unbacked assets is too small in both states, and the optimal policy rule requires the central bank to always provide additional stores of value in the form of real balances: differently from before, however, this supply of real balances now fluctuates with the bubble, rising when the bubble collapses and falling when the bubble grows. As before, the economy begins in the fundamental state with $b_t = 0$ (top-left panel): the central bank, however, supplies real balances so that the stock of unbacked assets equals $x^*_{\mathcal{P}}$. When the economy transitions to the bubbly state, the bubble grows but real balances are reduced to keep the total stock of unbacked assets unchanged. Since the total supply of unbacked assets is higher under the optimal policy than under the passive policy at all times, the capital stock and output are lower as well. Consumption, however, is always higher! The reason should be clear

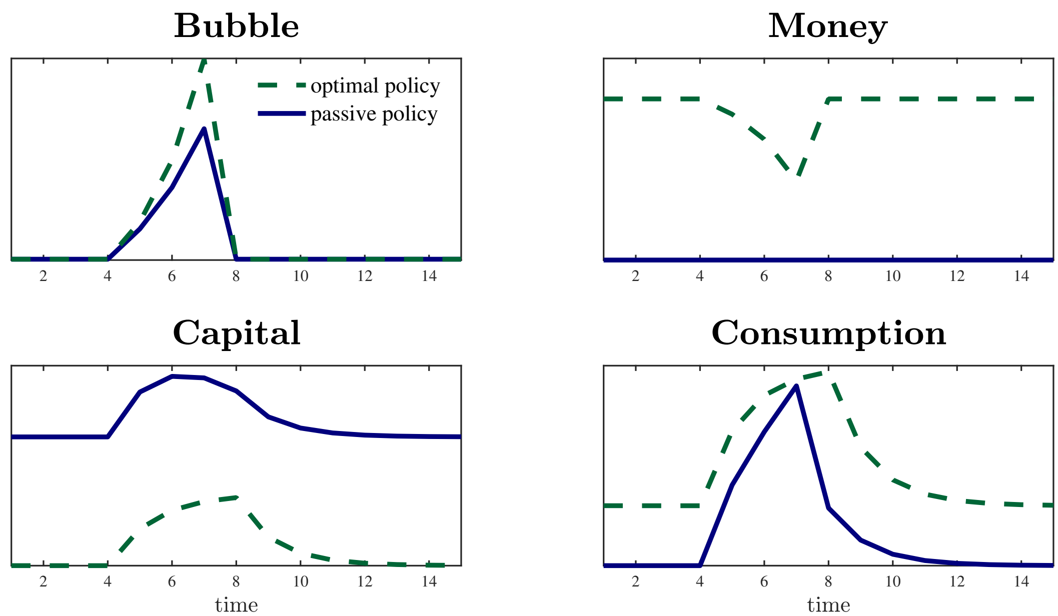

FIGURE 6: Illustrates the equilibrium dynamics of the economy under stochastic market psychology and optimal monetary policy.

by now: by adjusting the supply of real balances, the optimal policy is eliminating inefficient investment and thereby raising the resources available for consumption.

To conclude, we turn to implementation. In the proof of Proposition 3, we show that the central bank can guarantee that the equilibrium supply of unbacked assets is constant and equal to $x^*_{\mathcal{P}}$ with an appropriate choice of sequence $\{\mu^*_t\}$. Moreover, in the stationary equilibrium, these money growth rates are given by:

$$\mu^*_{t+1} = \frac{1-\alpha}{\alpha} \cdot [1 + \phi \cdot (\varepsilon + n_t) - x^*_{\mathcal{P}}] \cdot \frac{E_t\{x^*_{\mathcal{P}} - b_{t+1}\}}{x^*_{\mathcal{P}} - b_t}. \tag{38}$$

Intuitively, when the bubble grows during the bubbly episode, the central bank raises money growth in order to reduce the supply of stores of value. When the bubble bursts and the private stores of value disappear, the central bank instead reduces money growth in order to make money attractive as a store of value. A crucial feature of the policy, which we discuss further in Section 5, is that – by stabilizing the asset supply at $x^*_{\mathcal{P}}$ – it guarantees that $\mu^*_t \geq 1$ for all t and h^t, i.e., the central bank never needs "fiscal resources" to implement its target.

Albeit simple, the above market psychology provides the simplest environment in which to analyze the cyclical implications of optimal monetary policy rule. It should be clear, however, that these macroeconomic effects extend to market psychologies with richer stochastic structure.

4.3 Discussion

We live in a bubbly world, characterized by low interest rates and large booms and busts in asset prices, such as those experienced in recent decades by the US, the Eurozone and Japan. This raises a set of questions. How should monetary policy be conducted in this world? How is the bursting of bubbles linked to the emergence of liquidity traps and to the growth of central bank balance sheets? How is the central bank's ability to supply assets limited by the presence of bubbles?

The theory developed here enables us to address these questions. By limiting the supply of backed assets, financial frictions depress the interest rate and open the door for asset bubbles to arise. Bubbles are useful because they provide additional – albeit unbacked – assets. But bubbles are driven by market psychology, so that they may be potentially volatile and their size can be suboptimal. In such a world, we have shown that there is a novel and powerful role for monetary policy: to complement unbacked assets supplied by the private sector. We have also characterized an optimal policy that fully stabilizes the economy's supply of unbacked assets. In order to implement this policy, the central bank must expand its supply of assets – and thus its balance sheet – when the bubble is small and contract it when the bubble is large. Such a policy raises the welfare of all generations by eliminating inefficient investments.

Though powerful, these results have been derived under a set of restrictive assumptions regarding the central bank's fiscal backing and its ability to distribute seigniorage. In the next section, we relax these assumptions and discuss how doing so affects our conclusions. Before going there, however, we discuss one specific assumption that underlies our analysis: namely, that the central bank cannot affect market psychology. Formally, this implies that the central bank takes the market psychology as given and is constrained to design policies that are consistent with it, and it cannot – for instance – use monetary policy to "prick" bubbles. This assumption has been largely motivated by our general theme of constraining monetary policy as much as possible. But what if we assume instead that the central bank could "shape" the market psychology?

Consider, for instance, that we alter the "timing" of the model so that the central bank sets its policy before market psychology is determined. While in our baseline framework it is market psychology that restricts the set of feasible monetary policies, we can now think of monetary policy as restricting the set of feasible market psychologies. Indeed, through the appropriate design of policy, the central bank can therefore rule out certain bubbles. To see this, suppose for instance that the central bank sets $\mu_t = 1$ for all t and h^t, *independently* of the market psychology. It is then easy to show that this policy rules out all stationary bubbles in equilibrium. The reason is that, under this policy, the return on money is so high that the value of unbacked assets x_t would necessarily follow an explosive path should bubbles arise. This policy captures the popular narrative that, by raising interest rates for long enough, the central bank can prick bubbles.

Of course, this raises a number of additional questions. How credible is the central bank's choice of policy? Does it actually need to raise interest rates in equilibrium, or would off-equilibrium threats suffice? A thorough treatment of these issues would exceed the scope of this paper. But our brief discussion does highlight how, in the bubbly world, the effects of policy crucially depend on whether and how they affect market psychology.

5 On the design of a central bank

Thus far, we have stacked the cards against monetary policy, making restrictive assumptions about the functioning of the central bank. In particular, we assumed that (i) the central bank lacks fiscal backing, i.e., its seigniorage must be non-negative; and (ii) the central bank distributes seigniorage to the old, i.e., it is unable to transfer or lend these resources to financially constrained entrepreneurs. We now explore the consequences of relaxing these assumptions.

5.1 Fiscal backing

When the central bank lacks fiscal backing, the monetary policies it can implement are limited to those having non-negative seigniorage, i.e., satisfying $\mu_t \geq 1$ for all t and h^t. Despite this, we have shown that the central bank can fully stabilize the supply of unbacked assets.

This result may strike the reader as surprising. How is it possible for the central bank to control the value of real balances, and thus of unbacked assets, without having access to real resources? Part of the answer lies in the presence of money holders, who provide an arbitrarily small but stable demand for money. Embedded in a bubbly world, in which there is potential demand for stores of value, the presence of money holders rules out equilibria with hyper-inflationary paths in which the real value of money collapses to zero. But, more importantly, even if money is guaranteed to have positive value, how can the central bank manage it at will, i.e., increase the value of money when the bubble is small and decrease it when the bubble is large? The answer is inflation.

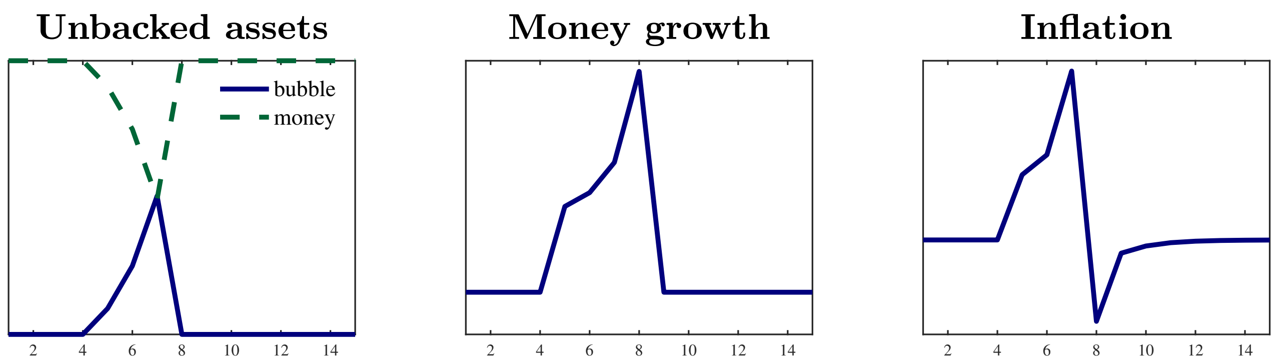

FIGURE 7: Illustrates the equilibrium dynamics of unbacked assets, money growth and the inflation rate under stochastic market psychology and optimal monetary policy.

Figure 7 reproduces the evolution of the economy in Figure 6, but now showing the equilibrium dynamics of unbacked assets, money growth and the inflation rate. Recall that the policy perfectly stabilizes the supply of unbacked assets; as a result, money balances fall when the bubble grows and rise when the bubble bursts. The figure illustrates that part of this adjustment is attained through fluctuations in inflation, i.e., part of the fall (rise) of real balances is attained through an increase (decrease) of the inflation rate. In our model, this is irrelevant as there are no costs associated to inflation volatility. But in the presence of such costs – due, for instance, to nominal rigidities – such a policy could be costly.

What if the central bank wants to avoid inflation volatility? Then, stabilizing the asset supply may require fiscal backing. To see this, consider that the bubble grows today and our policy mandates a reduction in real balances. According to our current policy rule, the central bank can attain this reduction by promising a higher rate of money growth – and thus a higher expected rate of inflation – which reduces the demand for money balances today. If there is a limit to how much inflation can rise, however, it may well be that the only way for the central bank to reduce real balances today is by actually contracting the nominal money supply. But orchestrating such "buy backs" of money balances clearly requires access to fiscal resources.

5.2 Distribution of Seigniorage

We have assumed that the central bank distributes all seigniorage revenue lump-sum to the old.[16] As a result, while the wealth effect of bubble creation expanded investment by redistributing resources to young entrepreneurs, the wealth effect of money creation did not. This was a useful benchmark assumption, which restricted as much as possible the powers of the central bank. We now relax it in two different ways. We first allow the central bank to directly distribute seigniorage revenues to entrepreneurs, thereby directly expanding investment. We then allow the central bank to lend directly to entrepreneurs, which can be interpreted as an "asset purchase scheme" in which seigniorage is used to purchase credit contracts.

5.2.1 Transfer of seigniorage to entrepreneurs

Consider the extreme case where the central bank is able to distribute seigniorage directly to young entrepreneurs. Formally, the law of motion of the capital stock in Equation (30) becomes:

$$k_{t+1} = \frac{1 - \frac{m_t}{\mu_t} - b_t + \phi \cdot \left(\varepsilon + n_t + \frac{\mu_t - 1}{\mu_t} \cdot m_t\right)}{1 + \phi} \cdot \frac{1-\alpha}{\gamma} \cdot k_t^{\alpha}, \tag{39}$$

where $\frac{\mu_t - 1}{\mu_t} \cdot m_t$ is the real value of seignorage and captures the expansionary effect of transferring seigniorage revenues directly to entrepreneurs. The effects of money now mirror those of bubbles: old bubbles b_t and old money $\frac{m_t}{\mu_t}$ have an overhang effect and crowd out investment, whereas new bubbles n_t and new money $\frac{\mu_t - 1}{\mu_t} \cdot m_t$ have a wealth effect and crowd in investment.

The first observation from Equation (40) is that monetary policy becomes more powerful if the central bank can distribute seigniorage directly to entrepreneurs. Through these transfers, monetary policy can expand the capital stock and mimic the effects of bubble creation. As we argue next, however, there are also dangers associated to the transfer of seigniorage to entrepreneurs: it may open the door for equilibrium indeterminacy in the value of money.

To see this, consider the law of motion of real balances in the absence of bubbles (i.e., $b_t = n_t = 0$ for all t and h^t), and for a given some constant rate of money growth μ:

$$m_t = \max\left\{ \upsilon, \left[1 - m_t + \phi \cdot \varepsilon + \mathcal{I} \cdot (1+\phi) \cdot \frac{\mu - 1}{\mu} \cdot m_t\right] \cdot \frac{1-\alpha}{\alpha} \cdot \frac{E_t m_{t+1}}{\mu} \right\}, \tag{40}$$

where υ is the small demand by money holders and $\mathcal{I}$ is the indicator function that takes value of one if seigniorage is distributed to entrepreneurs and zero otherwise.

[16] One interpretation is that the central bank rebates these revenues to the treasury, which then distributes them through tax rebates to the old.

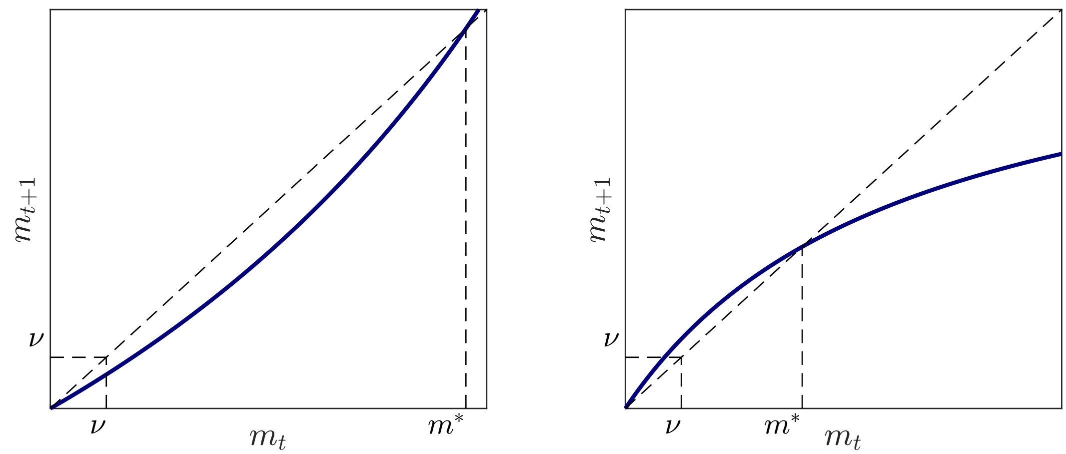

FIGURE 8: Illustrates the law of motion for money balances in the absence of bubbles, in two different scenarios. The left panel depicts our benchmark case where the central bank rebates its seignorage revenue to the old, whereas the right panel depicts the case where the central bank rebates the seignorage revenue to the young entrepreneurs.

When $\mathcal{I} = 0$, we are back to our baseline specification, since Equation (40) collapses to Equation (29) as υ becomes small. In this case, the second term in the brackets defines $E_t m_{t+1}$ as a *convex* function of m_t, as illustrated in the left panel of Figure 8. This in turn is sufficient to ensure that there is a unique value of money balances, denoted by m^*, which is consistent with competitive equilibrium. If money balances were either above or below m^* in any period, then they would eventually either be on an explosive path or decline below υ in finite time, both of which are inconsistent with competitive equilibrium.

Things change dramatically, however, when $\mathcal{I} = 1$. In this case, the second term in the brackets in Equation (40) may define $E_t m_{t+1}$ as a *concave* function of m_t, as illustrated in the right panel of Figure 8.[17] Thus, the equilibrium value of money may then be indeterminate. As the figure shows, there is always a stationary equilibrium in which money is not held as a store of value, i.e., $m_t = \upsilon$ for all t and h^t, since then the discounted value of money is less than υ. But, there also exists a stationary equilibrium in which money is held as a store of value and its value is $m^* > \upsilon$.[18] That is, the monetary policy rule above is no longer decisive.

The intuition for this result is simple. In the first equilibrium, the capital stock is low, and the interest rate is therefore high (relative to the return on money). Thus, agents do not demand

[17] The necessary and sufficient condition for this map to be concave is $1 + \phi \cdot \varepsilon - \mu \cdot \frac{\alpha}{1-\alpha} > \upsilon \cdot (\phi \cdot (\mu - 1) - 1)$.

[18] There is also a continuum of non-stationary equilibria in which m_t converges gradually to m^* from below.

real balances as a store of value and seigniorage revenues and thus central bank transfers to entrepreneurs are low, which indeed confirms the low investment and capital stock. In the second equilibrium, instead, the capital stock is high, and the interest rate is therefore low. This boosts the demand for real balances and thus seigniorage revenues, which – given their distribution to entrepreneurs – confirms the high investment and capital stock.

What is going on? When the central bank distributes seigniorage to entrepreneurs, an expansion in real balances may boost economic activity. But, since higher economic activity also increases the demand for real balances, this may lead to indeterminacy in the value of money. Of course, the central bank can potentially choose a more sophisticated distribution scheme for its seigniorage revenues so as to avoid such problems. The bottom line, however, is that although the ability to transfer seigniorage to entrepreneurs makes monetary policy more powerful, it also introduces the danger of indeterminacy that must be taken into account when thinking about the design of monetary policy.

5.2.2 Asset purchases

We have stressed throughout that monetary policy in our setting is powerful because it can affect the supply of assets available to the private sector. This is different from a standard balance sheet expansion, in which the central bank issues some assets and purchases others, leaving the total supply available to the private sector unchanged. The distinction is key, as it turns out. Although monetary policy interventions that change the supply of stores of value affect capital accumulation, output, and aggregate consumption, policy interventions that merely exchange some stores of value for others do not.

To see this, consider an equilibrium sequence $\{b_t, n_t, m_t, \mu_t\}$ – where $\{\mu_t\}$ is the optimal monetary policy of Proposition 3 – and suppose that the monetary policy is modified through a balance sheet expansion. In particular, the central bank expands the supply of real balances at time t by an amount $\omega_t \cdot k_t^{\alpha}$, and uses all the proceeds to purchase credit contracts from entrepreneurs. At time $t+1$, in turn, the central bank uses the income generated by these contracts to purchase back an amount $\omega_{t+1} \cdot k_{t+1}^{\alpha}$ of real balances. In this manner, the central bank expands its balance sheet at time t and contracts it at time $t+1$. We want to show that, given the market psychology and the monetary policy rule $\{\mu_{t+1}\}$, the original equilibrium sequence $\{b_t, n_t, m_t, \mu_t\}$ is still an equilibrium after the intervention.

Under the proposed intervention, the equilibrium conditions at time t become:

$$b_t + n_t = [1 - m_t - b_t + \phi \cdot (\varepsilon + n_t)] \cdot \frac{1-\alpha}{\alpha} \cdot E_t b_{t+1}, \tag{41}$$

$$m_t + \omega_t = [1 - m_t - b_t + \phi \cdot (\varepsilon + n_t)] \cdot \frac{1-\alpha}{\alpha} \cdot \left[E_t \left\{ \frac{m_{t+1}}{\mu_{t+1}} + \widehat{\pi}_{t+1}^{-1} \cdot \omega_t \cdot \frac{k_t^\alpha}{k_{t+1}^\alpha} \right\} \right], \tag{42}$$

where $\widehat{\pi}_{t+1}$ indicates the value of inflation after the intervention. The first observation is that the law of motion of the bubble is unchanged relative to Equation (28). The reason is that, as long as m_t and b_t do not change, the policy intervention leaves total investment and thus the interest rate unchanged. Hence, the equilibrium growth of the bubble is unaffected. The intervention does not affect investment because, when the central bank purchases credit contracts for an amount ω_t, savers reduce their lending to entrepreneurs by an equal amount and use these resources instead to hold the real balances injected by the central bank. In a sense, the only thing that changes in period t is that the central bank intermediates between savers and entrepreneurs.

The second observation is that, from Equations (29) and (42), it must hold that:

$$E_t \left\{ \widehat{\pi}_{t+1}^{-1} \right\} = E_t \left\{ \pi_{t+1}^{-1} \right\}. \tag{43}$$

In other words, if the equilibrium interest rate does not change, neither does the expected return on money balances if agents are to continue to hold real balances as a store of value.

To confirm that $\{b_t, n_t, m_t, \mu_t\}$ is indeed an equilibrium of the economy after the intervention, it only remains to show that the path of realized inflation $\widehat{\pi}_{t+1}$ satisfies Equation (43). To see this, note that real balances in period $t+1$ are given by:

$$m_{t+1} = \frac{\mu_{t+1} \cdot m_t + \omega_t}{\widehat{\pi}_{t+1}} \cdot \frac{k_t^\alpha}{k_{t+1}^\alpha} - \omega_{t+1}, \tag{44}$$

i.e., real balances at $t+1$ equal the real value of "old money" plus the new money issued by the central bank, where the latter includes the "buy backs" ω_{t+1} entailed by the intervention. From the budget constraint of the central bank, these "buy backs" are given by:

$$\omega_{t+1} = \frac{R_{t+1}^K}{1+\phi} \cdot \omega_t \cdot \frac{k_t^\alpha}{k_{t+1}^\alpha}, \tag{45}$$

i.e., buy backs are equal to the revenues that the central bank earns on the credit contracts that it purchased at t. Combining Equations (44) and (45), we have that:

$$\widehat{\pi}_{t+1}^{-1} = \frac{\pi_{t+1}^{-1} \cdot \mu_{t+1} \cdot m_t + E_t\{\widehat{\pi}_{t+1}^{-1}\} \cdot \omega_t}{\mu_{t+1} \cdot m_t + \omega_t}. \tag{46}$$

Finally, taking expectations and using the fact that μ_{t+1} is known at time t (see Equation (58)), we confirm that Equation (43) is satisfied.

This shows that a balance sheet expansion does not affect the evolution of the bubble, real balances, interest rates or investment. But it does affect realized inflation in period $t+1$, as Equation (46) shows. The reason is that, even though the central bank's revenues in period $t+1$ suffice *in expectation* to buy back the real balances that it injected in period t, this is not true in all states. In particular, the central bank is unable to buy back ω_t in states in which $\pi_{t+1}^{-1} > E_t\{\pi_{t+1}^{-1}\}$ in the original equilibrium. Ex post, the return to money holdings is so high in these states that the income from the credit contracts held by the central bank is insufficient to buy back the additional money balances that it injected at time t. The only way for real balances to remain unchanged relative to the original equilibrium is through higher inflation, i.e., $\widehat{\pi}_{t+1}^{-1} < \pi_{t+1}^{-1}$. The opposite happens in states in which $\pi_{t+1}^{-1} < E_t\{\pi_{t+1}^{-1}\}$, in which case the intervention implies that $\widehat{\pi}_{t+1}^{-1} > \pi_{t+1}^{-1}$.

We have thus shown that that the intervention (ω_t, ω_{t+1}) does not affect the equilibrium $\{b_t, n_t, m_t, \mu_t\}$. By affecting realized inflation, however, the intervention does have redistributive effects between old savers and old entrepreneurs. But in our framework this redistribution is irrelevant for the evolution of capital, output, and aggregate consumption.

6 Concluding remarks

This paper has developed a macroeconomic framework of rational bubbles, credit and monetary policy. A central feature of the framework is the presence of financial frictions, which limit the supply of backed assets and create space for unbacked assets to emerge. These unbacked assets can be issued by private agents, in which case we refer to them as bubbles. Bubbles can be beneficial because they alleviate the scarcity of assets. But they are also inherently fragile because they are driven by market psychology.

In this bubbly world, we have shown that monetary policy plays a key role as a provider of unbacked assets. When the supply of private unbacked assets is inefficiently low, the central bank can step in and provide additional assets through a balance sheet expansion. Indeed, we show that it is optimal for the central bank to actively manage its balance sheet in order to stabilize the economy's total supply of unbacked assets at an optimal level. Moreover, we show that this policy does not require the central bank to have fiscal backing.

Our analysis presents a first step towards understanding the optimal conduct of monetary policy in a bubbly world. However, much is still to be done. The policy studied here, for instance, may lead to high inflation volatility. This is not a problem for us because we have abstracted from nominal rigidities, which are prevalent in New-Keynesian models. What would change if we introduced nominal rigidities? Do they generate a trade-off between the optimal provision of assets by the central bank and the traditional price-stability objective of monetary policy? If so, under which conditions is one objective likely to dominate over the other? We believe that our framework is a useful starting point for future research aiming at answering these crucial questions.

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A Appendix

Proof of Proposition 1. The set $\mathcal{E}$ contains all non-negative sequences $\{b_t, n_t, m_t, \mu_t\}$ that satisfy Equations (16)-(17)-(18) and the constraints $\mu_t \geq 1$ and $m_t + b_t < 1$ for all t and h^t. We want to characterize the set $\mathcal{E}$ as a function of the three relevant model parameters: α, ϕ and ε. Our strategy is to first characterize a subset $\mathcal{E}_0 \subseteq \mathcal{E}$ of sequences that satisfy three convenient assumptions. We then show that the results for $\mathcal{E}_0$ extend to $\mathcal{E}$.

We make three assumptions that minimize, for each t and h^t, the largest possible value for b_{t+1}:

1. If there is a history h^{t+1} in which the bubble is smaller than expected, i.e. $b_{t+1} < E_t b_{t+1}$; there must be another history h^{t+1} in which it is larger, i.e., $b_{t+1} > E_t b_{t+1}$. Thus, we assume that $b_{t+1} = E_t b_{t+1}$.

2. New bubbles n_t have opposing effects on b_{t+1}. Define:

$$n_t(\chi) = \arg\min E_t b_{t+1} = \begin{cases} 0 & \text{if } b_t \leq \frac{1+\phi\cdot\varepsilon - m_t}{1+\phi} \\ 1-\varepsilon - m_t - b_t - \chi & \text{if } \frac{1+\phi\cdot\varepsilon - m_t}{1+\phi} < b_t < 1-\varepsilon - m_t \\ 0 & \text{if } b_t \geq 1-\varepsilon - m_t \end{cases}, \quad (47)$$

   for some small $\chi > 0$, which implies some value $b_{t+1}(\chi)$ for the bubble. Then, to minimize the bubble, simply take the limiting bubble $b_{t+1} \equiv \lim_{\chi\to 0} b_{t+1}(\chi)$.

3. Larger values for m_t imply larger values for b_{t+1}. Thus, we assume that $\{\mu_t\}$ is such that $m_t = 0$ in each t and h^t; see Lemma 1 for a proof that such a sequence $\{\mu_t\}$ always exists. Recall that we are looking at equilibria as the share of money holders goes to zero, $\upsilon \to 0$.

Under Assumptions 1-3, we obtain the following law of motion for b_t:

$$b_{t+1} \begin{cases} = \frac{\alpha}{1-\alpha}\cdot\frac{b_t}{1+\phi\cdot\varepsilon - b_t} & \text{if } b_t \leq \frac{1+\phi\cdot\varepsilon}{1+\phi} \\ = \frac{\alpha}{1-\alpha}\cdot\frac{1-\varepsilon}{(1+\phi)\cdot(1-b_t)} & \text{if } \frac{1+\phi\cdot\varepsilon}{1+\phi} < b_t < 1-\varepsilon \\ \in \left[\frac{\alpha}{1-\alpha}\cdot\frac{b_t}{(1+\phi)\cdot\varepsilon}, \frac{\alpha}{1-\alpha}\cdot\frac{b_t}{\varepsilon}\right] & \text{if } b_t = 1-\varepsilon \\ = \frac{\alpha}{1-\alpha}\cdot\frac{b_t}{1-b_t} & \text{if } b_t > 1-\varepsilon \end{cases}. \quad (48)$$

Let $\mathcal{E}_0$= $\{\{b_t, n_t, m_t, \mu_t\} \in \mathcal{E} : b_{t+1} = E_t b_{t+1},\ n_t = \arg\min E_t b_{t+1}$ and $m_t = 0\}$. This set contains all sequences that satisfy the law of motion in Equation (48) and the constraint $b_t < 1$ for all

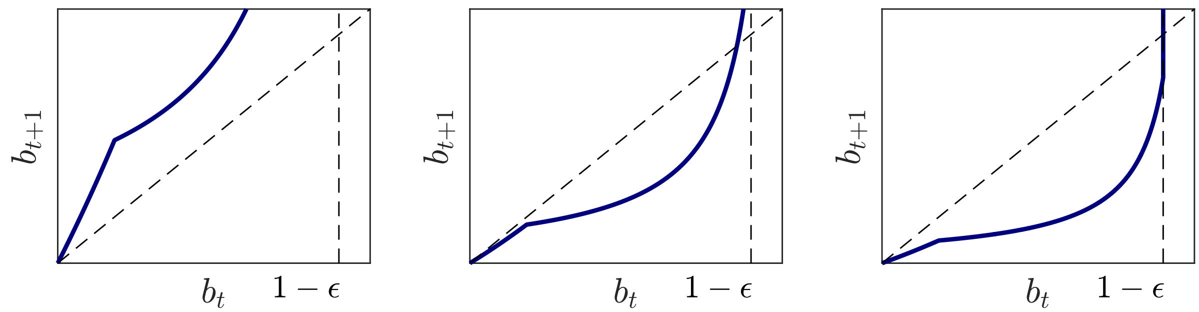

FIGURE 9

t and h^t. Figure 9 depicts the map $b_t \mapsto b_{t+1}$ implied by the law of motion in Equation (48), under the three different parameter conditions stated in Proposition 1.

- The left panel shows the case in which:

$$\frac{\alpha}{1-\alpha} > \max\left\{1+\phi\cdot\varepsilon,\ \frac{1}{4}\cdot\frac{1+\phi}{1-\varepsilon}\right\}. \tag{49}$$

  This parametric condition ensures that the map $b_t \mapsto b_{t+1}$ lies entirely above the 45 degree line. Thus, any initial value $b_0 > 0$ would generate an explosive path for the bubble. Hence, all sequences in $\mathcal{E}_0$ are such that $b_t = 0$ for all t and h^t.

- The middle panel shows the case in which:

$$(1+\phi)\cdot\varepsilon < \frac{\alpha}{1-\alpha} < \max\left\{1+\phi\cdot\varepsilon,\ \frac{1}{4}\cdot\frac{1+\phi}{1-\varepsilon}\right\} \tag{50}$$

  This parametric condition ensures that the map $b_t \mapsto b_{t+1}$ lies above the 45 degree whenever $b_t \geq 1-\varepsilon$. Thus, any initial value $b_0 \in [0, b^{\max}]$, where $b^{\max}$ is the largest fixed point of the map, generates a non-explosive path for the bubble. As a result, sequences such that $b_t > 0$ for some t and h^t are also in $\mathcal{E}_0$. Since $b^{\max} < 1-\varepsilon$, it must be the case that $b_t + n_t \leq 1-\varepsilon$ for all t and h^t. Hence, the economy must always remain in the first regime described in Section 2.2, where $R_{t+1} = \frac{R^K_{t+1}}{1+\phi}$.

- The bottom panel shows the case in which:

$$\frac{\alpha}{1-\alpha} < (1+\phi)\cdot\varepsilon. \tag{51}$$

This parametric condition ensures that the map $b_t \mapsto b_{t+1}$ lies entirely below the 45 degree line whenever $b_t < 1-\varepsilon$. Thus, any initial value $b_0 \in [0, 1-\varepsilon]$ generates a non-explosive path for the bubble. As a result, sequences such that $b_t \geq 1-\varepsilon$ for some t and h^t are also in $\mathcal{E}_0$. Hence, the economy can be outside of the first regime described in Section 2.2, where $R_{t+1} > \frac{R^K_{t+1}}{1+\phi}$.

The next observation is that violations of Assumptions 1-3 shift the map $b_t \mapsto b_{t+1}$ upwards. Thus, if a bubbly equilibrium does not exist when Assumptions 1-3 hold, it cannot exist when these assumptions are violated. Similarly, if a bubbly equilibrium does not exist outside of the first regime described in Section 2.2 when Assumptions 1-3 hold, it cannot exist outside of the first regime when these assumptions are violated. Thus, we can extend our results from the set $\mathcal{E}_0$ to the set $\mathcal{E}$.

Finally, when a bubbly equilibrium does not exist, it must also be that $m_t = 0$ for all t and h^t. By arguments similar to above, the most favorable conditions for an equilibrium with $m_t > 0$ for some t and h^t to exist is that $\mu_{t+1} = 1$ and $m_{t+1} = E_t\{m_{t+1}\}$ for all t and h^t. If $\frac{\alpha}{1-\alpha} > \max\left\{1+\phi\cdot\varepsilon,\ \frac{1}{4}\cdot\frac{1+\phi}{1-\varepsilon}\right\}$, then $b_t = n_t = 0$ as we have shown above. But then, by inspection of Equations (17)-(18), the law of motion $m_t \mapsto m_{t+1}$ implies an explosive path for m_t, starting from any $m_0 > 0$, as it is the same law of motion as for b_t but with n_t set to zero. If $\frac{\alpha}{1-\alpha} > (1+\phi)\cdot\varepsilon$ and the economy is outside of the first regime described in Section 2.2, then $b_t = n_t = 0$ as we have shown above. But then, by inspection of Equations (17)-(18), the law of motion $m_t \mapsto m_{t+1}$ is explosive starting from any $m_0 \geq 1-\varepsilon$, a contradiction. ■

Proof of Proposition 2. Using the arguments in the proof of Proposition 1, it is clear that the market psychology $\mathcal{P}$ is feasible whenever $\mathcal{E}^{\mathcal{M},\mathcal{P}} \subset \mathcal{E}^{\mathcal{P}}$ contains sequences $\{b_t, n_t, m_t, \mu_t\}$ with $m_t = 0$ for all t and h^t. In what follows, therefore, without loss of generality we set $m_t = 0$ for all t and h^t.

Suppose that $\mathcal{P}$ is feasible but that $\Gamma_s < 0$ for some s. The law of motion for the bubble, given in Equation (16), along the sample path in which $u_t = u_s$ and $n_t = n_s$ for all t is:

$$b_{t+1} = u_s \cdot \frac{\alpha}{1-\alpha} \cdot \frac{b_t + n_s}{1+\phi\cdot(\varepsilon+n_s) - b_t}, \tag{52}$$

where we have used the fact that the economy is always in the first regime describe in Section 2.2. Since $\Gamma_s < 0$, it is easy to check that the map $b_t \mapsto b_{t+1}$ induced by Equation (52) lies everywhere above the 45 degree line. Hence, along this sample path the bubble is explosive, a contradiction.

Suppose that $\mathcal{P}$ is feasible but that $b_L > b_H$. Consider the following sample path for the bubble. For $t \in \{0, ..., T\}$, set $u_t = u_s$ and $n_t = n_s$ corresponding to state s such that $b_{L,s} = b_L$. For $t > T$, set $u_t = u_{s'}$ and $n_t = n_{s'}$ corresponding to state s' such that $b_{H,s'} = b_H$. Observe that, for T large enough, it must be that $b_T \approx b_L > b_H$. But then, the map $b_t \mapsto b_{t+1}$ for law of motion of the bubble must lie above the 45 degree line for $b_t > b_H$ and $t > T$. Hence, along this sample path the bubble is explosive, a contradiction.

Suppose that $\mathcal{P}$ is feasible but that $\beta > b_H$. But then the map $b_t \mapsto b_{t+1}$ for the law of motion of the bubble along the sample path in which $u_t = u_s$ and $n_t = n_s$ for all $t > 0$ and corresponding to state s such that $b_{H,s} = b_H$ lies above the 45 degree line for $b_t > b_H$. Hence, along this sample path the bubble is explosive, a contradiction.

Finally, suppose that $\Gamma_s \geq 0$ for all s, $b_0 \leq b_H$ and $b_H \geq b_L$. But then, it is straighforward to show that Equation (52) implies $b_t < 1$ for all t and h^t, i.e., $\mathcal{P}$ is feasible. ■

Proof of Proposition 3. Fix a market psychology $\mathcal{P}$ and consider the candidate asset supply $x^*_{\mathcal{P}}$, given by Proposition 3. We first show that there does not exist another feasible sequence $\{x_t\}$ for the total supply of unbacked assets that Pareto-improves on $x^*_{\mathcal{P}}$, i.e., such that it strictly increases the consumption of some generation without reducing it for another.

Case 1. Suppose that $\max_s \Omega(n_s) < b_H$, so that $x^*_{\mathcal{P}} = \max\{b_L, \max_s \Omega(n_s)\}$. Consider a sequence of policy changes $\{\delta_t\}$ to the asset supply such that $\delta_0 > 0$.[19] To ensure that these policy changes produce a Pareto-improvement, the consumption of all subsequent generations cannot be smaller after the policy changes:

$$[\alpha + (x^*_{\mathcal{P}} + \delta_t) \cdot (1 - \alpha)] \cdot \prod_{\tau=0}^{t-1} [1 + \phi \cdot (\varepsilon + n_\tau) - (x^*_{\mathcal{P}} + \delta_\tau)]^{\alpha \cdot (t-\tau)} \geq$$

$$\geq [\alpha + x^*_{\mathcal{P}} \cdot (1 - \alpha)] \cdot \prod_{\tau=0}^{t-1} [1 + \phi \cdot (\varepsilon + n_\tau) - x^*_{\mathcal{P}}]^{\alpha \cdot (t-\tau)} \quad (53)$$

for all t and h^t (see Equations (30)-(31)). Since it is without loss of generality to consider sequences $\{\delta_t\}$ such that the above inequality holds with strict equality, a sequence of policy

[19] A non-trivial sequence of policy changes must have a non-zero term at some point. What is required is that the first non-zero term of the sequence be strictly positive. Otherwise, the policy changes cannot a produce a Pareto improvement since the first generation affected is worse off. If the first non-zero term happens in period $t \neq 0$, all the arguments that follow go through after a simple re-labelling of the time index.

changes $\{\delta_t\}$ produces a Pareto-improvement if and only if:[20]

$$\delta_{t+1} = \left[\left(\frac{1+\phi\cdot(\varepsilon+n_t)-x^*_{\mathcal{P}}}{1+\phi\cdot(\varepsilon+n_t)-(x^*_{\mathcal{P}}+\delta_t)}\cdot\frac{\alpha+(x^*_{\mathcal{P}}+\delta_t)\cdot(1-\alpha)}{\alpha+x^*_{\mathcal{P}}\cdot(1-\alpha)}\right)^{\alpha}-1\right]\cdot\left(\frac{\alpha}{1-\alpha}+x^*_{\mathcal{P}}\right) \tag{54}$$

for all t and h^t.

We now ask whether such a perturbation is feasible, i.e., whether it does not imply an explosive path for the new sequence $\{x^*_{\mathcal{P}}+\delta_t\}$. Define the following function:

$$\Delta\left(\delta_t; x^*_{\mathcal{P}}, n_t\right) = \left[\left(\frac{1+\phi\cdot(\varepsilon+n_t)-x^*_{\mathcal{P}}}{1+\phi\cdot(\varepsilon+n_t)-(x^*_{\mathcal{P}}+\delta_t)}\cdot\frac{\alpha+(x^*_{\mathcal{P}}+\delta_t)\cdot(1-\alpha)}{\alpha+x^*_{\mathcal{P}}\cdot(1-\alpha)}\right)^{\alpha}-1\right]\cdot\left(\frac{\alpha}{1-\alpha}+x^*_{\mathcal{P}}\right) \tag{55}$$

which is the RHS of Condition (54) and note that $\Delta\left(\cdot; x^*_{\mathcal{P}}, n_t\right)$ is increasing and continuously differentiable. Therefore, if $\lim_{t\to\infty}\delta_{t+1}$ is infinite for all $\delta_0>0$, then there does not exist a feasible sequence $\{\delta_t\}$ that satisfies Condition (54). If $x^*_{\mathcal{P}}$ is such that for t sufficiently large $\frac{\partial\Delta}{\partial\delta_t}|_{\delta_t=0}\geq 1$ for all h^t, then $\lim_{t\to\infty}\delta_{t+1}$ is infinite for any $\delta_0>0$; hence, this implies that $x^*_{\mathcal{P}}$ is Pareto efficient. But, since $\frac{\partial\Delta}{\partial\delta_t}|_{\delta_t=0}=\frac{\alpha}{1-\alpha}\cdot\frac{1+\phi\cdot(\varepsilon+n_t)\cdot(1-\alpha)}{1+\phi\cdot(\varepsilon+n_t)-x^*_{\mathcal{P}}}$, this is equivalent to saying that for t sufficiently large $x^*_{\mathcal{P}}\geq\Omega(n_t)$ for all h^t, which holds by assumption.

Case 2. Suppose that $\max_{s\in\mathcal{S}}\Omega(n_s)>b_H$, so that $x^*_{\mathcal{P}}=b_H$. Consider an alternative sequence $\{\widehat{x}_t\}$ for the total supply of unbacked assets and let t_0 be the first period in which $\widehat{x}_{t_0}>b_H$ along some history h^{t_0}.[21] Suppose that there were a continuation sample path along which $\widehat{x}_t>b_H$ for $t\geq t_0$. Then the law of motion of the bubble along this sample path would be:

$$\begin{aligned} b_{t+1} &= u_{t+1}\cdot\frac{\alpha}{1-\alpha}\cdot\frac{b_t+n_t}{1+\phi\cdot(\varepsilon+n_t)-\widehat{x}_t} && (56)\\ &> u_{t+1}\cdot\frac{\alpha}{1-\alpha}\cdot\frac{b_t+n_t}{1+\phi\cdot(\varepsilon+n_t)-b_H}, && (57)\end{aligned}$$

which must be explosive (see Proposition 2), a contradiction. Thus, the asset supply cannot exceed b_H indefinitely.

Suppose instead that there were continuation sample paths such that $\widehat{x}_{t_0+q}>b_H$ for all $0\leq q<\tau$ but that $\widehat{x}_{t_0+\tau}\leq b_H$, for some $\tau>0$. Take the sample path with largest such τ, which must be finite by our previous argument. Consider the expected consumption of the generation born at time $\tau-1$ along this sample path. These agents expect the asset supply to be less than b_H with probability one in period τ. Moreover, since along this sample path the

[20] To do this, for each history h^t, divide the condition for period $t+1$ by its counterpart for period t raised to the exponent α. Then re-arrange to solve for δ_{t+1}.

[21] Pareto-improvement over the asset supply $x^*_{\mathcal{P}}$ requires that $\widehat{x}_{t_0}>x^*_{\mathcal{P}}$ (with positive probability); otherwise, the old agents in the period where the change in the asset supply occurs would be worse off.

asset supply has been greater than b_H until period τ, it must be that the capital stock and thus wages have been lower as well than with asset supply b_H (see Equation (30)). From Equation (31), it follows that the expected consumption of these agents must be strictly lower than with asset supply b_H. Therefore, $\{\widehat{x}_t\}$ does not generate a Pareto-improvement over b_H.

Next, we construct a monetary policy rule that is decisive in implementing the desired supply for unbacked assets. Fix the market psychology $\mathcal{P}$ and consider the candidate equilibrium in which the value of unbacked assets is $x^*_{\mathcal{P}}$ for all t and h^t. Let b^*_t and m^*_t denote the associated candidate equilibrium values of the bubble and money balances respectively, so that $b^*_t + m^*_t = x^*_{\mathcal{P}}$. Define a sequence $\{\mu^*_t\}$ as follows:

$$\mu^*_{t+1} = \frac{1-\alpha}{\alpha} \cdot (1 + \phi \cdot (\varepsilon + n_t) - x^*_{\mathcal{P}}) \cdot \frac{E_t \left\{ x^*_{\mathcal{P}} - b^*_{t+1} \right\}}{x^*_{\mathcal{P}} - b^*_t} \tag{58}$$

for all t and h^t. Note that, given the market psychology, $\{\mu^*_t\}$ is a sequence of numbers that depends only on the history of the shocks $\{u_t, n_t\}$. From Equations (28)-(29), given the sequence $\{\mu^*_t\}$, we see that the candidate values for the bubble and money balances are indeed part of competitive equilibrium. We next show that the competitive equilibrium is unique, which boils down to showing that, given the market psychology $\mathcal{P}$ and the sequence $\{\mu^*_t\}$, the equilibrium value of money balances is pinned down uniquely. We proceed by contradiction.

Suppose to the contrary that, given the market psychology and the monetary policy rule, there is another competitive equilibrium in which in some period t_0 we have that $m_{t_0} > m^*_{t_0}$ for the first time. Note that $m_t = m^*_t$ and $b_t = b^*_t$ for $t \leq t_0$. From Equation (29), it must be that:

$$\begin{aligned} E_{t_0}\left\{m_{t_0+1}\right\} &= \frac{\alpha}{1-\alpha} \cdot \frac{\mu^*_{t_0+1} \cdot m_{t_0}}{1 + \phi \cdot (\varepsilon + n_{t_0}) - (b_{t_0} + m_{t_0})} \\ &\geq \frac{1 + \phi \cdot (\varepsilon + n_{t_0}) - \left(b^*_{t_0} + m^*_{t_0}\right)}{1 + \phi \cdot (\varepsilon + n_{t_0}) - (b_{t_0} + m_{t_0})} \cdot m_{t_0} \cdot \frac{E_{t_0} m^*_{t_0+1}}{m^*_{t_0}} \\ &= \left(1 + \gamma^+_{t_0+1}\right) \cdot m_{t_0} \cdot \frac{E_{t_0} m^*_{t_0+1}}{m^*_{t_0}}, \end{aligned} \tag{59}$$

where, because $b_{t_0} \geq b^*_{t_0}$ and $m_{t_0} > m^*_{t_0}$, we have that:

$$\gamma^+_{t_0+1} = \frac{(b_{t_0} + m_{t_0}) - \left(b^*_{t_0} + m^*_{t_0}\right)}{1 + \phi \cdot (\varepsilon + n_{t_0}) - (b_{t_0} + m_{t_0})} > 0. \tag{60}$$

Thus, there exists a (continuation) state at $t_0 + 1$ that occurs with positive probability such that $b_{t_0+1} \geq b^*_{t_0+1}$ (see Equation (28)) and such that:

$$\frac{m_{t_0+1}}{m^*_{t_0+1}} \geq \left(1+\gamma^+_{t_0+1}\right) \cdot \frac{m_{t_0}}{m^*_{t_0}}, \tag{61}$$

which also implies that $m_{t_0+1} > m^*_{t_0+1}$. Proceeding inductively, we can construct a sample path in which, for $t > t_0$, we have that $b_t \geq b^*_t$:

$$\frac{m_{t+1}}{m^*_{t+1}} \geq \left(1+\gamma^+_{t+1}\right) \cdot \frac{m_t}{m^*_t}, \tag{62}$$

which implies $m_{t+1} > m^*_{t+1}$ and where:

$$\gamma^+_{t+1} = \frac{(b_t + m_t) - (b^*_t + m^*_t)}{1+\phi \cdot (\varepsilon + n_t) - (b_t + m_t)} > 0. \tag{63}$$

Since along this path $m_t = \frac{m_t}{m^*_t} \cdot m^*_t = \left(\Pi_{j=0}^{t-t_0-1}\left(1+\gamma^+_{t_0+j}\right) \cdot \frac{m_{t_0}}{m^*_{t_0}}\right) \cdot m^*_t > \left(1+\gamma^+_{t_0+1}\right) \cdot \frac{m_{t_0}}{m^*_{t_0}} \cdot m^*_t$, it follows that:

$$\inf_{t \geq t_0} \gamma^+_{t+1} \geq \frac{\left(1+\gamma^+_{t_0+1}\right) \cdot \frac{m_{t_0}}{m^*_{t_0}} - 1}{1+\phi \cdot (\varepsilon + \max_{s \in \mathcal{S}} n_s)} \cdot \upsilon > 0. \tag{64}$$

Thus, $\lim_{t\to\infty} \Pi_{j=0}^{t-t_0-1}\left(1+\gamma^+_{t_0+j}\right) = \infty$ and therefore $\lim_{t\to\infty} \frac{m_t}{m^*_t} = \infty$, which implies that $\lim_{t\to\infty} m_t = \infty$. Thus, this sample path for money balances is explosive, a contradiction.

Second, suppose to the contrary that, given the market psychology and the monetary rule, there is another competitive equilibrium, in which in some period t_0 we have that $m_{t_0} < m^*_{t_0}$ for the first time. Note that $m_t = m^*_t$ and $b_t = b^*_t$ for $t < t_0$. Hence, it must be the case that:

$$\begin{aligned} E_{t_0}\{m_{t_0+1}\} &\leq \frac{\alpha}{1-\alpha} \cdot \frac{\mu^*_{t_0+1} \cdot m_{t_0}}{1+\phi \cdot (\varepsilon + n_{t_0}) - (b_{t_0} + m_{t_0})} \\ &\leq \frac{1+\phi \cdot (\varepsilon + n_{t_0}) - \left(b^*_{t_0} + m^*_{t_0}\right)}{1+\phi \cdot (\varepsilon + n_{t_0}) - (b_{t_0} + m_{t_0})} \cdot m_{t_0} \cdot \frac{E_{t_0} m^*_{t_0+1}}{m^*_{t_0}} \\ &= \left(1-\gamma^-_{t_0+1}\right) \cdot m_{t_0} \cdot \frac{E_{t_0} m^*_{t_0+1}}{m^*_{t_0}}, \end{aligned} \tag{65}$$

where the first inequality is an equality if $m_{t_0} > \upsilon \approx 0$, and where, because $b_{t_0} \leq b^*_{t_0}$ and $m_{t_0} < m^*_{t_0}$, we have that:

$$\gamma^-_{t_0+1} = \frac{\left(b^*_{t_0} + m^*_{t_0}\right) - (b_{t_0} + m_{t_0})}{1+\phi \cdot (\varepsilon + n_{t_0}) - (b_{t_0} + m_{t_0})} > 0. \tag{66}$$

Thus, there must exist a (continuation) state at $t_0 + 1$ that occurs with positive probability such that $b_{t_0+1} \leq b^*_{t_0+1}$ (again, see Equation (28)) and such that:

$$\frac{m_{t_0+1}}{m^*_{t_0+1}} \leq \left(1 - \gamma^-_{t_0+1}\right) \cdot \frac{m_{t_0}}{m^*_{t_0}}, \tag{67}$$

which also implies that $m_{t_0+1} < m^*_{t_0+1}$. Proceeding inductively, we can construct a sample path in which, for $t > t_0$, we have that $b_t \leq b^*_t$:

$$\frac{m_{t+1}}{m^*_{t+1}} \leq \left(1 - \gamma^-_{t+1}\right) \cdot \frac{m_t}{m^*_t}, \tag{68}$$

which implies $m_{t+1} < m^*_{t+1}$ and where:

$$\gamma^-_{t+1} = \frac{(b^*_t + m^*_t) - (b_t + m_t)}{1 + \phi \cdot (\varepsilon + n_t) - (b_t + m_t)} > 0. \tag{69}$$

Since along this path $m_t = \frac{m_t}{m^*_t} \cdot m^*_t = \left(\Pi_{j=0}^{t-t_0-1} \left(1 - \gamma^-_{t_0+j}\right) \cdot \frac{m_{t_0}}{m^*_{t_0}}\right) \cdot m^*_t < \left(1 - \gamma^-_{t_0+1}\right) \cdot \frac{m_{t_0}}{m^*_{t_0}} \cdot m^*_t$, it follows that:

$$\inf_{t \geq t_0} \gamma^-_{t+1} \geq \frac{1 - \left(1 - \gamma^-_{t_0+1}\right) \cdot \frac{m_{t_0}}{m^*_{t_0}}}{1 + \phi \cdot (\varepsilon + \max_{s \in \mathcal{S}} n_s)} \cdot \upsilon > 0. \tag{70}$$

Thus, $\lim_{t\to\infty} \Pi_{j=0}^{t-t_0-1} \left(1 - \gamma^-_{t_0+j}\right) = 0$ and therefore $\lim_{t\to\infty} \frac{m_t}{m^*_t} = 0$, which implies that $\lim_{t\to\infty} m_t = 0 < \upsilon$. Thus, money balances are eventually below the money holders' demand, a contradiction.

Finally, we are left to verify that the sequence $\{\mu^*_t\}$ is such that seignorage is non-negative at all times, i.e. $\mu^*_t \geq 1$ for all t and h^t. Using the law of motion for the bubble in Equation (28) and combining it with the monetary policy rule in Equation (58), we have that:

$$\mu^*_{t+1} = \frac{x^*_{\mathcal{P}} \cdot [1 + \phi \cdot (\varepsilon + n_t) - x^*_{\mathcal{P}}] \cdot \frac{1-\alpha}{\alpha} - (b^*_t + n_t)}{x^*_{\mathcal{P}} - b^*_t}, \tag{71}$$

where

$$b^*_{t+1} = u_{t+1} \cdot \frac{\alpha}{1-\alpha} \cdot \frac{b^*_t + n_t}{1 + \phi \cdot (\varepsilon + n_t) - x^*_{\mathcal{P}}}. \tag{72}$$

Let $\widetilde{b}_{L,s}$ denote the smallest solution to:

$$\widetilde{b}_{L,s} = u_s \cdot \frac{\alpha}{1-\alpha} \cdot \frac{\widetilde{b}_{L,s} + n_s}{1 + \phi \cdot (\varepsilon + n_s) - x^*_{\mathcal{P}}}, \tag{73}$$

and let $\widetilde{b}_L \equiv \max_{s \in \mathcal{S}} \widetilde{b}_{L,s}$. Note that by construction $\widetilde{b}_L \leq x^*_{\mathcal{P}}$. Now, note that in any state s, we have:

$$1+\phi\cdot(\varepsilon+n_s)-x^*_{\mathcal{P}}=\frac{\alpha}{1-\alpha}\cdot u_s\cdot\frac{\widetilde{b}_{L,s}+n_s}{\widetilde{b}_{L,s}}. \tag{74}$$

Plugging this into the monetary rule, we have:

$$\begin{aligned}\mu^*_{t+1} &= \frac{x^*_{\mathcal{P}}\cdot u_s\cdot\frac{\widetilde{b}_{L,s}+n_s}{\widetilde{b}_{L,s}}-(b^*_t+n_t)}{x^*_{\mathcal{P}}-b^*_t}\\ &= \frac{x^*_{\mathcal{P}}-b^*_t+x^*_{\mathcal{P}}\cdot\left[u_s\cdot\frac{\widetilde{b}_{L,s}+n_s}{\widetilde{b}_{L,s}}-1\right]-n_t}{x^*_{\mathcal{P}}-b_t}. \end{aligned}\tag{75}$$

Let $S_\eta=\{s:n_s=\eta\}$ denote the set of all states in the state space in which bubble creation is equal to some η. From Equation (73), for all $s,s'\in S_\eta\subset\mathcal{S}$, we have:

$$u_s\cdot\frac{\widetilde{b}_{L,s}+\eta}{\widetilde{b}_{L,s}}=u_{s'}\cdot\frac{\widetilde{b}_{L,s'}+\eta}{\widetilde{b}_{L,s'}}. \tag{76}$$

Moverover, since $E_t u_{t+1}=1$, if there is a negative bubble return shock in some state, i.e. $u_s<1$, there must also be a positive one in another state, i.e. $u_{s'}>1$. This implies that, for any $n_t=\eta$ and any $s\in S_\eta\subset\mathcal{S}$, we have:

$$\begin{aligned} x^*_{\mathcal{P}}\cdot\left[u_s\cdot\frac{\widetilde{b}_{L,s}+\eta}{\widetilde{b}_{L,s}}-1\right]-\eta &= x^*_{\mathcal{P}}\cdot\left[u_{s'}\cdot\frac{\widetilde{b}_{L,s'}+\eta}{\widetilde{b}_{L,s'}}-1\right]-\eta && (77)\\ &\geq x^*_{\mathcal{P}}\cdot\frac{\eta}{\widetilde{b}_{L,s'}}-\eta && (78)\\ &\geq \left(\frac{\widetilde{b}_L}{\widetilde{b}_{L,s'}}-1\right)\cdot\eta>0, && (79)\end{aligned}$$

where to obtain the last inequality, we used the fact that $\widetilde{b}_L\leq x^*_{\mathcal{P}}$. But then, by inspection of Equation (75), we see that $\mu^*_t\geq 1$ for all t and h^t. ■

Lemma 1 *Given any feasible market psychology $\mathcal{P}$, there exists a sequence $\{\mu_t\}$ such that in equilibrium $m_t=0$ for all t and h^t.*

Proof. Fix a feasible market psychology $\mathcal{P}(\beta,\mathcal{S},\mathcal{T})$, and let $b^*_0=\beta$ and $\{u_s,n_s\}$ denote the initial condition for the bubble and the bubble return- and creation-shocks associated with this market psychology, respectively. Consider now a candidate competitive equilibrium in which $m_t=0$ for all t and h^t, and let $\{b^*_t,n_t\}$ denote the associated equilibrium sequence for the bubble, computed using Equation (16), for a given history of bubble-shocks. Consider a sequence $\{\mu_t^*\}$ that satisfies:

$$\mu_{t+1}^* > \frac{1-\alpha}{\alpha} \cdot (1 + \phi \cdot (\varepsilon + n_t) - b_t^*) \tag{80}$$

for all t and h^t. Thus, $\{\mu_t^*\}$ is a stochastic sequence whose sample path depends solely on $b_0^* = \beta$ and the realized history of the bubble-shocks, $\{u_t, n_t\}$. By inspection of Equations (28)-(29), we see that the candidate values for the bubble and money balances are indeed part of a competitive equilibrium. We next show that this equilibrium is unique, and we proceed to do so by contradiction.

Suppose to the contrary that, given the sequence $\{\mu_t^*\}$ satisfying condition (80), there exists another competitive equilibrium in which, in some period t_0, we have $m_{t_0} > 0$ for the first time. Let $\{b_t, n_t\}$ denote the associated equilibrium sequence for the bubble, again computed using Equation (28). Notice that by assumption we have that $m_t = 0$ and $b_t = b_t^*$ for all $t \leq t_0$. From Equations (28)-(29), it must be that:

$$\begin{aligned} E_{t_0}\{m_{t_0+1}\} &= \frac{\alpha}{1-\alpha} \cdot \frac{\mu_{t_0+1}^* \cdot m_{t_0}}{1 + \phi \cdot (\varepsilon + n_{t_0}) - (b_{t_0} + m_{t_0})} \\ &> \frac{1 + \phi \cdot (\varepsilon + n_{t_0}) - b_{t_0}^*}{1 + \phi \cdot (\varepsilon + n_{t_0}) - (b_{t_0} + m_{t_0})} \cdot m_{t_0} \\ &\equiv (1 + \gamma_{t_0+1}) \cdot m_{t_0}, \end{aligned} \tag{81}$$

where, because $b_{t_0} \geq b_{t_0}^*$ and $m_{t_0} > 0$, we have that:

$$\gamma_{t_0+1} = \frac{(b_{t_0} + m_{t_0}) - b_{t_0}^*}{1 + \phi \cdot (\varepsilon + n_{t_0}) - (b_{t_0} + m_{t_0})} > 0. \tag{82}$$

Thus, there exists a (continuation) state at $t_0 + 1$ that must occur with positive probability such that $b_{t_0+1} \geq b_{t_0+1}^*$ (see Equation (28)) and such that:

$$m_{t_0+1} > (1 + \gamma_{t_0+1}) \cdot m_{t_0}. \tag{83}$$

Proceeding inductively, we can construct a sample path in which, for $t > t_0$, we have $b_t \geq b_t^*$ and:

$$m_{t+1} \geq (1 + \gamma_{t+1}) \cdot m_t, \tag{84}$$

and where:

$$\gamma_{t+1} = \frac{(b_t + m_t) - b_t^*}{1 + \phi \cdot (\varepsilon + n_t) - (b_t + m_t)} > 0. \tag{85}$$

Since along this path $m_t > m_{t_0}$ and $b_t \geq b_t^*$, it follows that:

$$\inf_{t \geq t_0} \gamma_{t+1} \geq \frac{m_{t_0}}{1 + \phi \cdot (\varepsilon + \max_{s \in \mathcal{S}} n_s)} > 0. \tag{86}$$

Thus, it must be that $\lim_{t \to \infty} m_t = \infty$, a contradiction. ■