Universitat Pompeu Fabra Barcelona
Department of Economics and Business

Economics Working Paper Series

Working Paper No. 1532

Menu costs, uncertainty cycles, and the propagation of nominal shocks

Isaac Baley
Julio A. Blanco

July 2016

Menu Costs, Uncertainty Cycles, and the Propagation of Nominal Shocks*

Isaac Baley[†] **Julio A. Blanco** [‡]

CLICK HERE FOR LATEST VERSION

July 11, 2016

Abstract

Nominal shocks have long-lasting effects on real economic activity, beyond those implied by standard models that target the average frequency of price adjustment in micro data. This paper develops a price-setting model that explains this gap through the interplay of menu costs and uncertainty about idiosyncratic productivity. Uncertainty arises from firms' inability to distinguish between permanent and transitory productivity changes. Upon the arrival of a productivity shock, a firm's uncertainty spikes up and then fades with learning until the next shock arrives. These uncertainty cycles, when paired with menu costs, generate recurrent episodes of high adjustment frequency followed by episodes of low adjustment frequency at the firm level. A decreasing hazard rate of price adjustment results, as in the data. Taking into account this pricing behavior amplifies the persistence and reduces the pass-through of nominal shocks.

JEL: D8, E3, E5

Keywords: Menu costs, uncertainty, information frictions, monetary policy, hazard rates.

*Previously circulated as "Learning to Price." We are especially thankful to Virgiliu Midrigan and Laura Veldkamp for their advice and to three anonymous referees for their constructive comments. We also thank Fernando Álvarez, Rudi Bachmann, Anmol Bhandari, Jarda Borovička, Katka Borovičková, Olivier Coibion, Mark Gertler, Ricardo Lagos, John Leahy, Francesco Lippi, Robert E. Lucas, Rody Manuelli, Cynthia-Marie Marmo, Simon Mongey, Joseph Mullins, Emi Nakamura, Gastón Navarro, Ricardo Reis, Tom Sargent, Edouard Schaal, Ennio Stacchetti, Venky Venkateswaran, as well as seminar participants at 4th Ifo Conference on Macroeconomics and Survey Data 2013, Midwest Economics Association 2013, Society of Economic Dynamics 2013, ASSA Meetings 2015, Stanford Institute for Theoretical Economics 2015, Econometric Society Meetings 2015, 40 Simposio de la Asociación Española de Economía, Barcelona GSE Summer Forum 2016, New York University, NYU Stern School of Business, Princeton University, Washington University St. Louis, St. Louis Fed, Federal Reserve Board, University of Toronto, Einaudi Institute, CREI, Pompeu Fabra, Bank of International Settlements, Singapore Management University, Carnegie Mellon, UC Davis, University of Melbourne, University of Sydney, Banco de México, ITAM, Oxford University, and Universitat Autònoma de Barcelona for very useful comments and suggestions. Julio A. Blanco gratefully acknowledges the hospitality of the St. Louis Fed where part of this paper was completed.

[†]Universitat Pompeu Fabra and Barcelona GSE. isaac.baley@upf.edu; http://www.isaacbaley.com

[‡]University of Michigan. jablanco@umich.edu; https://sites.google.com/site/julioandresblanco1984/home

1 Introduction

How do nominal shocks propagate, affecting prices and output? This classic question in monetary economics is largely motivated by an empirical puzzle. Nominal shocks have very persistent effects on real output, lasting up to twelve quarters (Christiano, Eichenbaum and Evans (2005), Romer and Romer (2004), Galí and Gertler (1999)). Microdata shows that prices change every two to three quarters on average (Nakamura and Steinsson (2008), Klenow and Kryvtsov (2008)). When standard frameworks like Calvo, Taylor, and menu cost models are calibrated to match this average frequency of price adjustment, they do not generate the large persistence of output response that follows a nominal shock.

In this paper we argue that firms' hazard rate of price adjustment – the probability of adjusting their price since its last adjustment – is a key statistic to assess the flexibility of the aggregate price level, and that the output response to nominal shocks depends largely on its shape. Specifically, a model that targets a decreasing hazard rate amplifies the persistence of the output response. We develop a price-setting model that generates a decreasing hazard rate through the interplay of menu costs and uncertainty about firms' idiosyncratic productivity. When we match the hazard rate in the micro data, the persistence of monetary shocks is amplified in our model with respect to a Calvo pricing model. Furthermore, our model predicts behavior differences between young and old prices, which is consistent with micro evidence.

The price-setting problem involves nominal and informational frictions. The starting point is the framework by Álvarez, Lippi and Paciello (2011), where firms face a menu cost to adjust their prices and are uncertain about their level of productivity.[1] In particular, we assume that the firms receive permanent and transitory shocks to their idiosyncratic productivity, but they cannot distinguish between types of shocks. Because firms must pay a menu cost with each adjustment, it is optimal to ignore transitory shocks and only respond to permanent shocks. Therefore firms estimate the permanent component of their productivity. Firms use Bayes law to estimate and we call the conditional variance of the estimates firm uncertainty. As in any problem with fixed adjustment costs, the decision rule takes the form of an inaction region, in which the firm adjusts her price only if she receives shocks that make it worth paying the menu cost. In this case, the inaction region also depends on firm uncertainty.

Our framework's contribution is a structure of productivity shocks that gives rise to firm uncertainty cycles, defined as recurrent episodes of high uncertainty followed by episodes of low uncertainty. The key to generate these cycles are infrequent and large shocks to permanent idiosyncratic productivity – or regime changes – where the timing but not the magnitude of the shock is known. That is, a firm knows when a regime change has occurred, but not the sign or the size of the change. It is also assumed that these shocks have the potential to push productivity either upwards or downwards, but in expectation they have no effect. Examples are changes in the supply chain or the cost structure, changes in the fiscal or regulatory environment, new competitors, the introduction of a new technology, product turnover, and access to new markets, among others. Large and infrequent idiosyncratic shocks to productivity were first introduced in menu cost models by Gertler and Leahy (2008) and then used by Midrigan (2011) as a way to account for the empirical patterns of pricing behavior such as fat tails in price change distributions.

[1] In Álvarez, Lippi and Paciello (2011) firms pay an observation cost to see their true productivity level; here we make the observation cost infinite and the true state is never fully revealed. The Appendix of that paper discusses this particular case in an environment where the information friction does not have effects in steady state.

In our model, the infrequent first moment shocks paired with the information friction give rise to second moment shocks in beliefs or uncertainty shocks.[2] When a regime change shock hits, uncertainty spikes up; then it fades with learning until it jumps again with the arrival of the next shock; these are the uncertainty cycles.

Uncertainty, inaction regions, and decreasing hazard Our theoretical contribution is twofold. First, we contribute to the filtering literature by extending the Kalman-Bucy filter to an environment where the state follows a general jump-diffusion process. Second, we characterize analytically the dynamic inaction region and several price statistics as a function of uncertainty. This involves solving a stopping time problem together with a signal extraction problem. This analytical characterization allows for understanding how uncertainty shapes pricing decisions. The model is very general and can be applied to a variety of environments with non-convex adjustment costs and idiosyncratic uncertainty shocks.

The mechanism that generates a decreasing hazard rate comes from the combination of the uncertainty cycles and a positive relationship between uncertainty and adjustment frequency. This positive relationship is subtle as uncertainty has two opposing effects on frequency. Higher uncertainty means that the firm does not trust her current estimates of permanent productivity, and thus she optimally puts a high Bayesian weight on new observations that contain transitory shocks. Estimates become more volatile and the probability of leaving the inaction region and adjusting the price increases. This is known as the "volatility effect" and it has a positive effect on the adjustment frequency. This volatility arises from belief uncertainty, not from fundamental shocks. As a reaction to the volatility effect, which triggers more price changes and menu costs payments, the optimal policy calls for saving menu costs by widening the inaction region. This is known as "option value effect" (Barro (1972) and Dixit (1991)), and it has a negative effect on the adjustment frequency. However, the widening of the inaction region does not compensate for the increase in volatility. Overall, the volatility effect dominates and higher uncertainty yields higher adjustment frequency. When this relationship is paired with uncertainty cycles, we obtain adjustment frequency cycles as well: firms alternate between periods of high frequency with periods of low frequency, in other words, price changes get clustered in some periods instead of evenly spread across time. This gives rise to the decreasing hazard rate of price adjustment.

With respect to the positive relationship between uncertainty and adjustment frequency, Bachmann, Born, Elstner and Grimme (2013) use survey data collected from German firms to document a positive relationship between the variance of firm-specific forecast errors on sales – a measure of firm-level belief uncertainty – and the individual adjustment frequency. Vavra (2014) and Karadi and Reiff (2014) exploit a version of this positive relationship in menu cost models where productivity shocks volatility follows exogenous autoregressive processes. Both belief uncertainty as in our model and fundamental volatility shocks generate higher adjustment frequency in a menu cost model; however, we show that the decreasing hazard cannot be generated by an autoregressive stochastic process for fundamental volatility.[3]

Regarding decreasing hazard rates of price adjustment, these are documented in several datasets, covering different countries and different periods. For instance, decreasing hazards are documented by

[2] Senga (2014) uses of a similar mechanism in a model of investment and misallocation, in which firms occasionally experience a shock that forces them to start learning afresh about their productivity.

[3] In the Appendix we compare the hazard rates from our learning model and a model with autoregressive volatility and show that the later always produces an increasing hazard rate.

Nakamura and Steinsson (2008) using monthly BLS data for consumer and producer prices, Campbell and Eden (2014) using retailer weekly scanner data, Eden and Jaremski (2009) using Dominik's weekly scanner data, Dhyne *et al.* (2006) using monthly CPI data for Euro zone countries, and Cortés, Murillo and Ramos-Francia (2012) for CPI data in Mexico. Most of these papers use the mixed proportional hazard model to construct estimates, which Álvarez, Borovičková and Shimer (2015) argue is a convenient statistical representation of the pricing data. They control for observed and unobserved heterogeneity and also filter discounts out, these are known sources of potential downward bias in the slope of hazard rates. There are other alternative explanations for decreasing hazard rates of price adjustment; examples are discounts in Kehoe and Midrigan (2015), mean reverting shocks in Nakamura and Steinsson (2008), experimentation in Bachmann and Moscarini (2011), introduction of new products in Argente and Yeh (2015), price plans in Álvarez and Lippi (2015), and rational inattention in Matějka (2015). Below, we provide additional support for the our theory using cross-sectional implications of our learning model.

Decreasing hazard and propagation of monetary shocks Why does a decreasing hazard rate imply more persistent monetary shock effects on output? To answer this question, it is key to recognize two observations. First, a decreasing hazard rate generates cross-sectional heterogeneity. At the firm level, a falling hazard is equivalent to having time-varying adjustment frequency; in the aggregate, it implies that there are different types of firms: high frequency firms and low frequency firms. Second, a firm's *first* price change after a monetary shock takes care of incorporating the monetary shock into her price and, in the absence of complementarities, it is the only price change that matters for the accounting of monetary effects. Any price changes after the first one are the result of idiosyncratic shocks that cancel out in the aggregate and do not contribute to changes in the aggregate price level. When a monetary shock arrives, the high frequency firms will incorporate almost immediately the monetary shock with their first price change; but the monetary shock will have effects until the low frequency firms have made their first price adjustment. Therefore, the heterogeneity generated by a decreasing hazard makes the aggregate price level less responsive to monetary shocks compared to an aggregate price level where every firm faces the same average frequency.

The following simplified example highlights the main mechanisms in our framework. Suppose there is a continuum of firms and two states for uncertainty, high and low; assume that half of the firms are in each state. High uncertainty firms change their price during N consecutive periods and then become low uncertainty firms with probability one; this switch in firm type captures the learning process. Low uncertainty firms do not change their price and with probability $1/N$ they become high uncertainty firms; this switch in firm type captures the regime changes. In steady state, the aggregate adjustment frequency is equal to $1/2$. Now suppose there is a monetary shock. To measure the output effects, let us keep track of the mass of firms that have not adjusted their price. On impact, $1/2$ of the firms (all high uncertainty firms) change their price and the output effect is equal to $1/2$ (all low uncertainty firms). In subsequent periods, all high uncertainty firms adjust again, but we do not count these price changes towards the effect of the monetary shock because these respond only to idiosyncratic shocks. Then the low uncertainty firms that become high uncertainty (a fraction $1/N$ of firms) adjust and incorporate the monetary shock. Therefore, the output effect is $1/2(1-1/N)$, which is equal to the mass of low uncertainty firms that have not switched yet. Continuing in this way, the output effect τ periods after the impact of the monetary shock is given by $1/2(1-1/N)^{\tau}$. The persistence of the output response is driven by N, which is the number of periods that firms remain characterized by high uncertainty (the speed of learning). Now let us compare this stylized economy with learning to a Calvo economy with the same aggregate frequency, which is generated with a random probability of adjustment of 1/2. On impact, the output effects also equal to 1/2, but in subsequent periods the response is $1/2(1-1/2)^{\tau}$. Therefore, as long as $N > 2$, the economy with learning has more persistence than the Calvo economy.

Heterogeneity in adjustment frequency has been analyzed as a source of non-neutrality before. For instance, Carvalho (2006) and Nakamura and Steinsson (2010) find larger non-neutralities in sticky price models with exogenous heterogeneity in sector level adjustment frequency. Heterogeneity in our setup arises endogenously in ex-ante identical firms that churn between high and low levels of uncertainty. Importantly, this type of heterogeneity does not refer to different types of firms, but to different uncertainty states within each firm. Therefore, our mechanism does not rely on survivor bias to generate a decreasing hazard.[4] The regime change shocks are crucial to produce a non-degenerate distribution of uncertainty that keeps heterogeneity active in steady state. Without regime changes, uncertainty becomes constant and equal across firms in steady state, as Álvarez, Lippi and Paciello (2011) recognize. The model collapses to that of Golosov and Lucas (2007) without heterogeneity and where money is highly neutral.

Larger persistence of output response to monetary shocks To give a quantitative assessment of the impact of monetary shocks implied by the model, we study a general equilibrium economy with a continuum of firms that solve the price-setting problem with menu costs and idiosyncratic uncertainty cycles. The environment includes a representative household that provides labor in exchange for a wage, consumes a bundle of goods produced by the firms, and holds real money balances. We solve for the steady state of this economy and calibrate the parameters using US micro pricing data. We focus on matching the statistics produced by Nakamura and Steinsson (2008) with CPI data from the Bureau of Labor Statistics. We target three factors jointly: the average adjustment frequency, the dispersion of the price change distribution, and the decreasing hazard rate. In particular, we use the slope of the hazard rate to calibrate the volatility of the transitory shocks that give rise to the information friction. This approach of using a price statistic to recover information parameters was first suggested in Jovanovic (1979), and Borovičková (2013) uses it to calibrate a signal-noise ratio in a labor market framework.

In the calibrated economy we study the effect of a small unanticipated increase in the money supply. In equilibrium this monetary shock increases wages and gives incentives to firms to increase their prices. As a baseline case, we assume that the monetary shock is perfectly observable and then relax this assumption. The results show that the output response to the monetary shock is more persistent in our model than in alternative models. The larger persistence generated in the baseline model only relies on information frictions regarding idiosyncratic conditions; the arrival of the aggregate nominal shock is perfectly observed by firms. Even though this model performs well in terms of the long-run effects of the monetary shock by increasing persistence, it has shortcomings with respect to its short-run response. On impact of the monetary shock, there is an overshooting in the adjustment frequency that makes the monetary shock's total effect too small. Furthermore, this overshoot is not observed in the data.

[4] Survivor bias emerges when computing hazards in populations with heterogenous types as noted by Kiefer (1988) and studied in an economy with different Calvo agents as in Álvarez, Burriel and Hernando (2005).

To address this issue, we consider an extension of the model that incorporates an additional information friction. We assume that there is a fraction of firms that does not observe the monetary shock's arrival. These type of constraints on the information set regarding aggregate shocks are at the core of the pricing literature with information frictions that started with Lucas (1972) and has been recently explored by Mankiw and Reis (2002), Woodford (2009), Maćkowiak and Wiederholt (2009), Hellwig and Venkateswaran (2009), and Álvarez, Lippi and Paciello (2011). These firms apply the same learning technology to filter the monetary shock as they do to filter their idiosyncratic permanent productivity shocks. Upon the impact of the monetary shock, there will be initial forecast errors that disappear over time. The persistence of forecast errors increases the persistence of the output response. Under this assumption, the output response is significantly amplified compared to the case with the observable monetary shock.

Aggregate uncertainty, forecast errors, and persistence The model also predicts that unobserved monetary shocks have less effects when aggregate uncertainty is high. We interact the monetary shock with a synchronized uncertainty shock across all firms. In more uncertain times, firms place a higher weight on new information, forecast errors disappear faster, and the monetary shock is quickly incorporated into prices; this reduces the persistence of the average forecast error, and in turn, the persistence of the output response. This relationship between uncertainty and forecast errors is novel and there is empirical evidence in this respect. For instance, Coibion and Gorodnichenko (2015) compares the dynamics of forecast errors during periods of high economic volatility (as the 70's and 80's) with periods of low economic volatility (as the late 90's). It concludes that information rigidities are higher during periods of low uncertainty than higher uncertainty. The joint dynamics of uncertainty, prices, and forecast errors implied by our model provide a theoretical framework to think about this piece of evidence. Furthermore, we show how forecast errors can be disciplined with micro-price data.

The negative relationship between the effects of monetary shocks and aggregate uncertainty is also documented empirically in various studies. Pellegrino (2015) finds weaker real effects of monetary policy shocks during periods of high uncertainty, and even more, it finds that prices respond more to a monetary shock during times of greater firm-level uncertainty. Aastveit, Natvik and Sola (2013) shows that monetary shocks produce less output effects when various measures of economic uncertainty are high; and other papers find differential effects of monetary shocks in good and bad times, where bad times are associated with periods of high uncertainty, as Caggiano, Castelnuovo and Nodari (2014), Tenreyro and Thwaites (2015), Mumtaz and Surico (2015). Finally, Vavra (2014) uses BLS data to document that the cross-sectional dispersion of the price change distribution (a potential measure of aggregate uncertainty) is larger during recessions, implying higher price level flexibility and lower effects of monetary policy.

Age dependent pricing An interesting prediction of our learning model is that price age, defined as the time elapsed since its last change, is a determinant of the size and frequency of its next adjustment. Young prices – or recently set, mostly by firms who are highly uncertain at the time of the change – and old prices – set many periods ago by firms which are currently certain about their productivity– will exhibit different behavior. In particular, young (and uncertain) prices are more likely to be reset than older (and certain) prices. Furthermore, as the inaction region decreases with uncertainty and price age, young prices changes will tend to be larger and more dispersed compared to older prices. These predictions are documented by Campbell and Eden (2014) using weekly scanner data. They find that young prices (set less than three weeks ago) are relatively more dispersed and more likely to be reset than older prices. Further evidence regarding age dependence is documented in Baley, Kochen and Sámano (2015) who find a negative relationship between price age and exchange rate passthrough using Mexican CPI data: conditional on adjustment, older prices incorporate a smaller fraction of the exchange rate depreciation since the last change. Our results on age dependence are in line with those in Carvalho and Schwartzman (2015), which shows that in time-dependent sticky price models, monetary non-neutrality is larger if older prices are disproportionately less likely to change.

2 Firm problem with nominal rigidities and information frictions

We develop a model that combines an inaction problem arising from a non-convex adjustment cost together with a signal extraction problem. Although the focus here is on pricing decisions, the model is easy to generalize to other settings. We contribute in three ways. First, we provide filtering equations for a state that has both continuous and jump processes. Second, we derive closed form decision rules that take the form of a time-varying inaction region that reflects the uncertainty dynamics. Lastly, we characterize micro-price statistics and some aggregation results, also in closed form.

2.1 Environment

Consider a profit maximizing firm that chooses the price at which to sell her product, subject to idiosyncratic cost shocks. She must pay a menu cost θ in units of product every time she changes the price. We assume that in the absence of the menu cost, the firm would like to set a price that makes her markup – price over marginal cost – constant. The cost shocks –and therefore her markup– are not perfectly observed, only noisy signals are available to the firm. She chooses the timing of the adjustments as well as the new reset markups. Time is continuous and the firm discounts the future at a rate r.

Quadratic loss function Let μ_t be the markup gap, defined as the log difference between the current markup and the optimal markup obtained from a static problem without menu costs. Firms incur an instantaneous quadratic loss as the markup gap moves away from zero:

$$\Pi(\mu_t) = -B\mu_t^2, \qquad B > 0$$

Quadratic profit functions are standard in price setting models, such as Barro (1972) and Caplin and Leahy (1997), and can be motivated as second order approximations of more general profit functions.

Markup gap process The markup gap μ_t follows a jump-diffusion process as in Merton (1976)

$$d\mu_t = \sigma_f dW_t + \sigma_u u_t dq_t \tag{1}$$

where W_t is a Wiener process, $u_t q_t$ is a compound Poisson process with the Poisson counter's intensity λ, and σ_f and σ_u are the respective volatilities. When $dq_t = 1$, the markup gap receives a Gaussian innovation $u_t \sim \mathcal{N}(0,1)$. The process q_t is independent of W_t and u_t. Analogously, the markup gap can be also expressed as

$$\mu_t = \sigma_f W_t + \sigma_u \sum_{\kappa=0}^{q_t} u_\kappa$$

where $\{\kappa\}$ are the number of times when $dq_t = 1$ and $\sum_{\kappa=0}^{q_t} u_\kappa$ is a compound Poisson process. Note that $\mathbb{E}[\mu_t] = 0$ and $\mathbb{V}[\mu_t] = (\sigma_f^2 + \lambda\sigma_u^2)t$. This process for markup gaps nests two specifications that are benchmarks in the literature:

i) small *frequent* shocks modeled as the Wiener process W_t with small volatility σ_f; these shocks are the driving force in standard menu cost models, such as Golosov and Lucas (2007)[5];

ii) large *infrequent* shocks modeled through the Poisson process q_t with large volatility σ_u. These shocks produce a leptokurtic distribution of price changes and are used in Gertler and Leahy (2008) and Midrigan (2011) to capture the fat tailed price change distribution in the data.

Signals Firms do not observe their markup gaps directly. They receive continuous noisy observations denoted by s_t. The noisy signals about the markup gap evolve according to

$$ds_t = \mu_t dt + \gamma dZ_t \tag{2}$$

where the signal noise Z_t follows a Wiener process, independent from W_t. The volatility parameter γ measures the information friction's size. Note that the underlying state, μ_t, enters as the drift of the signal. This representation makes the filtering problem tractable as the signal process has continuous paths.[6]

Information set We assume that a firm knows if there has been an infrequent large shock to her markup – our notion of regime change–, but not the size of the innovation u_t. Therefore, the information set at time t is given by the σ-algebra generated by the history of signals s as well as the realizations of the Poisson counter q:

$$I_t = \sigma\{s_r, q_r;\ r \leq t\}$$

These regime changes reflect innovations in the economic environment that, given the information available to her, a firm cannot assign a sign or magnitude to in terms of the effects it will have on her markup. These shocks may push the firm's optimal price, and therefore its markup gap, either upwards or downwards: in expectation the firm thinks the shock will have no effect as the mean of the innovation u_t is zero. For analytical traction, we assume that the firm knows the arrival of a regime change. This allows us to keep the problem within a finite dimensional state Gaussian framework, as we show in Proposition (1), where only the first two moments of posterior distributions are needed for the firm's decision problem. Another approach would be to assume a finite number of markup gaps and keep track of their probability distribution, and use the techniques of hidden state Markov models pioneered by Hamilton (1989). Other

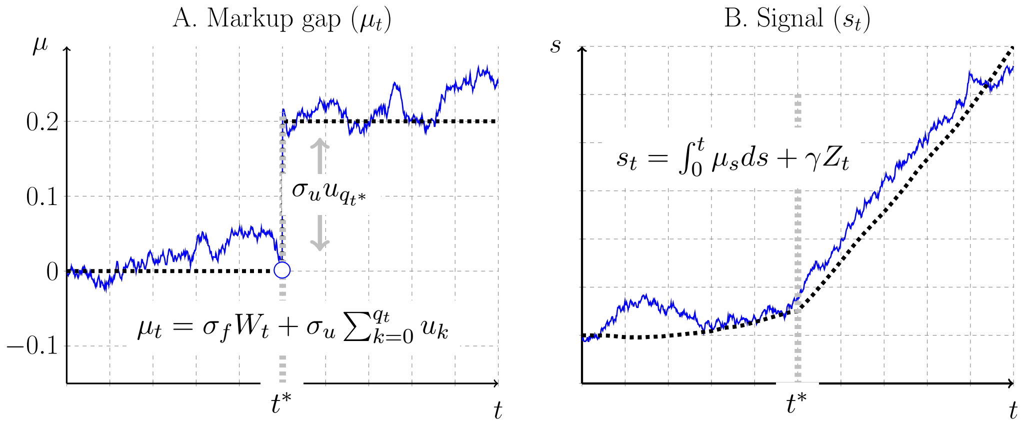
[5]Golosov and Lucas (2007) use a mean reverting process for productivity instead of a random walk. Still, our results concerning small frequent shocks will be compared with their setup.

[6]Rewrite the signal as $s_t = \int_0^t \mu_s ds + \gamma Z_t$ that is the sum of an integral and a Wiener process, and therefore it is continuous. See Øksendal (2007) for details about filtering problems in continuous time.

methods that would solve the filtering problem without our assumptions involve approximations as in the Kim (1994) filter or infinite dimensional states as in particle filters, which are not suitable for solving the inaction problem.

Figure I illustrates the evolution of the markup gap and the signal process. It assumes that there is a regime change at time t^*. At that moment, the average *level* of the markup gap jumps to a new value; nevertheless, the signal has continuous paths and only its *slope* changes to a new average value.

Figure I: ILLUSTRATION OF THE PROCESS FOR THE MARKUP GAP AND THE SIGNAL.



Panel A describes a sample path of the markup gap. The dotted black line describes the compound Poisson process and the blue line describes the markup gap (the sum of the compound Poisson process and the Wiener process). t^* is the date of an increase in the Poisson counter. Panel B describes a sample path for the signal. The dotted black lines describes the drift and the solid blue line describes the signal (the sum of the drift and the local volatility).

2.2 Filtering problem

This section describes the filtering problem and derives the laws of motion for estimates and estimation variance, our measure of uncertainty. The key challenge is to keep the finite state properties of the Gaussian model and apply Bayesian estimation in a jump-diffusion framework. Álvarez, Lippi and Paciello (2011) analyze the filtering problem without the jumps and they show that the steady state of such a model is equal to a perfect information model. Our contribution extends the Kalman–Bucy filter beyond the standard assumption of Brownian motion innovations. We are able to represent the posterior distribution of markup gaps $\mu_t|\mathcal{I}_t$ as a function of mean and variance. To our knowledge, this is a novel result in the filtering literature.

Firms make estimates in a Bayesian way by optimally weighing new information contained in signals against old information from previous estimates. This is a passive learning technology in the sense that firms process the information that is available to them, but they cannot make any action to change the quality of the signals; this contrasts with the active learning models in Bachmann and Moscarini (2011), Willems (2013), and Argente and Yeh (2015) where firms learn the elasticity of their demand by experimenting with price changes.

Estimates and uncertainty Let $\hat{\mu}_t \equiv \mathbb{E}[\mu_t|I_t]$ be the best estimate (in a mean-squared error sense) of the markup gap and let $\Sigma_t \equiv \mathbb{E}[(\mu_t - \hat{\mu}_t)^2|I_t]$ be its variance. Firm level uncertainty is defined as $\Omega_t \equiv \frac{\Sigma_t}{\gamma}$, which is the estimation variance normalized by the signal volatility. Proposition 1 below establishes the laws of motion for estimates and uncertainty for our drift-less case. In the Appendix we provide the generalization of the Kalman-Bucy filter to a jump-diffusion process with drift.

Proposition 1 (**Filtering equations**). *Let the markup gap and the signal evolve according to the following processes:*

$$\begin{aligned} (state) \qquad d\mu_t &= \sigma_f dW_t + \sigma_u u_t t dq_t, \qquad && \mu_0 \sim \mathcal{N}(a,b) \\ (signal) \qquad ds_t &= \mu_t dt + \gamma dZ_t, && s_0 = 0 \end{aligned}$$

where W_t, Z_t are Wiener processes, q_t is a Poisson process with intensity λ, $u_t \sim \mathcal{N}(0,1)$, and a, b are constants. Let the information set be given by $\mathcal{I}_t = \sigma\{s_r, q_r; r \leq t\}$, and define the markup estimate $\hat{\mu}_t \equiv \mathbb{E}[\mu_t|\mathcal{I}_t]$ and the estimation variance $\Sigma_t \equiv \mathbb{V}[\mu_t|\mathcal{I}_t] = \mathbb{E}[(\mu_t - \hat{\mu}_t)^2|\mathcal{I}_t]$. Finally, define firm uncertainty as the estimation variance normalized by the signal noise: $\Omega_t = \frac{\Sigma_t}{\gamma}$. Then the posterior distribution of markups is Gaussian $\mu_t|\mathcal{I}_t \sim \mathcal{N}(\hat{\mu}_t, \gamma\Omega_t)$, where $(\hat{\mu}_t, \Omega_t)$ satisfy

$$d\hat{\mu}_t = \Omega_t d\hat{Z}_t, \qquad \hat{\mu}_0 = a \tag{3}$$

$$d\Omega_t = \frac{\sigma_f^2 - \Omega_t^2}{\gamma} dt + \frac{\sigma_u^2}{\gamma} dq_t, \qquad \Omega_0 = \frac{b}{\gamma} \tag{4}$$

$\hat{Z}_t$ is the innovation process given by $d\hat{Z}_t = \frac{1}{\gamma}(ds_t - \hat{\mu}_t dt) = \frac{1}{\gamma}(\mu_t - \hat{\mu}_t)dt + dZ_t$ and it is one-dimensional Wiener process under the probability distribution of the firm independent of dq_t.

Proof. All proofs are given in the Appendix. □

The proof consists of three steps. First, we show that the solution to the system of stochastic differential equations in (1) and (2), conditional on the history of Poisson shocks, follows a Gaussian process; second, we show that $\mu_t|\mathcal{I}_t$ is a Gaussian random variable where its mean and variance can be obtained as the limit of a discrete sampling of observations; and third, we show that the laws of motion of markup estimates and uncertainty obtained with discrete sampling converge to the system given by (3) and (4). We now discuss each filtering equation with detail.

Higher uncertainty implies more volatile estimates Equation (3) says that the estimate $\hat{\mu}_t$ is a Brownian motion driven by the innovation process $\hat{Z}_t$ with stochastic volatility[7] given by Ω_t. The stochastic process $\hat{Z}_t$ is the difference between the markup gap estimate and the signal, which under the firm's information set is a Wiener process independent of dq_t. We can see this property using a discrete time approximation of the estimates process in (3) and the signal process in (2).

Consider a small period of time Δ. As a discrete process, the markup gap estimate at time $t + \Delta$ is given by the Bayesian convex combination of the previous estimate $\hat{\mu}_t$ and the signal change $s_t - s_{t-\Delta}$

[7]In Section 3 we discuss the differences between our model with stochastic volatility that arises from learning Ω_t and a model with fundamental stochastic volatility $\sigma_f(t)$ as in Vavra (2014).

(see Appendix for a formal proof)

$$\hat{\mu}_{t+\Delta}\Delta = \underbrace{\frac{\gamma}{\Omega_t\Delta + \gamma}}_{\text{weight on prior estimate}} \hat{\mu}_t\Delta + \underbrace{\left(1 - \frac{\gamma}{\Omega_t\Delta + \gamma}\right)}_{\text{weight on signal}} (s_t - s_{t-\Delta}) \tag{5}$$

A discrete time approximation of the signal is given by:

$$s_t = s_{t-\Delta} + \mu_t\Delta + \gamma\sqrt{\Delta}\epsilon_t, \quad \epsilon_t \sim \mathcal{N}(0,1) \tag{6}$$

Substituting (6) into (5) and rearranging we obtain:

$$\hat{\mu}_{t+\Delta} - \hat{\mu}_t = \frac{\Omega_t}{\Omega_t\Delta + \gamma} \underbrace{\left((\mu_t - \hat{\mu}_t)\Delta + \gamma\sqrt{\Delta}\epsilon_t\right)}_{\to\, \gamma d\hat{Z}_t} \tag{7}$$

Since the estimate $\hat{\mu}_t$ is unbiased, the term inside parentheses has all the properties of a Wiener process. Therefore, $\hat{\mu}_t$ follows an Itō process with local variance given by Ω_t.

The same approximation in (5) makes evident that, when uncertainty is high, the estimates put more weight on the signals than on the previous estimate. This means that the estimate incorporates more information about the current markup μ_t; in other words learning is faster, but it also brings more white noise ϵ_t into the estimation. Therefore, estimates become more volatile with high uncertainty. This effect will be key in our discussion of firms' responsiveness to monetary shocks, as with high uncertainty the markup estimates will incorporate the monetary shock faster and responsiveness will be larger.

Uncertainty cycles Regarding the evolution of uncertainty, Equation (4) shows that it is composed of a deterministic and a stochastic component, where the latter is active whenever the markup gap receives a regime change. Let's study each component separately. In the absence of regime changes ($\lambda = 0$), uncertainty Ω_t follows a deterministic path which converges to the constant volatility of the continuous shocks σ_f, i.e. the fundamental volatility of the markup gap. The deterministic convergence is a result of the learning process: as time goes by, estimation variance decreases until the only volatility left is fundamental. In the model with regime changes ($\lambda > 0$), uncertainty jumps up on impact with the arrival of regime change and then decreases deterministically until the arrival of a new regime that will push uncertainty up again. The time series profile of uncertainty features a saw-toothed profile that never stabilizes due to the recurrent nature of these shocks. If the arrival of the infrequent shocks were not known and instead the firm had to filter their arrival as well, uncertainty would feature a hump-shaped profile instead of a jump. Although uncertainty never settles down, it is convenient to characterize the level of uncertainty such that its expected change is equal to zero, $\mathbb{E}\left[d\Omega_t \middle| \mathcal{I}_t\right] = 0$. It is equal to the variance of the state $\mathbb{V}[\mu_t] = \Omega^{*2}t$, hence we call this "fundamental" uncertainty with a value of $\Omega^* \equiv \sqrt{\sigma_f^2 + \lambda\sigma_u^2}$. The next section shows that the ratio of current to fundamental uncertainty is a key determinant of decision rules and price statistics.

Further comments on the filtering problem A notable characteristic of this filtering problem is that point estimates, as well as the signals and innovations, have continuous paths even though the underlying state is discontinuous. The continuity of these paths comes from two facts. First, changes in the state affect the slope of the innovations and signals but not their levels; second, the expected size of an infrequent shock u_t is zero. As a consequence of the continuity, markup estimations are not affected by the arrival of a regime change; only uncertainty features jumps. It is also worth noticing that both the filtered estimates $\mu_t|\mathcal{I}_t$ and smoothed estimates $\mu_{t-\delta}|\mathcal{I}_t$, for any $\delta > 0$ are Gaussian. In contrast, the predicted estimate $(\mu_{t+\delta}|\mathcal{I}_t)$ is not. For instance, in the case $\sigma_f = 0$, the predicted markup has Laplace distribution with fat tails. We focus our attention on the filtered estimate since it is the only input in our firm's decision problem. We leave for further research the analysis of other estimates.

2.3 Decision rules

With the filtering problem at hand, this section derives the price adjustment decision of the firm.

Sequential problem Let $\{\tau_i\}_{i=1}^{\infty}$ be the series of dates where the firm adjusts her markup gap and $\{\mu_{\tau_i}\}_{i=1}^{\infty}$ the series of reset markup gaps on the adjusting dates. Given an initial condition μ_0, the law of motion for markup gaps, and the filtration $\{\mathcal{I}_t\}_{t=0}^{\infty}$, the sequential problem of the firm is described by:

$$\max_{\{\mu_{\tau_i},\tau_i\}_{i=1}^{\infty}} -\mathbb{E}\left[\sum_{i=0}^{\infty} e^{-r\tau_{i+1}}\left(\theta + \int_{\tau_i}^{\tau_{i+1}} e^{-r(s-\tau_{i+1})} B\mu_s^2 \, ds\right)\right] \tag{8}$$

The sequential problem is solved recursively as a stopping time problem using the Principle of Optimality (see Øksendal (2007) and Stokey (2009) for details). This is formalized in Proposition 2. The firm's state has two components: the point estimate of the markup gap $\hat{\mu}$ and the level of uncertainty Ω attached to that estimate. Given her current state $(\hat{\mu}_t, \Omega_t)$, the firm policy consists of (i) a stopping time τ, which is a measurable function with respect to the filtration $\{\mathcal{I}_t\}_{t=0}^{\infty}$; and (ii) the new markup gap μ'.

Proposition 2 (Stopping time problem). *Let $(\hat{\mu}_0, \Omega_0)$ be the firm's current state immediately after the last markup adjustment. Also let $\bar{\theta} = \frac{\theta}{B}$ be the normalized menu cost. Then the optimal stopping time and reset markup gap (τ, μ') solve the following problem:*

$$V(\hat{\mu}_0, \Omega_0) = \max_{\tau} \mathbb{E}\left[\int_0^{\tau} -e^{-rs}\hat{\mu}_s^2 ds + e^{-r\tau}\Big(-\bar{\theta} + \max_{\mu'} V(\mu', \Omega_\tau)\Big)\Big|\mathcal{I}_0\right] \tag{9}$$

subject to the filtering equations in Proposition 1.

Observe in Equation (9) that the estimates enter directly into the instantaneous return, while uncertainty affects only the continuation value. To be precise, uncertainty does have a negative effect on current profits that reflects the firm's permanent ignorance about her true productivity. However, this loss is constant and can be treated as a sunk cost; thus it is set to zero.

Inaction region The solution to the stopping time problem is characterized by an inaction region $\mathcal{R}$ such that the optimal time to adjust is given by the first time that the state falls outside such region:

$$\tau = \inf\{t > 0 : (\mu_t, \Omega_t) \notin \mathcal{R}\}$$

Since the firm has two states, the inaction region is two-dimensional, in the space of markup gap estimations and uncertainty. Let $\bar{\mu}(\Omega)$ denote the inaction region's border as a function of uncertainty. The inaction region is described by the set:

$$\mathcal{R} = \{(\mu, \Omega) : |\mu| \leq \bar{\mu}(\Omega)\}$$

The symmetry of the inaction region around zero is inherited from the specification of the stochastic process, the quadratic profits, and zero inflation. Notice that this is a non-standard inaction problem since it is two-dimensional. In order to solve it, we derive the Hamilton-Jacobi-Bellman equation, the value matching condition, and, following Øksendal and Sulem (2010), we ensure that the standard smooth pasting condition is satisfied by both states. Proposition 3 formalizes these points.

Proposition 3 (**HJB Equation**). *Let $V(\hat{\mu}, \Omega)$ be the value of the firm and V_x denote the derivative of V with respect to x. For all states inside the inaction region $\mathcal{R}$, V satisfies:*

1. *the Hamilton-Jacobi-Bellman (HJB) equation:*

$$rV(\hat{\mu}, \Omega) = -\hat{\mu}^2 + \left(\frac{\sigma_f^2 - \Omega^2}{\gamma}\right) V_\Omega(\hat{\mu}, \Omega) + \frac{\Omega^2}{2} V_{\hat{\mu}^2}(\hat{\mu}, \Omega) + \lambda \left[V\left(\hat{\mu}, \Omega + \frac{\sigma_u^2}{\gamma}\right) - V(\hat{\mu}, \Omega)\right]$$

2. *the value matching condition that sets equal the value of adjusting and not adjusting at the border:*

$$V(0, \Omega) - \bar{\theta} = V(\bar{\mu}(\Omega), \Omega)$$

3. *two smooth pasting conditions, one for each state:* $V_{\hat{\mu}}(\bar{\mu}(\Omega), \Omega) = 0, \quad V_\Omega(\bar{\mu}(\Omega), \Omega) = V_\Omega(0, \Omega.$

A key property of the HJB is the lack of interaction terms between uncertainty and markup gap estimates. This property is implied by the passive learning process in which the firm cannot change the quality of the information flow by changing her markup. Using the HJB equation and other conditions, Proposition 4 gives an analytical characterization of the inaction region's border $\bar{\mu}(\Omega)$. The proof uses a Taylor expansion of the value function.[8]

Proposition 4 (**Inaction region**). *For r and $\bar{\theta}$ be small, the border of the inaction region $\bar{\mu}(\Omega)$ is approximated by*

$$\bar{\mu}(\Omega) = \left(\frac{6\bar{\theta}\Omega^2}{1 + \mathcal{L}^{\bar{\mu}}(\Omega)}\right)^{1/4}, \quad \textit{with} \quad \mathcal{L}^{\bar{\mu}}(\Omega) = \frac{\Omega^2 - \Omega^{*2}}{\gamma} \frac{3}{2} \left(6\bar{\theta}\Omega^{*2}\right)^{1/4} \tag{10}$$

The elasticity of $\bar{\mu}(\Omega)$ with respect to Ω is equal to

$$\mathcal{E}(\Omega) \equiv \frac{1}{2} - \frac{3}{\gamma} \left(6\bar{\theta}\Omega^{*2}\right)^{1/4} \Omega^2 \tag{11}$$

Lastly, the reset markup gap is equal to $\hat{\mu}' = 0$.

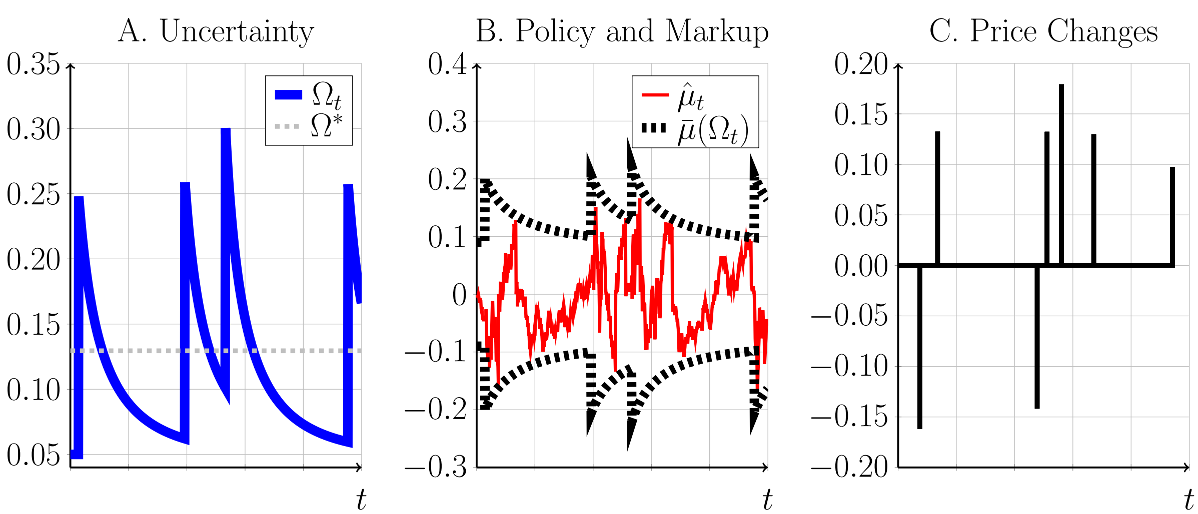
[8] In the Online Appendix we show that this is a good approximation.

Higher uncertainty implies wider inaction region The numerator of the inaction region $\bar{\mu}(\Omega)$ in equation (10) is increasing in uncertainty and captures the well-known *option value effect* (see Barro (1972) and Dixit (1991)). As a result of belief dynamics, the option value is time-varying and driven by uncertainty. In the denominator there is a new factor $\mathcal{L}^{\bar{\mu}}(\Omega)$ that amplifies or dampens the option value effect depending on the ratio of current uncertainty to fundamental uncertainty $\frac{\Omega}{\Omega^*}$. When current uncertainty is high with respect to its average level $\left(\frac{\Omega^2}{\Omega^{*2}} > 1\right)$, uncertainty is expected to decrease ($\mathbb{E}[d\Omega] < 0$) and therefore future option values also decrease. This feeds back into the current inaction region shrinking it as $\mathcal{L}^{\bar{\mu}}(\Omega) > 0$. Analogously, when uncertainty is low with respect to its average level $\left(\frac{\Omega^2}{\Omega^{*2}} < 1\right)$, it is expected to increase ($\mathbb{E}[d\Omega] > 0$) and thus the option values in the future also increase. This feeds back into current bands that get expanded as $\mathcal{L}^{\bar{\mu}}(\Omega) < 0$.

The overall effect of uncertainty on the inaction region also depends on the size of the menu cost and the signal noise. The expression (10) shows that small menu costs θ paired with large signal noise γ make the factor $\mathcal{L}^{\bar{\mu}}(\Omega)$ close to zero, implying that the elasticity of the inaction region with respect to uncertainty $\mathcal{E}(\Omega)$ in (11) is close to $1/2$ and thus the inaction region is increasing in uncertainty. This implies that the size of prices changes done by uncertain firms will be larger.

Figure II shows a particular firm realization for the parametrization we will use in our quantitative exercise, which has small menu costs $\bar{\theta}$ and large signal noise γ. Panel A shows the evolution of uncertainty, which follows a saw-toothed profile: it decreases monotonically with learning until a regime change happens and makes uncertainty jump up; then, learning brings uncertainty down again. The dashed horizontal line is the average fundamental uncertainty Ω^*. Panel B plots the estimate of the markup gap and the inaction region. This path is inherited by the inaction region because the calibration makes the inaction region increasing in uncertainty. Finally, Panel C shows the magnitude of price changes. These changes are triggered when the markup gap estimate touches the border of the inaction region.

Figure II: Sample paths for one firm.



Panel A: Uncertainty (solid line) and fundamental uncertainty (horizontal dotted line). Panel B: Markup gap estimate (solid line) and inaction region (dotted line). Panel C: Price changes.

Note that without regime changes, uncertainty would converge to the fundamental volatility in the long run, i.e., $\Omega \to \sigma_f$. The inaction region would also become constant and akin to that of a steady state model without information frictions, namely $\bar{\mu} = \left(6\bar{\theta}{\sigma_f}^2\right)^{1/4}$. That is the case analyzed in the Online Appendix in Álvarez, Lippi and Paciello (2011). As that paper shows, such a model collapses to that of Golosov and Lucas (2007) where there is no price change size dispersion, since all firms would have the same inaction region. Therefore, both the regime changes and the information friction are key to generate the cross-sectional variation in price setting that arises from the heterogenous uncertainty.

How does uncertainty affect the adjustment frequency? Notice that price changes appear to be clustered over time, that is, there are recurrent periods with high adjustment frequency followed by periods of low adjustment frequency. Figure II shows that after a regime change arrives, the estimation becomes more volatile, which increases the probability of hitting the bands and changing the price. As a response to higher volatility and to save on menu costs, the inaction region becomes wider, which reduces the probability of a price change. Therefore, we have two opposite forces acting on the adjustment frequency. Since the elasticity of the inaction region with respect to uncertainty is less than unity, the volatility effect dominates and higher uncertainty brings more price changes. We formalize these observations in the following section on price statistics.

3 Uncertainty and micro-price statistics

In this section we characterize analytically two price statistics that are crucial to understand the economy's response to aggregate nominal shocks: the expected duration of prices and the hazard rate of price adjustment. First, we focus on price statistics *conditional* on a level of uncertainty, and we shed light on the role of uncertainty in pricing behavior. We show that higher uncertainty decreases price duration (increases the adjustment frequency) and that the hazard rate of price adjustment is decreasing for firms with a high level of uncertainty. Furthermore, we show that the slope of the hazard rate is determined by the volatility of the signal noise. To obtain these results, we require an elasticity of the inaction region with respect to uncertainty that is less than unity.

Second, we aggregate the conditional statistics to generate the *unconditional* statistics that we observe in the data. For aggregation, we use the renewal distribution of uncertainty, which is the distribution of uncertainty of adjusting firms. We show that this renewal distribution puts more weight on high levels of uncertainty than does the steady state distribution of uncertainty. This implies that aggregate statistics reflect the behavior of highly uncertain firms, and therefore, decreasing hazard rates are also observed in the aggregate.

3.1 Expected time

In Proposition 5 we formalize a positive relationship between adjustment frequency and uncertainty, as observed in Figure II. It is followed by Proposition 6 which formalizes a positive relationship between adjustment frequency and uncertainty dispersion. These relationships prove to be very useful to back out an unobservable state – uncertainty – with observable price statistics.

Proposition 5 (**Conditional Expected Time**). *Let r and $\bar{\theta}$ be small. The expected time for the next price change conditional on the state, denoted by $\mathbb{E}[\tau \big| \hat{\mu}, \Omega]$, is approximated as:*

$$\mathbb{E}[\tau \big| \hat{\mu}, \Omega] = \frac{\bar{\mu}(\Omega)^2 - \hat{\mu}^2}{\Omega^2} \left(1 + \mathcal{L}^\tau(\Omega)\right) \quad \textit{where} \quad \mathcal{L}^\tau(\Omega) \equiv \left(\frac{\Omega}{\Omega^*} - 1\right) \left(1 - \mathcal{E}(\Omega^*)\right) \left(\frac{4\gamma(6\bar{\theta})^{1/2}}{\gamma + 2(6\bar{\theta})^{1/2}}\right) \tag{12}$$

If the elasticity of the inaction region with respect to uncertainty is lower than unity and signal noise is large, then the expected time between price changes (i.e. $\mathbb{E}[\tau \big| 0, \Omega]$) is a decreasing and convex function of uncertainty.

The expected time between price changes has two terms. The first term $\frac{\bar{\mu}(\Omega)^2 - \hat{\mu}^2}{\Omega^2}$ is standard, and it states that the closer the current markup gap is to the border of the inaction region, then the shorter the expected time for the next adjustment. This term is decreasing in uncertainty with an elasticity larger than unity in absolute value, and it is time-varying. The second term $\mathcal{L}^\tau(\Omega)$ amplifies or dampens the first effect depending on the level of uncertainty, and it has an elasticity equal to unity with respect to uncertainty. Therefore, uncertainty's overall effect on the expected time to adjustment is negative: a firm with high uncertainty is going to change the price more frequently than a firm with low uncertainty.

As mentioned in the introduction, there is empirical evidence of this positive relationship between uncertainty and adjustment frequency. Bachmann, Born, Elstner and Grimme (2013) use German survey data to document a positive relationship between firm-level belief uncertainty, measured as the variance of sales' forecast errors, and the individual adjustment frequency; Vavra (2014) uses BLS micro-price data to document a positive relationship between the cross-sectional dispersion of price changes – another measure of uncertainty – and the individual frequency of price changes.

Proposition 6 establishes a positive relationship between uncertainty dispersion and adjustment frequency, and between uncertainty dispersion and price change dispersion. It generalizes Proposition 1 found in Álvarez, Le Bihan and Lippi (2014) for general Ω_t and it demonstrates a very intuitive link between uncertainty dispersion and price statistics. The key point is that observable price statistics provide a way to back-out the level of heterogeneity in an unobserved state, uncertainty.

Proposition 6 (**Uncertainty and Frequency**). *The following relationship between uncertainty dispersion, average price duration, and price change dispersion holds:*

$$\mathbb{E}[\Omega^2] = \frac{\mathbb{V}[\Delta p]}{\mathbb{E}[\tau]} \tag{13}$$

Holding fixed uncertainty's cross-sectional dispersion in the left-hand side, expression (13) establishes a positive link between average price duration and price change dispersion. Prices either change often for small amounts or rarely for large amounts. This implication of menu cost models can be tested empirically, for instance, using price statistics from different sectors. As an alternative way to read this relationship, consider a fixed price change dispersion; then heterogeneity in uncertainty and average price duration are negatively related. Underlying these results is a Jensen inequality and the fact that frequency decreases with price age. We turn next into characterizing the hazard rate, which is a dynamic measure of adjustment frequency.

3.2 Hazard rate

Let $h_\tau(\Omega)$ be the conditional hazard rate of price adjustment. It is the probability of changing the price at date τ since the last price change and conditional on a current level of uncertainty Ω. It is computed as $h_\tau(\Omega) \equiv \frac{f(\tau|\Omega)}{\int_\tau^\infty f(s|\Omega)ds}$, where $f(s|\Omega)$ is the conditional distribution of stopping times. It reflects the probability of exiting the inaction region, or first passage time. Without loss of generality, assume the last adjustment occurred at time $t = 0$ and denote price duration with $\tau > 0$. The hazard rate is a function of two objects:

i) estimate's unconditional variance: this is the variance of the estimate at a future date τ from a time $t = 0$ perspective, which we denote by $\mathcal{V}_\tau(\Omega_0)$

$$\hat{\mu}_\tau|\mathcal{I}_0 \sim \mathcal{N}(0, \mathcal{V}_\tau(\Omega_0))$$

ii) expected path of the inaction region $\bar{\mu}(\Omega)$ given the information available at time $t = 0$.

An analytical characterization of the hazard rate is provided in Proposition (7). The key message is that the concavity of the unconditional variance $\mathcal{V}_\tau(\Omega_0)$ determines the shape of the hazard function, because it measures how fast learning occurs. The presence of infrequent shocks only changes the level of the hazard rate but not its slope; thus we characterize the hazard rate assuming no infrequent shocks ($\lambda = 0$). Furthermore, the inaction region is assumed to be constant. This is also a valid assumption since what matters for the hazard rate is the inaction region's size relative to the volatility of the uncontrolled process. The validity of both assumptions is explored in the Online Appendix where we compute the numerical hazard rate.

Proposition 7 (Conditional Hazard Rate). *Without loss of generality, assume the last price change occurred at $t = 0$ and let $\Omega_0 > \sigma_f$ be the level of uncertainty. There are no infrequent shocks ($\lambda = 0$) and a constant inaction region $\bar{\mu}(\Omega_\tau) = \bar{\mu}_0$. Denote derivatives with respect to τ with a prime ($h'_\tau \equiv \partial h/\partial \tau$).*

1. *The estimate's unconditional variance, denoted by $\mathcal{V}_\tau(\Omega_0)$, is given by:*

$$\mathcal{V}_\tau(\Omega_0) = \sigma_f^2 \tau + \mathcal{L}_\tau^{\mathcal{V}}(\Omega_0) \tag{14}$$

where $\mathcal{L}_\tau^{\mathcal{V}}(\Omega_0) \equiv \gamma(\Omega_0 - \Omega_\tau)$, with $\mathcal{L}_0^{\mathcal{V}}(\Omega_0) = 0$, $\lim_{\tau\to\infty} \mathcal{L}_\tau^{\mathcal{V}}(\Omega_0) = \gamma(\Omega_0 - \sigma_f)$, and equal to:

$$\mathcal{L}_\tau^{\mathcal{V}}(\Omega_0) = \gamma\Omega_0 - \gamma\sigma_f \left(\frac{\frac{\Omega_0}{\sigma_f} + \tanh\left(\frac{\sigma_f}{\gamma}\tau\right)}{1 + \frac{\Omega_0}{\sigma_f}\tanh\left(\frac{\sigma_f}{\gamma}\tau\right)} \right)$$

2. *$\mathcal{V}_\tau(\Omega_0)$ is increasing and concave in duration τ: $\mathcal{V}'_\tau(\Omega_0) > 0$ and $\mathcal{V}''_\tau(\Omega_0) < 0$. Furthermore, the following cross-derivatives with initial uncertainty are positive:*

$$\frac{\partial \mathcal{V}_\tau(\Omega_0)}{\partial \Omega_0} > 0, \qquad \frac{\partial \mathcal{V}'_\tau(\Omega_0)}{\partial \Omega_0} > 0, \qquad \frac{\partial |\mathcal{V}''_\tau(\Omega_0)|}{\partial \Omega_0} > 0$$

3. *The hazard of adjusting the price at date τ, conditional on Ω_0, is characterized by:*

$$h_\tau(\Omega_0) = \frac{\pi^2}{8} \underbrace{\frac{\mathcal{V}'_\tau(\Omega_0)}{\bar{\mu}_0^2}}_{\textit{decreasing in } \tau} \underbrace{\Psi\left(\frac{\mathcal{V}_\tau(\Omega_0)}{\bar{\mu}_0^2}\right)}_{\textit{increasing in } \tau} \tag{15}$$

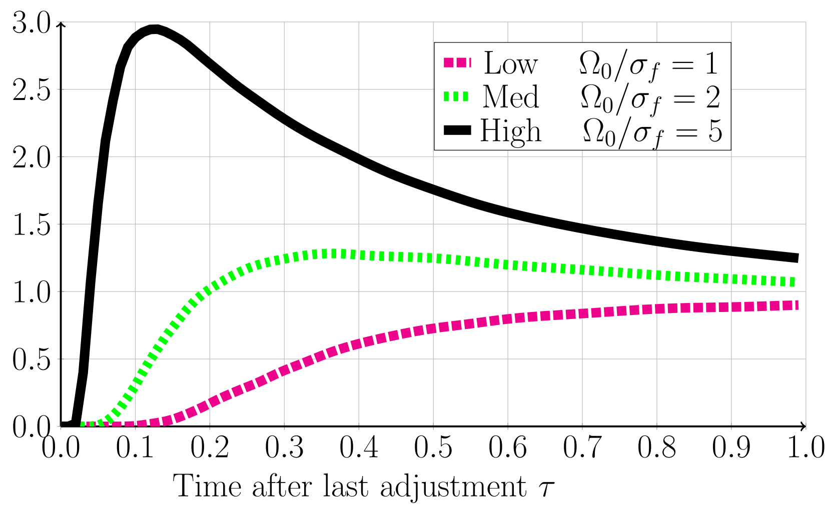
where $\Psi(x) \geq 0$, $\Psi(0) = 0$, $\Psi'(x) > 0$, $\lim_{x\to\infty} \Psi(x) = 1$, first convex then concave, and it is given by:

$$\Psi(x) = \frac{\sum_{j=0}^{\infty} \alpha_j \exp(-\beta_j x)}{\sum_{j=0}^{\infty} \frac{1}{\alpha_j} \exp(-\beta_j x)}, \quad \alpha_j \equiv (-1)^j(2j+1), \quad \beta_j \equiv \frac{\pi^2}{8}(2j+1)^2$$

4. *Exists a $\tau^*(\Omega_0)$ such that the slope of the hazard rate is negative for $\tau > \tau^*(\Omega_0)$, and $\tau^*(\Omega_0)$ is decreasing in Ω_0.*

Estimate's unconditional variance $\mathcal{V}_\tau(\Omega_0)$ in (14) captures the evolution of uncertainty. The first term, $\sigma_f^2\tau$, refers to the linear time trend that comes from the fact that fundamental shocks follow a Brownian Motion. The second term, $\mathcal{L}_\tau^{\mathcal{V}}(\Omega_0)$, is an additional source of variance coming from imperfect information. The second point in Proposition (7) establishes that higher initial uncertainty increases the level, slope, and concavity of this additional variance. In other words, higher initial uncertainty brings higher expected gains from learning. In the third point, we show that the hazard rate with imperfect information is given by the product of $\Psi(x)$, an increasing function of τ, times the derivative of the unconditional variance $\mathcal{V}'_\tau$, a decreasing function of τ. The function $\Psi(x)$ characterizes the hazard rate with perfect information as derived in Álvarez, Lippi and Paciello (2011), which in turn uses a transformation of the stopping time density by Kolkiewicz (2002). Therefore, there are two opposing forces acting upon the slope of the hazard rate. Finally, the fourth point states that there exists a date after which the hazard is downward sloping. If the initial uncertainty is larger with respect to its lower bound σ_f, then the decreasing force becomes stronger and the hazard's slope is negative for a larger range of price durations. Figure (III) illustrates the hazard rate for different initial conditions Ω_0.

Figure III: Hazard Rate Conditional on Initial Uncertainty



Decreasing hazard and noise volatility The economics behind the decreasing hazard rate are as follows. Because of learning, firm uncertainty decreases with time and the weight given to new observations in the forecasting process decreases too. Since the markup gap estimates' volatility is reduced, the probability of adjusting also decreases. A firms expects to transition from high uncertainty and frequent adjustment to low uncertainty and infrequent adjustments. The speed of the transition is determined by the level of information frictions as captured by the noise volatility γ. If noise volatility is high, a firm will take a long time after a regime switch to learn her new level of permanent productivity. Both uncertainty and adjustment frequency remain high for many periods and the hazard rate is flat; in contrast, when the noise volatility is low, a firm learns quickly her new level of permanent productivity, both uncertainty and adjustment frequency fall after a few periods, and the hazard rate is relatively steep. Therefore γ can be chosen to match the shape of the hazard rate.

3.3 Belief uncertainty vs. stochastic volatility

The uncertainty shocks in this paper contrast with the stochastic volatility processes for productivity used in Vavra (2014) and Karadi and Reiff (2014). Our definition of uncertainty concerns idiosyncratic beliefs; it is the conditional variance of the estimates of markup gaps. The volatility of the markup process is a known constant Ω^*; it is the realizations which are unknown. In these other papers, there is perfect information but the volatility of the markup process is stochastic. Regardless of the structure, however, the positive relationship between the frequency of price changes and the uncertainty (or volatility) faced by the firm is maintained.

A natural question arises: can we distinguish our model of endogenous uncertainty with one of exogenous stochastic fundamental volatility? The answer is yes: a model with stochastic volatility generates an increasing hazard rate, while the learning model generates a decreasing hazard rate.

The exercise is documented in detail in the Appendix. We calibrate the stochastic volatility process to match the frequency of price changes in the data, as well as the autocorrelation and cross-sectional dispersion of volatility/uncertainty in the learning model. We obtain an increasing hazard rate for the stochastic volatility model. This is robust to changes in the persistence of the stochastic volatility process. The reason for this result lies in the dynamics for volatility. The AR(1) process produces smooth changes in volatility, whereas the Poisson shock we have large changes in uncertainty. The price change distribution obtained with the stochastic volatility model has much lower dispersion and kurtosis (thinner tails) compared to the learning model.

3.4 Aggregation

In the data we observe unconditional statistics. These moments are equal to the weighted average of the conditional statistics, where the weights are given by the renewal distribution of uncertainty. The renewal distribution is the stationary distribution of uncertainty *conditional* on price adjustment: it is the uncertainty faced by adjusting firms. Such distribution is different from the *unconditional* steady state distribution of uncertainty, which is the uncertainty in the entire cross-section. Importantly, micro price statistics are the outcomes of aggregation using the renewal distribution of uncertainty. This section characterizes analytically the ratio between these two distributions.

The distribution of price adjuster uncertainty– the renewal distribution – is difficult to compute analytically because of the jump process. Nevertheless, we can characterize the ratio between the renewal distribution and marginal distribution over uncertainty to show that it is increasing in uncertainty. The next proposition formalizes this result.

Proposition 8 (**Renewal distribution**). *Let $f(\hat{\mu}, \Omega)$ be the joint density of markup gaps and uncertainty in the population of firms. Let $r(\Omega)$ be denote the density of uncertainty conditional on adjusting, or renewal distribution. Assume the inaction region is increasing in uncertainty (i.e. $\bar{\mu}'(\Omega) > 0$). Then we have the following results:*

- *For each $(\hat{\mu}, \Omega)$, we can write the joint density as $f(\hat{\mu}, \Omega) = h(\Omega) g(\hat{\mu}, \Omega)$, where $g(\hat{\mu}, \Omega)$ is the density of markup gap estimates conditional on uncertainty and $h(\Omega)$ is the marginal density of uncertainty.*

- *The ratio between the renewal and marginal distributions of uncertainty is approximated by*

$$\frac{r(\Omega)}{h(\Omega)} \propto |g_{\hat{\mu}}(\bar{\mu}(\Omega), \Omega)| \Omega^2 \tag{16}$$

  where $g(\mu, \Omega)$ solves the following differential equation

$$\frac{\Omega^2 - \Omega^{*2}}{\gamma} g_{\Omega}(\hat{\mu}, \Omega) + \frac{\Omega^2}{2} g_{\hat{\mu}^2}(\hat{\mu}, \Omega) = 0$$

  with border conditions:

$$g(\bar{\mu}(\Omega), \Omega) = 0 \qquad \int_{-\bar{\mu}(\Omega)}^{\bar{\mu}(\Omega)} g(\mu, \Omega) d\mu = 1$$

- *If $\Omega = \Omega^*$, then the ratio is proportional to the inverse of the expected time between price adjustments. Then if the inaction region's elasticity to uncertainty is lower than unity, the ratio is an increasing function of uncertainty:*

$$\frac{r(\Omega^*)}{h(\Omega^*)} \propto \frac{\Omega^{*2}}{\bar{\mu}(\Omega^*)^2} = \frac{1}{\mathbb{E}[\tau|(0, \Omega^*)]} \tag{17}$$

The last point of Proposition (8) states that if inaction regions are relatively flat with respect to uncertainty, as it is the case, the renewal distribution is biased towards high levels of uncertainty. This implies that micro-price statistics will reflect more intensively the pricing behavior of highly uncertain firms. In the particular case of the hazard rate, the average hazard rate is decreasing because it puts a higher weight on the decreasing hazard rate of high uncertainty firms compared to the increasing hazard rate of low uncertainty firms.

4 General Equilibrium model

In this section we develop a standard general equilibrium framework with monopolistic firms that face the pricing-setting problem with menu costs and information frictions studied in the previous sections. We extend the environment in Golosov and Lucas (2007) to include the information friction and characterize the steady state of the economy. Then we calibrate the model to match several micro price statistics from CPI data in the US. In particular, we calibrate the signal noise to match the slope of the hazard rate of price adjustment in the data. Finally, as an orthogonal check to our model, we refer to evidence from US scanner data and Mexican CPI data that confirms the age dependence in pricing implied by our model.

4.1 Model

Environment Time is continuous. There is a representative consumer, a continuum of monopolistic firms, and a monetary authority.

Representative Household The household has preferences over consumption C_t, labor N_t, and real money holdings $\frac{M_t}{P_t}$, where P_t is the aggregate price level. She discounts the future at rate $r > 0$.

$$\mathbb{E}_0\left[\int_0^\infty e^{-rt}\Big(\log C_t - N_t + \log\frac{M_t}{P_t}\Big)dt\right] \tag{18}$$

Consumption consists of a continuum of imperfectly substitutable goods indexed by z bundled together with a CES aggregator as

$$C_t = \left(\int_0^1 \Big(A_t(z)c_t(z)\Big)^{\frac{\eta-1}{\eta}} dz\right)^{\frac{\eta}{\eta-1}} \tag{19}$$

where $\eta > 1$ is the elasticity of substitution across goods and $c_t(z)$ is the amount of goods purchased from firm z at price $p_t(z)$. The ideal price index is the minimum expenditure necessary to deliver one unit of the final consumption good, and is given by:

$$P_t \equiv \left[\int_0^1 \left(\frac{p_t(z)}{A_t(z)}\right)^{1-\eta} dz\right]^{\frac{1}{1-\eta}} \tag{20}$$

In the consumption bundle and the price index, $A_t(z)$ reflects the quality of the good, with higher quality providing larger marginal utility of consumption but at a higher price. Quality shocks are firm specific and will be described fully in the firm's problem below. The household has access to complete financial markets. The budget includes income from wages W_t, profits Π_t from the ownership of all firms, and the opportunity cost of holding cash R_tM_t, where R_t is the nominal interest rate. Let Q_t be the stochastic discount factor, or valuation in nominal terms of one unit of consumption in period t. Thus the budget constraint reads:

$$\mathbb{E}_0\left[\int_0^\infty Q_t\left(P_tC_t + R_tM_t - W_tN_t - \Pi_t\right)dt\right] \le M_0 \tag{21}$$

The household problem is to choose consumption of the different goods, labor supply and money holdings to maximize preferences (18) subject to (19), (20) and (21).

Monopolistic Firms On the production side, there is a continuum of firms indexed by $z \in [0,1]$. Each firm produces and sells her product in a monopolistically competitive market. They own a linear technology that uses labor as its only input: producing $y_t(z)$ units of good z requires $l_t(z) = y_t(z)A_t(z)$ units of labor, so that the marginal nominal cost is $A_t(z)W_t$ (higher quality $A_t(z)$ requires more labor input). The assumption that the quality shock enters both the production function and the marginal utility of the household is done for tractability as it helps to condense the numbers of states of the firm into one, the markup, as in Midrigan (2011). Each firm sets a nominal price $p_t(z)$ and satisfies all demand at this posted price. Given the current price $p_t(z)$, the consumer's demand $c_t(z)$, and current quality $A_t(z)$, the instantaneous nominal profits of firm z are equal to the difference between nominal revenues and nominal costs:

$$\Pi(p_t(z), A_t(z)) = c_t(p_t(z), A_t(z))\Big(p_t(z) - A_t(z)W_t\Big) \tag{22}$$

Firms maximize their expected stream of profits, which is discounted at the same rate of the consumer Q_t. They choose either to keep the current price or to change it, in which case they must pay a menu cost θ and reset the price to a new optimal one. Let $\{\tau_i(z)\}_{i=1}^{\infty}$ be a series of stopping times, that is, dates where firm z adjusts her price. The sequential problem of firm z is given by:

$$V(p_0(z), A_0(z)) = \max_{\{p_{\tau_i}(z), \tau_i(z)\}_{i=1}^{\infty}} \mathbb{E}\left[\sum_{i=0}^{\infty} Q_{\tau_{i+1}(z)}\left(-\theta + \int_{\tau_i(z)}^{\tau_{i+1}(z)} \frac{Q_s}{Q_{\tau_{i+1}(z)}} \Pi(p_{\tau_i(z)}, A_s(z))ds\right)\right] \tag{23}$$

with initial conditions $(p_0(z), A_0(z))$ and subject to the quality process described next.

Quality process Firm z's log quality $a_t(z) \equiv \ln A_t(z)$ evolves as the following jump-diffusion process which is idiosyncratic and independent across z:

$$da_t(z) = \sigma_f W_t(z) + \sigma_u u_t(z) dq_t(z) \tag{24}$$

where $W_t(z)$ is a Wiener process and $u_t(z)q_t(z)$ is a compound Poisson process with arrival rate λ and Gaussian innovations $u_t(z) \sim \mathcal{N}(0,1)$ as in the previous sections. As before, firms do not observe their quality directly, and they do not learn it from observing their wage bill or revenues either. The only source of information are noisy signals about quality together with the information that a regime change has hit them. The noisy signals $s_t(z)$ evolve as

$$ds_t(z) = a_t(z)dt + \gamma dZ_t(z) \tag{25}$$

where $Z_t(z)$ is an independent Brownian motion for each firm z and γ is signal noise. Each information set is $\mathcal{I}_t(z) = \sigma\{s_r(z), q_r(z); r \le t\}$. The parameters $\{\sigma_f, \sigma_u, \lambda, \gamma\}$ are identical across firms.

Money supply The monetary authority keeps money supply constant at a level $\bar{M}$.

Equilibrium An equilibrium is a set of stochastic processes for (i) consumption strategies $c_t(z)$, labor supply N_t, and money holdings M_t for the household, (ii) pricing functions $p_t(z)$, and (iii) prices W_t, R_t, Q_t, P_t such that the household and all firms optimize and markets clear at each date.

4.2 Characterization of steady state equilibrium

Household optimality The first order conditions of the household problem establish: nominal wages as a proportion of the (constant) money stock $W_t = r\bar{M}$; the stochastic discount factor as $Q_t = e^{-rt}$; and demand for good z as $c_t(z) = A_t(z)^{\eta-1}\left(\frac{p_t(z)}{P_t}\right)^{-\eta} C_t$.

Constant aggregate prices The equilibrium with constant money supply implies a constant nominal wage $W_t = W$ and a constant nominal interest rate equal to the household's discount factor $R_t = 1 + r$. The ideal price index in (20) is also a constant $P_t = P$. Then nominal expenditure is also constant $P_t C_t = PC = M = W$. Therefore, there is no uncertainty in aggregate variables.

Back to quadratic losses Given the strategy of the consumer $c_t(z)$ and defining markups as $\mu_t(z) \equiv \frac{p_t(z)}{A_t(z)W}$, the instantaneous profits can be written as a function of markups alone:

$$\Pi(p_t(z), A_t(z)) \quad = \quad K \mu_t(z)^{-\eta}\Big(\mu_t(z) - 1\Big)$$

where $K \equiv M\left(\frac{W}{P}\right)^{1-\eta}$ is a constant in steady state. A second order approximation to this expression produces a quadratic form in the markup gap, defined as $\mu_t(z) \equiv \log(\mu_t(z)/\mu^*)$, i.e. the log-deviations of the current markup to the unconstrained markup $\mu^* \equiv \frac{\eta}{\eta-1}$:

$$\Pi(\mu_t(z)) = C - B\mu_t(z)^2$$

where the constants are $C \equiv K\eta^{-\eta}(\eta-1)^{\eta-1}$ and $B \equiv \frac{1}{2}K\frac{(\eta-1)^{\eta}}{\eta^{\eta-1}}$. The constant C does not affect the decisions of the firm and it is omitted for the calculations of decision rules; the constant B captures the curvature of the original profit function. This quadratic problem is the same as 8.

Markup gap estimation and uncertainty The markup gap is equal to

$$\mu_t(z) = \log p_t(z) - a_t(z) - \log W - \log \mu^*$$

When the price is kept fixed (inside the inaction region), the markup gap is driven completely by the productivity process: $d\mu_t(z) = -da_t(z)$. When there is a price adjustment, the markup process is reset to its new optimal value and then it will again follow the quality process. By symmetry of the Brownian motion without drift and the mean zero innovations of the Poisson process, we have that $da_t(z) = -da_t(z)$. Given the quality and signal processes in (24) and (25), together with $d\mu_t(z) = da_t(z)$, we obtain the same filtering equations as in Proposition 1, but now each process is indexed by z and is independent across firms.

$$\begin{aligned} d\hat{\mu}_t(z) &= \Omega_t(z) dX_t(z) \\ d\Omega_t(z) &= \frac{\sigma_f^2 - \Omega_t^2(z)}{\gamma} dt + \frac{\sigma_u^2}{\gamma} dq_t(z) \end{aligned}$$

where $X_t(z)$ is a standard Brownian motion for every z, just as before.

4.3 Data and calibration

The model is solved numerically as a discrete time version of the continuous time model described above. For the calibration, we use price statistics reported in Nakamura and Steinsson (2008), who use BLS monthly data for a sample that is representative of consumer and producer prices except services, controlling for heterogeneity and sales. The sample is restricted to regular price changes, that is, with sales filtered out. These statistics are also consistent with Dominick's database reported in Midrigan (2011). The targets for calibration are the expected time between price changes, the standard deviation of price changes and the hazard rate.

The calibration is at weekly frequency and then the price statistics are aggregated to match the monthly price statistics in the data. The discount factor is set to $\frac{1}{1+r} = 0.96^{1/52}$ to match an annual risk free rate of 4%. Following the empirical evidence in Zbaracki *et al.* (2004) and Levy *et al.* (1997), the normalized menu cost is set to $\bar{\theta} = 0.064$ in all models so that the expected menu cost payments ($\frac{1}{\mathbb{E}[\tau]}\theta$) represent 0.5% of the average revenue. The CES elasticity of substitution between goods is set to $\eta = 6$ in order to match an average markup of 20%.

We consider three alternative calibrations that allow us to highlight the properties of our model. The first calibration shuts down the information friction ($\gamma = 0$) and the regime changes ($\lambda = 0$), and the only parameter σ_f is set to match the adjustment frequency. Price change distribution has zero dispersion and kurtosis of 1. The second calibration also shuts down the information frictions ($\gamma = 0$) and the frequent shocks ($\sigma_f = 0$), keeping the regime changes active. Its two parameters λ and σ_u match the frequency and the dispersion of price changes. The price change distribution features fatter tails with a kurtosis of 1.5. The third is the full model with information frictions that has an additional parameter to calibrate, the signal noise, which is set to match the shape of the hazard rate[9]. The volatility of the frequent shocks, σ_f, is set very close to zero so that the minimum level of uncertainty is also close to zero and small price changes may occur. The price change distribution has even fatter tails, with a kurtosis of 1.9.

Notice that the arrival rate of the Poisson shocks in the imperfect information model is 80% smaller than in the perfect information model. Nevertheless, both models generate the same expected time between price changes. The reason is that one Poisson shock produces many more price changes in the imperfect information model because of the decreasing hazard rate. A lower arrival rate is key to higher persistence of the output response to monetary shocks, as we show in the next section.

Price statistics Panel A in Figure IV shows the ergodic distribution of price changes for the BLS data in Nakamura and Steinsson (2008) and the three parametrizations of the model. The symmetry of the distribution comes from the assumption of zero inflation and the symmetry of the stochastic process. The baseline model of perfect information and only small frequent shocks generates a price change distribution concentrated at the borders of the inaction region. The models with regime changes, with and without information frictions, are able to match better the fatter tails and larger dispersion of the empirical distribution of price changes. Panel B in figure IV plots the hazard rate of price adjustment. The model with perfect information and only small shocks features an increasing hazard rate: after a price adjustment, it takes time for the small shocks to accumulate in the markup gap and trigger a

[9]The Online Appendix shows how the slope of the unconditional hazard rate varies with different choices of γ without changing the price change distribution. We also show that the two parameters σ_f and σ_u are well identified.

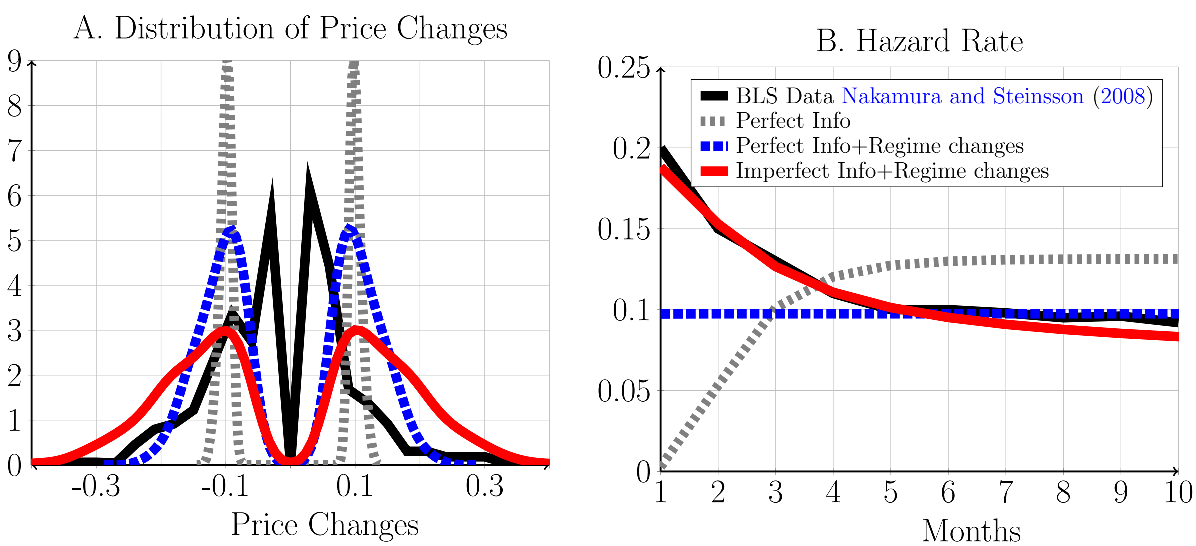
Table I: Model parameters and targets

Target	Data	Model		
		Perfect Info		Info Frictions
	Monthly BLS	No infreq shocks	Infreq shocks	
$\mathbb{E}[\tau]$	2.3 quarters	$\sigma_f = 0.0165$	$\lambda = 0.055$	$\lambda = 0.016$
$std[\lvert\Delta p\rvert]$	0.08		$\sigma_u = 0.08$	$\sigma_u = 0.21$
$min\lvert\Delta p\rvert$	≈ 0		$\sigma_f = 0$	$\sigma_f = 0.0005$
$h(\tau)$	$slope < 0$			$\gamma = 0.22$

CPI data from Nakamura and Steinsson (2008). For the slope of the hazard rate $h(\tau)$ see Figure IV.

price change. The model with perfect information and regime changes produces an flat hazard: the probability of changing the price is constant as it reflects the constant arrival rate of the Poisson shocks that trigger price changes. Therefore, it works as a Calvo model. Finally, the model with information frictions generates the decreasing hazard rate. Note that by calibrating one parameter, the signal noise γ, we can match very well the shape of the hazard rate for a large span of durations.

Figure IV: Distribution of price changes and hazard rate of price adjustments



The model has some difficulty in matching small price changes because the minimum price change is bounded by the size of the menu cost. In the Online Appendix, we extend the baseline model to the so-called CalvoPlus model in Nakamura and Steinsson (2010), in which there are random opportunities to adjust prices without the menu cost. This extended model generates small price changes. Small price changes can also be generated by introducing economies of scope through multi-product firms as in see Midrigan (2011) and Álvarez and Lippi (2014). However, as noted by Eichenbaum, Jaimovich, Rebelo and Smith (2014), small price changes might be the result of measurement errors and not a reason to dismiss a menu cost model.

4.4 Uncertainty and price age

Our model generates a very tight connection between the age of price and firm uncertainty, where age is measured as the number of periods that a price has remained unchanged. High uncertainty firms are more likely to be charging young prices, while low uncertainty firms are more likely to be charging old prices. Therefore, price age becomes a determinant of the size and dispersion of price changes as well as the adjustment frequency. In particular, our model predicts that young (uncertain) prices are larger, more dispersed, and more likely to be reset than older (certain) prices.

These predictions are documented by Campbell and Eden (2014) using weekly scanner data. They define a young price if its age is less than three weeks and an old price if its age is more than four weeks. They find that conditional on adjustment, young prices have double the dispersion of old prices (15% vs. 7%) and that price changes in the extreme tails of the price change distribution tend to be young. Regarding the frequency, they find that young prices are three times more likely to be changed than old prices (36% vs 13%). We compute analogous numbers in our model, defining young prices to be in the 25th quartile of the price age distribution and old prices to be in the 75th quartile. We obtain that dispersion of young price changes is one and half times larger than that of old prices, and that adjustment frequency is twice as large for young prices. Interestingly, the uncertainty faced by young prices is also twice the uncertainty faced by old prices, thus the relative adjustment frequency seems to be informative about the relative uncertainty faced by firms.

Further evidence regarding age dependence in pricing is documented in Baley, Kochen and Sámano (2015). Using detailed CPI data from Mexico, they show that adjustment frequency and price change dispersion falls with the age of the price. Furthermore, they document a negative relationship between price age and exchange rate passthrough: conditional on adjustment, older prices incorporate a smaller fraction of the exchange rate depreciation occurred since the last change. Specifically, exchange-rate passthrough is 50% smaller for six-month old prices compared to one-month old prices. These results point towards relevant age dependence in the responsiveness of prices to aggregate shocks. We explore this responsiveness in the next section within our framework.

5 Propagation of nominal shocks

What are the macroeconomic consequences of our pricing model with information frictions? Specifically, how does output respond to an aggregate nominal shock? In order to answer these questions, we conduct three exercises. In the first exercise, we compute the response of output to a unanticipated permanent monetary shock. We find that information frictions amplify the persistence of output response compared to a Calvo economy. This is because of the heterogeneity in uncertainty that arises from matching the decreasing hazard rate. In the second exercise, the monetary shock interacts with an uncertainty shock that is synchronized across all firms. We find that output responses are smaller and less persistent when aggregate uncertainty is higher. Lastly, we explore the relevance of price age – a proxy for firm uncertainty – in explaining the responsiveness of prices to the monetary shock. We find that old prices are less responsive to nominal shocks compared to young prices, which is consistent with empirical observations.

5.1 Output response to an unanticipated monetary shock

In the first exercise, we compute the impulse-response function of output to a one-time unanticipated small shock to money supply. This monetary shock is fully observed by all firms and thus we say that is it disclosed. Starting from a zero inflation steady state at $t = 0$, we shock the economy with a permanent increase in the money supply of a small size δ, such that $\log M_t = \log \bar{M} + \delta,\ t \geq 0$. Since wages are proportional to the money supply, the shock translates directly into a wage increase. In turn, the wage increase brings down all markups by δ. Given that the monetary shock is disclosed, markup estimates also fall by δ as they are updated by the full amount of the monetary shock:

$$\hat{\mu}_0(z) = \hat{\mu}_{-1}(z) - \delta, \quad \forall z$$

Response of aggregate price level and output Even though markup gap estimates get updated immediately, prices will only be changed when these estimates fall outside the respective inaction regions. The price index in (20) can be written in terms of the markup gaps by multiplying and dividing by the nominal wages and using the definition of markup gap:

$$P_t = W_t \left[\int_0^1 \left(\frac{p_t(z)}{W_t A_t(z)} \right)^{1-\eta} dz \right]^{\frac{1}{1-\eta}} = W_t \left[\int_0^1 \mathrm{\mu}_t(z)^{1-\eta} dz \right]^{\frac{1}{1-\eta}} = W_t \mu^* \left[\int_0^1 \left(e^{\mu_t(z)} \right)^{1-\eta} dz \right]^{\frac{1}{1-\eta}}$$

Taking the log difference from steady state, approximating the integral, and substituting the wage deviation $\ln\left(\frac{W_t}{\overline{W}}\right) = \delta$, we obtain the price deviations from steady state denoted by $\tilde{P}_t$:

$$\tilde{P}_t \equiv \ln\left(\frac{P_t}{\overline{P}}\right) \approx \delta + \int_0^1 \mu_t(z) dz \ \approx\ \delta + \int_0^1 (\mu_t(z) - \hat{\mu}_t(z)) + \hat{\mu}_t(z)\, dz \ \approx\ \delta + \int_0^1 \hat{\mu}_t(z)\, dz \tag{26}$$

We arrive to the last equality by noticing that the forecast error $\mu_t(z) - \hat{\mu}_t(z)$ is *iid* across firms and therefore the average forecast error is equal to zero. Expression (26) states that the price level will deviate from its steady state value as long as some firms have not adjusted their price.

To compute the output response to the monetary shock, we use the equilibrium condition that output equals the real wage. Therefore, putting together the wage and price level deviations from steady state, output deviations are given by the negative of the cross-sectional average of markup gap estimates:

$$\tilde{Y}_t \equiv \ln\left(\frac{Y_t}{\overline{Y}}\right) = \delta - \tilde{P}_t = -\int_0^1 \hat{\mu}_t(z)\, dz \tag{27}$$

In a frictionless world, all firms would increase their price in δ to reflect the higher marginal costs, implying that $\hat{\mu}_t(z) = 0$ for all firms. The monetary shock would have no output effects. With the menu costs and the information frictions, the price level will fail to fully reflect the monetary shock and there will be real effects.

During the transition to the new steady state, there are general equilibrium effects arising from changes in the average markup in the economy that affect individual policies. However in Álvarez and Lippi (2014)'s Proposition 7, it is demonstrated that in this type of framework, such general equilibrium effects can be ignored. Following this result, we compute price responses using the steady state policies.

Figure (V) shows the impulse-response of output to a monetary shock of size $\delta = 1\%$ for the three calibrations outlined in the previous section. We also report two statistics: the area under the impulse-response function, $\mathcal{M} \equiv \int_0^\infty \tilde{Y}_t \, dt$, which is a measure of the total output effect; and the half-life of the impulse response. Columns (1) to (3) of Table II report these statistics.

Figure V: Output response to a positive monetary shock of size $\delta = 1\%$.

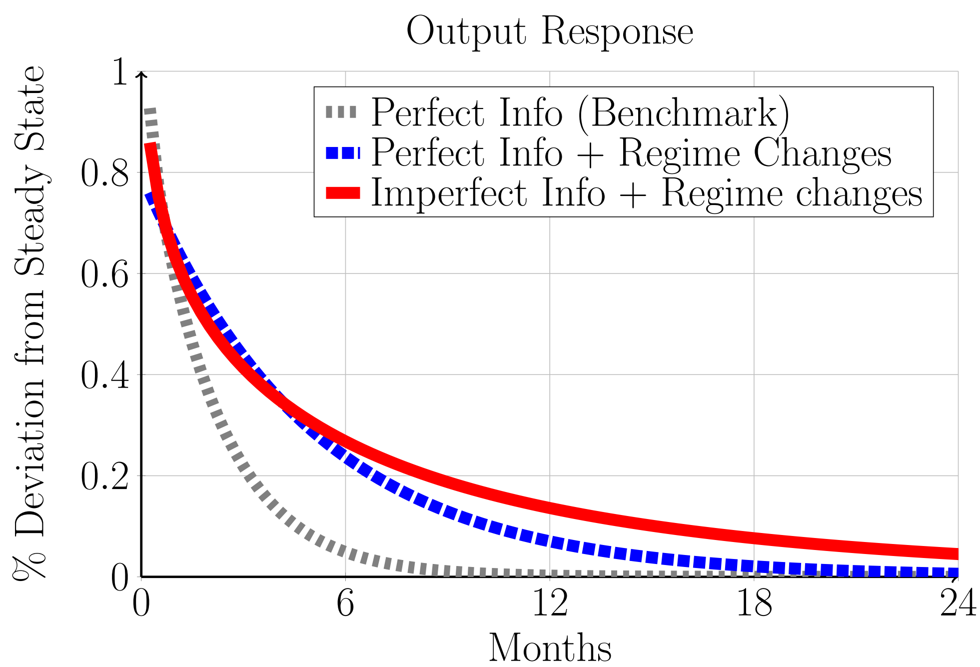


Table II: Output response to monetary shocks (as a multiple of the first column).

		Perfect Info		Info Frictions	
		(1)	(2)	(3)	(4)
Output Effect		Benchmark	Regime changes	Disclosed	Undisclosed
Total effect	($\mathcal{M}$)	1.00	2.16	2.97	6.47
Half-life	($t_{0.5}$)	1.00	1.67	1.33	5.17

For the first column, the total output effects are $\mathcal{M} = 1.74\%$ and the half-life is 1.5 months.

Let us first consider the models with perfect information. For the first calibration with only small frequent shocks (Column 1 of Table II), an increase of 1% in the money supply generates a total output effect of $\mathcal{M} = 1.74\%$, and it has a half-life of 1.5 months. These numbers are set as a benchmark to compare the statistics of the other models. The small and short-lived output response is the result of a large selection effect as highlighted by Golosov and Lucas (2007). The firms that are more likely to adjust their price after the monetary shock are those with the largest desired price changes; their adjustments offset any potential monetary effects. The second calibration (Column 2 of Table II), which introduces the regime changes, features more than two times the total output effects and 1.7 times the half-life of the first model. By matching the dispersion of the price change distribution, this model generates a flat hazard rate; it is akin to a Calvo economy where all firms have the same probability to adjust their prices regardless of their state. By breaking the selection effect, it obtains a larger non-neutrality of monetary shocks as in Gertler and Leahy (2008) and Midrigan (2011). The third calibration with information frictions (Column 3 of Table II) is able to generate almost three times the total output effects of the benchmark model.

As Álvarez, Le Bihan and Lippi (2014) points out, higher kurtosis is associated with higher effects of monetary shocks. That paper derives a formula that computes the cumulative output effect after a monetary shock as an increasing function of kurtosis and a decreasing function of adjustment frequency. Since all our calibrations match the same adjustment frequency and are identical in other parameters, the only relative difference in output effects should be driven by the relative kurtosis of their price change distributions. Recall that the kurtosis of the price change distributions were 1, 1.5 and 1.9 respectively. Thus our results confirm the predictions of such formula, as the output effects are increasing in kurtosis[10]

Despite the increased output effects obtained with the third calibration, the half-life is only slightly larger than in the benchmark and under that of the Calvo-like model. We can observe this tension in Figure (V), which shows that the impulse-response of the information frictions model crosses the impulse-response of the Calvo model after five months. Both the larger output effects and the shorter half-life are the result of having a large mass of firms with low uncertainty in steady state. Low uncertainty firms have small inaction regions, so the impact of the monetary shock triggers many price changes. This resembles the selection effect of the benchmark model. In fact, the adjustment frequency overshoots by 80% compared to its steady state level (see Panel C of Figure VI below) and reduces the output effect drastically during the first months. However, even with the frequency overshoot, the model with information frictions still obtains a larger output effect. The reason for this is that there are low uncertainty firms that did not adjust on impact, and they will only incorporate the monetary shock when they receive a regime change. This delay increases the persistence.

The frequency overshoot after a monetary shock is not observed in the data, as the aggregate frequency is very stable (Nakamura and Steinsson (2008), Klenow and Kryvtsov (2008)) or slightly countercyclical (Vavra (2014)). To address this issue, we consider an extension of the model where firms only observe a fraction $\alpha \in [0, 1]$ of the monetary shock, and their markup gap estimates are only partially updated[11]:

$$\hat{\mu}_0(z) = \hat{\mu}_{-1}(z) - \alpha\delta$$

We assume that firms filter the monetary shock using the same learning technology they use to estimate their permanent productivity shocks. Upon the impact of the monetary shock, there will be an initial forecast error $f_0(z)$ which then evolves as follows:

$$f_0(z) = (1 - \alpha)\delta, \qquad f_{t+1}(z) = \omega(\Omega_t(z))f_t(z)$$

[10]The one caveat to applying the formula by Álvarez, Le Bihan and Lippi (2014) in our model is that there is a large reaction of the adjustment frequency on the impact of the monetary shock, while this formula assumes that frequency does not change.

[11]The CalvoPlus model with random menu costs developed in the Online Appendix also reduces the frequency overshoot from 80% to 50%.

where $\omega(\Omega_t(z))$ denotes the Bayesian weight used by firm z with uncertainty $\Omega_t(z)$ to update her estimate. The average forecast error is given by:

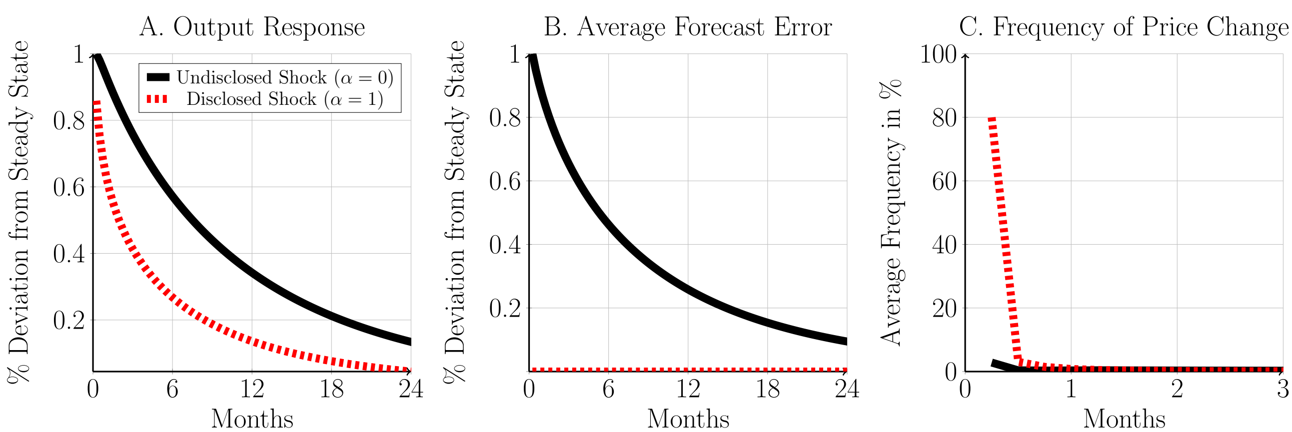
$$F_0 = (1-\alpha)\delta, \qquad F_{t+1} = F_t \left(\int_0^1 \omega(\Omega_t(z))\ dz \right)$$

An equivalent assumption that delivers the same average forecast error is that a fraction $\alpha \in [0,1]$ of firms that does not observe the monetary shock. With these definitions, we can write the output deviation from steady as the negative of average markup gaps (as in (27)) plus the aggregate forecast error as follows:

$$\tilde{Y}_t = -\int_0^1 \hat{\mu}_t(z)\ dz + F_t \tag{28}$$

Figure VI compares the output response, the average forecast error, and the adjustment frequency for a fully disclosed ($\alpha = 1$) and a fully undisclosed ($\alpha = 0$) monetary shock. From Panel A and columns 3 and 4 of Table II, we observe that the output effect is more than doubled and the half-life more than tripled when moving from disclosed to undisclosed shock. There are two forces that contribute to this amplification. First, the frequency overshoot disappears (Panel C). The low uncertainty firms that reacted immediately with an undisclosed shock, now have the largest forecast errors. Second, there is additional persistence coming from the average forecast error (Panel B), which is the result of heterogeneity in uncertainty. High uncertainty firms put a high weight $\omega(\Omega_t(z))$ on signals and incorporate the monetary shock quickly into their estimates; whereas low uncertainty firms put a low weight on signals and take a long time to incorporate the monetary shock, increasing the persistence of the average forecast error.

Figure VI: Undisclosed vs. Disclosed Monetary Shock



5.2 Aggregate uncertainty and nominal shocks

The second exercise explores the output response to a monetary shock when it occurs at the same time as an aggregate uncertainty shock. The motivation for this exercise is to provide an explanation to the empirical finding that monetary policy is less effective when economic uncertainty is higher. The strategy is for an undisclosed monetary shock to interact with an exogenous and synchronized uncertainty shock of size $\kappa\bar{\Omega}$, where $\bar{\Omega} \equiv \mathbb{E}_z[\Omega(z)]$ is the average uncertainty across firms and $\kappa \in \{0, 1, 4\}$. The magnitude and persistence of uncertainty shocks are comparable to the ones documented in Bloom (2009) and Jurado, Ludvigson and Ng (2015).

Figure VII shows the output impulse-response, the average forecast error, and average uncertainty for each experiment and Table III reports the statistics. Panel A shows that a monetary shock paired with a small uncertainty shock reduces the output response by half; and if it is paired with a high uncertainty shock, the output effects are significantly reduced. Besides the positive relationship between adjustment frequency and uncertainty that underlies all our exercises, aggregate uncertainty shocks reduce the persistence of forecast errors. This can be seen in Panel B, which shows that the average forecast error F_t converges faster to zero when uncertainty is higher. Finally, in Panel C we observe that the uncertainty shocks are short-lived, as average uncertainty converges back to its steady state level after a few months.

Figure VII: Responses to Monetary Shock and Synchronized Uncertainty Shock

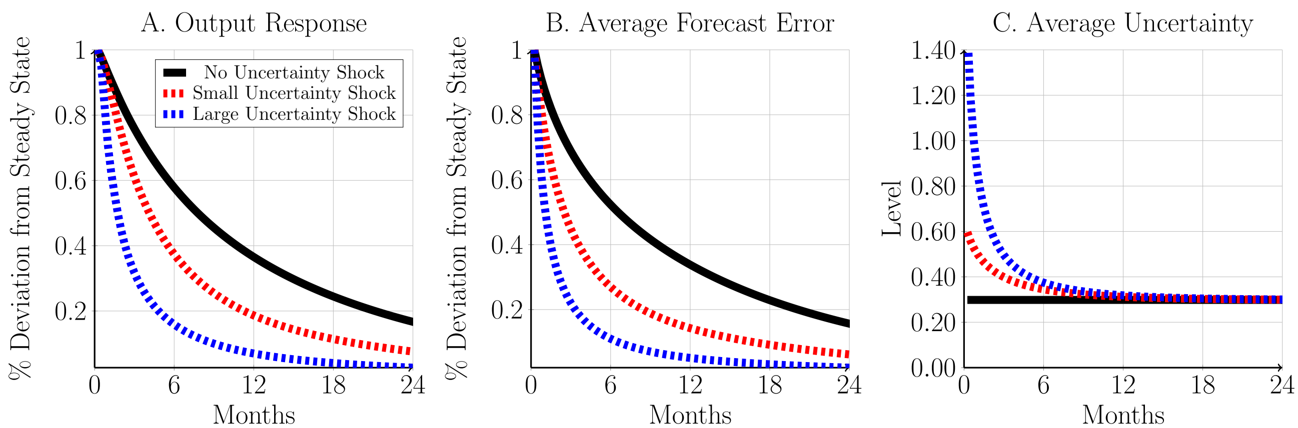


Table III: Monetary and synchronized uncertainty shock (multiple of benchmark)

Output Effect	Monetary + Uncertainty Shock		
	No Ω shock $\kappa = 0$	Small Ω shock $\kappa = 1$	Large Ω shock $\kappa = 4$
Total effect $(\mathcal{M})$	6.47	4.00	1.96
Half-life $(t_{0.5})$	5.17	2.74	1.13

As multiples of benchmark case, reported in Column (1) in Table III.

5.3 Price age and responsiveness to monetary shocks

The third and last exercise explores the relevance of price age – a proxy for firm uncertainty – in explaining the responsiveness of prices to the monetary shock. For this purpose, we follow the methodology used to estimate the pass-though of exchange rates shocks into prices as in Gopinath, Itskhoki and Rigobón (2010) and Berger and Vavra (2013).

We consider a random walk process for the log deviations of money supply from its steady state, $\ln M_{t+1} = \ln M_t + \sigma_M \varepsilon_{t+1}$, with $\varepsilon_{t+1} \sim \mathcal{N}(0,1)$. We consider a volatility $\sigma_M = 0.001$ at weekly frequency. The monetary shock is observable. We then generate a panel of prices for $N = 10,000$ firms denoted with i and for $T = 1000$ periods denoted with t. Price spells are denoted with k. For each firm, we will record the dates of a price change $\{\tau_k^i\}_k$, the duration of each price spell $\{age_{t,k}^i\}_k$, and the size of the price change $\{\Delta p_{t,k}^i\}_k$. For each spell k, price $age_{t,k}^i$ is computed as the difference between t and the date of her last price change $\tau_{t,k}^i$; and the cumulative nominal shock $\Delta_c M_t^i$ is measured as the money supply deviations from steady state between price changes: $\Delta_c M_t^i \equiv \ln M_t - \ln M_{\tau_i}^i$.

Our specification regresses the size of price changes into the cumulative monetary shock, the price's age at the moment of the price change, and an interaction term.

$$\Delta p_t^i = \beta \Delta_c M_t^i + \delta_0 age_t^i + \delta_1 (age_t^i \times \Delta_c M_t^i) + \epsilon_t^i \tag{29}$$

The coefficient on the cumulative monetary shock, β, measures the pass-through of the nominal shock into the price and it is expected to be close to one as the monetary shocks are observable. Our learning model predicts a negative coefficient δ_0 for age – as older prices have smaller Ss bands–, and a negative coefficient δ_1 for the interaction term – as older prices are less responsive to the monetary shock–. Since all our firms are equal in their stochastic processes, we do not include fixed effects. Results from different specifications of this regression are reported in Table IV. All coefficients are significative at 1%, considering robust standard errors.

Table IV: Price Age and Nominal Pass-Through (Benchmark calibration)

Variable	(1)	(2)	(3)
Monetary shock (β)	1.06***	1.06***	1.325***
Age (δ_0)		0.0001***	0.001***
Age $\times$ Monetary shock (δ_1)			−0.028***
R^2	0.29	0.29	0.35
N	1,222,676		

Superscript *** denotes statistical significance at 1% level, considering robust standard errors.

The results are in line with our model's predictions. The estimated passthrough coefficient $\hat{\beta}$ is close to one when it is the only regressor and when age is included, and it goes above one in the full specification. The estimated coefficient for the interaction of age and passthrough $\hat{\delta}_1$ is negative and economically important: passthrough is 50% smaller for 24 weeks old price compared to a 4 weeks old price. Lastly, the estimated coefficient for age $\hat{\delta}_0$ is not economically important.

6 Conclusions

Central banks around the world use models that produce large and persistent output responses to monetary shocks at business cycle frequency. These models have two main building blocks, namely Calvo pricing and strategic complementarities, which together generate the desired inertia in inflation. However, the evidence on such mechanisms is controversial. The model of menu costs and uncertainty cycles developed in this paper is an alternative to those assumptions that generates persistent output responses, while also explaining micro evidence on decreasing hazard rates and age dependent price statistics. Furthermore, the model can explain recent evidence regarding the effectiveness of monetary policy during highly uncertain times and the way in which forecast error dynamics are shaped by uncertainty.

In this paper, productivity regime changes are assumed to be independent across firms. An interesting extension is the introduction of correlated regime changes across firms, which would capture aggregate shocks such as tax changes. Another extension could consider regime changes coming from a source within the firm, for instance, arising from the innovation process. In this case, the creation and life-cycle of a product would be linked to its pricing decision and the pricing of other products in the firm. Finally, given its tractability, our learning framework with regime changes could be used to study the impact of government policy changes or disaster risk on financial markets, complementing the literature on learning in financial markets (see Pastor and Veronesi (2009) for a survey).

References

Aastveit, K. A., Natvik, G. J. and Sola, S. (2013). Economic uncertainty and the effectiveness of monetary policy. *Norges Bank Working Paper 17*.

Álvarez, F., Borovičková, K. and Shimer, R. (2015). The proportional hazard model: Estimation and testing using price change and labor market data. *Working Paper*.

Álvarez, F., Le Bihan, H. and Lippi, F. (2014). *Small and Large Price Changes and the Propagation of Monetary Shocks*. Working Paper 20155, National Bureau of Economic Research.

— and Lippi, F. (2014). Price setting with menu cost for multi-product firms. *Econometrica*, **82** (1), 89–135.

Álvarez, F. and Lippi, F. (2015). *Price plans and the real effects of monetary policy*. Tech. rep.

Álvarez, F., Lippi, F. and Paciello, L. (2011). Optimal price setting with observation and menu costs. *The Quarterly Journal of Economics*, **126** (4), 1909–1960.

Álvarez, L. J., Burriel, P. and Hernando, I. (2005). Do decreasing hazard functions for price changes make any sense? *ECB Working Paper*.

Argente, D. and Yeh, C. (2015). *A Menu Cost Model with Price Experimentation*. Working paper.

Bachmann, R., Born, B., Elstner, S. and Grimme, C. (2013). *Time-Varying Business Volatility, Price Setting, and the Real Effects of Monetary Policy*. Working Paper 19180, National Bureau of Economic Research.

— and Moscarini, G. (2011). *Business Cycles and Endogenous Uncertainty*. 2011 Meeting Papers 36, Society for Economic Dynamics.

Baley, I., Kochen, F. and Sámano, D. (2015). Price duration and pass-through. *Mimeo, Banco de México*.

Barro, R. J. (1972). A theory of monopolistic price adjustment. *The Review of Economic Studies*, **39** (1), 17–26.

Berger, D. and Vavra, J. S. (2013). *Volatility and Pass-through*. Working Paper 19651, National Bureau of Economic Research.

Bloom, N. (2009). The impact of uncertainty shocks. *Econometrica*, **77** (3), 623–685.

Borovičková, K. (2013). *What Drives Labor Market Flows?* Working paper, New York University, Mimeo.

Caggiano, G., Castelnuovo, E. and Nodari, G. (2014). *Uncertainty and Monetary Policy in Good and Bad Times*. Tech. rep., Dipartimento di Scienze Economiche" Marco Fanno".

Campbell, J. R. and Eden, B. (2014). Rigid prices: Evidence from us scanner data. *International Economic Review*, **55** (2), 423–442.

Caplin, A. and Leahy, J. (1997). Aggregation and optimization with state-dependent pricing. *Econometrica: Journal of the Econometric Society*, pp. 601–625.

Carvalho, C. (2006). Heterogeneity in price stickiness and the real effects of monetary shocks. *The BE Journal of Macroeconomics*, **6** (3), 1–58.

— and Schwartzman, F. (2015). Selection and monetary non-neutrality in time-dependent pricing models. *Journal of Monetary Economics*, **76**, 141 – 156.

Christiano, L. J., Eichenbaum, M. and Evans, C. L. (2005). Nominal rigidities and the dynamic effects of a shock to monetary policy. *Journal of Political Economy*, **113** (1), pp. 1–45.

Coibion, O. and Gorodnichenko, Y. (2015). Information rigidity and the expectations formation process: A simple framework and new facts. *American Economic Review*, **105** (8), 2644–78.

Cortés, J., Murillo, J. A. and Ramos-Francia, M. (2012). Evidencia de los micro datos del inpc respecto al proceso de formación de precios.

Davis, M. H. (1977). Linear estimation and stochastic control. *Press, London*.

Dhyne, E., Alvarez, L. J., Le Bihan, H., Veronese, G., Dias, D., Hoffmann, J., Jonker, N., Lunnemann, P., Rumler, F. and Vilmunen, J. (2006). Price changes in the euro area and the united states: Some facts from individual consumer price data. *The Journal of Economic Perspectives*, **20** (2), 171–192.

Dixit, A. (1991). Analytical approximations in models of hysteresis. *The Review of Economic Studies*, **58** (1), 141–151.

Eden, B. and Jaremski, M. S. (2009). *Rigidity, Dispersion and Discreteness in Chain Prices*. Working paper, Vanderbilt University.

Eichenbaum, M., Jaimovich, N., Rebelo, S. and Smith, J. (2014). How frequent are small price changes? *American Economic Journal: Macroeconomics*, **6** (2), 137–55.

Galí, J. and Gertler, M. (1999). Inflation dynamics: A structural econometric analysis. *Journal of monetary Economics*, **44** (2), 195–222.

Gertler, M. and Leahy, J. (2008). A phillips curve with an ss foundation. *Journal of Political Economy*, **116** (3), 533–572.

Golosov, M. and Lucas, R. E. (2007). Menu costs and phillips curves. *Journal of Political Economy*, pp. 171–199.

Gopinath, G., Itskhoki, O. and Rigobón, R. (2010). Currency choice and exchange rate pass-through. *The American economic review*, **100** (1), 304–336.

Hamilton, J. D. (1989). A New Approach to the Economic Analysis of Nonstationary Time Series and the Business Cycle. *Econometrica*, **57** (2), 357–84.

Hellwig, C. and Venkateswaran, V. (2009). Setting the right prices for the wrong reasons. *Journal of Monetary Economics*, **56**, S57–S77.

Jovanovic, B. (1979). Job Matching and the Theory of Turnover. *Journal of Political Economy*, **87** (5), 972–90.

Jurado, K., Ludvigson, S. C. and Ng, S. (2015). Measuring uncertainty. *American Economic Review*, **105** (3), 1177–1216.

Karadi, P. and Reiff, A. (2014). Menu costs, aggregate fluctuations, and large shocks. *CEPR Discussion Paper No. DP10138*.

Kehoe, P. and Midrigan, V. (2015). Prices are sticky after all. *Journal of Monetary Economics*, **75**, 35 – 53.

Kiefer, N. M. (1988). Economic duration data and hazard functions. *Journal of economic literature*, **26** (2), 646–679.

Kim, C.-J. (1994). Dynamic linear models with markov-switching. *Journal of Econometrics*, **60** (1-2), 1–22.

Klenow, P. J. and Kryvtsov, O. (2008). State-dependent or time-dependent pricing: Does it matter for recent u.s. inflation? *Quarterly Journal of Economics*, **123** (3), 863–904.

Kolkiewicz, A. W. (2002). Pricing and hedging more general double-barrier options. *Journal of Computational Finance*, **5** (3), 1–26.

Kushner, H. J. and Dupuis, P. G. (2001). *Numerical Methods for Stochastic Control Problems in Continuous Time.* Springer, 2nd edn.

Levy, D., Bergen, M., Dutta, S. and Venable, R. (1997). The magnitude of menu costs: Direct evidence from large u. s. supermarket chains. *The Quarterly Journal of Economics*, **112** (3), 791–824.

Lucas, R. E. (1972). Expectations and the neutrality of money. *Journal of economic theory*, **4** (2), 103–124.

Maćkowiak, B. and Wiederholt, M. (2009). Optimal Sticky Prices under Rational Inattention. *American Economic Review*, **99** (3), 769–803.

Mankiw, N. G. and Reis, R. (2002). Sticky information versus sticky prices: a proposal to replace the new keynesian phillips curve. *The Quarterly Journal of Economics*, **117** (4), 1295–1328.

Matějka, F. (2015). Rationally inattentive seller: Sales and discrete pricing. *The Review of Economic Studies*.

Merton, R. C. (1976). Option pricing when underlying stock returns are discontinuous. *Journal of financial economics*, **3** (1), 125–144.

Midrigan, V. (2011). Menu cost, multiproduct firms, and aggregate fluctuations. *Econometrica*, **79** (4), 1139–1180.

Mumtaz, H. and Surico, P. (2015). The transmission mechanism in good and bad times. *International Economic Review*, **56** (4), 1237–1260.

Nakamura, E. and Steinsson, J. (2008). Five Facts about Prices: A Reevaluation of Menu Cost Models. *The Quarterly Journal of Economics*, **123** (4), 1415–1464.

— and — (2010). Monetary non-neutrality in a multisector menu cost model. *The Quarterly journal of economics*, **125** (3), 961–1013.

Øksendal, B. (2007). *Stochastic Differential Equations.* Springer, 6th edn.

Øksendal, B. K. and Sulem, A. (2010). *Applied stochastic control of jump diffusions*, vol. 498. Springer.

Pastor, L. and Veronesi, P. (2009). Learning in financial markets. *Annual Review of Financial Economics*, **1** (1), 361–381.

Pellegrino, G. (2015). Uncertainty And Monetary Policy In The US: A Journey Into Non-Linear Territory. *"Marco Fanno" Working Papers*, (0184).

Romer, C. D. and Romer, t. H. (2004). A new measure of monetary shocks: Derivation and implications. *American Economic Review*, **94** (4), 1055–1084.

Senga, T. (2014). A new look at uncertainty shocks: Imperfect information and misallocation.

Stokey, N. (2009). *The Economics of Inaction*. Princeton University Press.

Tenreyro, S. and Thwaites, G. (2015). Pushing on a string: Us monetary policy is less powerful in recessions. *CEPR Discussion Paper No. DP10786*.

Vavra, J. (2014). Inflation dynamics and time-varying volatility: New evidence and an ss interpretation. *The Quarterly Journal of Economics*, **129** (1), 215–258.

Willems, T. (2013). Actively learning by pricing: A model of an experimenting seller. *Available at SSRN 2381430*.

Woodford, M. (2009). Information-constrained state-dependent pricing. *Journal of Monetary Economics*, **56**, S100–S124.

Zbaracki, M. J., Ritson, M., Levy, D., Dutta, S. and Bergen, M. (2004). Managerial and Customer Costs of Price Adjustment: Direct Evidence from Industrial Markets. *The Review of Economics and Statistics*, **86** (2), 514–533.

A Appendix

Preliminaries: Infinitesimal generator and its adjoint operator

($\mathcal{A}$) **Infinitesimal generator.** The infinitesimal generator of $(\hat{\mu}, \Omega)$ denoted by $\mathcal{A}$, applied to a continuous bounded function $\phi : \mathcal{R}^\circ \to \mathbb{R}$ is given by

$$\mathcal{A}\phi(X(t)) \equiv \lim_{dt \downarrow 0} \frac{\mathbb{E}[\phi(X(t+dt)) - \phi(X(t))]}{dt}$$

For our problem, the generator is given by:

$$\mathcal{A}\phi(\hat{\mu}_t, \Omega_t) = \frac{\sigma_f^2 - \Omega_t^2}{\gamma}\phi_\Omega(\hat{\mu}_t, \Omega_t) + \frac{\Omega_t^2}{2}\phi_{\hat{\mu}^2}(\hat{\mu}_t, \Omega_t) + \lambda\left[\phi\left(\hat{\mu}, \Omega_t + \frac{\sigma_u^2}{\gamma}\right) - \phi(\hat{\mu}, \Omega_t)\right] \tag{A.1}$$

A key property of our generator $\mathcal{A}$ is the lack of interaction terms between uncertainty and markup gap estimates. This property is implied by the passive learning process in which the firm cannot change the quality of the information flow by changing her markup.

Proof. To obtain the generator $\mathcal{A}$, first we get a formula for a jump-diffusion process analogous to Itø's formula. We follow Theorem 1.16 of Øksendal and Sulem (2010). Let $B(t)$ be an m-dimensional Brownian motion and $\{N(dt)\}$ are l independent Poisson random measures each with intensity λ_j. Then consider a multidimensional Itō-Lévy process $X(t)$, where each component is given by

$$dX_i(t) = \alpha_i(t)dt + \sum_{j=1}^{m}\sigma_{ij}(t)dB_j(t) + \sum_{j=1}^{l}\int_{\mathbb{R}}\gamma_{ij}(t)N_j(dt)$$

Let $X^c(t)$ be the continuous part of $X(t)$ (obtained by removing the jumps). Also let $Y(t) = \phi(t, X(t))$. Then changes in $Y(t)$ arise from increments in $X^c(t)$ plus the jumps coming from N:

$$\begin{aligned} dY(t) &= \frac{\partial \phi}{\partial t}(t, X(t))dt + \sum_{i=1}^{n}\frac{\partial \phi}{\partial x_i}(t, X(t))[\alpha_i(t)dt + \sigma_i(t)dB_t] + \frac{1}{2}\sum_{i,j=1}^{n}(\sigma\sigma')_{ij}(t)\frac{\partial^2\phi}{\partial x_i \partial x_j}(t, X(t))dt \\ &+ \sum_{k=1}^{l}\int_{\mathbb{R}}\left\{[\phi(t, X(t^-) + \gamma^k(t)] - \phi(t, X(t^-))\right\}N_k(dt) \end{aligned}$$

where γ^k is column k of the $n \times l$ matrix γ and σ_i is row i of σ. To apply this formula in our context, use

$$X(t) = [\hat{\mu}_t,\ \Omega_t]',\ B(t) = [d\hat{Z}_t,\ 0]',\ N(t) = [0\ q(t)]',\ \alpha_1(t) = 0,\ \alpha_2(t) = \frac{\sigma_f^2 - \Omega^2}{\gamma},\ \sigma_{11}(t) = \Omega_t,\ \gamma_1(t) = \frac{\sigma_u^2}{\gamma}$$

and all other entries equal to zero. Finally, apply expectations and obtain (A.1) (note that $\mathbb{E}[N_1(dt)] = \lambda dt$). ☐

($\mathcal{A}^*$) **Adjoint operator.** The adjoint of $\mathcal{A}$, denoted by $\mathcal{A}^*$, is such that $< \mathcal{A}\phi, f > = < \phi, \mathcal{A}^* f >$, where $<, >$ denotes the $\mathcal{L}^2$-inner product. It is given by

$$\mathcal{A}^* f(\hat{\mu}, \Omega) = -\frac{\sigma_f^2 - \Omega^2}{\gamma} f_\Omega(\hat{\mu}, \Omega) + \frac{2\Omega}{\gamma} f(\hat{\mu}, \Omega) + \frac{\Omega^2}{2} f_{\hat{\mu}^2}(\hat{\mu}, \Omega) + \lambda\left[f\left(\hat{\mu}, \Omega - \frac{\sigma_u^2}{\gamma}\right) - f(\hat{\mu}, \Omega)\right] \tag{A.2}$$

Proof. To obtain the adjoint operator, let us apply the definition.

$$< \mathcal{A}\phi, f > = \int_{\sigma_f}^{\infty}\int_{|\hat{\mu}| \le \bar{\mu}(\Omega)}\left\{\frac{\sigma_f^2 - \Omega^2}{\gamma}\phi_\Omega(\hat{\mu}, \Omega) + \frac{\Omega^2}{2}\phi_{\hat{\mu}^2}(\hat{\mu}, \Omega) + \lambda\left[\phi\left(\hat{\mu}, \Omega + \frac{\sigma_u^2}{\gamma}\right) - \phi(\hat{\mu}, \Omega)\right]\right\} f(\hat{\mu}, \Omega)d\hat{\mu}d\Omega$$

Let us simplify each integral and isolate $\phi(\hat{\mu}, \Omega)$ from other terms. We highlight it in bold to make it easier to track.

(i) The first integral is computed by integration by parts with respect to Ω. We also assume that $\lim_{x\to\infty}\phi(\hat{\mu},x)=0$.

$$\begin{aligned}\int\int \phi_\Omega(\hat{\mu},\Omega)\frac{\sigma_f^2-\Omega^2}{\gamma}f(\hat{\mu},\Omega)d\hat{\mu}d\Omega &= \underbrace{\int \phi(\hat{\mu},x)\frac{\sigma_f^2-x^2}{\gamma}f(\hat{\mu},x)\Big|_{\sigma_f}^{\infty}d\hat{\mu}}_{=0}-\int\int\frac{\partial}{\partial\Omega}\left(\frac{\sigma_f^2-\Omega^2}{\gamma}f(\hat{\mu},\Omega)\right)\phi(\hat{\mu},\Omega)d\hat{\mu}d\Omega \\ &= \int\int\left(-\frac{\sigma_f^2-\Omega^2}{\gamma}f_\Omega(\hat{\mu},\Omega)+\frac{2\Omega}{\gamma}f(\hat{\mu},\Omega)\right)\boldsymbol{\phi(\hat{\mu},\Omega)}d\hat{\mu}d\Omega\end{aligned}$$

(ii) The second integral is computed integrating by parts twice with respect to $\hat{\mu}$:

$$\begin{aligned}\int\int\frac{\Omega^2}{2}\phi_{\hat{\mu}^2}(\hat{\mu},\Omega)f(\hat{\mu},\Omega)d\hat{\mu}d\Omega &= \int\frac{\Omega^2}{2}\left[f(x,\Omega)\phi_{\hat{\mu}}(x,\Omega)-f_{\hat{\mu}}(x,\Omega)\phi(x,\Omega)\Big|_{-\bar{\mu}(\Omega)}^{\bar{\mu}(\Omega)}+\int f_{\hat{\mu}^2}(\hat{\mu},\Omega)\phi(\hat{\mu},\Omega)d\hat{\mu}\right]d\Omega \\ &= \int\int\frac{\Omega^2}{2}f_{\hat{\mu}^2}(\hat{\mu},\Omega)\boldsymbol{\phi(\hat{\mu},\Omega)}d\hat{\mu}d\Omega\end{aligned}$$

where the first term is equal to zero since $f(\bar{\mu}(\Omega),\Omega)=f(-\bar{\mu}(\Omega),\Omega)=0$ and $\phi(\bar{\mu}(\Omega),\Omega)=\phi(-\bar{\mu}(\Omega),\Omega)=0$.

(iii) For the third integral, we split the Ω domain in two disjoint sets and use a change of variable to rewrite it as:

$$\begin{aligned}\int\int\lambda\left[\phi\left(\hat{\mu},\Omega+\frac{\sigma_u^2}{\gamma}\right)-\phi(\hat{\mu},\Omega)\right]f(\hat{\mu},\Omega)d\hat{\mu}d\Omega &= \int_{\sigma_f+\frac{\sigma_u^2}{\gamma}}^{\infty}\int_{|\hat{\mu}|\le\bar{\mu}(\Omega)}\lambda\left[f\left(\hat{\mu},\Omega-\frac{\sigma_u^2}{\gamma}\right)-f(\hat{\mu},\Omega)\right]\phi(\hat{\mu},\Omega)\,d\hat{\mu}d\Omega \\ &- \int_{\sigma_f}^{\sigma_f+\frac{\sigma_u^2}{\gamma}}\int_{|\hat{\mu}|\le\bar{\mu}(\Omega)}f(\hat{\mu},\Omega)\phi(\hat{\mu},\Omega)d\hat{\mu}d\Omega \\ &= \int\int\lambda\left[f\left(\hat{\mu},\Omega-\frac{\sigma_u^2}{\gamma}\right)-f(\hat{\mu},\Omega)\right]\boldsymbol{\phi(\hat{\mu},\Omega)}d\hat{\mu}d\Omega\end{aligned}$$

For the second equality, notice that f's second argument only takes positive values. We define f to be equal to zero outside its domain, and therefore $f\left(\hat{\mu},\Omega-\frac{\sigma_u^2}{\gamma}\right)\phi(\hat{\mu},\Omega)=0$ for all $\Omega\in[\sigma_f,\sigma_f+\frac{\sigma_u^2}{\gamma}]$. Therefore, we can add the missing terms and integrate over the complete domain.

Putting all the integrals together we recover the adjoint operator A^*:

$$\int\int\underbrace{\left\{-\frac{\sigma_f^2-\Omega^2}{\gamma}f_\Omega(\hat{\mu},\Omega)+\frac{2\Omega}{\gamma}f(\hat{\mu},\Omega)+\frac{\Omega^2}{2}f_{\hat{\mu}^2}(\hat{\mu},\Omega)+\lambda\left[f\left(\hat{\mu},\Omega-\frac{\sigma_u^2}{\gamma}\right)-f(\hat{\mu},\Omega)\right]\right\}}_{A^*}\boldsymbol{\phi(\hat{\mu},\Omega)}d\hat{\mu}d\Omega = <\phi,\mathcal{A}^*f>$$

□

Proposition 1 is proved in a more general setup, where the state has a non-zero drift.

Proposition 1 (**Filtering Equations, Including Drift**). *Let the following processes define the state and the signal*

$$\begin{aligned}
(state) \quad d\mu_t &= F\mu_t dt + \sigma_f dW_t + \sigma_u u_t dq_t \\
(observation) \quad ds_t &= G\mu_t dt + \gamma dZ_t \\
(initial\ conditions\ for\ state) \quad \mu_0 &\sim \mathcal{N}(a,b) \\
(initial\ conditions\ for\ observations) \quad s_0 &= 0 \\
where \quad W_t, Z_t \sim Wiener\ Process, \quad q_t &\sim Poisson(\lambda), \quad u_t \sim \mathcal{N}(0,1)
\end{aligned} \tag{A.3}$$

Let the information set (with continuous sampling) be $\mathcal{I}_t = \sigma\{s_h, q_h : h \in [0,t]\}$. Then the posterior distribution of the state is Normal, i.e. $\mu_t|\mathcal{I}_t \sim \mathcal{N}(\hat{\mu}_t, \Sigma_t)$, where the posterior mean $\hat{\mu}_t \equiv \mathbb{E}[\mu_t|\mathcal{I}_t]$ and posterior variance $\Sigma_t \equiv \mathbb{E}[(\mu_t - \hat{\mu}_t)^2|\mathcal{I}_t]$ satisfy the following stochastic processes:

$$\begin{aligned}
d\hat{\mu}_t &= \left(F - \tfrac{G^2\Sigma_t}{\gamma^2}\right)\hat{\mu}_t dt + \tfrac{G\Sigma_t}{\gamma^2} ds_t, \qquad \hat{\mu}_0 = a \\
d\Sigma_t &= \left(2F\Sigma_t + \sigma_f^2 - \tfrac{G^2\Sigma_t^2}{\gamma^2}\right) dt + \sigma_u^2 dq_t, \quad \Sigma_0 = b
\end{aligned} \tag{A.4}$$

Furthermore, the first filtering equation can be written as

$$d\hat{\mu}_t = F\hat{\mu}_t dt + \frac{G^2\Sigma_t}{\gamma} d\hat{Z}_t$$

where $\hat{Z}_t$ is the innovation process given by $d\hat{Z}_t = \frac{1}{\gamma}(ds_t - \hat{\mu}_t dt) = \frac{1}{\gamma}(\mu_t - \hat{\mu}_t)dt + dZ_t$ and it is one-dimensional Wiener process under the probability distribution of the firm independent of dq_t.

Finally, using the definition of uncertainty $\Omega_t \equiv \gamma\Sigma_t$, and substituting $F = 0$ and $G = 1$, we obtain the filtering equations used in the text:

$$d\hat{\mu}_t = \Omega_t d\hat{Z}_t, \qquad \hat{\mu}_0 = a \tag{A.5}$$

$$d\Omega_t = \frac{\sigma_f^2 - \Omega_t^2}{\gamma} dt + \frac{\sigma_u^2}{\gamma} dq_t, \qquad \Omega_0 = \frac{b}{\gamma} \tag{A.6}$$

Proof. The strategy of the proof has three steps, each established in a Lemma.

(I) We show that the solution $M_t \equiv [\mu_t, s_t]$ to the system of stochastic differential equations in (A.3), conditional on the history of Poisson shocks $\mathcal{Q}_t = \sigma\{q_r | r \leq t\}$, follows a Gaussian process.

(II) $\mu_t|\mathcal{I}_t$ is Normal and can be obtained as the limit of a discrete sampling of observations;

(III) The recursive estimation formulas obtained with discrete sampling converge to (A.4).[12]

Now we elaborate on the three steps.

Lemma 1. *Let $M_t \equiv [\mu_t, s_t]$ be the solution to (A.3) and $\mathcal{Q}_t = \sigma\{q_r | r \leq t\}$. Then $M_t|\mathcal{Q}_t$ is Normal.*

Proof. Fix a realization ω and let $N_t(\omega)$ be the quantity of jumps between 0 and t, which is a number known at t. Applying Picard iterative process to (A.3) and considering the initial conditions, we obtain the following sequences

$$\begin{aligned}
\mu_t^{k+1} &= \mu_0 + F\int_0^t \mu_\tau^k d\tau + \sigma_f W_t + \sigma_f \sum_{i=1}^{N_t(\omega)} u_i \\
s_t^{k+1} &= G\int_0^t \mu_\tau^k d\tau + \gamma Z_t
\end{aligned}$$

Assume that μ_t^0 is Normal. As an induction hypothesis, assume that $M_r^k|\mathcal{Q}_t \equiv [\mu_r^k, s_r^k|\mathcal{Q}_t]$ is Normal for all $r \leq t$. Note that (μ_0, W_r, Z_r) are Normal random variables independent of $\mathcal{Q}_t$; the term $\sum_{i=1}^{N_r(\omega)} u_i|\mathcal{Q}_t$ is Normal since it is a fixed sum of $N_r(\omega)$ Normal random variables; and finally, the term $\int_0^r \mu_\tau^k d\tau$ is a Riemann integral of Normal variables by the induction

[12] In the Online Appendix, we derive additional details and a formal convergence proof.

hypothesis. Given that the linear combination of Normals is Normal, then $M_r^{k+1}|\mathcal{Q}_t \equiv [\mu_r^{k+1}, s_r^{k+1}|\mathcal{Q}_t]$ is Normal as well for $r \leq t$. Therefore, for each $r \leq t$, we have a sequence of Normal random variables $\{M_r^k|\mathcal{Q}_t\}_{k=0}^{\infty}$.

To show Normality of $M_t|\mathcal{Q}_t$, notice that $M_r^k|\mathcal{Q}_t = M_r^k|\mathcal{Q}_r$ and $M_r^k|\mathcal{Q}_r$ converges in L^2 to M_r (wee chapter 5 of Øksendal (2007)). Since the limit in L^2 of Normal variables is Normal, M_t is Normal. Therefore the solution to the system of stochastic differential equations, conditional to the history of Poisson shocks, i.e. $M_t|\mathcal{Q}_t$, is a Gaussian process. □

Lemma 2. *The conditional distribution of the state $\mu_t|\mathcal{I}_t$ is Normal, $\mu_t|\mathcal{I}_t \sim \mathcal{N}\left(\mathbb{E}\left[\mu_t|\mathcal{I}_t\right], \mathbb{E}\left[\left(\mu_t - \mathbb{E}\left[\mu_t|\mathcal{I}_t\right]\right)^2|\mathcal{I}_t\right]\right)$, and the conditional mean and variance can be obtained as the limit of a discrete sampling of observations.*

Proof. Let $\Delta \equiv \frac{1}{2^n}$ and define an increasing sequence of σ-algebras $\{\mathcal{I}_t^n\}_{n=0}^{\infty}$ using the dyadic set as follows:

$$\mathcal{I}_t^n = \sigma\left\{s_r, q_h : r \in \{0, \Delta, 2\Delta, 3\Delta, \dots\}, \ r \leq t, \ h \in [0,t]\right\}$$

Let $M_t^n \equiv \mu_t|\mathcal{I}_t^n$ be the estimate at time t produced with discrete sampling. The following properties are true.

(i) For each n, M_t^n is a Normal random variable. By the previous Lemma $(\mu_t, s_{r_1}, s_{r_2}, \dots, s_{r_n})|\mathcal{Q}_t$ is Normal; by properties of Normals, M_t^n is also Normal.

(ii) For each n, M_t^n has finite variance. This is a direct implication of Normality.

(iii) Let $\mathcal{I}_t^{\infty} \equiv \sigma\{U_{n=1}^{\infty} I_t^n\}$ be the σ-algebra generated by the union of the discrete sampling information sets. For each t, M_t^n converges to some limit $M_t^{\infty} \equiv \mu_t|\mathcal{I}_t^{\infty}$ as $n \to \infty$. Since $\mathcal{I}_t^n$ is a increasing sequence of σ-algebras, by the Law of Iterated Expectations M_t^n is a martingale with finite variance, therefore it converges in L^2. Given that the limit of Normal random variables is Normal, the limit M_t^{∞} is a Normal random variable as well.

$$M_t^n \to_{L^2} M_t^{\infty} \ \sim \ \mathcal{N}(\mathbb{E}\left[\mu_t|\mathcal{I}_t^{\infty}\right], \mathbb{E}\left[\left(\mu_t - \mathbb{E}\left[\mu_t|\mathcal{I}_t\right]\right)^2|\mathcal{I}_t^{\infty}\right]$$

Since signals s_t are continuous (in particular left-continuous) and the dyadic set is dense in the interval $[0,t]$, the information set obtained as the limit of the discrete sampling is equal to the information set obtained with continuous sampling: $\mathcal{I}_t^{\infty} = \sigma\left\{s_h, q_h : h \in [0,t]\right\}$. Therefore, the estimate obtained with the limit of discrete sampling converges (in L^2) to the estimate with continuos sampling (see Davis (1977) for more details in this topic).

$$M_t^{\infty} \to_{L^2} \mu_t|\mathcal{I}_t \ \sim \ \mathcal{N}\left(\mathbb{E}\left[\mu_t|\mathcal{I}_t\right], \mathbb{E}\left[\left(\mu_t - \mathbb{E}\left[\mu_t|\mathcal{I}_t\right]\right)^2|\mathcal{I}_t\right]\right)$$

□

Lemma 3. *Let $\Delta \equiv \frac{1}{2^n}$ and define $\mathcal{I}_t^{n,*}$ as the information set before measurement (used to construct predicted estimates)*

$$\mathcal{I}_t^{n,*} = \sigma\left\{s_{r-1}, q_h \,|r \in \{0, \Delta, 2\Delta, 3\Delta, \dots\}, \ r \leq t, \ h \in [0,t]\right\}$$

and define $\hat{\mu}_t^n = \mathbb{E}[\mu_t|\mathcal{I}_t^{n,*}]$ and $\Sigma_t^n = \mathbb{E}[(\mu_t - \hat{\mu}_t)^2|\mathcal{I}_t^{n,*}]$. Then the laws of motion of $\{\hat{\mu}_t^n, \Sigma_t^n\}$ converge weakly to the solution of (A.4), namely the laws of motion for $\{\hat{\mu}_t, \Sigma_t\}$, where $\hat{\mu}_t \equiv \mathbb{E}[\mu_t|\mathcal{I}_t]$ and $\Sigma_t \equiv \mathbb{E}[(\mu_t - \hat{\mu}_t)^2|\mathcal{I}_t]$.

Proof. Before we derive the processes for the estimate and its conditional variance, an explanation of why we use the information set $\mathcal{I}_t^{n,*}$ instead of $\mathcal{I}_t^n$ is due. The reason is convenience, as the first information set produces independent recursive formulas for the predicted estimate $\mu_t|\sigma\{U_{i=1}^{\infty} I_t^{n,*}\}$ and it is easier to show its convergence. Let us show that the union of information sets are equal, i.e. $\sigma\{U_{i=1}^{\infty} I_t^n\} = \sigma\{U_{i=1}^{\infty} I_t^{n,*}\}$, and thus the way we construct the limit is innocuous. Trivially, we have that $\sigma\{U_{i=1}^{\infty} I_t^{n,*}\} \subset \sigma\{U_{i=1}^{\infty} I_t^n\}$. For the reverse to be true $\sigma\{U_{i=1}^{\infty} I_t^n\} \subset \sigma\{U_{i=1}^{\infty} I_t^{n,*}\}$, it is sufficient to show that signals s are continuous, since left-continuous filtrations of continuous process are always continuous. To show that signals are continuous, notice that they can be written as $s_t = \int_0^t \mu_s ds + \gamma Z_t$, which is an integral of a finite set of discontinuities plus a Wiener process, and thus they are continuous.

Now let us derive the laws of motion. Considering an interval Δ, then the processes in (A.3) can be written as

$$\begin{aligned}
\mu_t &= \mu_{t-\Delta} + F\int_{t-\Delta}^{t}\mu_\tau d\tau + \sqrt{\Delta\sigma_f^2}\epsilon_t + \sigma_u u_t(q_t - q_{t-\Delta}), \qquad \mu_{-\Delta} \sim \mathcal{N}(\hat{\mu}_\Delta, \Sigma_\Delta) \\
s_t &= s_{t-\Delta} + G\int_{t-\Delta}^{t}\mu_\tau d\tau + \sqrt{\Delta\gamma^2}\eta_t, \qquad s_0 = 0 \\
(q_t - q_{t-\Delta}) &\sim_{i.i.d} \begin{cases} 1 & \text{with probability } 1 - e^{-\lambda\Delta} - o(\Delta^2) \\ 0 & \text{with probability } e^{-\lambda\Delta} - o(\Delta^2) \\ > 1 & \text{with probability } o(\Delta^2) \end{cases} \\
\epsilon_t, \eta_t, u_t &\sim_{i.i.d} \mathcal{N}(0,1)
\end{aligned}$$

First order approximations of the integral yield $\int_{t-\Delta}^{t}\mu_\tau d\tau = \mu_{t-\Delta}\Delta + \xi_t = \mu_t\Delta + \tilde{\xi}_t$, where ξ_t and $\tilde{\xi}_t$ are Normal random variables conditional on $\mathcal{Q}_t$, with $\mathbb{E}[\xi_t] = o(\Delta^2)$, $\mathbb{E}[\xi_t^2] = o(\Delta^2)$, $\mathbb{E}[\tilde{\xi}_t] = o(\Delta^2)$ and $\mathbb{E}[\tilde{\xi}_t] = o(\Delta^2)$. Substituting these approximations above, we can express the laws of motion for μ, s as follows:

$$\begin{aligned}
\mu_t &= (1 + F\Delta)\mu_{t-\Delta} + \sqrt{\Delta\sigma_f^2}\epsilon_t + \sigma_u u_t(q_t - q_{t-\Delta}) + o(\Delta^2) \\
s_t &= s_{t-\Delta} + G\Delta\mu_t + \sqrt{\Delta\gamma^2}\eta_t + o(\Delta^2)
\end{aligned}$$

Since the model is Gaussian, we use the Kalman Filter to estimate the conditional mean $\hat{\mu}_t^n = \mathbb{E}[\mu_t|\mathcal{I}_t^{n,*}]$ and variance $\Sigma_t^n = \mathbb{E}[(\mu_t - \hat{\mu}_t)^2|\mathcal{I}_t^{n,*}]$. The recursive formulas are

$$\begin{aligned}
\hat{\mu}_{t+\Delta}^n &= (1 + \Delta F)\,\hat{\mu}_t^n + K_t^n\,(s_t - s_{t-\Delta} - \Delta G(1 + \Delta F)\hat{\mu}_t^n) + o(\Delta^2) \\
\Sigma_{t+\Delta}^n &= (1 + \Delta F)^2\frac{\Sigma_t^n\gamma^2}{\Sigma_t^n G^2\Delta + \gamma^2} + \sigma_f^2\Delta + (q_{t+\Delta} - q_t)\sigma_u^2 + o(\Delta^2) \\
K_t^n &= (1 + \Delta F)\,\frac{\Sigma_t^n G}{\Sigma_t^n G^2\Delta + \gamma^2}
\end{aligned}$$

Notice that since u_t has mean zero, the known arrival of a Poisson shock does not affect the estimate. However, it does affect the variance by adding a shock of size σ_u^2. Rearranging and doing some algebra, the previous system can be written as

$$\begin{aligned}
\hat{\mu}_{t+\Delta}^n - \hat{\mu}_t^n &= \left(F - G\varphi^I(\Delta)\right)\hat{\mu}_t\Delta + \varphi^I(\Delta)\,(s_t - s_{t-\Delta}) + o(\Delta^2), \qquad \varphi^I(\Delta) \equiv \frac{\Sigma_t^n G}{\Sigma_t^2 G^2\Delta + \gamma^2} \\
\Sigma_{t+\Delta}^n - \Sigma_t^n &= \left(\varphi^{II}(\Delta) + \sigma_f^2\right)\Delta + (q_{t+\Delta} - q_t)\sigma_u^2 + o(\Delta^2), \qquad \varphi^{II}(\Delta) \equiv \left[\frac{\gamma^2(2F + F^2\Delta) - G^2\Sigma_t^n}{\Sigma_t^n G^2\Delta + \gamma^2}\right]\Sigma_t^n
\end{aligned}$$

Taking the limit as $n \to \infty$ (or $\Delta \to 0$), we see that $\varphi^I(\Delta) \to \frac{\Sigma_t G}{\gamma^2}$ and $\varphi^{II}(\Delta) \to 2F\Sigma_t - \frac{G^2\Sigma_t^2}{\gamma^2}$, which yield exactly the same laws of motion that can be obtained with the continuous time Kalman-Bucy filter. Therefore, the laws of motion obtained with discrete sampling are locally consistent with the continuous time filtering equations in (A.4) (see Online Appendix for more details, where we follow closely Theorem 1.1, Chapter 10 of Kushner and Dupuis (2001).). □

To conclude the proof, use the structure of the signal to rewrite the law of motion in innovation representation as

$$d\hat{\mu}_t = F\hat{\mu}_t dt + \frac{G\Sigma_t}{\gamma}\left(\frac{G}{\gamma}(\mu_t - \hat{\mu}_t)\,dt + dZ_t\right) = F\hat{\mu}_t dt + \frac{G\Sigma_t}{\gamma}d\hat{Z}_t$$

where $d\hat{Z}_t \equiv \frac{G}{\gamma}(\mu_t - \hat{\mu}_t)\,dt + dZ_t$ is the innovation process. We now show $d\hat{Z}_t$ is a Wiener process. Applying the law of iterated expectations:

$$\mathbb{E}[(\mu_t - \hat{\mu}_t)|\sigma\{\hat{\mu}_s : s \le t\}] = \mathbb{E}\left[\mathbb{E}\left[(\mu_t - \hat{\mu}_t)|\,\mathcal{I}_t\right]|\sigma\{\hat{\mu}_s : s \le t\}\right] = \mathbb{E}[(\hat{\mu}_t - \hat{\mu}_t)|\sigma\{\hat{\mu}_s : s \le t\}] = 0$$

Since $\mathbb{E}[(\mu_t - \hat{\mu_t})|\sigma\{\hat{\mu}_s : s \le t\}] = 0\ \forall t$ and dZ_t is a Wiener process, we apply corollary 8.4.5 of Øksendal (2007) and conclude that $d\hat{Z}_t$ is a Weiner process as well.

□

Proposition 2 **(Stopping time problem).** *Let* $(\hat{\mu}_0, \Omega_0)$ *be the firm's current state immediately after the last markup adjustment. Also let* $\bar{\theta} = \frac{\theta}{B}$ *be the normalized menu cost. Then the optimal stopping time and reset markup gap* (τ, μ') *solve the following problem:*

$$V(\hat{\mu}_0, \Omega_0) = \max_{\tau} \mathbb{E}\left[\int_0^{\tau} -e^{-rs}\hat{\mu}_s^2 ds + e^{-r\tau}\Big(-\bar{\theta} + \max_{\mu'} V(\mu', \Omega_\tau)\Big)\Big|\mathcal{I}_0\right] \tag{9}$$

subject to the filtering equations in Proposition 1.

Proof. Let $\{\tau_i\}_{i=1}^{\infty}$ be the series of dates where the firm adjusts her markup gap and $\{\mu_i\}_{i=1}^{\infty}$ the series of reset markup gaps. Given an initial condition μ_0 and a law of motion for the markup gaps, the sequential problem of the firm is expressed as follows:

$$\max_{\{\mu_{\tau_i}, \tau_i\}_{i=1}^{\infty}} \mathbb{E}\left[\sum_{i=0}^{\infty} e^{-r\tau_{i+1}}\left(-\bar{\theta} - \int_{\tau_i}^{\tau_{i+1}} e^{-r(t-\tau_{i+1})}\mu_t^2 dt\right)\right] \tag{A.7}$$

Using the definition of variance, we can write the condition expectation of the markup gap at time t as:

$$\mathbb{E}[\mu_t^2|\mathcal{I}_t] = \mathbb{E}[\mu_t|\mathcal{I}_t]^2 + \mathbb{V}[\mu_t|\mathcal{I}_t] = \hat{\mu}_t^2 + \mathbb{V}[\mu_t|\mathcal{I}_t] = \hat{\mu}_t^2 + (\sigma_f^2 + \lambda\sigma_u^2)t = \hat{\mu}_t^2 + \Omega^{*2}t$$

where in the last equality we use the definition of fundamental uncertainty Ω^*. Use the Law of Iterated Expectations in (A.7) to take expectation given the information set at time t. Use the decomposition above to write the problem in terms of estimates:

$$\begin{aligned}
&\mathbb{E}\left[\sum_{i=0}^{\infty} e^{-r\tau_{i+1}}\left(-\bar{\theta} - \int_{\tau_i}^{\tau_{i+1}} e^{-r(t-\tau_{i+1})}\mathbb{E}\left[\mu_t^2\Big|\mathcal{I}_t\right] dt\right)\right] \\
&\mathbb{E}\left[\sum_{i=0}^{\infty} e^{-r\tau_{i+1}}\left(-\bar{\theta} - \int_{\tau_i}^{\tau_{i+1}} e^{-r(t-\tau_{i+1})}(\hat{\mu}_t^2 + \Omega^{*2}t) dt\right)\right] \\
&\mathbb{E}\left[\sum_{i=0}^{\infty} e^{-r\tau_{i+1}}\left(-\bar{\theta} - \int_{\tau_i}^{\tau_{i+1}} e^{-r(t-\tau_{i+1})}\hat{\mu}_t^2 dt\right)\right] - \underbrace{\Omega^{*2}\mathbb{E}\left[\sum_{i=0}^{\infty}\int_{\tau_i}^{\tau_{i+1}} te^{-rt}dt\right]}_{sunk\ cost}
\end{aligned}$$

The last term in the previous expression is a constant number, and it arises from the fact that the firm will never learn the true realization of the markup gap. It is considered a sunk cost in the firm's problem since she cannot take any action to alter its value; therefore, we can ignore it from her problem. To compute its value, note that the term inside the expectation is equal to:

$$\sum_{i=0}^{\infty}\int_{\tau_i}^{\tau_{i+1}} te^{-rt}dt \quad = \quad \sum_{i=0}^{\infty}\left[\frac{e^{-r\tau_i}(1+r\tau_i) - e^{-r\tau_{i+1}}(1+r\tau_{i+1})}{r^2}\right] = \frac{e^{-r\tau_0}(1+r\tau_0)}{r^2}$$

where the sum is telescopic and all terms except the first cancel out. Therefore, the sunk cost term becomes:

$$-\Omega^{*2}\mathbb{E}\left[\frac{e^{-r\tau_0}(1+r\tau_0)}{r^2}\right] < \infty$$

Using the previous results, the sequential problem in (A.7) can be written in terms of estimates instead of the true realizations:

$$\max_{\{\mu_{\tau_i}, \tau_i\}_{i=1}^{\infty}} \mathbb{E}\left[\sum_{i=0}^{\infty} e^{-r\tau_{i+1}}\left(-\bar{\theta} - \int_{\tau_i}^{\tau_{i+1}} e^{-r(t-\tau_{i+1})}\hat{\mu}_t^2 dt\right)\right]$$

Given the stationarity of the problem and the stochastic processes, we apply the Principle of Optimality to the sequential problem and express it as a sequence of stopping time problems where the state is given by $(\hat{\mu}_t, \Omega_t)$:

$$V(\hat{\mu}_0, \Omega_0) = \max_{\tau} \mathbb{E}\left[\int_0^{\tau} -e^{-rt}\hat{\mu}_t^2 dt + e^{-r\tau}[-\bar{\theta} + \max_{\mu'} V(\mu', \Omega_\tau)]\right]$$

subject to the filtering equations. □

Proposition 3 (HJB Equation). *Let $V(\hat{\mu},\Omega)$ be the value of the firm and V_x denote the derivative of V with respect to x. For all states inside the inaction region $\mathcal{R}$, V satisfies:*

1. *the Hamilton-Jacobi-Bellman (HJB) equation:*

$$rV(\hat{\mu},\Omega) = -\hat{\mu}^2 + \left(\frac{\sigma_f^2 - \Omega^2}{\gamma}\right) V_\Omega(\hat{\mu},\Omega) + \frac{\Omega^2}{2} V_{\hat{\mu}^2}(\hat{\mu},\Omega) + \lambda \left[V\left(\hat{\mu},\Omega + \frac{\sigma_u^2}{\gamma}\right) - V(\hat{\mu},\Omega)\right]$$

2. *the value matching condition that sets equal the value of adjusting and not adjusting at the border:*

$$V(0,\Omega) - \bar{\theta} = V(\bar{\mu}(\Omega),\Omega)$$

3. *two smooth pasting conditions, one for each state:* $V_{\hat{\mu}}(\bar{\mu}(\Omega),\Omega) = 0, \quad V_\Omega(\bar{\mu}(\Omega),\Omega) = V_\Omega(0,\Omega.$

Proof. Start from the recursive representation of the value function as a stopping time problem derived in Proposition 2.

$$\begin{aligned} V(\hat{\mu}_0,\Omega_0) &= \max_\tau \mathbb{E}\left[\int_0^\tau -e^{-rt}\hat{\mu}_t^2 dt + e^{-r\tau}[-\bar{\theta} + \max_{\mu'} V(\mu',\Omega_\tau)]\right] \\ d\hat{\mu}_t &= \Omega_t d\hat{Z}_t \\ d\Omega_t &= \frac{\sigma_f^2 - \Omega_t^2}{\gamma} dt + \frac{\sigma_u^2}{\gamma} dq_t \end{aligned}$$

To obtain the HJB, consider the value function inside the continuation region. Then for a small interval dt we can write:

$$\begin{aligned} V(\hat{\mu}_t,\Omega_t) &= -\hat{\mu}_t^2 dt + e^{-rdt}\mathbb{E}[V(\hat{\mu}_{t+dt},\Omega_{t+dt})] \\ (1-e^{-rdt})V(\hat{\mu}_t,\Omega_t) &= -\hat{\mu}_t^2 dt + e^{-rdt}\mathbb{E}[V(\hat{\mu}_{t+dt},\Omega_{t+dt}) - V(\hat{\mu}_t,\Omega_t)] \\ rV(\hat{\mu}_t,\Omega_t)dt &= -\hat{\mu}_t^2 dt + (1-rdt)\mathbb{E}[V(\hat{\mu}_{t+dt},\Omega_{t+dt}) - V(\hat{\mu}_t,\Omega_t)] \\ rV(\hat{\mu}_t,\Omega_t) &= -\hat{\mu}_t^2 + \lim_{dt\downarrow 0}(1-rdt)\frac{\mathbb{E}[V(\hat{\mu}_{t+dt},\Omega_{t+dt}) - V(\hat{\mu}_t,\Omega_t)]}{dt} \\ rV(\hat{\mu}_t,\Omega_t) &= -\hat{\mu}_t^2 + \mathcal{A}V(\hat{\mu}_t,\Omega_t) \end{aligned}$$

where in the second line we have subtracted $e^{-rdt}V(\hat{\mu}_t,\Omega_t)$ from both sides, in the third line we have approximated e^{-rdt} with $1-rdt$, in the fourth line we divide by dt and then take the limit $dt \to 0$, and finally in the fifth line we recognized the definition of the generator. Substituting the generator $\mathcal{A}$ from (A.1) we obtain the HJB equation:

$$rV(\hat{\mu},\Omega) = -\hat{\mu}^2 + \left(\frac{\sigma_f^2 - \Omega^2}{\gamma}\right) V_\Omega(\hat{\mu},\Omega) + \frac{1}{2}\Omega^2 V_{\hat{\mu}^2}(\hat{\mu},\Omega) + \lambda \left[V\left(\hat{\mu},\Omega + \frac{\sigma_u^2}{\gamma}\right) - V(\hat{\mu},\Omega)\right]$$

The value matching condition that sets equal the value of adjusting and not adjusting at the border:

$$V(\bar{\mu}(\Omega),\Omega) = V(0,\Omega) - \bar{\theta}$$

To derive the smooth pasting conditions, we follow Øksendal and Sulem (2010) and get two two smooth pasting conditions, one for each state, by taking the derivative on both sides of the value matching:

$$V_{\hat{\mu}}(\bar{\mu}(\Omega),\Omega) = 0, \qquad V_\Omega(\bar{\mu}(\Omega),\Omega) = V_\Omega(0,\Omega)$$

□

Proposition 4 (**Inaction region**). *For r and $\bar{\theta}$ be small, the border of the inaction region $\bar{\mu}(\Omega)$ is approximated by*

$$\bar{\mu}(\Omega) = \left(\frac{6\bar{\theta}\Omega^2}{1+\mathcal{L}^{\bar{\mu}}(\Omega)}\right)^{1/4}, \quad with \quad \mathcal{L}^{\bar{\mu}}(\Omega) = \frac{\Omega^2 - \Omega^{*2}}{\gamma}\frac{3}{2}\left(6\bar{\theta}\Omega^{*2}\right)^{1/4} \tag{10}$$

The elasticity of $\bar{\mu}(\Omega)$ with respect to Ω is equal to

$$\mathcal{E}(\Omega) \equiv \frac{1}{2} - \frac{3}{\gamma}\left(6\bar{\theta}\Omega^{*2}\right)^{1/4}\Omega^2 \tag{11}$$

Lastly, the reset markup gap is equal to $\hat{\mu}' = 0$.

Proof. The plan for the proof is as follows. First we use the optimality conditions to derive a differential equation that characterizes the border of the inaction region as a function of uncertainty. Second, we approximate the solution of the inaction region. For this proof and the rest to follow we use the following notation for partial derivatives:

$$V_{\hat{\mu}} \equiv \frac{\partial V}{\partial \hat{\mu}}, \qquad V_{\hat{\mu}^2} \equiv \frac{\partial^2 V}{\partial \hat{\mu}^2}, \qquad V_{\Omega} \equiv \frac{\partial V}{\partial \Omega}, \qquad V_{\hat{\mu}^2,\Omega} \equiv \frac{\partial^3 V}{\partial \Omega \partial \hat{\mu}^2}$$

1. **Optimality conditions:** The optimality conditions of the problem are given by:

$$rV(\hat{\mu},\Omega) = -\hat{\mu}^2 + \lambda\left[V\left(\hat{\mu},\Omega+\frac{\sigma_u^2}{\gamma}\right) - V(\hat{\mu},\Omega)\right] + \frac{\sigma_f^2-\Omega^2}{\gamma}V_{\Omega}(\hat{\mu},\Omega) + \frac{\Omega^2}{2}V_{\hat{\mu}^2}(\hat{\mu},\Omega) \tag{A.8}$$

$$V(\bar{\mu}(\Omega),\Omega) = V(0,\Omega) - \bar{\theta} \tag{A.9}$$

$$V_{\mu}(\bar{\mu}(\Omega),\Omega) = 0 \tag{A.10}$$

$$V_{\Omega}(\bar{\mu}(\Omega),\Omega) = V_{\Omega}(0,\Omega) \tag{A.11}$$

2. **Taylor approximation of V and value matching**. For a given level of uncertainty Ω, we do a 4^{th} order Taylor expansion on the first argument of V around zero:

$$V(\hat{\mu},\Omega) = V(0,\Omega) + \frac{V_{\hat{\mu}^2}(0,\Omega)}{2!}\hat{\mu}^2 + \frac{V_{\hat{\mu}^4}(0,\Omega)}{4!}\hat{\mu}^4$$

Odd terms do not appear due to the symmetry of the value function around 0. Evaluating at the border and combining with the value matching condition (A.9) we obtain:

$$-\bar{\theta} = V_{\hat{\mu}^2}(0,\Omega)\frac{\bar{\mu}(\Omega)^2}{2} + V_{\hat{\mu}^4}(0,\Omega)\frac{\bar{\mu}(\Omega)^4}{24} \tag{A.12}$$

3. **Taylor approximation of V_{μ} and smooth pasting**. For a given level of uncertainty Ω, we do a 3^{rd} order Taylor expansion on the first argument of V_{μ} around zero:

$$V_{\mu}(\hat{\mu},\Omega) = V_{\hat{\mu}^2}(0,\Omega)\hat{\mu} + \frac{V_{\hat{\mu}^4}(0,\Omega)}{3!}\hat{\mu}^3$$

Again the odd derivatives are zero. Evaluate at the border, multiply both sides by $\frac{\bar{\mu}(\Omega)}{2}$ and combine with the smooth pasting condition (A.10) to obtain:

$$0 \quad = \quad V_{\hat{\mu}^2}(0,\Omega)\frac{\bar{\mu}(\Omega)^2}{2} + V_{\hat{\mu}^4}(0,\Omega)\frac{\bar{\mu}(\Omega)^4}{12} \tag{A.13}$$

4. **Inaction border (as a function of V)**: Combine the relationships between the 2nd and 4th derivatives of V in (A.12) and (A.13):

$$\bar{\theta} = \bar{\mu}(\Omega)^4\frac{V_{\hat{\mu}^4}(0,\Omega)}{24} = -\bar{\mu}(\Omega)^2\frac{V_{\hat{\mu}^2}(0,\Omega)}{4} \tag{A.14}$$

From the previous equality, we obtain an expression for the border of inaction:

$$\bar{\mu}(\Omega) = \left(\frac{24\bar{\theta}}{V_{\hat{\mu}^4}(0,\Omega)}\right)^{1/4} \tag{A.15}$$

5. **Characterize** $V_{\hat{\mu}^4}(0,\Omega)$: Taking second derivatives of the HBJ in (A.8) with respect to $\hat{\mu}$:

$$\begin{aligned} rV_{\hat{\mu}^2}(\hat{\mu},\Omega) &= -2+\lambda\left(V_{\hat{\mu}^2}\left(\mu,\Omega+\frac{\sigma_u^2}{\gamma}\right)-V_{\hat{\mu}^2}(\hat{\mu},\Omega)\right)+\frac{\sigma_f^2-\Omega^2}{\gamma}V_{\hat{\mu}^2\Omega}(\hat{\mu},\Omega)+\frac{\Omega^2}{2}V_{\hat{\mu}^4}(\hat{\mu},\Omega) \\ &= -2-\frac{\Omega^2-\Omega^{*2}}{\gamma}V_{\hat{\mu}^2\Omega}(\hat{\mu},\Omega)+\frac{\Omega^2}{2}V_{\hat{\mu}^4}(\hat{\mu},\Omega) \end{aligned}$$

where we have used a Taylor approximation of the second argument of $V_{\hat{\mu}^2}\left(\hat{\mu},\Omega+\frac{\sigma_u^2}{\gamma}\right)$ around Ω:

$$V_{\hat{\mu}^2}\left(\hat{\mu},\Omega+\frac{\sigma_u^2}{\gamma}\right)=V_{\hat{\mu}^2}\left(\hat{\mu},\Omega\right)+V_{\hat{\mu}^2\Omega}\left(\hat{\mu},\Omega\right)\frac{\sigma_u^2}{\gamma}$$

and substituted the definition of fundamental uncertainty Ω^*.

Taking the limit $r\to 0$, evaluating at $\hat{\mu}=0$, and rearranging:

$$V_{\hat{\mu}^4}(0,\Omega)=\frac{4}{\Omega^2}\left(1+\frac{\Omega^2-\Omega^{*2}}{\gamma}\frac{V_{\hat{\mu}^2\Omega}(0,\Omega)}{2}\right)=\frac{4}{\Omega^2}\left(1+\mathcal{L}^{\bar{\mu}}(\Omega)\right)$$

where we define the learning component as

$$\mathcal{L}^{\bar{\mu}}(\Omega)\equiv\frac{\Omega^2-\Omega^{*2}}{\gamma}\frac{V_{\hat{\mu}^2\Omega}(0,\Omega)}{2} \tag{A.16}$$

Therefore, the border of the inaction region in (A.15) is given by:

$$\bar{\mu}(\Omega)=\left(\frac{6\bar{\theta}\Omega^2}{1+\mathcal{L}^{\bar{\mu}}(\Omega)}\right)^{1/4} \tag{A.17}$$

6. **Characterize** $V_{\hat{\mu}^2\Omega}(0,\Omega)$. Recall from (A.14) that

$$-\bar{\theta}=\bar{\mu}(\Omega)^2\frac{V_{\hat{\mu}^2}(0,\Omega)}{4}$$

Take derivative with respect to Ω on both sides and then multiply by $\frac{\bar{\mu}(\Omega)}{2}$:

$$\begin{aligned} 0 &= \bar{\mu}(\Omega)^2\frac{V_{\hat{\mu}^2\Omega}(0,\Omega)}{4}+\bar{\mu}(\Omega)\frac{V_{\hat{\mu}^2}(0,\Omega)}{2}\bar{\mu}_\Omega(\Omega) \\ &= \bar{\mu}(\Omega)^3\frac{V_{\hat{\mu}^2\Omega}(0,\Omega)}{8}+\bar{\mu}(\Omega)^2\frac{V_{\hat{\mu}^2}(0,\Omega)}{4}\bar{\mu}_\Omega(\Omega) \\ &= \bar{\mu}(\Omega)^3\frac{V_{\hat{\mu}^2\Omega}(0,\Omega)}{8}-\bar{\theta}\bar{\mu}_\Omega(\Omega) \end{aligned}$$

In the last line we have substituted again (A.15). Rearranging, we obtain the expression

$$V_{\hat{\mu}^2\Omega}(0,\Omega)=\frac{8\bar{\theta}}{\bar{\mu}(\Omega)^3}\bar{\mu}_\Omega(\Omega)$$

Substituting back into (A.16), we obtain that the border of the inaction region is given by:

$$\begin{aligned} \bar{\mu}(\Omega) &= \left(\frac{6\bar{\theta}\Omega^2}{1+\mathcal{L}^{\bar{\mu}}(\Omega)}\right)^{1/4}, \qquad \bar{\mu}(\Omega^*)=\left(6\bar{\theta}\Omega^{*2}\right)^{1/4} \\ \mathcal{L}^{\bar{\mu}}(\Omega) &= \frac{\Omega^2-\Omega^{*2}}{\gamma}\bar{\mu}_\Omega(\Omega)\frac{4\theta}{\bar{\mu}(\Omega)^3}, \qquad \mathcal{L}^{\bar{\mu}}(\Omega^*)=0 \end{aligned}$$

This is an Abel differential equation of the first kind[13]. Unfortunately, no closed-form solution is available. Therefore, we proceed to approximate it around fundamental uncertainty Ω^*.

[13] An Abel differential equation of the first kind takes the form $y'=f_3(x)y^3+f_2(x)y^2+f_1(x)y+f_0(x)$, and in our case $y\equiv\bar{\mu}$, $x\equiv\Omega$, $f_3(x)\equiv-\frac{\gamma}{4\bar{\theta}}\frac{1}{\Omega^2-\Omega^{*2}}$, $f_2(x)\equiv-\frac{3\gamma}{2}\frac{\Omega^2}{\Omega^2-\Omega^{*2}}$, and $f_1(x)=f_0\equiv 0$

7. **Approximation of policy around Ω^*.** Using the condition $\bar{\mu}(\Omega^*) = \left(6\bar{\theta}\Omega^{*2}\right)^{1/4}$, we have that

$$\frac{d\bar{\mu}(\Omega^*)}{d\Omega} 4\bar{\mu}(\Omega^*) \left(\bar{\mu}(\Omega^*)^2 + \Omega^* \frac{2\bar{\theta}}{\gamma}\right) = 6\bar{\theta}\Omega^{*2} \tag{A.18}$$

Given that $\left(\bar{\mu}(\Omega^*)^2 + \frac{2\Omega^*}{\gamma}\bar{\theta}\right) \approx \bar{\mu}(\Omega^*)^2$, we have that

$$\frac{d\bar{\mu}(\Omega^*)}{d\Omega} = \frac{3}{2}\left(6\bar{\theta}\Omega^{*2}\right)^{1/4} \tag{A.19}$$

Therefore

$$\bar{\mu}(\Omega) = \left(\frac{6\bar{\theta}\Omega^2}{1 + \mathcal{L}^{\bar{\mu}}(\Omega)}\right)^{1/4} \qquad \mathcal{L}^{\bar{\mu}}(\Omega) = \frac{\Omega^2 - \Omega^{*2}}{\gamma}\frac{3}{2}\left(6\bar{\theta}\Omega^{*2}\right)^{1/4} \tag{A.20}$$

Define $\Gamma(\Omega) \equiv \frac{V_{\hat{\mu}^2,\Omega}(0,\Omega)}{2}$ and $\Lambda(\Omega) \equiv \Gamma(\Omega)\frac{\Omega^2 - \Omega^{*2}}{\gamma}$, which are characterized next.

8. **Characterize $\Gamma(\Omega)$.** The cross-derivative $\Gamma(\Omega) \equiv \frac{V_{\hat{\mu}^2,\Omega}(0,\Omega)}{2} = \frac{\partial}{\partial\Omega}\frac{V_{\hat{\mu}^2}(0,\Omega)}{2}$ is given by:

$$\begin{aligned}
\frac{\partial}{\partial\Omega}\frac{V_{\hat{\mu}^2}(0,\Omega)}{2} &= \frac{\partial}{\partial\Omega}\left[-\frac{V_{\hat{\mu}^4}(0,\Omega)\bar{\mu}(\Omega)^2}{12}\right] \\
&= \frac{\partial}{\partial\Omega}\left[-\frac{\left(1 + \frac{V_{\hat{\mu}^2,\Omega}(0,\Omega)}{2}\frac{\Omega^2 - \Omega^{*2}}{\gamma}\right)}{3\Omega^2}\Omega\bar{\theta}^2\left(1 + \frac{V_{\hat{\mu}^2,\Omega}(0,\Omega)}{2}\frac{\Omega^2 - \Omega^{*2}}{\gamma}\right)^{-1/2}\right] \\
&= \frac{\partial}{\partial\Omega}\left[-\frac{12\bar{\theta}^2}{\Omega}\left(1 + \frac{V_{\hat{\mu}^2,\Omega}(0,\Omega)}{2}\frac{\Omega^2 - \Omega^{*2}}{\gamma}\right)^{1/2}\right]
\end{aligned}$$

Using the definition of $\Gamma(\Omega)$ write the previous equation recursively as:

$$\Gamma(\Omega) = \frac{\partial}{\partial\Omega}\left[-\frac{12\bar{\theta}^2}{\Omega}\left(1 + \Gamma(\Omega)\frac{\Omega^2 - \Omega^{*2}}{\gamma}\right)^{1/2}\right] \tag{A.21}$$

9. **Characterize $\Lambda(\Omega)$:** Note that: $\Lambda(\Omega^*) = 0$, $\Lambda'(\Omega^*) = 2\frac{\Omega^*}{\gamma}\Gamma(\Omega^*)$, and using (A.21) $\Gamma(\Omega^*) = \frac{12\bar{\theta}^2}{\Omega^{*2}}$. A first order Taylor approximation of $\Lambda(\Omega)$ around Ω^* yields:

$$\Lambda(\Omega) = \Lambda(\Omega^*) + \Lambda'(\Omega^*)(\Omega - \Omega^*) = 2\Omega^*\Gamma(\Omega^*)\frac{\Omega - \Omega^*}{\gamma} = \frac{24\bar{\theta}^2}{\gamma}\left[\frac{\Omega}{\Omega^*} - 1\right]$$

10. Finally, substitute $\Lambda(\Omega)$ into the cutoff value to obtain:

$$\bar{\mu}(\Omega) \approx (6\bar{\theta}\Omega^2)^{1/4}\left(1 + \frac{24\bar{\theta}^2}{\gamma}\left[\frac{\Omega}{\Omega^*} - 1\right]\right)^{-1/4} \tag{A.22}$$

11. **Compute elasticity** Now we compute the elasticity of the cutoff with respect to uncertainty, which is given by $\mathcal{E}_{\phi,\Omega} \equiv \frac{\partial \ln \bar{\mu}(\Omega)}{\partial \ln \Omega}$. Applying logs to (A.22) we obtain:

$$\ln\bar{\mu}(\Omega) = \frac{1}{2}\ln\Omega - \frac{1}{4}\ln\left(1 + \frac{3}{2\gamma}\left(6\bar{\theta}\Omega^{*2}\right)^{1/4}\left[\Omega^2 - \Omega^{*2}\right]\right)$$

Since $\ln(1+x) \approx x$ for x small, taking the derivative we get the result:

$$\mathcal{E}_{\phi,\Omega} \equiv \frac{1}{2} - \frac{3}{\gamma}\left(6\bar{\theta}\Omega^{*2}\right)^{1/4} \tag{A.23}$$

12. **Smooth pasting condition for Ω is implied by (A.15):**

First, let us compute the derivative of the inaction region with respect to uncertainty:

$$\bar{\mu}_\Omega(\Omega) = \frac{d}{d\Omega}\left(\frac{4\bar{\theta}}{-V_{\hat{\mu}^2}(0,\Omega)}\right)^{1/2} = \frac{\bar{\mu}(\Omega)}{2}\frac{1}{V_{\hat{\mu}^2}(0,\Omega)}V_{\hat{\mu}^2\Omega}(0,\Omega)$$

Recall from (A.14) that

$$-\bar{\theta} = \bar{\mu}(\Omega)^2 \frac{V_{\hat{\mu}^2}(0,\Omega)}{4}$$

Take derivative with respect to Ω on both sides, then substitute the expression for elasticity above, and rearrange:

$$\begin{aligned}
0 &= \frac{d}{d\Omega}\bar{\theta} = \frac{d}{d\Omega}\left(\frac{V_{\hat{\mu}^2}(0,\Omega)\bar{\mu}(\Omega)^2}{4}\right) \\
&= V_{\hat{\mu}^2\Omega}(0,\Omega)\frac{\bar{\mu}(\Omega)^2}{4} + \frac{\bar{\mu}(\Omega)}{2}\bar{\mu}_\Omega(\Omega)V_{\hat{\mu}^2}(0,\Omega) \\
&= V_{\hat{\mu}^2\Omega}(0,\Omega)\frac{\bar{\mu}(\Omega)^2}{4} + \frac{\bar{\mu}(\Omega)}{2}\left[\frac{\bar{\mu}(\Omega)}{2}\frac{1}{V_{\hat{\mu}^2}(0,\Omega)}V_{\hat{\mu}^2}(0,\Omega)V_{\hat{\mu}^2\Omega}(0,\Omega)\right] \\
&= V_{\hat{\mu}^2\Omega}(0,\Omega)\frac{\bar{\mu}(\Omega)^2}{4} + \frac{\bar{\mu}(\Omega)^2}{4}V_{\hat{\mu}^2\Omega}(0,\Omega) \\
&= V_{\hat{\mu}^2\Omega}(0,\Omega)\left(\frac{\bar{\mu}(\Omega)^2}{2!} + \frac{\bar{\mu}(\Omega)^4}{4!}\right) \\
&= \frac{d}{d\Omega}\left(V_{\hat{\mu}^2}(\hat{\mu},\Omega) - V_{\hat{\mu}^2}(0,\Omega)\right)\Big|_{\hat{\mu}=\bar{\mu}(\Omega)} \\
&= V_\Omega(\bar{\mu}(\Omega),\Omega) - V_\Omega(0,\Omega)
\end{aligned}$$

□

Proposition 5 **(Conditional Expected Time).** *Let r and $\bar{\theta}$ be small. The expected time for the next price change conditional on the state, denoted by $\mathbb{E}[\tau\big|\hat{\mu},\Omega]$, is approximated as:*

$$\mathbb{E}[\tau\big|\hat{\mu},\Omega] = \frac{\bar{\mu}(\Omega)^2 - \hat{\mu}^2}{\Omega^2}\left(1+\mathcal{L}^\tau(\Omega)\right) \quad \textit{where} \quad \mathcal{L}^\tau(\Omega) \equiv \left(\frac{\Omega}{\Omega^*}-1\right)\left(1-\mathcal{E}(\Omega^*)\right)\left(\frac{4\gamma(6\bar{\theta})^{1/2}}{\gamma+2(6\bar{\theta})^{1/2}}\right) \tag{12}$$

If the elasticity of the inaction region with respect to uncertainty is lower than unity and signal noise is large, then the expected time between price changes (i.e. $\mathbb{E}[\tau\big|0,\Omega]$) is a decreasing and convex function of uncertainty.

Proof. Let $T(\hat{\mu},\Omega)$ denote the expected time for the next price change given the current state, i.e. $\mathbb{E}[\tau\big|\hat{\mu},\Omega]$. The proof consists of four steps. First, we establish the HJB equation for $T(\hat{\mu},\Omega)$ and its corresponding border condition. We apply a first order approximation to the HJB equation on the second state to compute the value with uncertainty jump. Second, we do a second order Taylor approximation of $T(\hat{\mu},\Omega)$ around $(0,\Omega)$, and substitute both the HJB and the border condition into this approximation. This delivers an expression for the expected time that depends on two multiplicative terms: (i) the distance between the markup gap estimate and the border of the inaction region, normalized by uncertainty; and (ii) a term that measures the effect of uncertainty changes into the expected time. Third, we approximate term (ii) around fundamental uncertainty Ω^*. Lastly, we show that if $\mathcal{E}<1$, then time for between price adjustments $T(0,\Omega)$ is decreasing in uncertainty.

1. **HJB equation, jump approximation, and border condition**. Consider a small interval dt. Then $T(\hat{\mu},\Omega)$ can be written recursively as:

$$T(\hat{\mu}_t,\Omega_t) = 1dt + \mathbb{E}\left[T(\hat{\mu}_{t+dt},\Omega_{t+dt})\right]$$

Passing T to the right hand side, dividing by dt and taking the limit $dt\to 0$:

$$0 = 1 + \lim_{dt\downarrow 0}\frac{\mathbb{E}\left[T(\hat{\mu}_{t+dt},\Omega_{t+dt}) - T(\hat{\mu}_t,\Omega_t)\right]}{dt}$$

Recognizing the definition of the generator, we obtain the following HJB equation:

$$0 = 1 + \mathcal{A}T(\hat{\mu},\Omega)$$

Substituting the infinitesimal generator $\mathcal{A}$ from (A.1) we obtain:

$$0 = 1 + \lambda\left[T\left(\hat{\mu},\Omega+\frac{\sigma_u^2}{\gamma}\right) - T(\hat{\mu},\Omega)\right] + \frac{(\sigma_f^2-\Omega^2)}{\gamma}T_\Omega(\hat{\mu},\Omega) + \frac{\Omega^2}{2}T_{\hat{\mu}^2}(\hat{\mu},\Omega)$$

We approximate the uncertainty jump with a linear approximation to the second state:

$$T\left(\hat{\mu},\Omega+\frac{\sigma_u^2}{\gamma}\right) \approx T(\hat{\mu},\Omega) + \frac{\sigma_u^2}{\gamma}T_\Omega(\hat{\mu},\Omega)$$

Substituting the approximation and using the definition of fundamental uncertainty Ω^*, we obtain:

$$0 = 1 + \frac{\Omega^{*2}-\Omega^2}{\gamma}T_\Omega(\hat{\mu},\Omega) + \frac{\Omega^2}{2}T_{\hat{\mu}^2}(\hat{\mu},\Omega) \tag{A.24}$$

The border condition states that at the border of action, the expected time is equal to zero:

$$T(\bar{\mu}(\Omega),\Omega) = 0 \tag{A.25}$$

2. **Approximation of $T(\hat{\mu},\Omega)$.** A second order Taylor approximation of $T(\hat{\mu},\Omega)$ in the first state around $\hat{\mu}=0$ yields:

$$T(\hat{\mu},\Omega) = T(0,\Omega) + \frac{T_{\hat{\mu}^2}(0,\Omega)}{2}\hat{\mu}^2 \tag{A.26}$$

   - To compute $T(0,\Omega)$, we evaluate (A.26) at $(\bar{\mu}(\Omega),\Omega)$ and use the border condition in (A.25):

$$T(0,\Omega) = -\frac{T_{\hat{\mu}^2}(0,\Omega)}{2}\bar{\mu}(\Omega)^2$$

- To compute $T_{\hat{\mu}^2}(0,\Omega)/2$, we evaluate the HJB in (A.24) at $(0,\Omega)$ and solve for it:

$$\frac{T_{\hat{\mu}^2}(0,\Omega)}{2} = -\frac{1}{\Omega^2}\left[1 + T_\Omega(0,\Omega)\frac{\Omega^{*2}-\Omega^2}{\gamma}\right]$$

Substitute both terms into the Taylor approximation and rearrange:

$$T(\hat{\mu},\Omega) = \frac{\bar{\mu}(\Omega)^2 - \hat{\mu}^2}{\Omega^2}\left(1 + \mathcal{L}^\tau(\Omega)\right) \tag{A.27}$$

where $\mathcal{L}^\tau(\Omega) \equiv T_\Omega(0,\Omega)\frac{\Omega^{*2}-\Omega^2}{\gamma}$ measures the effect of uncertainty changes on the expected time and $\mathcal{L}_\tau(\Omega^*) = 0$.

3. **Approximation around** Ω^*. A first order Taylor approximation of $\mathcal{L}_\tau(\Omega)$ around Ω^* yields:

$$\mathcal{L}^\tau(\Omega) = \mathcal{L}^\tau(\Omega^*) + \mathcal{L}^\tau_\Omega(\Omega^*)(\Omega - \Omega^*) = -\frac{2\Omega^*}{\gamma}T_\Omega(0,\Omega^*)(\Omega-\Omega^*)$$

To characterize $T_\Omega(0,\Omega^*)$, take the partial derivative of (A.27) with respect to Ω and evaluate it at $(0,\Omega^*)$:

$$T_\Omega(\Omega^*,0) = -\frac{2\bar{\mu}(\Omega)^2}{\Omega^{*3}}(1-\mathcal{E}(\Omega^*))\Big(1 + \frac{2}{\gamma}\frac{\bar{\mu}(\Omega^*)^2}{\Omega^*}\Big)^{-1} = -\frac{2(1-\mathcal{E}(\Omega^*))}{\Omega^{*2}}\Big(\frac{2\gamma(6\bar{\theta})^{1/2}}{\gamma + 2(6\bar{\theta})^{1/2}}\Big)$$

where $\mathcal{E}(\Omega^*)$ is the elasticity of the inaction region at Ω^*. Substitute back into $\mathcal{L}^\tau(\Omega)$ and arrive to

$$\mathcal{L}^\tau(\Omega) = 2\left(\frac{\Omega}{\Omega^*} - 1\right)(1-\mathcal{E}(\Omega^*))\left(\frac{2\gamma(6\bar{\theta})^{1/2}}{\gamma + 2(6\bar{\theta})^{1/2}}\right)$$

Finally, we arrive at the result

$$T(\hat{\mu},\Omega) = \frac{\bar{\mu}(\Omega)^2 - \hat{\mu}^2}{\Omega^2}\left[1 + A\left(\frac{\Omega}{\Omega^*} - 1\right)\right]$$

where $A \equiv 2(1-\mathcal{E}(\Omega^*))\left(\frac{2\gamma(6\bar{\theta})^{1/2}}{\gamma+2(6\bar{\theta})^{1/2}}\right)$ is a positive constant since the elasticity $\mathcal{E}(\Omega^*)$ is lower than unity. Furthermore, A is close to zero for small menu costs and large signal noise, as in our calibration.

4. **Decreasing and convex in uncertainty.** The expected time *between* price changes is equal to $T(0,\Omega)$:

$$T(0,\Omega) = \frac{\bar{\mu}(\Omega)^2}{\Omega^2}\left[1 + A\left(\frac{\Omega}{\Omega^*} - 1\right)\right]$$

Its first derivative with respect to uncertainty is given by:

$$\frac{\partial T(0,\Omega)}{\partial \Omega} = \frac{\bar{\mu}(\Omega)^2}{\Omega^3}\left(2(\mathcal{E}(\Omega) - 1)\left[1 + A\left(\frac{\Omega}{\Omega^*} - 1\right)\right] + A\frac{\Omega}{\Omega^*}\right)$$

If A is close to zero (as it is the case with small menu costs and large signal noise) we obtain:

$$\frac{\partial T(0,\Omega)}{\partial \Omega} = -2\frac{\bar{\mu}(\Omega)^2}{\Omega^3}(1-\mathcal{E}(\Omega)) < 0$$

which is negative because the elasticity $\mathcal{E}(\Omega)$ is lower than unity. Finally, the second derivative

$$\frac{\partial^2 T(0,\Omega)}{\partial \Omega^2} = 4\frac{\bar{\mu}(\Omega)^2}{\Omega^4}\left[\left(\frac{3}{2} - \mathcal{E}(\Omega)\right)(1-\mathcal{E}(\Omega)) + \frac{\Omega}{2}\mathcal{E}'(\Omega))\right] > 0$$

which is positive since the elasticity is lower than unity and increasing in uncertainty. Therefore, the expected time is decreasing and convex in uncertainty.

□

Proposition 6 (**Uncertainty and Frequency**). *The following relationship between uncertainty dispersion, average price duration, and price change dispersion holds:*

$$\mathbb{E}[\Omega^2] = \frac{\mathbb{V}[\Delta p]}{\mathbb{E}[\tau]} \tag{13}$$

Proof. See Proposition 1 in Álvarez, Le Bihan and Lippi (2014) for a derivation of this result for the case of a fixed uncertainty level $\Omega_t = \sigma$. Here we extend the proof for the case of stochastic uncertainty, but most of the steps are analogous to their proof.

The process for markup estimation is given by $d\hat{\mu}_t = \Omega_t dB_t$. Using Itō's Lemma we have that

$$d(\hat{\mu}_t^2) = \Omega_t^2 dt + 2\mu_t \Omega_t dB_t$$

Therefore $d(\hat{\mu}_t^2) - \Omega_t^2 dt$ is a martingale. Using the Optional Sampling Theorem we have that

$$\mathbb{E}\left[\hat{\mu}_\tau^2 - \int_0^\tau \Omega_s^2 ds \Big| (\mu, \Omega) = (0, \tilde{\Omega})\right] = 0 \tag{A.28}$$

since the stochastic process inside the $\hat{\mu}_t^2 - \int_0^t \Omega_s^2 ds$ is a martingale. Therefore we can write (A.28) as:

$$\mathbb{E}\left[\hat{\mu}_\tau^2 \Big| (\mu_0, \Omega_0) = (0, \tilde{\Omega})\right] = \mathbb{E}\left[\int_0^\tau \Omega_s^2 ds \Big| (\mu_0, \Omega_0) = (0, \tilde{\Omega})\right]$$

Now we will integrate both sides using the renewal density $r(\Omega)$ (See proposition 8). This is the distribution of uncertainty of adjusting firms. The LHS is equal to the expectation of the square of price changes or the variance of price changes since the mean is zero

$$\int_0^\infty \mathbb{E}\left[\hat{\mu}_\tau^2 \Big| (\mu_0, \Omega_0) = (0, \tilde{\Omega})\right] dS(\tilde{\Omega}) = \mathbb{E}[(\Delta p)^2] = \mathbb{V}[(\Delta p)] \tag{A.29}$$

On the RHS, note that $\int_0^\infty \mathbb{E}[\int_0^\tau \Omega_s^2 ds | (\mu_0, \Omega_0) = (0, \tilde{\Omega})] r(\tilde{\Omega}) d\tilde{\Omega}$ is the expected local time $\mathbb{L}$ for the payoff function Ω_s^2. Following Stokey (2009), we express the local time in state domain instead of the time domain:

$$\begin{aligned}
\int_0^\infty \mathbb{E}\left[\int_0^\tau \Omega_s^2 ds \Big| (\mu_0, \Omega_0) = (0, \tilde{\Omega})\right] r(\tilde{\Omega}) d\tilde{\Omega} &= \int_0^\infty \left(\int_{\hat{\mu},\Omega} \mathbb{L}(0, \tilde{\Omega}; \hat{\mu}, \Omega)\Omega^2 d\hat{\mu} d\Omega\right) r(\tilde{\Omega}) d\tilde{\Omega} \\
&= \int_{\hat{\mu},\Omega} \left(\int_0^\infty \mathbb{L}(0, \tilde{\Omega}; \hat{\mu}, \Omega) r(\tilde{\Omega}) d\tilde{\Omega}\right) \Omega^2 d\hat{\mu} d\Omega \\
&= \mathbb{E}[\tau] \int_{\hat{\mu},\Omega} \left(\int_0^\infty \frac{\mathbb{L}(0, \tilde{\Omega}; \hat{\mu}, \Omega)}{\mathbb{E}[\tau]} r(\tilde{\Omega}) d\tilde{\Omega}\right) \Omega^2 d\hat{\mu} d\Omega \\
&= \mathbb{E}[\tau] \int_{\hat{\mu},\Omega} \Omega^2 f(\hat{\mu}, \Omega) d\hat{\mu} d\Omega \\
&= \mathbb{E}[\tau]\mathbb{E}[\Omega^2] \qquad \text{(A.30)}
\end{aligned}$$

where f is the joint density. Putting together (A.29) and (A.30) we get the result:

$$\frac{\mathbb{V}[\Delta p]}{\mathbb{E}[\tau]} = \mathbb{E}[\Omega^2]$$

□

Proposition 7 (**Conditional Hazard Rate**). *Without loss of generality, assume the last price change occurred at $t = 0$ and let $\Omega_0 > \sigma_f$ be the level of uncertainty. There are no infrequent shocks ($\lambda = 0$) and a constant inaction region $\bar{\mu}(\Omega_\tau) = \bar{\mu}_0$. Denote derivatives with respect to τ with a prime ($h'_\tau \equiv \partial h/\partial\tau$).*

1. *The estimate's unconditional variance, denoted by $\mathcal{V}_\tau(\Omega_0)$, is given by:*

$$\mathcal{V}_\tau(\Omega_0) = \sigma_f^2\tau + \mathcal{L}_\tau^{\mathcal{V}}(\Omega_0) \tag{14}$$

where $\mathcal{L}_\tau^{\mathcal{V}}(\Omega_0) \equiv \gamma(\Omega_0 - \Omega_\tau)$, with $\mathcal{L}_0^{\mathcal{V}}(\Omega_0) = 0$, $\lim_{\tau\to\infty} \mathcal{L}_\tau^{\mathcal{V}}(\Omega_0) = \gamma(\Omega_0 - \sigma_f)$, and equal to:

$$\mathcal{L}_\tau^{\mathcal{V}}(\Omega_0) = \gamma\Omega_0 - \gamma\sigma_f\left(\frac{\frac{\Omega_0}{\sigma_f} + \tanh\left(\frac{\sigma_f}{\gamma}\tau\right)}{1 + \frac{\Omega_0}{\sigma_f}\tanh\left(\frac{\sigma_f}{\gamma}\tau\right)}\right)$$

2. *$\mathcal{V}_\tau(\Omega_0)$ is increasing and concave in duration τ: $\mathcal{V}'_\tau(\Omega_0) > 0$ and $\mathcal{V}''_\tau(\Omega_0) < 0$. Furthermore, the following cross-derivatives with initial uncertainty are positive:*

$$\frac{\partial\mathcal{V}_\tau(\Omega_0)}{\partial\Omega_0} > 0, \qquad \frac{\partial\mathcal{V}'_\tau(\Omega_0)}{\partial\Omega_0} > 0, \qquad \frac{\partial|\mathcal{V}''_\tau(\Omega_0)|}{\partial\Omega_0} > 0$$

3. *The hazard of adjusting the price at date τ, conditional on Ω_0, is characterized by:*

$$h_\tau(\Omega_0) = \frac{\pi^2}{8}\underbrace{\frac{\mathcal{V}'_\tau(\Omega_0)}{\bar{\mu}_0^2}}_{\text{decreasing in }\tau}\ \underbrace{\Psi\left(\frac{\mathcal{V}_\tau(\Omega_0)}{\bar{\mu}_0^2}\right)}_{\text{increasing in }\tau} \tag{15}$$

where $\Psi(x) \geq 0$, $\Psi(0) = 0$, $\Psi'(x) > 0$, $\lim_{x\to\infty}\Psi(x) = 1$, first convex then concave, and it is given by:

$$\Psi(x) = \frac{\sum_{j=0}^{\infty}\alpha_j \exp(-\beta_j x)}{\sum_{j=0}^{\infty}\frac{1}{\alpha_j}\exp(-\beta_j x)}, \quad \alpha_j \equiv (-1)^j(2j+1), \quad \beta_j \equiv \frac{\pi^2}{8}(2j+1)^2$$

4. *Exists a $\tau^*(\Omega_0)$ such that the slope of the hazard rate is negative for $\tau > \tau^*(\Omega_0)$, and $\tau^*(\Omega_0)$ is decreasing in Ω_0.*

Proof. Assume $\lambda = 0$, initial conditions $(\hat{\mu}_0, \Omega) = (0, \Omega_0)$, and a constant inaction region at $\bar{\mu}_0 \equiv \bar{\mu}(\Omega) = \bar{\mu}(\Omega_0)$. Without loss of generality, we assume the last price change occurred at $t = 0$. First we derive expressions for two objects that will be part of the estimate's unconditional variance: the state's unconditional variance $\mathbb{E}_0[\mu_\tau^2]$ and the estimate's conditional variance Σ_τ. All moments are conditional on the initial conditions, but we do not make it explicit for simplicity.

1. **State's unconditional variance** Since the state evolves as $d\mu_\tau = \sigma_f dW_\tau$, we have that $\mu_\tau = \mu_0 + \sigma_f W_\tau$, with $W_0 = 0$ and $\mu_0 \sim \mathcal{N}(0, \Sigma_0)$. Therefore, the state's unconditional variance at time τ (after the last price change at $t = 0$) is given by:

$$\mathbb{E}_0[\mu_\tau^2] = \mathbb{E}_0[(\mu_0 + \sigma_f W_\tau)^2] = \mathbb{E}_0[\mu_0^2 + 2\mu_0\sigma_f\mathbb{E}_0[(W_\tau - W_0)] + \sigma_f^2\mathbb{E}_0[(W_\tau - W_0)^2] = \mathbb{E}_0[\mu_0^2] + \sigma_f^2\tau = \Sigma_0 + \sigma_f^2\tau \tag{A.31}$$

where we have use the properties of the Wiener process.

2. **Estimate's conditional variance**. The conditional forecast variance evolves as $d\Sigma_\tau = \left(\sigma_f^2 - \frac{\Sigma^2}{\gamma^2}\right)d\tau$. Assuming an initial condition Σ_0 such that $\Sigma_0 > \gamma\sigma_f$, the general solution to the differential equation is given by

$$\Sigma_\tau = \sigma_f\gamma\tanh\left[\sigma_f\gamma c + \frac{\sigma_f}{\gamma}\tau\right]$$

Evaluation at the initial condition, we get $\Sigma_0 = \sigma_f\gamma\tanh[\gamma\sigma_f c]$ and therefore $c = \frac{1}{\sigma_f\gamma}\tanh^{-1}\left(\frac{\Sigma_0}{\sigma_f\gamma}\right)$. Back into (2) and using properties of the hyperbolic tangent,

$$\Sigma_\tau = \sigma_f\gamma\tanh\left[\tanh^{-1}\left(\frac{\Sigma_0}{\sigma_f\gamma}\right) + \frac{\sigma_f}{\gamma}\tau\right] = \sigma_f\gamma\left(\frac{\frac{\Sigma_0}{\sigma_f\gamma} + \tanh\left(\frac{\sigma_f}{\gamma}\tau\right)}{1 + \frac{\Sigma_0}{\sigma_f\gamma}\tanh\left(\frac{\sigma_f}{\gamma}\tau\right)}\right) \tag{A.32}$$

Since $\tanh(0) = 0$ and $\tanh(+\infty) = 1$ we confirm that $\Sigma_\tau = \Sigma_0$ at $\tau = 0$ and $\lim_{\tau\to\infty}\Sigma_\tau = \sigma_f\gamma$.

3. **Estimate's unconditional variance.** Recall that the estimate follows $d\hat{\mu}_\tau = \Omega_\tau d\hat{Z}_\tau$. Since $\lambda = 0$, uncertainty evolves deterministically as $d\Omega_\tau = \frac{1}{\gamma}(\sigma_f - \Omega_\tau)$. Given the initial condition $\hat{\mu}_0 = 0$, the solution to the forecast equation is $\hat{\mu}_\tau = \int_0^\tau \Omega_s d\hat{Z}_s$. By definition of Itō's integral $\int_0^\tau \Omega_s d\hat{Z}_s = \lim_{(\tau_{i+1}-\tau_i)\to 0} \sum_{\tau_i} \Omega_{\tau_i}(\hat{Z}_{\tau_{i+1}} - \hat{Z}_{\tau_i})$. The increments' Normality and the fact that Ω_{τ_i} is deterministic imply that for each τ_i, $\Omega_{\tau_i}(\hat{Z}_{\tau_{i+1}} - \hat{Z}_{t_i})$ is Normally distributed as well. Since the limit of Normal variables is Normal, we have that markup gap's estimate at date τ, given information set $\mathcal{I}_0$, is also Normally distributed. Let $\mathcal{V}_\tau \equiv \mathbb{E}_0[\hat{\mu}_\tau^2]$ denote the estimate's unconditional variance, then $\hat{\mu}_\tau | \mathcal{I}_0 \sim \mathcal{N}(0, \mathcal{V}_\tau)$. To characterize $\mathcal{V}_\tau$, start from its definition and add and subtract μ_t:

$$\mathcal{V}_\tau \equiv \mathbb{E}_0[\hat{\mu}_\tau^2] = \mathbb{E}_0[\mu_\tau^2] + \mathbb{E}_0[(\hat{\mu}_\tau - \mu_t)^2] - 2\mathbb{E}_0[(\hat{\mu}_\tau - \mu_\tau)\mu_\tau] = \mathbb{E}_0[\mu_\tau^2] - \Sigma_\tau \quad \text{(A.33)}$$

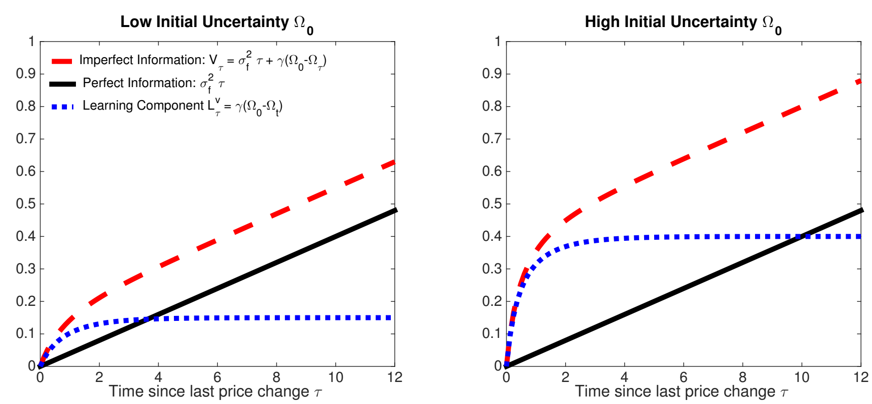
where we that $\mathbb{E}_0[(\hat{\mu}_\tau - \mu_\tau)\mu_\tau] = \mathbb{E}_0[(\hat{\mu}_\tau - \mu_\tau)^2] = \Sigma_t$, implied by the orthogonality of the innovation and the forecast: $\mu_\tau - \hat{\mu}_\tau \perp \hat{\mu}_\tau$. Substituting expressions (A.31) and (A.32) into (A.33) and using $\Omega_\tau = \gamma\Sigma_\tau$, we get:

$$\mathcal{V}_\tau = \sigma_f^2\tau + \gamma\left(\Omega_0 - \Omega_\tau\right) = \sigma_f^2\tau + \gamma\left(\Omega_0 - \sigma_f\left(\frac{\frac{\Omega_0}{\sigma_f} + \tanh\left(\frac{\sigma_f}{\gamma}\tau\right)}{1 + \frac{\Omega_0}{\sigma_f}\tanh\left(\frac{\sigma_f}{\gamma}\tau\right)}\right)\right) = \sigma_f^2\tau + \mathcal{L}_\tau^{\mathcal{V}} \quad \text{(A.34)}$$

where we define the learning component as:

$$\mathcal{L}_\tau^{\mathcal{V}} \equiv \gamma\left(\Omega_0 - \sigma_f\left(\frac{\frac{\Omega_0}{\sigma_f} + \tanh\left(\frac{\sigma_f}{\gamma}\tau\right)}{1 + \frac{\Omega_0}{\sigma_f}\tanh\left(\frac{\sigma_f}{\gamma}\tau\right)}\right)\right)$$

The hyperbolic tangent function is defined as $\tanh(x) \equiv \frac{e^x - e^{-x}}{e^x + e^{-x}}$, and has the following properties: $\tanh(0) = 0$, $\lim_{x\to\pm\infty}\tanh(x) = \pm 1$, $\tanh'(x) = 1 - \tanh^2(x)$.



Useful derivatives The first and second derivatives of the learning component with respect to τ are given by:

$$\frac{\partial \mathcal{L}_\tau^{\mathcal{V}}}{\partial \tau} = \sigma_f^2\left(\frac{\Omega_0}{\sigma_f} - 1\right)\frac{1 - \tanh^2\left(\frac{\sigma_f}{\gamma}\tau\right)}{\left[1 + \frac{\Omega_0}{\sigma_f}\tanh\left(\frac{\sigma_f}{\gamma}\tau\right)\right]^2} > 0$$

$$\frac{\partial^2 \mathcal{L}_\tau^{\mathcal{V}}}{\partial \tau^2} = -\frac{2\sigma_f}{\gamma}\tanh\left(\frac{\sigma_f}{\gamma}\tau\right)\left[1 + \frac{\Omega_0}{\sigma_f}\frac{1 - \tanh^2\left(\frac{\sigma_f}{\gamma}\tau\right)}{1 + \frac{\Omega_0}{\sigma_f}\tanh\left(\frac{\sigma_f}{\gamma}\tau\right)}\right]\frac{\partial \mathcal{L}_\tau^{\mathcal{V}}}{\partial \tau} < 0$$

The derivative of the learning component with respect to uncertainty is:

$$\frac{\partial \mathcal{L}_\tau^{\mathcal{V}}}{\partial \Omega_0} = \gamma - \frac{1 - \tanh^2\left(\frac{\sigma_f}{\gamma}\tau\right)}{\left[1 + \frac{\Omega_0}{\sigma_f}\tanh\left(\frac{\sigma_f}{\gamma}\tau\right)\right]^2}, \quad \text{positive for large } \gamma\text{, large } \Omega_0\text{, and large } \tau$$

Furthermore, the following relationship and signs hold:

$$\frac{\partial \mathcal{L}_\tau^{\mathcal{V}}}{\partial \tau} = \sigma_f^2\left(\frac{\Omega_0}{\sigma_f} - 1\right)\left(\gamma - \frac{\partial \mathcal{L}_\tau^{\mathcal{V}}}{\partial \Omega_0}\right), \qquad \frac{\partial^2 \mathcal{L}_\tau^{\mathcal{V}}}{\partial \tau \partial \Omega_0} > 0, \qquad \left|\frac{\partial^3 \mathcal{L}_\tau^{\mathcal{V}}}{\partial \tau^2 \partial \Omega_0}\right| > 0$$

4. **Stopping time distribution.** Let $F(\sigma_f^2\tau, \bar{\mu}_0)$ be the cumulative distribution of stopping times obtained from a problem with perfect information which considers a Brownian motion with unconditional variance of $\sigma_f^2\tau$, initial condition 0, and a symmetric inaction region $[-\bar{\mu}_0, \bar{\mu}_0]$. Following Kolkiewicz (2002) and Álvarez, Lippi and Paciello (2011)'s Online Appendix, the density of stopping times is given by:

$$f(\tau) = \frac{\pi}{2}x'(\tau)\sum_{j=0}^{\infty}\alpha_j \exp\left(-\beta_j x(\tau)\right), \qquad \text{where} \quad x(\tau) \equiv \frac{\sigma_f^2\tau}{\bar{\mu}_0^2}, \quad \alpha_j \equiv (2j+1)(-1)^j, \quad \beta_j \equiv (2j+1)^2\frac{\pi^2}{8}$$

The process $x(\tau)$ is equal to the ratio of volatility and the width of the inaction region. Since we assumed constant inaction regions, x only changes with volatility. In our case, the estimate's unconditional variance is given by $\mathcal{V}_\tau(\Omega_0)$. Using a change of variable, the distribution of stopping times becomes $F\left(\mathcal{V}_\tau(\Omega_0), \bar{\mu}_0\right)$ with density $f(\tau|\Omega_0) = f(\mathcal{V}_\tau(\Omega_0), \bar{\mu}_0)$. We can apply the previous formula using $x \equiv \frac{\mathcal{V}_\tau(\Omega_0)}{\bar{\mu}_0^2}$ and the same sequences of α_j and β_j.

5. **Hazard rate.** Given the stopping time distribution, the conditional hazard rate is computed using its definition:

$$h_\tau(\Omega_0) \equiv \frac{f(\tau|\Omega_0)}{\int_\tau^\infty f(s|\Omega)ds} = \frac{f(\mathcal{V}_\tau(\Omega_0), \bar{\mu}_0)}{\int_\tau^\infty f(\mathcal{V}_s(\Omega_0), \bar{\mu}_0)ds} = \frac{\mathcal{V}_\tau'(\Omega_0)\sum_{j=0}^\infty \alpha_j \exp\left(-\beta_j\frac{\mathcal{V}_\tau(\Omega_0)}{\bar{\mu}_0^2}\right)}{\int_\tau^\infty \mathcal{V}_s'(\Omega_0)\sum_{j=0}^\infty \alpha_j \exp\left(-\beta_j\frac{\mathcal{V}_s(\Omega_0)}{\bar{\mu}_0^2}\right)ds} \tag{A.35}$$

Let $u_j(s) \equiv \alpha_j \exp\left(-\beta_j\frac{\mathcal{V}_s(\Omega_0)}{\bar{\mu}_0^2}\right)$, then $du_j(s) \equiv \frac{-\alpha_j\beta_j}{\bar{\mu}_0^2}\mathcal{V}_s'(\Omega_0)\exp\left(-\beta_j\frac{\mathcal{V}_s(\Omega_0)}{\bar{\mu}_0^2}\right)ds$. Exchanging the summation with the integral, the denominator is equal to:

$$\sum_{j=0}^\infty \frac{-\bar{\mu}_0^2}{\beta_j}\int_\tau^\infty du_j(s)ds = \sum_{j=0}^\infty \frac{-\bar{\mu}_0^2}{\beta_j}u_j(s)\Big|_\tau^\infty = \sum_{j=0}^\infty \frac{\bar{\mu}_0^2}{\beta_j}u_j(\tau) = \bar{\mu}_0^2\sum_{j=0}^\infty\frac{\alpha_j}{\beta_j}\exp\left(-\beta_j\frac{\mathcal{V}_s(\Omega_0)}{\bar{\mu}_0^2}\right) = \frac{8\bar{\mu}_0^2}{\pi^2}\sum_{j=0}^\infty\frac{1}{\alpha_j}\exp\left(-\beta_j\frac{\mathcal{V}_s(\Omega_0)}{\bar{\mu}_0^2}\right)$$

where in the last equality we use $\frac{\alpha_j}{\beta_j} = \frac{(2j+1)(-1)^j}{(2j+1)^2\frac{\pi^2}{8}} = \frac{8}{\pi^2}(2j+1)^{-1}(-1)^j = \frac{8}{\pi^2}\frac{1}{\alpha_j}$. Substituting back into (A.35):

$$h_\tau(\Omega_0) = \frac{\pi^2}{8\bar{\mu}_0^2}\Psi\left(\frac{\mathcal{V}_\tau(\Omega_0)}{\bar{\mu}_0^2}\right)\mathcal{V}_\tau'(\Omega_0) \tag{A.36}$$

where we define $\Psi(x) \equiv \frac{\sum_{j=0}^\infty \alpha_j\exp\left(-\beta_j x\right)}{\sum_{j=0}^\infty\frac{1}{\alpha_j}\exp\left(-\beta_j x\right)}$ as in Álvarez, Lippi and Paciello (2011)'s Online Appendix. The function $\Psi(x)$ is increasing, first convex then concave, with $\Psi(0) = 0$ and $\lim_{x\to\infty}\Psi(x) = 1$.

6. **Hazard rate's slope.** Taking derivative of the hazard rate with respect to duration τ yields:

$$h_\tau' \propto \underbrace{\frac{\partial^2\mathcal{L}_\tau^\mathcal{V}}{\partial\tau^2}}_{<0}\underbrace{\Psi\left(\frac{\mathcal{V}_\tau}{\bar{\mu}_0^2}\right)}_{\to 1} + \underbrace{\left(\frac{\sigma_f^2 + \frac{\partial\mathcal{L}_\tau^\mathcal{V}}{\partial\tau}}{\bar{\mu}_0^2}\right)^2}_{>0}\underbrace{\Psi'\left(\frac{\mathcal{V}_\tau}{\bar{\mu}_0^2}\right)}_{\to 0}$$

For small τ, Ψ's derivative is very large and the second positive term dominates; as τ increases, the function Ψ and its derivative Ψ' converge to 1 and 0 respectively, and therefore the first term – which is negative – dominates. By the Intermediate Value Theorem, there exists a $\tau^*(\Omega_0)$ such that the slope is zero.

Taking the cross-derivative with respect to uncertainty and using the equivalence between derivatives stated above:

$$\frac{\partial h_\tau'}{\partial\Omega_0} \propto \underbrace{\Psi\left(\frac{\mathcal{V}_\tau}{\bar{\mu}_0^2}\right)}_{\to 1}\underbrace{\frac{\partial^3\mathcal{L}_\tau^\mathcal{V}}{\partial\tau^2\partial\Omega_0}}_{<0} + \underbrace{\Psi''\left(\frac{\mathcal{V}_\tau}{\bar{\mu}_0^2}\right)}_{\to 0^-}\underbrace{\left(\frac{\sigma_f^2 + \frac{\partial\mathcal{L}_\tau^\mathcal{V}}{\partial\tau}}{\bar{\mu}_0^2}\right)^2\frac{\partial\mathcal{L}_\tau^\mathcal{V}}{\partial\Omega_0}\frac{1}{\bar{\mu}_0^2}}_{>0} + \underbrace{\Psi'\left(\frac{\mathcal{V}_\tau}{\bar{\mu}_0^2}\right)}_{\to 0}\frac{1}{\bar{\mu}_0^2}\left[\underbrace{\frac{\partial^2\mathcal{L}_\tau^\mathcal{V}}{\partial\tau^2}\frac{\partial\mathcal{L}_\tau^\mathcal{V}}{\partial\Omega_0}}_{<0} + \underbrace{\frac{2\sigma_f^2}{\bar{\mu}_0^2}\frac{\partial^2\mathcal{L}_\tau^\mathcal{V}}{\partial\tau\partial\Omega_0}\left(1+\left(\frac{\Omega_0}{\sigma_f}-1\right)\left(\gamma - \frac{\partial\mathcal{L}_\tau^\mathcal{V}}{\partial\Omega_0}\right)\right)}_{>0}\right]$$

Since Ψ' and Ψ'' converge to 0 as τ increases, the first term dominates. Then the slope of the hazard rate becomes more negative as initial uncertainty increases. This means that the cutoff duration $\tau^*(\Omega_0)$ is decreasing with Ω_0.

□

Proposition 8 (**Renewal distribution**). *Let $f(\hat{\mu}, \Omega)$ be the joint density of markup gaps and uncertainty in the population of firms. Let $r(\Omega)$ be denote the density of uncertainty conditional on adjusting, or renewal distribution. Assume the inaction region is increasing in uncertainty (i.e. $\bar{\mu}'(\Omega) > 0$). Then we have the following results:*

- *For each $(\hat{\mu}, \Omega)$, we can write the joint density as $f(\hat{\mu}, \Omega) = h(\Omega) g(\hat{\mu}, \Omega)$, where $g(\hat{\mu}, \Omega)$ is the density of markup gap estimates conditional on uncertainty and $h(\Omega)$ is the marginal density of uncertainty.*
- *The ratio between the renewal and marginal distributions of uncertainty is approximated by*

$$\frac{r(\Omega)}{h(\Omega)} \propto |g_{\hat{\mu}}(\bar{\mu}(\Omega), \Omega)| \Omega^2 \tag{16}$$

  where $g(\mu, \Omega)$ solves the following differential equation

$$\frac{\Omega^2 - \Omega^{*2}}{\gamma} g_\Omega(\hat{\mu}, \Omega) + \frac{\Omega^2}{2} g_{\hat{\mu}^2}(\hat{\mu}, \Omega) = 0$$

  with border conditions:

$$g(\bar{\mu}(\Omega), \Omega) = 0 \qquad \int_{-\bar{\mu}(\Omega)}^{\bar{\mu}(\Omega)} g(\mu, \Omega) d\mu = 1$$

- *If $\Omega = \Omega^*$, then the ratio is proportional to the inverse of the expected time between price adjustments. Then if the inaction region's elasticity to uncertainty is lower than unity, the ratio is an increasing function of uncertainty:*

$$\frac{r(\Omega^*)}{h(\Omega^*)} \propto \frac{\Omega^{*2}}{\bar{\mu}(\Omega^*)^2} = \frac{1}{\mathbb{E}[\tau | (0, \Omega^*)]} \tag{17}$$

Proof. The strategy for the proof is as follows. We derive the Kolmogorov Forward Equation (KFE) of the joint ergodic distribution using the adjoint operator. Then we find the zeros of the KFE to characterize the ergodic distribution.

1. **Joint distribution.** Let $f(\hat{\mu}, \Omega) : [-\infty, \infty] \times [\sigma_f, \infty] \to \mathbb{R}$ be the ergodic density of markup estimates and uncertainty. Define the region:

$$\mathcal{R}^\circ \equiv \{(\hat{\mu}, \Omega) \in [-\infty, \infty] \times [\sigma_f, \infty] \quad \text{such that} \quad |\hat{\mu}| < \bar{\mu}(\Omega) \ \& \ \hat{\mu} \neq 0\} \tag{A.37}$$

   where $\bar{\mu}(\Omega)$ is the border of the inaction region. Thus $\mathcal{R}^\circ$ is equal to the continuation region except $\hat{\mu} \neq 0$. Then the function f has the following properties:

   a) f is continuous

   b) f is zero outside the continuation region. Given Ω, $f(x, \Omega) = 0 \ \forall x \notin (-\bar{\mu}(\Omega), \bar{\mu}(\Omega))$. In particular, it is zero at the borders of the inaction region:

$$f(-\bar{\mu}(\Omega), \Omega) = 0 = f(\bar{\mu}(\Omega), \Omega), \quad \forall \Omega$$

   c) f is a density: $\forall (\hat{\mu}, \Omega) \in \mathcal{R}^\circ$, we have that $f(\hat{\mu}, \Omega) \geq 0$ and $\int_{\Omega \geq \sigma_f} \int_{|\hat{\mu}| \leq \bar{\mu}(\Omega)} f(\hat{\mu}, \Omega) d\hat{\mu} d\Omega = 1$

   d) For any state $(\hat{\mu}, \Omega) \in \mathcal{R}^\circ$, f is a zero of the Kolmogorov Forward Equation (KFE):

$$A^* f(\hat{\mu}, \Omega) = 0$$

   Substituting the adjoint operator A^* obtained in (A.2) we write the KFE as:

$$-\frac{\sigma_f^2 - \Omega^2}{\gamma} f_\Omega(\hat{\mu}, \Omega) + \frac{2\Omega}{\gamma} f(\hat{\mu}, \Omega) + \frac{\Omega^2}{2} f_{\hat{\mu}^2}(\hat{\mu}, \Omega) + \lambda \left[f\left(\hat{\mu}, \Omega - \frac{\sigma_u^2}{\gamma}\right) - f(\hat{\mu}, \Omega) \right] = 0 \tag{A.38}$$

   We compute $f\left(\hat{\mu}, \Omega - \frac{\sigma_u^2}{\gamma}\right)$ with a first order Taylor approximation: $f\left(\hat{\mu}, \Omega - \frac{\sigma_u^2}{\gamma}\right) \approx f(\hat{\mu}, \Omega) - \frac{\sigma_u^2}{\gamma} f_\Omega(\hat{\mu}, \Omega)$. Substituting this approximation, collecting terms, and using the definition of fundamental uncertainty Ω^*, the KFE becomes:

$$\frac{2\Omega}{\gamma} f(\hat{\mu}, \Omega) + \frac{\Omega^2 - \Omega^{*2}}{\gamma} f_\Omega(\hat{\mu}, \Omega) + \frac{\Omega^2}{2} f_{\hat{\mu}^2}(\hat{\mu}, \Omega) = 0 \tag{A.39}$$

with two border conditions:

$$\forall\Omega \quad f(|\bar{\mu}(\Omega)|,\Omega)=0 \quad ; \quad \int_{\Omega\geq\sigma_f}\int_{|\hat{\mu}|\leq\bar{\mu}(\Omega)} f(\hat{\mu},\Omega)d\hat{\mu}d\Omega=1 \tag{A.40}$$

2. **Marginal density of uncertainty** Let $h(\Omega):[\sigma_f,\infty]\to\mathbb{R}$ be the uncertainty's ergodic density; it solves the following KFE

$$A^*h=\frac{2\Omega}{\gamma}h(\Omega)+\frac{\Omega^2-\Omega^{*2}}{\gamma}h_\Omega(\Omega)=0$$

and a border condition $\lim_{\Omega\to\infty}h(\Omega)=0$.

3. **Factorization of f.** For each $(\hat{\mu},\Omega)$, guess that we can write f as a product of the ergodic density of uncertainty h and a function g as follows:

$$f(\hat{\mu},\Omega)=h(\Omega)g(\hat{\mu},\Omega) \tag{A.41}$$

Substituting (A.41) into (A.39) and rearranging

$$\begin{aligned}
0 &= \frac{2\Omega}{\gamma}h(\Omega)g(\hat{\mu},\Omega)+\frac{\Omega^2-\Omega^{*2}}{\gamma}\left[h_\Omega(\Omega)g(\hat{\mu},\Omega)+h(\Omega)g_\Omega(\hat{\mu},\Omega)\right]+\frac{\Omega^2}{2}h(\Omega)g_{\hat{\mu}^2}(\hat{\mu},\Omega)\\
&= g(\hat{\mu},\Omega)\underbrace{\left[\frac{2\Omega}{\gamma}h(\Omega)+\frac{\Omega^2-\Omega^{*2}}{\gamma}h_\Omega(\Omega)\right]}_{\text{KFE for } h}+h(\Omega)\left[\frac{\Omega^2-\Omega^{*2}}{\gamma}g_\Omega(\hat{\mu},\Omega)+\frac{\Omega^2}{2}h(\Omega)g_{\hat{\mu}^2}(\hat{\mu},\Omega)\right]\\
&= \frac{\Omega^2-\Omega^{*2}}{\gamma}g_\Omega(\hat{\mu},\Omega)+\frac{\Omega^2}{2}g_{\hat{\mu}^2}(\hat{\mu},\Omega)
\end{aligned}$$

where in the second line we regroup terms and recognize the KFE for h, in the third line we set the KFE of h equal to zero because it is uncertainty's ergodic density and divide by h as it is assumed to be positive. To obtain the border conditions for g, substitute the decomposition (A.41) into (A.40):

$$\forall\Omega \quad h(\Omega)g(|\bar{\mu}(\Omega)|,\Omega)=0 \quad ; \quad \int_{\Omega\geq\sigma_f}\int_{|\hat{\mu}|\leq\bar{\mu}(\Omega)} h(\Omega)g(\hat{\mu},\Omega)d\hat{\mu}d\Omega=1 \tag{A.42}$$

Since $h>0$, we can eliminate it in the first condition and get a border condition for g:

$$g(|\bar{\mu}(\Omega)|,\Omega)=0$$

Then assume that for each Ω, g integrates to one. Use this assumption into the second condition:

$$\int_{\Omega\geq\sigma_f}h(\Omega)\left[\int_{|\hat{\mu}|\leq\bar{\mu}(\Omega)}g(\hat{\mu},\Omega)d\hat{\mu}\right]d\Omega=\int_{\Omega\geq\sigma_f}h(\Omega)d\Omega=1$$

Therefore, by the factorization method, the ergodic distribution h is also the marginal density $h(\Omega)=\int_{|\hat{\mu}|\leq\bar{\mu}(\Omega)}f(\hat{\mu},\Omega)d\hat{\mu}$ and g is the density of markup gap estimates conditional on uncertainty $g(\hat{\mu},\Omega)=f(\hat{\mu}|\Omega)=\frac{f(\hat{\mu},\Omega)}{h(\Omega)}$.

4. **Renewal density** The renewal density is the distribution of firm uncertainty conditional on a price adjustment. For each unit of time, the fraction of firms that adjusts at given uncertainty level is given by three terms (the terms multiplied by 2 take into account the symmetry of the distribution around a zero markup gap):

$$r(\Omega)\propto 2f(\bar{\mu}(\Omega),\Omega)\frac{\sigma_f^2-\Omega^2}{\gamma}+\lambda\int_{-\bar{\mu}\left(\Omega-\sigma_u^2/\gamma\right)}^{\bar{\mu}\left(\Omega-\sigma_u^2/\gamma\right)} f\left(\mu,\Omega-\frac{\sigma_u^2}{\gamma}\right)I\left(\hat{\mu}>\bar{\mu}\left(\Omega\right)\right)d\mu d\Omega+2|f_{\hat{\mu}}(\bar{\mu}(\Omega),\Omega)|\frac{\Omega^2}{2} \tag{A.43}$$

The first term counts price changes of firms at the border of the inaction region that suffer a deterministic decrease in uncertainty; by the border condition $f(\bar{\mu}(\Omega),\Omega)=0$, this term is equal to zero. The second term counts price changes due to jumps in uncertainty. These firms had an uncertainty level of $\Omega-\frac{\sigma_u^2}{\gamma}$ right before the jump; under the assumption that $\bar{\mu}(\Omega)$ is increasing in uncertainty, this term is also equal to zero since all markup estimates that were inside the initial inaction region remain inside the new inaction region. The last term counts price changes of firms at the border of the inaction region that suffer either a positive or negative change in the markup gap estimate (hence the absolute value). This term is the only one different from zero. Substituting the factorization of f, we obtain a simplified expression for the renewal distribution in terms of g:

$$\frac{r(\Omega)}{h(\Omega)} \propto |g_{\hat{\mu}}(\bar{\mu}(\Omega), \Omega)|\Omega^2 \tag{A.44}$$

5. **Characterize g when $\Omega = \Omega^*$.** If $\Omega = \Omega^*$, then the conditional distribution of markup gaps g can be further characterized:

$$g_{\mu^2}(\hat{\mu}, \Omega^*) = 0; \qquad g(\bar{\mu}(\Omega^*), \Omega^*) = 0; \qquad \int_{-\bar{\mu}(\Omega^*)}^{\bar{\mu}(\Omega^*)} g(\hat{\mu}, \Omega^*) d\hat{\mu} = 1 \qquad g \in \mathbb{C} \tag{A.45}$$

To solve this equation, integrate twice with respect to $\hat{\mu}$:

$$g(\hat{\mu}, \Omega^*) = |C|\hat{\mu} + |D|$$

To determine the constants $|C|$ and $|D|$, we use the border conditions:

$$\begin{aligned} 0 &= g(\bar{\mu}(\Omega^*), \Omega^*) = |C|\bar{\mu}(\Omega^*) + |D| \\ 1 &= \int_{-\bar{\mu}(\Omega^*)}^{\bar{\mu}(\Omega^*)} g(\hat{\mu}, \Omega^*) d\mu = \int_{-\bar{\mu}(\Omega^*)}^{\bar{\mu}(\Omega^*)} (|C|\hat{\mu} + |D|) d\mu = \left(\frac{|C|}{2}\hat{\mu}^2 + |D|\hat{\mu} \right) \Big|_{-\bar{\mu}(\Omega^*)}^{\bar{\mu}(\Omega^*)} = 2\bar{\mu}(\Omega^*)|D| \end{aligned}$$

From the second equality, we get that

$$D = \frac{1}{2\bar{\mu}(\Omega^*)}$$

Then substituting in the first equality:

$$|C| = -\frac{|D|}{\bar{\mu}(\Omega^*)} = -\frac{1}{2\bar{\mu}(\Omega^*)^2}$$

Lastly, since $g_{\mu^2}(\hat{\mu}, \Omega^*) \geq 0$, we obtain :

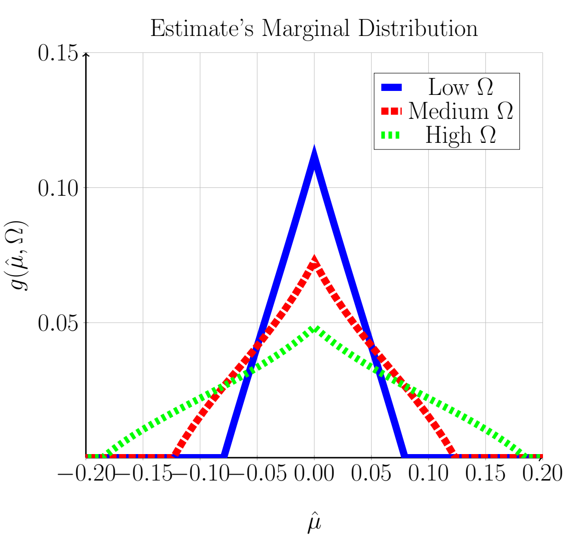
$$g(\mu, \hat{\Omega}) = \begin{cases} \frac{1}{2\bar{\mu}(\Omega^*)} \left(1 + \frac{\hat{\mu}}{\bar{\mu}(\Omega^*)}\right) & \text{if } \hat{\mu} \in [-\bar{\mu}(\hat{\Omega}), 0] \\ \frac{1}{2\bar{\mu}(\Omega^*)} \left(1 - \frac{\hat{\mu}}{\bar{\mu}(\Omega^*)}\right) & \text{if } \hat{\mu} \in (0, \bar{\mu}(\hat{\Omega})] \end{cases} \tag{A.46}$$

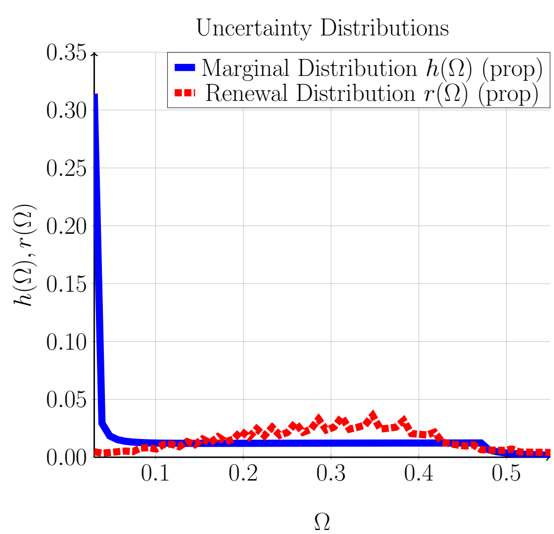
This is a triangular distribution in the $\hat{\mu}$ domain for each Ω (see next figure).

6. **Ratio when $\Omega = \Omega^*$.** By the previous result, the ratio of the renewal to marginal distributions at Ω^* is equal to:

$$\frac{r(\Omega^*)}{h(\Omega^*)} = |g_{\hat{\mu}}(\bar{\mu}(\Omega^*), \Omega^*)| \, \Omega^{*2} = \frac{\Omega^{*2}}{2\bar{\mu}(\Omega^*)^2} \tag{A.47}$$

Since the inaction region's elasticity to uncertainty is lower than unity, this ratio is increasing in uncertainty.





□