

The Linear Systems Approach to Linear Rational Expectations Models*

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Abstract

This paper considers linear rational expectations models from the linear systems point of view. Using a generalization of the Wiener-Hopf factorization, the linear systems approach is able to furnish very simple conditions for existence and uniqueness of both particular and generic linear rational expectations models. As applications of this approach, the paper provides results for existence of sequential solutions to block triangular systems and provides an exhaustive description of stationary and unit root solutions, including a generalization of Granger's representation theorem. In addition, the paper provides an innovative numerical solution to the Wiener-Hopf factorization and its generalization.

JEL Classification: C32, C51, C62, C63, C65.

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1 Introduction

Linear rational expectations models (LREMs) are the hallmark of modern macroeconomics and finance. Their distinct feature is that unlike classical linear systems, where the state of the system depends only on past and present values of the state and an exogenous process, the state in LREMs additionally depends on information used to formulate expectations about the future of the state. The main purpose of this paper is to situate the theory of LREMs within the framework of linear systems theory. It will be seen that, in addition to providing firm mathematical foundations for LREMs, linear systems theory provides a wide array of methods for tackling problems in LREM theory.

To be sure, linear system theory has had important applications in a number studies in the LREM literature, including Whiteman (1983), Broze et al. (1995), Funovits (2014), and Tan & Walker (2015). However, this paper makes a forceful point that the appropriate approach to LREM analysis is through a generalization of Wiener-Hopf factorization (WHF). This factorization, which has had applications in filtration (Anderson & Moore, 1979), stability analysis (Desoer & Vidyasagar, 2009), and optimal control (Youla, Bongiorno & Jabr, 1976; Youla, Jabr & Bongiorno, 1976), among many other areas in linear systems theory, has been used by Onatski (2006) to obtain conditions for existence and uniqueness of stable solutions to LREMs, both particular and generic.¹ However, the WHF cannot be employed in the context of unit roots and therefore cannot be applied to a number of macroeconomic and financial models.

Therefore, this paper begins by generalising the WHF as follows. WHF takes as inputs a suitably well-behaved matrix function (e.g. a matrix of rational functions) and a Cauchy contour (e.g. a circle). The existence of WHF is guaranteed whenever the matrix function has no zeros or poles on the contour (Gohberg & Krein, 1960; Gohberg & Fel'dman, 1974; Clancey & Gohberg, 1981; Gohberg et al., 2003). This paper proposes a generalization whereby one takes the limiting WHF with respect to contours that approach the contour of interest from the inside. This factorization is termed an Inner-Limit Wiener-Hopf Factorization (ILWHF). The paper provides a full treatment of the existence, uniqueness, and perturbation properties of ILWHF as well as its relationship to WHF.

¹A precursor to Onatski's paper in economics is the paper by Whiteman (1985), which uses WHF to solve an optimal control problem.

With this generalization in hand, the paper proceeds to provide existence and uniqueness results for both particular and generic LREMs, generalizing the results of Onatski (2006). The approach is closest in scope and generality to Sims (2002) (the connection is clarified in Section 6) in that it allows for stationary as well as non-stationary solutions and explosive solutions with heterogeneous growth rates. However, the paper takes great pains to rigorously define the solution space, the solution concept, as well as existence and uniqueness. It is demonstrated that the linear systems approach yields the simplest and most direct solution to LREMs in the literature. Moreover, the approach clarifies a number of ambiguities concerning non-uniqueness and the role played by information.

In order to demonstrate the power of the linear systems approach to LREMs, the paper provides a number of applications that would have been prohibitively difficult to undertake under any pre-existing framework for analysing LREMs.

The first application concerns block triangular LREMs. In classical linear systems, block triangular systems can be understood as two subsystems connected in sequence. This is, surprisingly, not the case for LREMs. The paper provides necessary and sufficient conditions for block triangular systems to yield block triangular (or sequential solutions). This result can provide valuable guidance to modellers. It also has demonstrable applications for linear transformations of LREM solutions that allow the researcher to condition out a set of variables and arrive at an LREM that includes only the variables of interest.

Next the paper considers the structure of LREM solutions under typical empirical assumptions. The paper describes the implications of rational expectations for the correlation structure of unique stationary solutions of LREMs extending classical results surveyed in Reinsel (2003). The paper also considers the implications for cointegration, providing conditions for the existence of cointegration as well as a representation theorem that generalizes Granger's representation theorem (Engle & Granger, 1987) to LREMs. The results generalize the treatments given in Broze et al. (1990), Binder & Pesaran (1995), and Juselius (2008).

The final contribution of this paper is an innovative numerical algorithm for computing the ILWHF (or WHF) that builds on both the linear systems literature and the LREM literature. The algorithm, which is implemented in Matlab and available on the author's website, is simple and works well away from the well-known regions of instability of ILWHF.

It will be clear to the reader that these applications are but the low hanging fruit of

the linear systems approach to LREMs and many more venues for research are given in the conclusion of the paper.

The paper is organized as follows. Section 2 introduces the ILWHF and develops its properties, including its relationship to WHF. Section 3 discusses existence and uniqueness of LREM solutions. Section 4 considers the problem of block triangular LREMs and their solutions. Section 5 discusses the implications of the linear systems approach to empirical modelling of data. Section 6 provides a numerical algorithm for obtaining the ILWHF. Section 7 is the conclusion to the paper. Finally, Section 8 provides the proofs of the results.

2 The Inner-Limit Wiener-Hopf Factorization

Linear system theory relies on a number of key factorizations including the Hermite form, the Smith form, and the Smith-McMillan form. The most natural one for LREMs, however, is the ILWHF. Here we develop its properties and its relationship to WHF.

Definition 2.1. $\mathbb{R}[z]$ is the set of polynomials in z with real coefficients. $\mathbb{R}(z)$ is the set of ratios of elements of $\mathbb{R}[z]$ with no common factors. $\mathbb{R}^{n \times m}[z]$ is the set of $n \times m$ matrices whose elements are in $\mathbb{R}[z]$. For $M(z) \in \mathbb{R}^{n \times m}[z]$, $\deg(M(z))$ is the highest power of z that appears in $M(z)$. $M(z) \in \mathbb{R}^{n \times n}[z]$ is said to be unimodular if $\det(M(z))$ is a non-zero constant. $\mathbb{R}^{n \times m}(z)$ is the set of $n \times m$ matrices whose elements are in $\mathbb{R}(z)$. $M(z) \in \mathbb{R}^{n \times n}(z)$ is said to be non-singular if $\det(M(z))$ is not identically zero. For non-negative integers p and q , the set of Laurent matrix polynomials, $M(z) = \sum_{i=-q}^p M_i z^i \in \mathbb{R}^{n \times n}(z)$, is denoted by $\mathbb{R}_{pq}^{n \times n}(z)$. \square

Recall that a polynomial of degree k has k finite zeros and a pole of order k at infinity. A ratio of two polynomials with no common factors of degrees k and m respectively has $\max\{k, m\}$ zeros and poles. If $k \geq m$ there are k finite zeros, m finite poles, and a pole of order $k - m$ at infinity. On the other hand, if $k \leq m$ there are k finite zeros, m finite poles, and a zero of order $m - k$ at infinity. See Section 2.1.4 of Ahlfors (1979) for more details.

For matrix rational functions, we follow the standard convention on zeros and poles (Kailath, 1980, Section 6.5.3). The set of finite zeros (resp. poles) of $M(z) \in \mathbb{R}^{n \times m}(z)$ are the set of finite zeros (resp. poles) of all the non-zero diagonal terms of its Smith-McMillan form. We then say that $M(z)$ has a zero (resp. pole) at infinity if $M(z^{-1})$ has a zero (resp. pole) at $z = 0$. It follows from the definition that $M(z) \in \mathbb{R}^{n \times m}(z)$ has a pole at $z_0 \in \mathbb{C} \cup \{\infty\}$ if

and only if some element of $M(z)$ has a pole at z_0 . It also follows that if $M(z) \in \mathbb{R}^{n \times n}(z)$ is non-singular, then it has a zero at $z_0 \in \mathbb{C} \cup \{\infty\}$ if and only if $M^{-1}(z)$ has a pole at z_0 .² Thus, $M(z)$ can have a zero and a pole at the same point. Note that if $M(z) \in \mathbb{R}^{n \times n}(z)$ is non-singular, then the finite and infinite zeros and poles of $\det(M(z))$ are finite and infinite zeros and poles of $M(z)$ (the reverse inclusion does not hold due to the possibility of cancellation).³

For $M(z) \in \mathbb{R}^{n \times n}[z]$, if ν_i denotes the degree of the i -th column of $M(z)$, then we may set $\Gamma_{\cdot i}$ to be the column vector whose elements are coefficients of z^{ν_i} in the i -th column of $M(z)$. There are many instances in linear system theory when we are interested in factoring z^{ν_i} from the i -th column of $M(z)$ to arrive at $N(z) = M(z)\text{diag}(z^{-\nu_1}, \dots, z^{-\nu_n}) \in \mathbb{R}^{n \times n}(z)$.⁴ Constructed in this way, $N(z)$ can be ensured to have no pole at infinity but it cannot be ensured to have no zero at infinity. $N(z)$ will have no zero at infinity if and only if $[\Gamma_{\cdot 1} \ \cdots \ \Gamma_{\cdot n}]$ is of full rank. If $M(z)$ satisfies this condition then it is said to be column proper (or column reduced). We recall, for future reference, that every non-singular $M(z) \in \mathbb{R}^{n \times n}[z]$ can be brought to column proper form by either left or right multiplication by a unimodular matrix (Wolovich, 1974, Theorem 2.5.14).

Given the linear system concepts above, we are now ready to proceed to the basic mathematical ideas that drive all of the results of this paper.

Definition 2.2. Let $M(z) \in \mathbb{R}^{n \times n}(z)$ be non-singular and $\rho > 0$. Let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, $\bar{\mathbb{D}} = \{z \in \mathbb{C} : |z| \leq 1\}$, and $\mathbb{D}^c = \{z \in \mathbb{C} : |z| \geq 1\} \cup \{\infty\}$. $M(z) = M_f(z)M_0(z)M_b(z)$ is an Inner-Limit Wiener-Hopf factorization (ILWHF) relative to $\rho\mathbb{T}$ if

- (i) $M_f(z) \in \mathbb{R}^{n \times n}(z)$ has no zeros or poles in $\rho\mathbb{D}^c$.
- (ii) $M_0(z) = \text{diag}(z^{\kappa_1}, \dots, z^{\kappa_n})$, where $\kappa_1 \geq \dots \geq \kappa_n$ are integers.
- (iii) $M_b(z) \in \mathbb{R}^{n \times n}(z)$ has no zeros or poles in $\rho\bar{\mathbb{D}}$.

We will refer to $M_f(z)$, $M_0(z)$, and $M_b(z)$ as the forward, null, and backward components of

²See Lemma 2.4.4 of Hannan & Deistler (2012) for the case $z_0 \in \mathbb{C}$. The result for the point at infinity follows a similar argument.

³Econometricians will be familiar with the importance of finite zeros and poles in LREM theory but perhaps less so of the importance of infinite zeros and poles. The importance of the latter was recognized only recently in the work of Funovits (2014).

⁴ $\text{diag}(a_1, \dots, a_n)$ is the $n \times n$ matrix with a_i as the i -th diagonal element and zero elsewhere.

$M(z)$ respectively.⁵ The exponents of z in $M_0(z)$ are called partial indices.

The Wiener-Hopf factorization (WHF) of $M(z)$ relative to $\rho\mathbb{T}$ is the special case of the above, defined only when $\det(M(z))$ has no zeros or poles on $\rho\mathbb{T}$. In that case (iii) can be strengthened to exclude $M_b(z)$ having any zeros or poles in $\rho\overline{\mathbb{D}}$. \square

Clearly, ILWHF extends WHF in allowing for zeros and poles on $\rho\mathbb{T}$, which are then appended to $M_b(z)$.⁶ This is of crucial importance for the analysis of econometric models, where $M(z)$ must be factorized relative to \mathbb{T} but may have zeros on \mathbb{T} (i.e. unit roots). Note that it is possible to state conditions (i) and (iii) in terms of conditions on poles only as:

(i)' $M_f(z), M_f^{-1}(z) \in \mathbb{R}^{n \times n}(z)$ have no poles in $\rho\mathbb{D}^c$.

(iii)' $M_b(z), M_b^{-1}(z) \in \mathbb{R}^{n \times n}(z)$ have no poles in $\rho\mathbb{D}$.

This form is closer to the WHF literature (Gohberg & Krein, 1960; Gohberg & Fel'dman, 1974; Clancey & Gohberg, 1981; Gohberg et al., 2003). It is also possible to state (i) and (iii) in terms of conditions on zeros only:

(i)'' $M_f(z), M_f^{-1}(z) \in \mathbb{R}^{n \times n}(z)$ have no zeros in $\rho\mathbb{D}^c$.

(iii)'' $M_b(z), M_b^{-1}(z) \in \mathbb{R}^{n \times n}(z)$ have no zeros in $\rho\mathbb{D}$.

This equivalence is made possible by our convention on zeros and poles. It would seem (to the author at least) that the choice of (i) and (iii) in Definition 2.2 is the simplest of the three. However, the alternative formulations will be very convenient.

Definition 2.2 (i) implies that the forward component and its inverse have a Laurent series expansion in $\rho\mathbb{D}^c$. Likewise, Definition 2.2 (iii) implies that the backward component and its inverse have a Taylor series expansion in $\rho\mathbb{D}$. Finally, note that the null component is determined completely by the partial indices.

Example 2.1. Let $M(z) = az^{-1} + b$, with $(0, 0) \neq (a, b) \in \mathbb{R}^2$. Then we can find the following

⁵The WHF literature uses the non-mnemonic notation $M_-(z)$ and $M_+(z)$ for $M_f(z)$ and $M_b(z)$ respectively. The reason for our change of notation will become apparent in the next section.

⁶Outer-Limit Wiener-Hopf Factorization could be defined analogously by appending the zeros and poles on $\rho\mathbb{T}$ to $M_f(z)$ instead. This, along with the ILWHF, can also be formulated relative to the more general class of Cauchy contours. However, we will have no use for these generalizations in this paper.

ILWHFs relative to $\rho\mathbb{T}$,

$$\begin{aligned} M_f(z) &= az^{-1} + b & M_0(z) &= 1, & M_b(z) &= 1, & \text{if } |a| < \rho|b| \\ M_f(z) &= 1, & M_0(z) &= z^{-1}, & M_b(z) &= a + bz, & \text{if } |a| \geq \rho|b|. \end{aligned}$$

If $a\rho^{-1} + b = 0$, $M(z)$ has no WHF relative to $\rho\mathbb{T}$. \square

Example 2.2. Let $M(z) = b + cz$, with $(0, 0) \neq (b, c) \in \mathbb{R}^2$. Then we can find the following ILWHFs relative to $\rho\mathbb{T}$,

$$\begin{aligned} M_f(z) &= bz^{-1} + c & M_0(z) &= z, & M_b(z) &= 1, & \text{if } |b| < \rho|c| \\ M_f(z) &= 1, & M_0(z) &= 1, & M_b(z) &= b + cz, & \text{if } |b| \geq \rho|c|. \end{aligned}$$

If $b + \rho c = 0$, $M(z)$ has no WHF relative to $\rho\mathbb{T}$. \square

Example 2.3. Let $M(z) = az^{-1} + b + cz$, with $0 \neq a \in \mathbb{R}$ and $0 \neq c \in \mathbb{R}$, and write it as $M(z) = cz^{-1}(z - \zeta_1)(z - \zeta_2)$. Then we may obtain the following ILWHFs relative to $\rho\mathbb{T}$.

$$\begin{aligned} M_f(z) &= 1 - \zeta_1 z^{-1}, & M_0(z) &= 1, & M_b(z) &= c(z - \zeta_2), & \text{if } |\zeta_1| < \rho \leq |\zeta_2| \\ M_f(z) &= 1, & M_0(z) &= z^{-1}, & M_b(z) &= c(z - \zeta_1)(z - \zeta_2), & \text{if } \rho \leq |\zeta_1|, |\zeta_2| \\ M_f(z) &= c(1 - \zeta_1 z^{-1})(1 - \zeta_2 z^{-1}), & M_0(z) &= z, & M_b(z) &= 1, & \text{if } |\zeta_1|, |\zeta_2| < \rho. \end{aligned}$$

If $a\rho^{-1}e^{-i\theta} + b + c\rho e^{i\theta} = 0$ for some $\theta \in [0, \pi]$, then $M(z)$ has no WHF relative to $\rho\mathbb{T}$. \square

Example 2.4. Let $M(z) = \begin{bmatrix} z^{-1} & 0 \\ 1 & 1-Rz \end{bmatrix}$. Then we can find the following ILWHFs relative to $\rho\mathbb{T}$

$$\begin{aligned} M_f(z) &= \begin{bmatrix} 0 & z^{-1} \\ z^{-1} & 1-Rz \end{bmatrix}, & M_0(z) &= \begin{bmatrix} z & 0 \\ 0 & 1 \end{bmatrix}, & M_b(z) &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, & \text{if } |R^{-1}|, 1 < \rho \\ M_f(z) &= \begin{bmatrix} z^{-1} & 1-Rz \\ 1 & 1 \end{bmatrix}, & M_0(z) &= I_2, & M_b(z) &= \begin{bmatrix} 1 & -(R-1)z \\ 0 & 1-z \end{bmatrix}, & \text{if } |R^{-1}| < \rho \leq 1 \\ M_f(z) &= \begin{bmatrix} z^{-1} & 0 \\ 0 & 1 \end{bmatrix}, & M_0(z) &= I_2, & M_b(z) &= \begin{bmatrix} 1 & 0 \\ 1 & 1-Rz \end{bmatrix}, & \text{if } 1 < \rho \leq |R^{-1}| \\ M_f(z) &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, & M_0(z) &= \begin{bmatrix} 1 & 0 \\ 0 & z^{-1} \end{bmatrix}, & M_b(z) &= \begin{bmatrix} 1 & 1-Rz \\ 1-z & 0 \end{bmatrix}, & \text{if } \rho \leq |R^{-1}|, 1. \end{aligned}$$

It is easily checked that the last two factorizations are the relevant ones when $R = 0$. $M(z)$ has no WHF whenever $\rho = 1$ or $\rho = |R^{-1}|$. \square

Our discussion so far suggests that ILWHF is a strict generalization of WHF. However, a more accurate characterization of the relationship between the two is given in the next result, which also explains where the ‘‘inner-limit’’ part of ILWHF comes from.

Proposition 2.1. Let $M(z) \in \mathbb{R}^{n \times n}(z)$ be non-singular and $\rho > 0$. Let $r\mathbb{T}$ encircle all the zeros and poles of $M(z)$ that are inside $\rho\mathbb{D}$ and $0 < r < \rho$.

- (i) If $M(z) = M_f(z)M_0(z)M_b(z)$ is a WHF relative to $r\mathbb{T}$, then it is also an ILWHF relative to $\rho\mathbb{T}$.
- (ii) If $N_f(z)N_0(z)N_b(z)$ is a WHF of $N(z) = M((r/\rho)z)$ relative to $\rho\mathbb{T}$, then $M(z) = M_f(z)M_0(z)M_b(z)$ with $M_f(z) = N_f((\rho/r)z)N_0(\rho/r)$, $M_0(z) = N_0(z)$, and $M_b(z) = N_b((\rho/r)z)$ is an ILWHF of $M(z)$ relative to $\rho\mathbb{T}$.

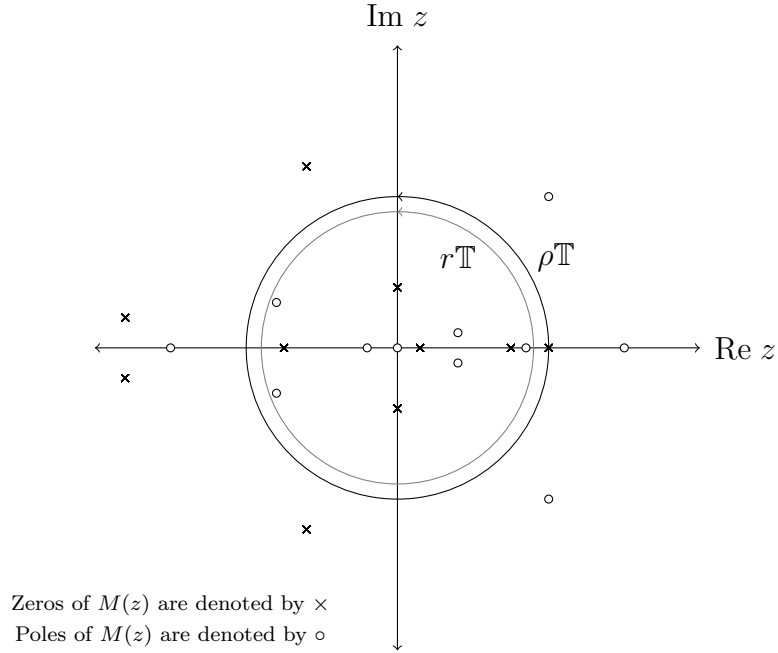
It follows from Proposition 2.1 (i) that an ILWHF is a WHF relative to any close-enough concentric circle inside $\rho\mathbb{T}$.⁷ Proposition 2.1 (ii) obtains an alternative derivation that amounts to a preliminary stretching of the complex plain that pushes any zeros or poles on $\rho\mathbb{T}$ outwards without letting any zeros or poles out of $\rho\mathbb{D}$, then obtaining the WHF, then contracting the complex plain to undo the effect of stretching. The r that appears in the proposition is illustrated in Figure 1.⁸ Proposition 2.1 will be very useful for our development because it will allow us to derive results for ILWHF relative to $\rho\mathbb{T}$ from analogous results for WHF relative to $r\mathbb{T}$.

Our first order of business is to prove the existence of ILWHF. Various proofs of the existence of WHF can be adopted to prove the existence of ILWHF using Proposition 2.1. The simplest approaches can be found in Gohberg et al. (1990) and Gohberg et al. (2003). However, in the appendix we detail a much more direct proof that utilizes the concept of column properness. The proof provides important information about the Smith canonical form of the backward component that will be important for results we derive in Section 5. It also forms the basis of the numerical implementation of the ILWHF presented in Section 6.

⁷Outer-Limit Wiener-Hopf factorization can also be shown to be a WHF relative to any close-enough concentric circle outside of $\rho\mathbb{T}$. It is also possible to define limits along more complicated sequences of contours. These generalizations of WHF do not seem to have received any attention in the linear operator theory literature. Feldman et al. (2002) consider the WHF relative to \mathbb{T} of a sequences of matrix functions that approaches a matrix function that has a zero on \mathbb{T} . In contrast, ILWHF keeps the matrix function fixed and takes a sequence of contours that converges from the inside.

⁸Whereas Proposition 2.1 (ii) exploits the geometry of the contour $\rho\mathbb{T}$, Proposition 2.1 (i) is generalizable to arbitrary Cauchy contours.

Figure 1: Contour of the Inner-Limit Wiener-Hopf Factorization.



Theorem 2.1. Let $M(z) \in \mathbb{R}^{n \times n}(z)$ be non-singular and $\rho > 0$, then an ILWHF exists for $M(z)$ relative to $\rho\mathbb{T}$.

Having proven existence, the next question that ought to be answered concerns the uniqueness of the factorization.

Theorem 2.2. Let $M(z) \in \mathbb{R}^{n \times n}(z)$ be non-singular and $\rho > 0$, then $M(z)$ has ILWHFs $M_f(z)M_0(z)M_b(z)$ and $\check{M}_f(z)\check{M}_0(z)\check{M}_b(z)$ relative to $\rho\mathbb{T}$ if and only if $M_0(z) = \check{M}_0(z)$, $\check{M}_f(z) = M_f(z)M_0(z)U^{-1}(z)M_0^{-1}(z)$, and $\check{M}_b(z) = U(z)M_b(z)$, where $U(z) \in \mathbb{R}^{n \times n}[z]$ is unimodular, $U_{ij}(z) = 0$ for $\kappa_i > \kappa_j$, $U_{ij}(z) \in \mathbb{R}$ for $\kappa_i = \kappa_j$, and $\deg(U_{ij}(z)) \leq \kappa_j - \kappa_i$ for $\kappa_i < \kappa_j$.

It follows from Theorem 2.2 that the partial indices of a non-singular matrix rational function are well defined and unique. Forward and backward components, on the other hand, are only determined up to a special class of unimodular transformations, which are block lower triangular with constant blocks on the diagonal and subdiagonal blocks of bounded degree. It easily checked that $M_0(z^{-1})U^{-1}(z^{-1})M_0^{-1}(z^{-1})$ is also unimodular block lower triangular with constant blocks on the diagonal and subdiagonal blocks of bounded degree. An important special case of the theorem occurs when the partial indices are all zero, in which

case $M_f(z)M_b(z)$ and $\check{M}_f(z)\check{M}_b(z)$ are ILWHFs of $M(z)$ if and only if there exists an invertible matrix $U \in \mathbb{R}^{n \times n}$ such that $M_f(z) = \check{M}_f(z)U^{-1}$ and $M_b(z) = U\check{M}_b(z)$. In that case, a unique choice of ILWHF can be obtained by setting either $M_f(\infty) = I_n$ or $M_b(0) = I_n$.

From here on, we will restrict attention to the set of Laurent polynomials, as this is the most relevant class for econometric applications. Here, the ILWHF takes a particularly simple form.

Theorem 2.3. Let $M(z) \in \mathbb{R}_{pq}^{n \times n}(z)$ be non-singular and $\rho > 0$, then $M(z)$ has an ILWHF $M_f(z)M_0(z)M_b(z)$ relative to $\rho\mathbb{T}$ if and only if:

- (i) $M_f(z^{-1}) \in \mathbb{R}^{n \times n}[z]$ and has no zeros in $\rho^{-1}\overline{\mathbb{D}}$.
- (ii) $M_0(z) = \text{diag}(z^{\kappa_1}, \dots, z^{\kappa_n})$, where $\kappa_1 \geq \dots \geq \kappa_n$ are integers.
- (iii) $M_b(z) \in \mathbb{R}^{n \times n}[z]$ and has no zeros in $\rho\mathbb{D}$.

Moreover $\deg(M_f(z^{-1})) \leq p + q$ and $\deg(M_b(z)) \leq p + q$. If the partial indices are all zero, $\deg(M_f(z^{-1})) \leq q$ and $\deg(M_b(z)) \leq p$.

Theorem 2.3 states that for a Laurent polynomial the forward component is a matrix polynomial in z^{-1} with no zeros in $\rho\mathbb{D}^c$, while the backward component is a matrix polynomial in z with no zeros in $\rho\mathbb{D}$.

A natural question that arises in relation to the ILWHF is whether the partial indices are stable under small perturbations of the matrix function. To answer this question, we endow $\mathbb{R}_{pq}^{n \times n}(z)$ with the metric $d(M(z), N(z)) = \sum_{i=-q}^p \|M_i - N_i\|$.⁹ Gohberg & Krein (1960) prove a general result that specializes in our context to the following statement: the set of all $M(z) \in \mathbb{R}_{pq}^{n \times n}(z)$ with $\det(M(z)) \neq 0$ for all $z \in \rho\mathbb{T}$ and $\kappa_1 \leq \kappa_n + 1$ is an open and dense subset of $\{M(z) \in \mathbb{R}_{pq}^{n \times n}(z) : \det(M(z)) \neq 0, z \in \rho\mathbb{T}\}$.¹⁰ The generalization for the ILWHF is given in the following result.

Theorem 2.4. For fixed $\rho > 0$, non-negative integers p and q , and $n \geq 1$, the set of all non-singular $M(z) \in \mathbb{R}_{pq}^{n \times n}(z)$ whose partial indices in ILWHFs relative to $\rho\mathbb{T}$ satisfy $\kappa_1 \leq \kappa_n + 1$ contains an open and dense subset.

⁹The choice of matrix norm, $\|\cdot\|$, here and elsewhere in this paper is immaterial.

¹⁰Gohberg et al. (2003) provide a simplified proof of this result.

Theorem 2.4 implies that a generic element of the non-singular elements of $\mathbb{R}_{pq}^{n \times n}(z)$ has partial indices that satisfy $\kappa_1 \leq \kappa_n + 1$ in the ILWHF relative to $\rho\mathbb{T}$. This has important implications for existence and uniqueness of solutions to generic LREMs.

3 Existence and Uniqueness of Solutions to LREMs

Having developed the mathematical machinery necessary to study LREMs, we now proceed to the specification and solution of these models. We first derive some preliminary results necessary for the construction of solutions. We then proceed to discussing existence and uniqueness. The role of information is strongly emphasized. Finally, the section closes with a discussion of solutions that exhibit exponential growth.

Now, in order to discuss existence and uniqueness, it is necessary to restrict the solution space and the space of exogenous processes (Pesaran, 1987, Section 5.3.2).

Definition 3.1. Given a probability space (Ω, \mathcal{A}, P) , let $\mathcal{L}^1(\Omega, \mathcal{A}, P)$ be the set of random variables Z defined on Ω with finite expected values, $EZ = \int_{\Omega} Z dP$, and let $\mathcal{S}^n(\Omega, \mathcal{A}, P)$ be the set of stochastic processes $X = \{X_t = (X_{1t}, \dots, X_{nt})' : X_{it} \in \mathcal{L}^1(\Omega, \mathcal{A}, P), i = 1, \dots, n, t \in \mathbb{Z}\}$, such that for any $|\theta| < 1$, $\lim_{t \rightarrow \infty} \theta^t E\|X_t\| = 0$. $X, \hat{X} \in \mathcal{S}^n(\Omega, \mathcal{A}, P)$ are said to be indistinguishable if $P(\hat{X}_t = X_t) = 1$ for all $t \in \mathbb{Z}$. When there is no danger of confusion, we will drop the reference to the probability space and simply write \mathcal{L}^1 and \mathcal{S}^n . \square

Our motivation for this class of processes is both empirical and mathematical. Empirically, \mathcal{S}^n includes large classes of stochastic processes of practical importance. All stable linear processes in \mathcal{L}^1 are included in \mathcal{S}^n , including all weakly stationary processes. Some unstable linear processes in \mathcal{L}^1 are also included in \mathcal{S}^n such as linear processes with unit roots. The usual deterministic processes such as dummies and polynomial trends as well as their interactions are also in \mathcal{S}^n . Note that although it is not possible to relax the condition of membership in \mathcal{L}^1 in Definition 3.1 as this is required in order to be able to take conditional expectations (Williams, 1991, Definition 9.2), only the first moment is required to exist and so \mathcal{S}^n also includes processes that exhibit heavy tails for example. Exponentially increasing process such as explosive linear processes are excluded from \mathcal{S}^n , however, we discuss solutions to LREMs in this class of processes later in this section.

The mathematical advantage of \mathcal{S}^n is that it is closed under all operations necessary for the study of LREMs. It is trivial to check that $X_1 \in \mathcal{S}^{n_1}$ and $X_2 \in \mathcal{S}^{n_2}$ if and only if $(X'_1, X'_2)' = \{(X'_{1t}, X'_{2t})' : t \in \mathbb{Z}\} \in \mathcal{S}^{n_1+n_2}$ and when $n_1 = n_2$, then $aX_1 + bX_2 = \{aX_{1t} + bX_{2t} : t \in \mathbb{Z}\} \in \mathcal{S}^{n_1}$ for all $a, b \in \mathbb{R}$. For $X \in \mathcal{S}^n$, the backward shift operator L is defined as $LX = \{LX_t = X_{t-1} : t \in \mathbb{Z}\}$. The operator that results from $p \geq 1$ applications of L is denoted by L^p . The forward shift operator is defined similarly as $L^{-q}X = \{L^{-q}X_t = X_{t+q} : t \in \mathbb{Z}\}$ for any $q \geq 1$ and $X \in \mathcal{S}^n$. The operator L^0 will be understood to be the identity map on \mathcal{S}^n . Clearly, $L^i X \in \mathcal{S}^n$ for all $i \in \mathbb{Z}$ and $X \in \mathcal{S}^n$. It follows that whenever $X \in \mathcal{S}^n$ and $M(z) = \sum_{i=-q}^p M_i z^i \in \mathbb{R}_{pq}^{n \times n}(z)$, then $M(L)X = \{M(L)X_t = \sum_{i=-q}^p M_i L^i X_t = \sum_{i=-q}^p M_i X_{t-i} : t \in \mathbb{Z}\} \in \mathcal{S}^n$. If $N(z) \in \mathbb{R}_{rs}^{n \times n}(z)$, then clearly $M(L)(N(L)X) = (M(L)N(L))X$ so associativity holds for these types of operators. However, we will need associativity of infinite sums of forward and backward shift operators and this will need to be verified before we can proceed.

Lemma 3.1. Given a probability space (Ω, \mathcal{A}, P) , let $Y \in \mathcal{S}^n(\Omega, \mathcal{A}, P)$, let $\mathcal{I} = \{\mathcal{I}_t \subset \mathcal{A} : t \in \mathbb{Z}\}$ be a filtration, and suppose that $N(z) \in \mathbb{R}^{n \times n}(z)$ has a Laurent series expansion $\sum_{i=0}^{\infty} N_i z^{-i}$ for $|z| > R$ with $0 < R < 1$.

- (i) $\sum_{i=0}^{\infty} N_i E(Y_{t+i} | \mathcal{I}_t) = E(N(L)Y_t | \mathcal{I}_t)$ a.s. for all $t \in \mathbb{Z}$.¹¹
- (ii) $\{E(N(L)Y_t | \mathcal{I}_t) : t \in \mathbb{Z}\} \in \mathcal{S}^n(\Omega, \mathcal{A}, P)$ and $N(L)Y \in \mathcal{S}^n(\Omega, \mathcal{A}, P)$.
- (iii) If $M(z) \in \mathbb{R}^{n \times n}(z)$ also has Laurent series expansion for $|z| > R$, then $M(L)(N(L)Y_t) = (M(L)N(L))Y_t$ a.s. for all $t \in \mathbb{Z}$ and $M(L)N(L)Y \in \mathcal{S}^n(\Omega, \mathcal{A}, P)$.

The “almost sure” ambiguities that appear in Lemma 3.1 come from two sources: (i) conditional expectations are defined only almost surely and that is “something one has to live with in general” (Williams, 1991, p. 85) and (ii) the asymptotic behaviour of $X \in \mathcal{S}^n$ is determined in expectation, thus any statement about its realization’s asymptotic behaviour can hold at most almost surely.

Lemma 3.2. Given a probability space (Ω, \mathcal{A}, P) , suppose $Y \in \mathcal{S}^n(\Omega, \mathcal{A}, P)$ and the initial conditions $\{\tilde{X}_t = (\tilde{X}_{1t}, \dots, \tilde{X}_{nt})' : \tilde{X}_{it} \in \mathcal{L}^1(\Omega, \mathcal{A}, P), i = 1, \dots, n, t < 0\}$ are given. Let $N(z) \in \mathbb{R}^{n \times n}[z]$ and $\det(N(z)) \neq 0$ for all $z \in \mathbb{D}$.

¹¹The abbreviation “a.s.” stands for “almost surely.”

(i) There exists $X \in \mathcal{S}^n(\Omega, \mathcal{A}, P)$ such that

$$X_t = \tilde{X}_t \quad \text{a.s.} \quad t < 0, \quad N(L)X_t = Y_t \quad \text{a.s.} \quad t \geq 0.$$

(ii) If $\hat{X} \in \mathcal{S}^n(\Omega, \mathcal{A}, P)$ is any other solution, then X and \hat{X} are indistinguishable.

The operations in Lemma 3.1 (i) and Lemma 3.2 (i) are known as forward and backward iteration respectively and form the basic techniques for the solution of LREMs. The lemmata have shown that \mathcal{S}^n is closed under these operations.

LREMs describe the behaviour of economic entities (e.g. firms and market participants) in response to observed and expected values of endogenous variables (e.g. prices and production levels) as well as exogenous variables (e.g. technology and government policy). These relationships are encoded into a formal LREM as

$$M_{-q}E_t X_{t+q} + \cdots + M_{-1}E_t X_{t+1} + M_0 X_t + M_1 X_{t-1} + \cdots + M_p X_{t-p} = \varepsilon_t, \quad t \geq 0. \quad (1)$$

Equation (1) is to be understood as a relationship between X_t , its past values $(X_{t-1}, \dots, X_{t-p})$, its expected values $(E_t X_{t+1}, \dots, E_t X_{t+q})$, and exogenous variables ε_t for each $t \geq 0$.¹² It is considered formal because we have not yet defined existence, uniqueness, or even the meaning of the expected values. To each formal LREM of the form (1) we will associate a Laurent polynomial $M(z) = \sum_{i=-q}^p M_i z^i \in \mathbb{R}_{pq}^{n \times n}(z)$.

An important subclass of (1) is the class of dynamic stochastic general equilibrium models, where the structural equations are obtained from an underlying dynamic optimization problems. Another important subclass is the set of models with $M_i = 0$ for $i < 0$, i.e. the set of structural VAR processes. Since ε can itself have a moving average representation, it also includes the set of all structural VARMA processes.

Example 3.1. A variant of the Cagan (1956) model relates the logarithm of the price level, X , to its expected value one period ahead and the money supply, ε , according to $X_t = \phi E_t X_{t+1} + \varepsilon_t$. Here, $M(z) = 1 - \phi z^{-1}$, which is a special case of the function considered in Example 2.1. □

Example 3.2. In the Hansen & Sargent (1980) model, the optimal level of employment of a factor of production, X , is related to exogenous economic forces, ε , by the LREM,

¹²Lagged expectations (i.e. terms of the form $E_{t-i} X_{t-i+j}$ for $i, j \geq 0$) are easily fit into this framework by expanding the state (Binder & Pesaran, 1995).

$aE_tX_{t+1} + bX_t + cX_{t-1} = \varepsilon_t$. Here, $M(z) = az^{-1} + b + cz$, which is the function we studied in Example 2.3. \square

Example 3.3. A variant of the Hall (1978) model has consumption, X_1 , and bond holdings, X_2 , determined by income, ε_2 , according to the system

$$\begin{aligned} X_{1t} &= E_t X_{1t+1} \\ X_{1t} + X_{2t} &= R X_{2t-1} + \varepsilon_{2t}, \end{aligned}$$

where $R - 1$ is the rate of interest. Here, $M(z) = \begin{bmatrix} z^{-1}-1 & 0 \\ 1 & 1-Rz \end{bmatrix}$, which was considered in Example 2.4. \square

Given the formal description above, the next order of business is to assign meaning to existence and uniqueness of a solution to the LREM.

Definition 3.2. Let (Ω, \mathcal{A}, P) be a given probability space. Given $\varepsilon \in \mathcal{S}^n(\Omega, \mathcal{A}, P)$, initial conditions $\{\tilde{X}_t = (\tilde{X}_{1t}, \dots, \tilde{X}_{nt})' : \tilde{X}_{it} \in \mathcal{L}^1(\Omega, \mathcal{A}, P), i = 1, \dots, n, t < 0\}$, and $M(z) \in \mathbb{R}_{pq}^{n \times n}(z)$, a solution to (1) is a pair (X, \mathcal{I}) such that:

- (i) $\mathcal{I} = \{\mathcal{I}_t \subset \mathcal{A} : t \in \mathbb{Z}\}$ is a filtration satisfying $\sigma(\tilde{X}_s : s \leq t) \subseteq \mathcal{I}_t$ for all $t < 0$ and $\varepsilon_t \in \mathfrak{m}\mathcal{I}_t$ for all $t \geq 0$.¹³
- (ii) $X \in \mathcal{S}^n(\Omega, \mathcal{A}, P)$ is adapted to \mathcal{I} .
- (iii) $X_t = \tilde{X}_t$ a.s. for all $t < 0$.
- (iv) $E(M(L)X_t | \mathcal{I}_t) = \varepsilon_t$ a.s. for all $t \geq 0$.

For a given filtration, \mathcal{I} , a solution (X, \mathcal{I}) is said to be unique if for any other solution (\hat{X}, \mathcal{I}) , X and \hat{X} are indistinguishable. A solution (X, \mathcal{I}) is said to be fundamental if $\mathcal{I} = \left\{ \mathcal{I}_t = \sigma(\varepsilon_s : 0 \leq s \leq t) \vee \sigma(\tilde{X}_s : s \leq \min\{t, -1\}) : t \in \mathbb{Z} \right\}$.¹⁴ \square

Similar to martingale theory (Williams, 1991), the solution involves the specification of a filtration. Condition (i) requires the filtration to contain the initial conditions and exogenous variables, i.e. the fundamental economic forces at play. Condition (ii) then requires X to

¹³For a σ -algebra $\mathcal{F} \subseteq \mathcal{A}$, $\mathfrak{m}\mathcal{F}$ is the set of finite dimensional random vectors measurable with respect to \mathcal{F} . For a collection of finite dimensional random vectors $\{Z_i, i \in I\}$, $\sigma(Z_i : i \in I)$ is the smallest σ -algebra with respect to which every Z_i is measurable.

¹⁴For $\mathcal{F}, \mathcal{G} \subset \mathcal{A}$, $\mathcal{F} \vee \mathcal{G}$ is the σ -algebra generated by $\mathcal{F} \cup \mathcal{G}$.

be a sub-exponential process that is always a function of the information available at hand.¹⁵ Condition (iii) requires the solution to satisfy whatever initial conditions are specified. Finally, condition (iv) requires the solution to satisfy the structural relationships specified in (1), where the formal terms $E_t X_{t+i}$ are now substituted by $E(X_{t+i}|\mathcal{I}_t)$. Note that the filtration of fundamental solutions is the smallest for which a solution to the LREM may exist.

With the notions of existence and uniqueness made explicit, we are now in a position to derive conditions for existence and uniqueness of the solution to LREMs.

Theorem 3.1. Let (Ω, \mathcal{A}, P) be a given probability space. Given $\varepsilon \in \mathcal{S}^n(\Omega, \mathcal{A}, P)$, initial conditions $\left\{ \tilde{X}_t = (\tilde{X}_{1t}, \dots, \tilde{X}_{nt})' : \tilde{X}_{it} \in \mathcal{L}^1(\Omega, \mathcal{A}, P), i = 1, \dots, n, t < 0 \right\}$, $M(z) \in \mathbb{R}_{pq}^{n \times n}(z)$, and filtration \mathcal{I} that satisfies Definition 3.2 (i), if $M(z)$ is non-singular and has an ILWHF relative to \mathbb{T} , $M_f(z)M_0(z)M_b(z)$, with partial indices $\kappa_1 \geq \dots \geq \kappa_n$, then the following holds:

- (i) If the partial indices of $M(z)$ are all zero, then there exists a unique solution to (1) (X, \mathcal{I}) generated recursively as

$$X_t = \tilde{X}_t, \quad t < 0, \quad M_b(L)X_t = E(M_f^{-1}(L)\varepsilon_t|\mathcal{I}_t), \quad t \geq 0. \quad (2)$$

- (ii) If the partial indices of $M(z)$ are non-positive and k are negative, then for every set of additional initial conditions $\{M_{b,i}(0)X_t \in m\mathcal{I}_t : n - k < i \leq n, 0 \leq t < -\kappa_i\} \subset \mathcal{L}^1(\Omega, \mathcal{A}, P)$, where $M_{b,i}(0)$ is the i -th row of $M_b(0)$, and every $\nu \in \mathcal{S}^k$ adapted to \mathcal{I} and satisfying $E(M_0(L)S\nu_t|\mathcal{I}_t) = 0$ a.s. for all $t \geq 0$ for $S = \begin{bmatrix} 0 \\ I_k \end{bmatrix}$, there exists a solution to (1) (X, \mathcal{I}) generated recursively as

$$X_t = \tilde{X}_t, \quad t < 0, \quad M_0(L)M_b(L)X_t = E(M_f^{-1}(L)\varepsilon_t|\mathcal{I}_t) + M_0(L)S\nu_t, \quad t \geq 0. \quad (3)$$

- (iii) If any partial index is positive, there exists an exogenous process and/or a set of initial conditions for which there is no solution to (1).

The assumptions of Theorem 3.1 are quite weak relative to the literature. Theorem 3.1 does not require ε to have a Wold decomposition (Whiteman, 1983; Tan & Walker, 2015), invertibility of M_0 (Broze et al., 1985, 1995; Binder & Pesaran, 1995, 1997), or a priori knowledge of the predetermined variables (Blanchard & Kahn, 1980). The result of Onatski (2006) is not nested above because he does not constrain $M(z)$ to be rational. However, when restricting

¹⁵The adaptedness conditions is akin to causality in time series analysis (Hannan & Deistler, 2012, pp. 4-5).

attention to Laurent polynomials, the conditions for existence and uniqueness in Theorem 3.1 generalize those found in Onatski (2006) because they allow for unit roots.

If all the partial indices are zero, then there exists a unique solution given by (2). The general form of the solution is in the form of an autoregressive process driven by current and expected values of ε .

If all the partial indices are non-positive and k are negative, any solution is determined only up to an arbitrary stochastic process ν satisfying

$$E((\nu_{1,t+|\kappa_{n-k+1}|}, \dots, \nu_{k,t+|\kappa_n|})' | \mathcal{I}_t) = 0 \quad \text{a.s.} \quad t \geq 0,$$

as well as arbitrary values of some linear combinations of X over the initial periods $t = 0, \dots, |\kappa_n|$. The process ν is often taken to be independent of ε and the initial conditions (this can always be arranged by expanding the underlying probability space) and is interpreted as a sunspot process that affects the system simply because it is believed (since it is adapted to \mathcal{I}) to play a role (Farmer, 1999). On the other hand, the indeterminacy of the initial values of some linear combinations of X is often overlooked in the literature (e.g. Lubik & Schorfheide (2003) never mentions it). That is because most treatments transform (3) to obtain the representation

$$M_b(L)X_t = M_0^{-1}(L)E(M_f^{-1}(L)\varepsilon_t | \mathcal{I}_t) + S\nu_t, \quad t \geq |\kappa_n|, \quad (4)$$

which masks this additional indeterminacy in X . Proper accounting of structural equations through time shows that the representation above holds only for $t \geq |\kappa_n|$. Notice that, when there is no uncertainty (i.e. when the filtration is given by $\mathcal{I} = \{\mathcal{I}_t = \mathcal{A} : t \in \mathbb{Z}\}$), ν does not enter into (3) although the indeterminacy of the initial values of X remains.

Finally, if any partial index is positive there is no solution in general, in the sense that one can always find exogenous processes and/or initial conditions that violate the structural equations. In fact, the proof of Theorem 3.1 (iii) makes clear that existence can only hold under very unnatural conditions where the exogenous process and/or initial conditions are restricted.

The solution concept advanced in Theorem 3.1 is a straightforward generalization to the multivariate setting of the univariate trick of factorizing an LREM into a part to iterate forwards and a part to iterate backwards. Multivariate extensions of univariate ideas invariably involve diagonalization, and this leads directly to the ILWHF utilized in Theorem 3.1. In fact,

vestiges of this trick appear in every single solution method in the LREM literature. Thus, an ILWHF is obtained implicitly in every single solution method in the literature. Note that the linear systems approach allows the researcher to obtain the VAR representations (2) and (4) directly without having to go through any rearrangement as in Klein (2000) and Sims (2002). The representations are, moreover, clearly the simplest and most compact in the literature.

Example 3.4. Consider the setting of Example 3.1 and let the initial conditions, \mathcal{I} , and ε satisfy the conditions of Definition 3.2. When $|a| < |b|$, the unique solution is (X, \mathcal{I}) with $X_t = \sum_{i=0}^{\infty} (-a/b)^i E(\varepsilon_{t+i} | \mathcal{I}_t)$ for $t \geq 0$ and $X_t = \tilde{X}_t$ for $t < 0$. When $|a| \geq |b|$, then for any $\nu \in \mathcal{S}^1$ satisfying $E(\nu_{t+1} | \mathcal{I}_t) = 0$ a.s. for all $t \geq 0$ and any $X_0 \in \mathcal{L}^1 \cap \mathfrak{m}\mathcal{S}_0$, there is a solution (X, \mathcal{I}) with $X_t = \begin{cases} \tilde{X}_t, & t < 0 \\ X_0, & t = 0 \\ -(b/a)X_{t-1} + a^{-1}\varepsilon_{t-1} + \nu_t, & t \geq 1 \end{cases}$. \square

Example 3.5. Consider the setting of Example 2.3 and let the initial conditions, \mathcal{I} , and ε satisfy the conditions of Definition 3.2. When $|\zeta_1| < 1 \leq |\zeta_2|$, the unique solution (X, \mathcal{I}) has $X_t = \zeta_2^{-1}X_{t-1} - (c\zeta_2)^{-1}E(\sum_{i=0}^{\infty} \zeta_1^i \varepsilon_{t+i} | \mathcal{I}_t)$ for $t \geq 0$ and $X_t = \tilde{X}_t$ for $t < 0$. When $|\zeta_1|, |\zeta_2| \geq 1$, for any $\nu \in \mathcal{S}^1$ satisfying $E(\nu_{t+1} | \mathcal{I}_t) = 0$ a.s. for all $t \geq 0$ and any $X_0 \in \mathcal{L}^1 \cap \mathfrak{m}\mathcal{S}_0$, there is a solution (X, \mathcal{I}) with $X_t = \begin{cases} \tilde{X}_t, & t < 0 \\ X_0, & t = 0 \\ -(b/a)X_{t-1} - (c/a)X_{t-2} + a^{-1}\varepsilon_{t-1} + \nu_t, & t \geq 1 \end{cases}$. Finally, when $|\zeta_1|, |\zeta_2| < 1$, if (X, \mathcal{I}) is a solution, then it must satisfy $X_{t-1} = E((L^{-1}M(L))^{-1}\varepsilon_t | \mathcal{I}_t)$ a.s. for all $t \geq 0$. However, the $t = 0$ equation cannot be ensured to hold and so there is no solution to this LREM in general. \square

Example 3.6. Consider the setting of Example 3.3 and let the initial conditions, \mathcal{I} , and ε satisfy the conditions of Definition 3.2. When $R > 1$, corresponding to a positive interest rate, the unique solution is given by (X, \mathcal{I}) with $X_t = \begin{bmatrix} 0 & R^{-1} \\ 1 & -L^{-1} \end{bmatrix} X_{t-1} + E\left(\begin{bmatrix} R^{-1} \\ 1-L^{-1} \end{bmatrix} \sum_{i=0}^{\infty} R^{-1-i} \varepsilon_{2,t+i} | \mathcal{I}_t\right)$ for $t \geq 0$ and $X_t = \tilde{X}_t$ for $t < 0$. If, on the other hand, $0 \leq R \leq 1$, so that the interest rate is negative, then for any $\nu \in \mathcal{S}^1$ satisfying $E(\nu_{t+1} | \mathcal{I}_t) = 0$ a.s. for all $t \geq 0$ and any $X_{10} \in \mathcal{L}^1 \cap \mathfrak{m}\mathcal{S}_0$, there is a solution generated recursively as $X_{1t+1} = X_{1t} + \nu_{t+1}$, $X_{1t} + X_{2t} - RX_{2t-1} = \varepsilon_{2t}$ for $t \geq 0$ and $X_t = \tilde{X}_t$ for $t < 0$. \square

The role played by \mathcal{I} is non-trivial and does not seem to have garnered sufficient attention in the literature. To see how it can make a significant difference to the solution, consider the following example.

Example 3.7. Consider the setup of Example 3.1 with $|\phi| < 1$. Suppose $m_1, m_2 \in \mathcal{S}^1$ are

i.i.d. and independent of each other and set $\varepsilon = m_1 + m_2$. Now define

$$\begin{aligned}\mathcal{I}_1 &= \left\{ \mathcal{I}_{1t} = \sigma(\varepsilon_s : 0 \leq s \leq t) \vee \sigma(\tilde{X}_s : s \leq \min\{t, -1\}) : t \in \mathbb{Z} \right\} \\ \mathcal{I}_2 &= \left\{ \mathcal{I}_{2t} = \mathcal{I}_{1t} \vee \sigma(m_{2t} : t \in \mathbb{Z}) : t \in \mathbb{Z} \right\} \\ \mathcal{I}_3 &= \left\{ \mathcal{I}_{3t} = \mathcal{A} : t \in \mathbb{Z} \right\}.\end{aligned}$$

Thus, \mathcal{I}_1 is the filtration of fundamental solutions, \mathcal{I}_2 correspond to the setting where, additionally, information about all current and future values of m_{2t} are known, and \mathcal{I}_3 corresponds to the case of no uncertainty. Now set

$$X_{1t} = m_{1t} + m_{2t}, \quad X_{2t} = m_{1t} + \sum_{i=0}^{\infty} \phi^i m_{2t+i}, \quad X_{3t} = \sum_{i=0}^{\infty} \phi^i (m_{1t+i} + m_{2t+i}), \quad t \geq 0$$

and $X_{it} = \tilde{X}_t$ for $t < 0$ and $i = 1, 2, 3$. Then we have three completely different solutions (X_1, \mathcal{I}_1) , (X_2, \mathcal{I}_2) , and (X_3, \mathcal{I}_3) each of which is unique. \square

Of course, if irrelevant information is added to the filtration, it is reasonable to expect it to have no effect on the solution.

Example 3.8. Suppose m_1 and m_2 are as in Example 3.7 and let $\varepsilon = m_1$ instead. Let

$$\begin{aligned}\mathcal{I}_1 &= \left\{ \mathcal{I}_{1t} = \sigma(m_{1s} : 0 \leq s \leq t) \vee \sigma(\tilde{X}_s : s \leq \min\{t, -1\}) : t \in \mathbb{Z} \right\} \\ \mathcal{I}_2 &= \left\{ \mathcal{I}_{2t} = \sigma((m_{1s}, m_{2s}) : 0 \leq s \leq t) \vee \sigma(\tilde{X}_s : s \leq \min\{t, -1\}) : t \in \mathbb{Z} \right\}.\end{aligned}$$

Then with $X_t = m_{1t}$ for $t \geq 0$ and $X_t = \tilde{X}_t$ for $t < 0$, we have that (X, \mathcal{I}_1) and (X, \mathcal{I}_2) are unique solutions to the LREM. \square

The key idea in the examples above is that filtrations factor into equivalence classes according to how they predict $M_f^{-1}(L)\varepsilon$. The next corollary follows directly from (2).

Corollary 3.1. If the partial indices are all zero and (X_1, \mathcal{I}_1) and (X_2, \mathcal{I}_2) are solutions, then X_1 and X_2 are indistinguishable if and only if $E(M_f^{-1}(L)\varepsilon_t | \mathcal{I}_{1t}) = E(M_f^{-1}(L)\varepsilon_t | \mathcal{I}_{2t})$ a.s. for all $t \geq 0$.

Corollary 3.1 defines an equivalence relationship between filtrations of solutions to (1) when the partial indices are all zero: \mathcal{I}_1 and \mathcal{I}_2 are equivalent if and only if both produce a.s. the same predictions of $M_f^{-1}(L)\varepsilon$, in which case they produce indistinguishable solutions. Recalling that the filtration of fundamental solutions is the smallest for which a solution to

(1) may exist, the equivalence class of the filtration of fundamental solutions is the set of filtrations that fail to Granger-cause $M_f^{-1}(L)\varepsilon$ at all horizons. Conversely, if we maintain that the partial indices are all non-positive, then if (X_1, \mathcal{I}_1) and (X_2, \mathcal{I}_2) are two solutions to (1) such that \mathcal{I}_1 and \mathcal{I}_2 give a.s. the same predictions of $M_f^{-1}(L)\varepsilon$, and X_1 and X_2 are not indistinguishable, then Corollary 3.1 implies that at least one of the partial indices must be negative. In words, irrelevant additional information leads to no change in the solution to an LREM if the partial indices are all zero and can lead to sunspot solutions only if the partial indices are non-positive and some are negative.

Next we consider existence and uniqueness of solutions to generic LREMs. Onatski (2006) proved a general result that specializes in our context to the existence of an open and dense subset $A \subset \{M(z) \in \mathbb{R}_{pq}^{n \times n}(z) : \det(M(z)) \neq 0, z \in \mathbb{T}\}$ such that the existence and uniqueness of any LREM with symbol $M(z) \in A$ is determined by the number of times $\det(M(z))$ winds around the origin as \mathbb{T} is traversed counter-clockwise. However, the winding number is not defined when $\det(M(z))$ has a zero on \mathbb{T} . We now show how this result can be extended.¹⁶

Theorem 3.2. Suppose the initial conditions, \mathcal{I} , and ε are as in Theorem 3.1. Let r be as in Proposition 2.1. Then, for a generic non-singular $M(z) \in \mathbb{R}_{pq}^{n \times n}(z)$ we have existence and uniqueness, existence but no uniqueness, or non-existence, according to whether $\det(M(z))$ winds around the origin zero, a negative, or a positive number of times respectively as $r\mathbb{T}$ is traversed counter-clockwise.

Corollary 3.2. For a generic $M(z)$ we have existence and uniqueness, existence but no uniqueness, or non-existence, according to whether $n_Z - n_P$ is zero, negative, or positive, where n_Z and n_P are the number of zeros and poles of $\det(M(z))$ (counting multiplicity) that are inside \mathbb{D} respectively.

Example 3.9. Consider the setting of Example 3.6, then $\det(M(z)) = (z^{-1} - 1)(1 - Rz)$. Onatski's original winding number index cannot be calculated for this system. However, $\det(M(z))$ has zeros at $\{1, R^{-1}\}$ and poles at $\{0, \infty\}$. Thus, Corollary 3.2 correctly predicts existence and uniqueness when $R > 1$ and existence but non-uniqueness when $0 \leq R \leq 1$. \square

If is worth heading Sims's warning in this context that some LREMs may be sufficiently

¹⁶Readers familiar with the stability theory of linear systems, will see similarity to the Nyquist criterion. This is due to the fact that both results rely on the argument principle in complex analysis (Ahlfors, 1979, Section 5.2).

restricted by theoretical considerations that they become non-generic (Sims, 2007). In this context, root counting rules of thumb can be misleading. Thus, strictly speaking, Example 3.9 is a misapplication of Theorem 3.2 as it should be applied only when all parameters are free of restrictions.

We close this section with a generalization of existence and uniqueness to spaces beyond \mathcal{S}^n . Theoretical considerations sometime warrant constructing solutions that exhibit exponential growth (Blanchard & Fischer, 1989, Chapter 5). In the univariate case, one can find the solutions simply by obtaining the ILWHF relative to $\rho\mathbb{T}$ with $0 < \rho < 1$.

Example 3.10. Consider the setting of Example 3.5 and suppose we would like to obtain solutions that exhibit a growth rate of up to ρ^{-1} , where $0 < \rho < 1$. When $|\zeta_1| < \rho \leq |\zeta_2|$, the unique solution (X, \mathcal{S}) has $X_t = \zeta_2^{-1}X_{t-1} - (c\zeta_2)^{-1}E(\sum_{i=0}^{\infty} \zeta_1^i \varepsilon_{t+i} | \mathcal{S}_t)$ for $t \geq 0$ and $X_t = \tilde{X}_t$ for $t < 0$. When $|\zeta_1|, |\zeta_2| \geq \rho$, for any $\nu \in \mathcal{S}^1$ satisfying $E(\nu_{t+1} | \mathcal{S}_t) = 0$ a.s. for all $t \geq 0$ and any $X_0 \in \mathcal{L}^1 \cap m\mathcal{S}_0$, there is a solution (X, \mathcal{S}) with $X_t = \begin{cases} \tilde{X}_t & t < 0 \\ X_0, & t = 0. \\ -(b/a)X_{t-1} - (c/a)X_{t-2} + a^{-1}\varepsilon_{t-1} + \nu_t, & t \geq 1 \end{cases}$. Finally, when $|\zeta_1|, |\zeta_2| < \rho$, there is no solution in general. \square

In the multivariate setting, the same logic as above applies albeit with a new subtlety. One can find solutions where X exhibits heterogeneous rates of growth, i.e. different components of X grow at different rates. The key insight to solving the multivariate problem is that an LREM can only produce solutions that exhibit exponential growth if the Laurent polynomial associated with the LREM has zeros in $\mathbb{D} \setminus \{0\}$. Thus, to produce these exponentially growing solutions, one proceeds as follows. Let $M(z) \in \mathbb{R}_{pq}^{n \times n}(z)$ and factorize it as $M(z) = \hat{M}(z)G(z)$, where $\hat{M}(z)$ is a square Laurent matrix polynomial and $G(z)$ is a matrix polynomial with all its zeros in $\mathbb{D} \setminus \{0\}$. Such a factorization is easily obtained using the Smith canonical form of $z^q M(z)$ (see the proof of Theorem 2.1). $G(z)$ may contain some or all of the zeros of $M(z)$ in $\mathbb{D} \setminus \{0\}$. We then obtain the ILWHF of $\hat{M}(z)$ relative to \mathbb{T} as $\hat{M}_f(z)\hat{M}_0(z)\hat{M}_b(z)$. Given initial conditions, \mathcal{S} , and ε that satisfy the conditions of Definition 3.2 and if all the partial indices of $\hat{M}(z)$ are non-positive, we may solve (1) in two steps. First, for a given filtration that satisfies Definition 3.2 (i), we obtain the solution (\hat{X}, \mathcal{S}) ,

$$\begin{aligned} \hat{M}_0(L)\hat{M}_b(L)\hat{X}_t &= E(\hat{M}_f^{-1}(L)\varepsilon_t | \mathcal{S}_t) + \hat{M}_0(L)S\nu_t, & t \geq 0 \\ \hat{X}_t &= G(L)\tilde{X}_t, & t < 0. \end{aligned}$$

where S and ν are as in Theorem 3.1 (ii). Next, we solve for X recursively in

$$\begin{aligned} G(L)X_t &= \hat{X}_t, & t \geq 0 \\ X_t &= \tilde{X}_t, & t < 0. \end{aligned}$$

Thus, while $\hat{X} \in \mathcal{S}^n$ by Theorem 3.1, $X \notin \mathcal{S}^n$ in general. The pair (X, \mathcal{J}) satisfies all of the conditions for an LREM solution in Definition 3.2, except membership in \mathcal{S}^n .

Example 3.11. Consider the setting of Example 3.3 and let the initial conditions, \mathcal{J} , and ε satisfy the conditions of Definition 3.2. Then we may write $M(z) = \hat{M}(z)G(z)$ with $\hat{M}(z) = \begin{bmatrix} z^{-1} & -1 & 0 \\ 1 & & 1 \end{bmatrix}$ and $G(z) = \begin{bmatrix} 1 & 0 \\ 0 & 1-Rz \end{bmatrix}$. An ILWHF of $\hat{M}(z)$ relative to \mathbb{T} is $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & z^{-1} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -z & 0 \end{bmatrix}$. It follows that $\hat{X}_{1t} + \hat{X}_{2t} = \varepsilon_{2t}$ and $\hat{X}_{1t+1} - \hat{X}_{1t} = \nu_{t+1}$ a.s. for $t \geq 0$, where $E(\nu_{t+1}|\mathcal{J}_t) = 0$ a.s. for all $t \geq 0$ and $\hat{X}_{10} \in \mathcal{L}^1 \cap \text{m}\mathcal{J}_0$. For $t \geq 1$, $\hat{X}_{1t} = \hat{X}_{10} + \sum_{s=1}^t \nu_s$ and $\hat{X}_{2t} = -\hat{X}_{10} - \sum_{s=1}^t \nu_s + \varepsilon_{2t}$. It follows that for $t \geq 1$, $X_{1t} = \hat{X}_{10} + \sum_{s=1}^t \nu_s$ and $X_{2t} = RX_{2t-1} - \hat{X}_{10} - \sum_{s=1}^t \nu_s + \varepsilon_{2t}$. Thus, in any solution of the LREM that exhibits exponential growth, it is bond holdings that experiences the growth, while consumption continues to exhibit its random walk behaviour. \square

A couple of comments are in order here. First, it is clear that the method above allows us to find any and all exponentially growing solutions to the LREM, including the cases discussed in Sims (2002) where one imposes exponential growth restrictions on certain linear combinations of X . Second, the logic above can be used to extract any non-zero set of zeros of $M(z)$ into $G(z)$. In particular, when $G(z)$ extracts a zero of $M(z)$ in \mathbb{D}^c , the resulting solution is the same as if we had followed the solution concept of Theorem 3.1. Thus, if one insists on using the WHF instead of the ILWHF, then the method above could be used to extract all the zeros on \mathbb{T} in $G(z)$ before applying the WHF. If $G(z)$ extracts all the non-zero zeros of $M(z)$, we obtain the solution found in Broze et al. (1995).

4 Block Triangular Systems

It is an interesting curiosity that a block triangular non-singular $M(z) \in \mathbb{R}_{pq}^{n \times n}(z)$ does not necessarily admit a block triangular ILWHF. We have already seen an instance of this in Example 2.4, where neither the forward nor the backward component were triangular in the economically relevant case $R > \rho = 1$. This is in stark contrast to classical dynamical systems

where a block triangular system can be view as two subsystems connected in sequence (i.e. the output of the first subsystem is the input of the second subsystem). This section seeks to answer the following questions: why is it that block triangular systems do not admit block triangular solutions in general? And which block triangular systems admit block triangular solutions? The method of factorizing non-singular lower triangular systems that we will present in this paper is adopted from barrier problem theory in complex analysis (see e.g. Section II.7 and Chapter IV of Clancey & Gohberg (1981)). In order to do this, we will need to introduce a few concepts.

Definition 4.1. For $M(z) \in \mathbb{R}(z)$, let $[M(z)]_{\rho\mathbb{D}^c} \in \mathbb{R}(z)$ be the part of the partial fractions representation of $M(z)$ that has no poles in $\rho\mathbb{D}^c$ and is zero at infinity. Similarly, let $[M(z)]_{\rho\mathbb{D}} \in \mathbb{R}(z)$ be the part of $M(z)$ that has no poles in $\rho\mathbb{D}$. For $M(z) \in \mathbb{R}^{n \times m}(z)$, $[M(z)]_{\rho\mathbb{D}^c}, [M(z)]_{\rho\mathbb{D}} \in \mathbb{R}^{n \times m}(z)$ are defined element-wise. Thus, any $M(z) \in \mathbb{R}^{n \times m}(z)$ decomposes uniquely as $M(z) = [M(z)]_{\rho\mathbb{D}} + [M(z)]_{\rho\mathbb{D}^c}$. \square

The operators $[\cdot]_{\rho\mathbb{D}}$ and $[\cdot]_{\rho\mathbb{D}^c}$ can be interpreted as projections on $\mathbb{R}^{n \times m}(z)$ (Clancey & Gohberg, 1981, p. 20). In particular, for any ILWHF relative to $\rho\mathbb{T}$, $M_f(z)M_0(z)M_b(z)$, we have that $[M_f(z)]_{\rho\mathbb{D}} = 0$, $[M_f(z)]_{\rho\mathbb{D}^c} = M_f(z)$, $[M_b(z)]_{\rho\mathbb{D}} = M_b(z)$, and $[M_b(z)]_{\rho\mathbb{D}^c} = 0$. To see how these operators are useful, consider the following simple generalization of Examples 2.1 and 3.1.

Example 4.1. Let $M(z) = \begin{bmatrix} 1-\chi z & 0 \\ -1 & 1-\phi z^{-1} \end{bmatrix}$ with $|\chi| \leq 1$ and $|\phi| < 1$. This corresponds to the case where the money supply follows a first order autoregressive process in Example 3.1. The diagonal elements of $M(z)$ are easily factorized with respect to \mathbb{T} and so it is natural to postulate an ILWHF with respect to \mathbb{T} of the form $M(z) = \begin{bmatrix} 1 & 0 \\ f(z) & 1-\phi z^{-1} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1-\chi z & 0 \\ b(z) & 1 \end{bmatrix}$, where $f(z)$ has no poles in \mathbb{D}^c and $b(z)$ has no poles in \mathbb{D} . In order to satisfy this equation we need $-1 = f(z)(1 - \chi z) + (1 - \phi z^{-1})b(z)$. Dividing both sides by $(1 - \chi z)(1 - \phi z^{-1})$, we obtain $\frac{-1}{(1-\chi z)(1-\phi z^{-1})} = \frac{f(z)}{1-\phi z^{-1}} + \frac{b(z)}{1-\chi z}$, the right hand side of which consists of $\frac{f(z)}{1-\phi z^{-1}}$, which has no poles in \mathbb{D}^c , and $\frac{b(z)}{1-\chi z}$, which has no poles in \mathbb{D} . If we further impose that $f(\infty) = 0$, then we may use the operators in Definition 4.1 to solve uniquely for both $f(z)$ and $b(z)$. Applying $[\cdot]_{\mathbb{D}^c}$, we obtain $\frac{f(z)}{1-\phi z^{-1}} = \left[\frac{-1}{(1-\chi z)(1-\phi z^{-1})} \right]_{\mathbb{D}^c} = \frac{\phi}{(\chi\phi-1)(z-\phi)}$, which implies that $f(z) = \frac{\phi z^{-1}}{\chi\phi-1}$. Applying $[\cdot]_{\mathbb{D}}$ on the other hand, yields $b(z) = \frac{1}{\chi\phi-1}$. It follows that an ILWHF of $M(z)$ with respect to \mathbb{T} is $M(z) = \begin{bmatrix} 1 & 0 \\ \frac{\phi z^{-1}}{\chi\phi-1} & 1-\phi z^{-1} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1-\chi z & 0 \\ \frac{1}{\chi\phi-1} & 1 \end{bmatrix}$. \square

The same logic used in the previous example can be attempted in Example 3.3 but will not yield a solution. The obstruction is explained in the following result.

Lemma 4.1. Let $M(z) \in \mathbb{R}_{pq}^{n \times n}(z)$ be non-singular and lower triangular and $\rho > 0$, then $M(z)$ has an ILWHF relative to $\rho\mathbb{T}$, $M_f(z)M_0(z)M_b(z)$, with lower triangular $M_f(z)$ and $M_b(z)$ if and only if the partial indices of the diagonal elements of $M(z)$ are in descending order.

In Example 2.4, for $R > 1$ the diagonal elements of $M(z)$ have partial indices of -1 and $+1$ in their ILWHFs with respect to \mathbb{T} respectively and, by Lemma 4.1, $M(z)$ does not admit a lower triangular factorization. In Example 4.1 on the other hand, the partial indices are both zero and so we are able to obtain a triangular factorization.

The generalization from the lower triangular to the block lower triangular case is now straightforward.

Theorem 4.1. Let $M(z) = \begin{bmatrix} M_{11}(z) & 0 \\ M_{21}(z) & M_{22}(z) \end{bmatrix} \in \mathbb{R}_{pq}^{n \times n}(z)$ be non-singular and $\rho > 0$, then $M(z)$ has an ILWHF relative to $\rho\mathbb{T}$, $M_f(z)M_0(z)M_b(z)$, with conformably partitioned $M_f(z) = \begin{bmatrix} M_{f,11}(z) & 0 \\ M_{f,21}(z) & M_{f,22}(z) \end{bmatrix}$ and $M_b(z) = \begin{bmatrix} M_{b,11}(z) & 0 \\ M_{b,21}(z) & M_{b,22}(z) \end{bmatrix}$ if and only if the the partial indices of $M_{11}(z)$ are no smaller than any partial index of $M_{22}(z)$.

To illustrate the power of the result and generalize the result in Example 4.1, consider the following example.

Example 4.2. An important example of a block lower triangular system occurs when ε in (1) is specified as a VARMA process, $A(L)\varepsilon_t = B(L)\eta_t$ for $t \geq 0$, where $\eta \in \mathcal{S}^n$, $A(z), B(z) \in \mathbb{R}^{n \times n}[z]$, and $A(z)$ has no zeros in \mathbb{D} . Then clearly $\varepsilon \in \mathcal{S}^n$. Now expand the state as $\hat{X} = (\eta', \varepsilon', X')'$, the exogenous process as $\hat{\varepsilon} = (\eta', 0', 0)'$, and set $\hat{M}(z) = \begin{bmatrix} I_n & 0 & 0 \\ -B(z) & A(z) & 0 \\ 0 & -I_n & M(z) \end{bmatrix}$. This system has the representation (1) with $M(z)$, X , and ε replaced by $\hat{M}(z)$, \hat{X} , and $\hat{\varepsilon}$ respectively. By Theorem 4.1, $\hat{M}(z)$ has a block lower triangular ILWHF relative to \mathbb{T} if and only if the partial indices of $M(z)$ are non-positive. To find the ILWHF, we can follow the

logic of the proof of Lemma 4.1 to arrive at

$$\begin{aligned} \hat{M}_f(z) &= \begin{bmatrix} I_n & 0 & 0 \\ -M_f(z)M_0(z)[[M_0^{-1}(z)M_f^{-1}(z)A^{-1}(z)]_{\mathbb{D}^c}B(z)]_{\mathbb{D}^c} & -M_f(z)M_0(z)[M_0^{-1}(z)M_f^{-1}(z)A^{-1}(z)]_{\mathbb{D}^c} & M_f(z) \end{bmatrix} \\ \hat{M}_0(z) &= \begin{bmatrix} I_n & & \\ 0 & I_n & \\ 0 & 0 & M_0(z) \end{bmatrix} \\ \hat{M}_b(z) &= \begin{bmatrix} & I_n & 0 & 0 \\ & -B(z) & A(z) & 0 \\ -[[M_0^{-1}(z)M_f^{-1}(z)A^{-1}(z)]_{\mathbb{D}^c}B(z)]_{\mathbb{D}} & -[M_0^{-1}(z)M_f^{-1}(z)A^{-1}(z)]_D A(z) & M_b(z) \end{bmatrix}. \end{aligned}$$

It follows that modifying the dynamics of the exogenous process has no effect on the autoregressive part of the LREM solution. It only changes the dependence on the exogenous process.¹⁷ \square

The above results imply that any block triangular LREM can be viewed as two subsystems connected in sequence if and only if the partial indices of the first subsystem are no smaller than any partial index of the second subsystem. This has important implications for modelling. For example, suppose a researcher wishes to model an economic system as interconnected ordered sectors, with each sector's output depending on exogenous shocks and inputs from previous sectors in the order (i.e. the network is in the shape of a directed acyclic graph), then such a system will be block triangular. If, moreover, the conditions of Theorem 4.1 are satisfied, then one may simply solve for the dynamics of each sector as an autonomous entity and then combine. If the conditions are not satisfied, then the behaviour upstream will have to anticipate the behaviour downstream and the dynamics become more complicated.

The results above also permit a discussion of linear transformations of LREM solutions. As is well known in the linear systems literature, a linear transformation of a VARMA process has a VARMA representation (Lütkepohl, 2005, Section 11.6). These results applied to (2) and (3) clearly imply that a linear transformation of an LREM solution also has a VARMA representation. However, given the results above, we can say more. If $M(z)$ in (1) is partitioned conformably with $X = (X'_1, X'_2)'$ as $M(z) = \begin{bmatrix} M_{11}(z) & M_{12}(z) \\ M_{21}(z) & M_{22}(z) \end{bmatrix}$, we can determine whether it is possible to condition out X_2 and arrive at an LREM in X_1 only. To see this,

¹⁷Note that the ILWHF relative to \mathbb{T} of $\begin{bmatrix} A(z) & 0 \\ -I_n & M(z) \end{bmatrix}$ can be read directly from the lower right $2n \times 2n$ submatrix of each factor.

note that there exists a unimodular $U(z) \in \mathbb{R}^{n \times n}[z]$ that brings $z^p M(z^{-1})$ to Hermite form (Hannan & Deistler, 2012, Lemma 2.2.2). Therefore, applying $E(U(L^{-1})(\cdot)|\mathcal{I}_t)$ to both sides of $E(M(L)X_t|\mathcal{I}_t) = \varepsilon_t$ a.s. eliminates the $M_{12}(L)$ block and yields an equivalent lower triangular system $E(U(L^{-1})M(L)X_t|\mathcal{I}_t) = E(U(L^{-1})\varepsilon_t|\mathcal{I}_t)$ a.s.¹⁸ If the Laurent polynomial associated with this new system now satisfies the conditions of Theorem 4.1, then it will be possible to solve for X_1 on its own.

Example 4.3. Consider the setup of Example 3.6 again and suppose we order capital first so that $M(z) = \begin{bmatrix} 1-Rz & 1 \\ 0 & z^{-1}-1 \end{bmatrix}$. We have seen that consumption could not be solved in isolation, but what about bond holdings? It is easily checked that $U(z) = \begin{bmatrix} z^{-1} & -1 \\ 1 & 0 \end{bmatrix}$ brings $zM(z^{-1})$ to Hermite form. Thus, applying $E(U(L^{-1})(\cdot)|\mathcal{I}_t)$ to both sides of Definition 3.2 (iv) yields $E\left(\begin{bmatrix} (L^{-1}-1)(1-RL) & 0 \\ 1-RL & 1 \end{bmatrix} X_t \middle| \mathcal{I}_t\right) = \begin{bmatrix} \varepsilon_{2t} \\ \varepsilon_{2t} \end{bmatrix}$. Clearly X_1 is uniquely solvable in this case. \square

Finally, we mention that the results of this section clearly have implications for exogeneity (Engle et al., 1983) and Granger causality (Granger, 1969). However, the connection is best left for future research.

5 Empirical LREMs

Empirical LREMs are typically specified as in Example 4.2 with an i.i.d. exogenous shock ε of zero mean and covariance matrix Σ and initial conditions independent of ε . Moreover, empirical analysis usually focuses on uniquely solvable (i.e. systems with partial indices equal to zero) fundamental solutions. For the purpose of this section, therefore, we will restrict attention to this subclass of LREMs. For a non-singular $M(z) \in \mathbb{R}_{pq}^{n \times n}(z)$ in this subclass, its ILWHF relative to \mathbb{T} is given by $M(z) = M_f(z)M_b(z)$ and there exists a unique solution to (1) satisfying

$$M_b(L)X_t = \varepsilon_t, \quad t \geq 0,$$

where we have further taken $M_f(\infty) = I_n$. By Theorem 2.3, $M_b(z) \in \mathbb{R}^{n \times n}[z]$ and is of degree at most p . We could of course add deterministic terms such as a constant, trend, or seasonal dummies. However, this simply burdens the notation without adding much to the discussion.

¹⁸Note that this operation is reversible as an application of $E(U^{-1}(L^{-1})(\cdot)|\mathcal{I}_t)$ retrieves the original system.

The stability of the solution can be read directly from the ILWHF relative to \mathbb{T} of $M(z)$. To see this note that the solution is unstable if and only if $M_b(z)$ has a zero on \mathbb{T} . However, for $|z_0| \geq 1$, $M_f(z_0)$ is invertible and, therefore, such a z_0 can be a zero of $M_b(z)$ if and only if it is also a zero of $M(z)$. Thus, the solution is stable if and only if $M(z)$ additionally has no zeros on \mathbb{T} .

Stationary Theory. Consider the first case in which $M(z)$ has no zeros on \mathbb{T} . It follows from elementary time series theory that it will be possible to choose initial conditions such that X is stationary and satisfies

$$X_t = M_b^{-1}(L)\varepsilon_t, \quad t \in \mathbb{Z}.$$

The spectral density matrix of the process is then immediately given as, $M_b^{-1}(e^{i\lambda})\Sigma M_b^{-1}(e^{-i\lambda})$.

Now consider the infinite set of equations $E(M(L)X_s|\mathcal{I}_t) = 0$ for $s > t \geq 0$. These may be arranged as

$$\underbrace{\begin{bmatrix} M_0 & M_{-1} & M_{-2} & \cdots \\ M_1 & M_0 & M_{-1} & \cdots \\ M_2 & M_1 & M_0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}}_{\Theta} \underbrace{\begin{bmatrix} E(X_{t+1}|\mathcal{I}_t) \\ E(X_{t+2}|\mathcal{I}_t) \\ E(X_{t+3}|\mathcal{I}_t) \\ \vdots \end{bmatrix}}_{F_{t+1|t}} + \underbrace{\begin{bmatrix} M_1 & M_2 & M_3 & \cdots \\ M_2 & M_3 & M_4 & \cdots \\ M_3 & M_4 & M_5 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}}_{\Psi} \underbrace{\begin{bmatrix} X_t \\ X_{t-1} \\ X_{t-2} \\ \vdots \end{bmatrix}}_{P_t} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix} \quad \text{a.s. } t \geq 0.$$

Note that $F_{t+1|t}$ and P_t are almost surely bounded sequences. On the other hand, Θ is a Toeplitz operator and Ψ is a Hankel operator, both of which map bounded sequences to bounded sequences.¹⁹ Each equation in the infinite set above determines a linear combination of expected values of X that is predictable by some linear combination of current and past values of X . Thus, the equation above can be interpreted as an infinite set of subspace Granger non-causality restrictions imposed by linear rational expectations (Al-Sadoon, 2014). Remarkably, this set of equations is sufficient for determining $F_{t+1|t}$. Since the partial indices of $M(z)$ are all zero and $M(z)$ has no zeros on \mathbb{T} , it follows from the theory of Toeplitz operators that Θ is invertible (Gohberg & Fel'dman, 1974, Corollary VIII.4.1). Therefore,

$$F_{t+1|t} = -\Theta^{-1}\Psi P_t, \quad \text{a.s. } t \geq 0,$$

¹⁹The boundedness of Θ follows from Theorem VIII.4.1 of (Gohberg & Fel'dman, 1974). Φ is zero everywhere, except for a finite dimensional block in its upper left corner, so it must be bounded.

It is well known in the time series literature that the covariance matrix between $F_{t+1|t}$ and P_t determines the linear structure of X (Reinsel, 2003, Section 3.1). Here, the relationship between the two is fixed by the LREM.

Note that a small modification of the argument above yields the LREM solution method of Shiller (1978) and Onatski (2006) as theirs involves thinking of the LREM as an infinite set of structural equations in the entire set of expected values of X and ε .

Non-Stationary Theory. Now consider the case in which $M(z)$ has a zero on \mathbb{T} . In particular, suppose there is a zero at $z = 1$ (seasonal unit roots follow a similar analysis). It is shown in the proof of Theorem 2.1 that the invariant polynomials of the Smith canonical form of $M_b(z)$ are precisely the backward components of the ILWHFs relative to \mathbb{T} of the invariant polynomials of the Smith canonical of $z^q M(z)$. That is, let the Smith canonical form of $z^q M(z)$ be $\Lambda(z) = \text{diag}(\Lambda_{11,f}(z)\Lambda_{11,0}(z)\Lambda_{11,b}(z), \dots, \Lambda_{nn,f}(z)\Lambda_{nn,0}(z)\Lambda_{nn,b}(z))$, where each diagonal term is expressed as an ILWHF relative to \mathbb{T} . Then the Smith canonical form of $M_b(z)$ is exactly $\text{diag}(\Lambda_{11,b}(z), \dots, \Lambda_{nn,b}(z))$. Now if we further factorize each $\Lambda_{ii,b}(z)$ as $\tilde{\Lambda}_{ii,b}(z)(z-1)^{\delta_i}$, then, by the definition of the Smith canonical, $\delta_1 \leq \dots \leq \delta_n$. Clearly, the factors $(z-1)$ appear in the Smith canonical forms of $z^q M(z)$ and $M_b(z)$ at exactly the same positions. Following Engle & Yoo (1991), suppose that $0 = \delta_1 = \dots = \delta_{n-r} < \delta_{n-r+1} = \dots = \delta_n = 1$ (higher order zeros at 1 follow a similar analysis). This implies that $M(z)$ specifies a cointegrated VAR with cointegration rank r . Since $M_f(z)$ has no zeros in \mathbb{D}^c , it must be the case that the right null space of $M(1)$ is exactly the right null space of $M_b(1)$, which is to say, the cointegration space of the solution to the LREM can be read directly from $M(z)$ as $\ker(M(1))$. This results generalize those by Binder & Pesaran (1995) and Juselius (2008), who consider a specification of the form considered in Example 4.2 and allow for unit roots only in $A(z)$ but not in $M(z)$. Thus, their results cannot apply to the consumption model or any other LREM that generates unit roots.

Given the above observations, we can follow the logic of Engle & Yoo (1991) to formulate an error correction form of an LREM as follows. Every element of $z^q M(z) - M(1)z^{q+1} \in \mathbb{R}^{n \times n}[z]$ is zero at $z = 1$. Thus, $z^q M(z) - M(1)z^{q+1} = P^*(z)(1 - z)$, where $\deg(P^*(z)) \leq p + q - 1$. It follows that $M(z) = M(1)z + M^*(z)(1 - z)$, where $M^*(z) = z^{-q}P^*(z) \in \mathbb{R}_{p-1,q}^{n \times n}(z)$. Thus, if $M(1) = \alpha\beta'$, where $\alpha, \beta \in \mathbb{R}^{n \times r}$ are each of full rank, we will be ensured that the LREM generates integrated data. Following the residue argument of Schumacher (1991), we will

ensure that the system generates data of order of integration no higher than 1 if $\alpha'_\perp M^*(1)\beta_\perp$ is non-singular. We are therefore, led to the general vector error correction form of an LREM,

$$M_{-q}^* E_t \Delta X_{t+q} + \dots + M_{-1}^* E_t \Delta X_{t+1} + \\ M_0^* \Delta X_t + \alpha \beta' X_{t-1} + M_1^* \Delta X_{t-1} + \dots + M_{p-1}^* \Delta X_{t-p+1} = \varepsilon_t, \quad t \geq 0.$$

The classical structural vector error correction model is clearly the special case of the above, where $M_i^* = 0$ for $i < 0$. Thus, we have proven the LREM extension to the Granger representation theorem (Engle & Granger, 1987). This expression is similar to that found in Broze et al. (1990), except that they arrange the forward terms as expectational errors rather than expectational innovations; they also do not provide conditions under which the order of integration is bounded by 1.

Example 5.1. Consider the setting of Example 3.3. $zM(z)$ has invariant polynomials 1 and $\frac{1}{R}(1-z)(1-Rz)z$. Thus, the system is cointegrated for all economically relevant values of R (i.e. $R > 1$). Since $M(1) = \begin{bmatrix} 0 & 0 \\ 1 & 1-R \end{bmatrix}$, the cointegration vector, $\beta = (1, 1-R)'$, is immediately evident. We may also choose $\alpha = (0, 1)'$. Now $zM(z) - M(1)z^2 = \begin{bmatrix} 1-z & 0 \\ z & z-Rz^2 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ z^2 & (1-R)z^2 \end{bmatrix} = (1-z) \begin{bmatrix} 1 & 0 \\ z & z \end{bmatrix}$. It follows that $M^*(z) = \begin{bmatrix} z^{-1} & 0 \\ 1 & 1 \end{bmatrix}$. If we choose $\alpha_\perp = (1, 0)'$ and $\beta_\perp = (R-1, 1)'$, then $\alpha'_\perp M^*(1)\beta_\perp = R-1 > 0$. \square

6 Computing the ILWHF

There is surprisingly little in the linear systems or the linear operators literature on the computation of WHFs. The methods used to prove Theorem 2.1 are non-starters due to their high complexity. Results using state space methods are available but, as far as the author is aware, these tend to impose restrictive assumptions on $M(z) \in \mathbb{R}_{pq}^{n \times n}(z)$ such as partial indices that are identically zero, strict properness, and non-singularity at infinity (Gohberg et al., 1993, 2003). Adukov (2008) suggested a method for obtaining the WHF that requires computing moments of $M^{-1}(z)$. We will derive a simpler solution that connects nicely to the proof of Theorem 2.1 and to the Sims (2002) method for solving LREMs. The algorithm is implemented in the Matlab program `ilwhf.m` accompanying this paper.

Let $M(z) \in \mathbb{R}_{pq}^{n \times n}(z)$ be non-singular and $\rho > 0$, then clearly $M(z) = M_f(z)M_0(z)M_b(z)$ is an ILWHF relative to $\rho\mathbb{T}$ if and only if $z^{q+1}M(z) = M_f(z)(z^{q+1}M_0(z))M_b(z)$ is also an

ILWHF relative to $\rho\mathbb{T}$. Therefore, we can obtain an ILWHF of $M(z)$ relative to $\rho\mathbb{T}$ from that of $z^{q+1}M(z)$. Note that although multiplying $M(z)$ by z^q is sufficient to turn it into a matrix polynomial for which polynomial operations are readily applicable, the fact that $\deg(z^{q+1}M(z)) \geq 1$ will serve an important purpose later on. Define the following matrices

$$\Gamma_0 = \left[\begin{array}{c|ccc} 0 & M_{-q} & \cdots & M_{p-1} \\ \hline 0 & & & \\ \vdots & & I_{n(p+q)} & \\ 0 & & & \end{array} \right], \quad \Gamma_1 = \left[\begin{array}{ccc|c} 0 & \cdots & 0 & M_p \\ \hline & & & 0 \\ & & -I_{n(p+q)} & \vdots \\ & & & 0 \end{array} \right],$$

and $\Gamma(z) = \Gamma_0 + \Gamma_1 z \in \mathbb{R}^{l \times l}[z]$, with $l = n(p+q+1)$. Then, $E(z)\Gamma(z) = \begin{bmatrix} z^{q+1}M(z) & 0 \\ 0 & I_{n(p+q)} \end{bmatrix} F(z)$, where

$$E(z) = \left[\begin{array}{c|ccc} I_n & E_1(z) & \cdots & E_{p+q}(z) \\ \hline 0 & & & \\ \vdots & & I_{n(p+q)} & \\ 0 & & & \end{array} \right], \quad F(z) = \begin{bmatrix} I_n & 0 & \cdots & 0 \\ -zI_n & I_n & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & -zI_n & I_n \end{bmatrix},$$

and $E_{p+q+1}(z) = -M_p$ and $E_i(z) = -M_{i-q-1} + zE_{i+1}(z)$ for $i = 1, \dots, p+q$. Since $\det(E(z)) = \det(F(z)) = 1$, it follows that $\det(z^{q+1}M(z)) = \det(\Gamma(z))$ so $\Gamma(z)$ is non-singular. Gohberg et al. (1990) refer to $E(z)\Gamma(z)F^{-1}(z)$ as a linearisation of $z^{q+1}M(z)$.

Our construction will proceed as in the proof of Theorem 2.1. We will factor $\Gamma(z)$ as $\Gamma_{\rho\mathbb{D}}(z)\Gamma_{\rho\mathbb{D}^c}(z)$, where $\Gamma_{\rho\mathbb{D}}(z) \in \mathbb{R}^{l \times l}[z]$ has zeros only in $\rho\mathbb{D}$ and $\Gamma_{\rho\mathbb{D}^c}(z) \in \mathbb{R}^{l \times l}[z]$ has zeros only in $\rho\mathbb{D}^c$. By Theorem 2.5.7 of Wolovich (1974), there exists a unimodular matrix $W(z) \in \mathbb{R}^{l \times l}[z]$ such that $E(z)\Gamma_{\rho\mathbb{D}}(z)W(z)$ is column proper. Let Π be a permutation matrix so that the column degrees of $E(z)\Gamma_{\rho\mathbb{D}}(z)W(z)\Pi$ are $\nu_1 \geq \dots \geq \nu_l \geq 0$. Then, an ILWHF relative to $\rho\mathbb{T}$ of $N(z) = \begin{bmatrix} z^{q+1}M(z) & 0 \\ 0 & I_{n(p+q)} \end{bmatrix} = N_f(z)N_0(z)N_b(z)$ is given by $N_f(z) = E(z)\Gamma_{\rho\mathbb{D}}(z)W(z)\Pi \text{diag}(z^{-\nu_1}, \dots, z^{-\nu_l})$, $N_0(z) = \text{diag}(z^{\nu_1}, \dots, z^{\nu_l})$, and $N_b(z) = \Pi^{-1}W^{-1}(z)\Gamma_{\rho\mathbb{D}^c}(z)F^{-1}(z)$. Theorem 2.2 then implies that $N_0(z) = \begin{bmatrix} z^{q+1}M_0(z) & 0 \\ 0 & I_{l-n} \end{bmatrix}$. Thus, the partial indices of $M(z)$, satisfy $\kappa_i = \nu_i - q - 1$ for $i = 1, \dots, n$ and $\nu_i = 0$ for $i = n+1, \dots, l$. We now claim that we may read $M_f(z)$ directly from the top left $n \times n$ block of $N_f(z)$ and likewise $M_b(z)$ from the top left $n \times n$ block of $N_b(z)$. To see this, note that the unimodular transformation that determines the set of all ILWHFs of $N(z)$ relative to $\rho\mathbb{T}$ in Theorem 2.2 takes the form $U(z) = \begin{bmatrix} U_{11}(z) & 0 \\ U_{21}(z) & U_{22} \end{bmatrix}$. Note that $U_{22} \in \mathbb{R}^{(l-n) \times (l-n)}$ is invertible and $U_{11}(z) \in \mathbb{R}^{n \times n}[z]$

is of the general type of unimodular transformation that $M(z) = \check{M}_f(z)\check{M}_0(z)\check{M}_b(z)$, with $\check{M}_f(z) = M_f(z)M_0(z)U_{11}^{-1}(z)M_0^{-1}(z)$, $\check{M}_0(z) = M_0(z)$, and $\check{M}_b(z) = U_{11}(z)M_b(z)$ is an IL-WHF relative to $\rho\mathbb{T}$. This characterization of $U(z)$ relies crucially on the fact that the non-zero partial indices of $N(z)$ are bounded below by 1, which is made possible by the extra power of z we mentioned earlier. Since $N(z) = \begin{bmatrix} M_f(z) & 0 \\ 0 & I_{l-n} \end{bmatrix} \begin{bmatrix} z^{q+1}M_0(z) & 0 \\ 0 & I_{l-n} \end{bmatrix} \begin{bmatrix} M_b(z) & 0 \\ 0 & I_{l-n} \end{bmatrix}$ is an IL-WHF relative to $\rho\mathbb{T}$. It follows that $N_f(z) = \begin{bmatrix} M_f(z) & 0 \\ 0 & I_{l-n} \end{bmatrix} \begin{bmatrix} M_0(z)U_{11}^{-1}(z)M_0^{-1}(z) & 0 \\ -U_{22}^{-1}U_{21}(z)U_{11}^{-1}(z)M_0^{-1}(z) & U_{22}^{-1} \end{bmatrix}$ and $N_b(z) = \begin{bmatrix} U_{11}(z) & 0 \\ U_{12}(z) & U_{22} \end{bmatrix} \begin{bmatrix} M_b(z) & 0 \\ 0 & I_{l-n} \end{bmatrix}$ for some unimodular $U(z)$ of the aforementioned form. But now note that the top left $n \times n$ blocks of $N_f(z)$ and $N_b(z)$ have the forms $\check{M}_f(z)$ and $\check{M}_b(z)$ respectively.

Now all that remains is to factorize $\Gamma(z)$. This can be accomplished using the real QZ decomposition. By Theorem VI.1.9 and Exercise IV.1.3 of Stewart & Sun (1990), there are orthogonal matrices $Q, Z \in \mathbb{R}^{l \times l}$ such that $Q\Gamma_0 Z = \begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ 0 & \Lambda_{22} \end{bmatrix}$ and $Q\Gamma_1 Z = \begin{bmatrix} \Omega_{11} & \Omega_{12} \\ 0 & \Omega_{22} \end{bmatrix}$ are partitioned conformably, $\det(\Lambda_{11} + \Omega_{11}z)$ has all its zeros in $\rho\mathbb{D}^c$ and $\det(\Lambda_{22} + \Omega_{22}z)$ has all its zeros in $\rho\mathbb{D}$ (thus, Ω_{22} is non-singular). It follows that $\Gamma(z) = Q'(\Lambda + \Omega z)Z' = \Gamma_{\rho\mathbb{D}}(z)\Gamma_{\rho\mathbb{D}^c}(z)$, where $\Gamma_{\rho\mathbb{D}}(z) = Q' \left(\begin{bmatrix} I_s & 0 \\ 0 & \Lambda_{22} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \Omega_{22} \end{bmatrix} z \right)$ and $\Gamma_{\rho\mathbb{D}^c}(z) = \left(\begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ 0 & I_{l-s} \end{bmatrix} + \begin{bmatrix} \Omega_{11} & \Omega_{12} \\ 0 & 0 \end{bmatrix} z \right) Z'$. The existence of this representation is proven under more general conditions in Gohberg & Kaashoek (1988). The notation here highlights the connection to the Sims (2002) method, which evidently implicitly computes an ILWHF.

The implementation of the algorithm in `ilwhf.m` uses the Matlab command `qz` with option `'real'` to obtain an initial QZ decomposition, then uses the Matlab command `ordqz` to obtain the final form discussed above. The column reduction part of the algorithm utilizes an implementation of the Geurts & Praagman (1996) correction to the Krishnarao & Chen (1984) algorithm, which requires a tolerance to be specified.

Example 6.1. Consider Example 2.1 with $R = 1.05$, corresponding to an interest rate of 5% per period. Then we compute the ILWHF as follows:

```
>> M(:, :, 1)=[1 0; 0 0]; M(:, :, 2)=[-1 0; 1 1]; M(:, :, 3)=[0 0; 0 -1.05]; q=1;
>> [Mf, Mb, kappa]=ilwhf(M, q);
>> Mf
Mf(:, :, 1) =
    -0.0408         0
         0    -0.2300
```

```

Mf(:, :, 2) =
    0.0389   -0.0110
    0.0000         0
>> Mb
Mb(:, :, 1) =
    24.4898   -1.2245
   -4.3471   -4.3471
Mb(:, :, 2) =
    0.0000    0.0000
    0.0000    4.5645
>> kappa
kappa =
     0     0

```

Here $M_f(z) = \text{Mf}(:, :, 1) + \text{Mf}(:, :, 2)z^{-1}$ and $M_b(z) = \text{Mb}(:, :, 1) + \text{Mb}(:, :, 2)z$. These are exactly equal to the factor we obtained in Example 2.4 up to a non-singular transformation. To obtain the factors in Example 2.4 one simply computes $M_f(z)M_b(0)$ and $M_b^{-1}(0)M_b(z)$. \square

Next we consider the stability of the algorithm.

Example 6.2. Take $M(z) = \begin{bmatrix} z & \epsilon \\ 0 & z^{-1} \end{bmatrix}$, whose partial indices are $\{+1, -1\}$ for $\epsilon = 0$ and $\{0\}$ for $\epsilon \neq 0$. Then with tolerance set at machine epsilon, we obtain the following output.

```

>> M(:, :, 1)=[0 0; 0 1]; M(:, :, 2)=[0 10^-15; 0 0]; M(:, :, 3)=[1 0; 0 0]; q=1;
>> [Mf,Mb,kappa]=ilwhf(M,q,eps);
>> kappa
kappa =
     0     0
>> M(:, :, 1)=[0 0; 0 1]; M(:, :, 2)=[0 10^-16; 0 0]; M(:, :, 3)=[1 0; 0 0]; q=1;
>> [Mf,Mb,kappa]=ilwhf(M,q,eps);
>> kappa
kappa =
     1    -1

```

Thus, the algorithm gives the correct partial indices for ϵ as small as 10^{-15} but provides incorrect partial indices when $\epsilon = 10^{-16}$. \square

Although the example above may be reassuring, the algorithm does fail in other situations where the system is near the boundary of the set of systems with $\kappa_1 \leq \kappa_n + 1$. In particular, the example given in Sims (2007) cannot be computed with the algorithm above using a reasonable tolerance. On the other hand, these problems occurs only when one is looking for trouble so to speak. Thus, the problem of formulating the optimal approach to computing the ILWHF must be left for future research.

It is worth emphasizing that even in the region where the algorithm is expected to work well (i.e. away from the set of systems with $\kappa_1 > \kappa_n + 1$), it should be used only if the factors of the ILWHF are of interest. If the researcher is interested only in obtaining the representations (2) or (3), then it is much quicker to use the Sims (2002) algorithm.

7 Conclusion

This paper has attempted to situate LREM theory in the wider linear systems literature by providing firm mathematical foundations for the former and bringing to bear the wide arsenal of techniques from the latter. In the remainder, we discuss possible venues for future research, some of which are already part of ongoing research.

First, this is the first in a series of papers that attempt to resolve some long standing econometric problems with LREMs. This includes characterizing the manifold structure of the parameter space of LREMSs, finding the set of observationally equivalent models, structural identification, estimation, inference, and specification analysis.

Second, the causal meaning of structural vector autoregressions has been explored recently in a number of papers (e.g. White & Lu (2010), White et al. (2011), and White & Pettenuzzo (2014)). The framework of this paper can elucidate the causal content of LREM and is taken up in White et al. (2015).

Third, the ILWHF is easy to generalize to non-rational functions meromorphic in a neighbourhood of a Cauchy contour as the limit with respect to sequences of contours that tend to the contour of interest from the inside. Finding the most general class of functions with respect to which such a generalization holds is an interesting question that deserves attention.

Because spectral factorization is a special case of ILWHF, such a theory could potentially provide important lower-level assumptions for fractionally integrated processes.

Fourth, continuous time LREMs also utilize a Wiener-Hopf factorization albeit relative to a different contour than we considered in this paper. The theory of continuous time LREMs would therefore follow almost word-for-word from the theory of this paper. However, it deserves further investigation as the mathematics of stochastic differential equations is substantially more involved than that of discrete time processes.

8 Appendix

Proof of Proposition 2.1. (i) $M_f(z)$ has no zeros or poles in $r\mathbb{D}^c$, thus it has no zeros or poles in $\rho\mathbb{D}^c$ by inclusion. On the other hand, $M_b(z)$ has no zeros or poles in $r\mathbb{D}$. If it had a zero or pole in $r\mathbb{D}^c \cap \rho\mathbb{D}$, this would translate to a zero or pole of $M(z)$ in that same region contradicting the definition of r . Thus $M_b(z)$ has no zeros or poles in $\rho\mathbb{D}$.

(ii) $N_f(z)$ has no zeros or poles in $\rho\mathbb{D}^c$, therefore $N_f((\rho/r)z)$ has no zeros or poles in $(r/\rho)\rho\mathbb{D}^c = r\mathbb{D}^c$ and, by inclusion, $N_f((\rho/r)z)$ has no zeros or poles in $\rho\mathbb{D}^c$. Thus $N_f((\rho/r)z)N_0(\rho/r)$ has no zeros or poles in $\rho\mathbb{D}^c$ either. On the other hand, $N_b(z)$ has no zeros or poles in $\rho\mathbb{D}$, thus $N_b((\rho/r)z)$ has no zeros or poles in $(r/\rho)\rho\mathbb{D} = r\mathbb{D}$. If $N_b((\rho/r)z)$ had a zero or pole in $r\mathbb{D}^c \cap \rho\mathbb{D}$, this would translate to a zero or pole of $M(z)$ in the same region, but this would contradict the definition of r . Thus, $N_b((\rho/r)z)$ has no zeros or poles in $\rho\mathbb{D}$. \square

Proof of Theorem 2.1. Let $q(z)$ be the greatest common denominator of all of the elements of $M(z)$, and let $P(z) = q(z)M(z)$. Then, an ILWHF of $M(z) = M_f(z)M_0(z)M_b(z)$ is obtained from any ILWHFs of $P(z) = P_f(z)P_0(z)P_b(z)$ and $q(z) = q_f(z)q_0(z)q_b(z)$ as $M_f(z) = P_f(z)/q_f(z)$, $M_0(z) = P_0(z)/q_0(z)$, and $M_b(z) = P_b(z)/q_b(z)$. Thus, the existence of an ILWHF for a non-singular rational matrix follows from the existence of an ILWHF for a non-singular polynomial matrix.

Let $P(z) \in \mathbb{R}[z]$. The result is trivial if $P(z)$ is a non-zero constant. Therefore assume $P(z)$ has a non-empty set of zeros, $\{\zeta_i\}$. Counting multiplicities, let κ_1 be the number of roots

of $P(z)$ inside $\rho\mathbb{D}$ and let K be the leading coefficient of $P(z)$.

$$P(z) = \underbrace{\prod_{|\zeta_i| < \rho} (z - \zeta_i)}_{P_{\rho\mathbb{D}}(z)} K \underbrace{\prod_{|\zeta_i| \geq \rho} (z - \zeta_i)}_{P_{\rho\mathbb{D}^c}(z)} = \underbrace{\prod_{|\zeta_i| < \rho} (1 - \zeta_i z^{-1})}_{P_f(z)} z^{\kappa_1} \underbrace{K \prod_{|\zeta_i| \geq \rho} (z - \zeta_i)}_{P_b(z)}.$$

It is clear that an ILWHF with respect to $\rho\mathbb{T}$ is obtained. Notice that two steps are required for the factorization, a polynomial factorization relative to $\rho\mathbb{T}$ into $P_{\rho\mathbb{D}}(z)P_{\rho\mathbb{D}^c}(z)$, followed by division of $P_{\rho\mathbb{D}}(z)$ by its degree.

The factorization of a non-singular $P(z) \in \mathbb{R}^{n \times n}[z]$ follows exactly the same logic. First, obtain the Smith form of $P(z) = U(z)\Lambda(z)V(z)$. Next, obtain the ILWHF relative to $\rho\mathbb{T}$ of the i -th diagonal element of $\Lambda(z)$ as $\Lambda_{ii}(z) = \Lambda_{iif}(z)\Lambda_{iio}(z)\Lambda_{iib}(z)$ and set

$$P_{\rho\mathbb{D}}(z) = U(z)\text{diag}(\Lambda_{11f}(z)\Lambda_{110}(z), \dots, \Lambda_{nnf}(z)\Lambda_{nn0}(z))$$

$$P_{\rho\mathbb{D}^c}(z) = \text{diag}(\Lambda_{11b}(z), \dots, \Lambda_{n nb}(z))V(z).$$

Thus, $P(z) = P_{\rho\mathbb{D}}(z)P_{\rho\mathbb{D}^c}(z)$, where $P_{\rho\mathbb{D}}(z) \in \mathbb{R}^{n \times n}[z]$ contains all the zeros of $P(z)$ that are in $\rho\mathbb{D}$ and $P_{\rho\mathbb{D}^c}(z) \in \mathbb{R}^{n \times n}[z]$ contains all the zeros in $\rho\mathbb{D}^c$. We may now attempt to divide each column of $P_{\rho\mathbb{D}}(z)$ by its degree to form $P_b(z)$. However, if $P_{\rho\mathbb{D}}(z)$ is not column reduced, this may result in a rational matrix that has a zero at infinity. Thus, let $W(z) \in \mathbb{R}^{n \times n}[z]$ be a unimodular matrix such that $P_{\rho\mathbb{D}}(z)W(z)$ is column proper (Wolovich, 1974, Theorem 2.5.7) and let $\Pi \in \mathbb{R}^{n \times n}$ be a permutation matrix such that $P_{\rho\mathbb{D}}(z)W(z)\Pi$ has column degrees $\kappa_1 \geq \dots \geq \kappa_n$. Then $P(z) = P_f(z)P_0(z)P_b(z)$, with $P_0(z) = \text{diag}(z^{\kappa_1}, \dots, z^{\kappa_n})$, $P_f(z) = P_{\rho\mathbb{D}}(z)W(z)\Pi P_0^{-1}(z)$, and $P_b(z) = \Pi^{-1}W^{-1}(z)P_{\rho\mathbb{D}^c}(z)$. To see that this is an ILWHF with respect to $\rho\mathbb{T}$, note that any finite zeros or poles of $P_f(z)$ occur only in $\rho\mathbb{D}$ and due to the column reduction step and subsequent division by the column degrees, $P_f(z)$ has no zeros or poles at infinity; on the other hand, $P_b(z)$ has all its zeros and poles in $\rho\mathbb{D}^c$. \square

Proof of Theorem 2.2. The result follows from Theorem I.1.2 of Clancey & Gohberg (1981) and Proposition 2.1. \square

Proof of Theorem 2.3. The “if” part. Since $M_f(z)$ is a polynomial in z^{-1} , its elements can have poles only at zero. Thus, none of the elements of $M_f(z)$ can have any poles in $\rho\mathbb{D}^c$. On the other hand, $\det(M_f(z^{-1})) \neq 0$ for all $z \in \rho^{-1}\overline{\mathbb{D}}$ if and only if $\det(M_f(z)) \neq 0$ for all $z \in \rho\mathbb{D}^c$ so $M_f^{-1}(z) = \frac{\text{adj}(M_f(z))}{\det(M_f(z))}$ can have no elements with poles in $\rho\mathbb{D}^c$. A similar argument proves that no element of $M_b(z)$ or $M_b^{-1}(z)$ can have any pole in $\rho\mathbb{D}$.

The “only if” part. This follows from the proof of Theorem 2.1.

The highest power of z achievable in the factorization, $\deg(M_b(z)) + \kappa_1$, must be bounded above by p because $M(z) \in \mathbb{R}_{pq}^{n \times n}(z)$. Thus $\kappa_1 \leq p$ and $\deg(M_b(z)) \leq p$ whenever $\kappa_1 = 0$. By a similar argument, $-\deg(M_f(z^{-1})) + \kappa_n \geq -q$, which implies that $\kappa_n \geq -q$ and $\deg(M_f(z^{-1})) \leq q$ whenever $\kappa_n = 0$. It follows that $\deg(M_b(z)) - q \leq \deg(M_b(z)) + \kappa_n \leq \deg(M_b(z)) + \kappa_1 \leq p$, which implies that $\deg(M_b(z)) \leq p + q$. The bound on $\deg(M_f(z^{-1}))$ is proven similarly. \square

Proof of Theorem 2.4. Let $\mathfrak{S}_{pq} = \{M(z) \in \mathbb{R}_{pq}^{n \times n}(z) : \det(M(z)) \neq 0, z \in \rho\mathbb{T}\}$, $\bar{\mathfrak{S}}_{pq} = \{M(z) \in \mathbb{R}_{pq}^{n \times n}(z) : M(z) \text{ is non-singular}\}$, $A = \{M(z) \in \mathfrak{S}_{pq} : \kappa_1 \leq \kappa_n + 1\}$, and $\bar{A} = \{M(z) \in \bar{\mathfrak{S}}_{pq} : \kappa_1 \leq \kappa_n + 1\}$. $A \subset \bar{A}$, since $\mathfrak{S}_{pq} \subset \bar{\mathfrak{S}}_{pq}$. We now claim that A is open and dense in $\bar{\mathfrak{S}}_{pq}$. The fact that it is open follows from the fact that A is open in \mathfrak{S}_{pq} (Gohberg & Krein, 1960) and the fact that \mathfrak{S}_{pq} is open in $\bar{\mathfrak{S}}_{pq}$. The latter fact follows from continuity of the set of zeros of a polynomial as a function of its own coefficients (Stewart & Sun, 1990, p. 166). The fact that A is dense in $\bar{\mathfrak{S}}_{pq}$, follows from the fact that A is dense in \mathfrak{S}_{pq} (Gohberg & Krein, 1960) and the fact that \mathfrak{S}_{pq} is dense in $\bar{\mathfrak{S}}_{pq}$. The latter fact follows from taking r to be as in Proposition 2.1, then $M((r/\rho)z) \in \mathbb{R}_{pq}^{n \times n}(z)$ has no zeros on ρT and can be made arbitrarily close to $M(z)$. \square

Proof of Lemma 3.1. Before we begin, we will need to state a useful inequality that will aid us here and in the next result: if $Z \in \mathcal{S}^n$, $t \in \mathbb{Z}$, and $\psi \in [0, 1)$, there is a constant $C(Z, t, \psi) > 0$ such that

$$\psi^{t+i} E \|Z_{t+i}\| \leq C(Z, t, \psi), \quad i \geq 0. \quad (5)$$

(i) The Monotone Convergence Theorem (Williams, 1991, Theorem 5.3) and inequality (5) imply that $E(\sum_{i=0}^{\infty} \|N_i\| \|Y_{t+i}\|) = \sum_{i=0}^{\infty} \|N_i\| E \|Y_{t+i}\| \leq \sum_{i=0}^{\infty} \|N_i\| |\varphi|^{-t-i} C(Y, t, |\varphi|) = C(Y, t, |\varphi|) |\varphi|^{-t} \sum_{i=0}^{\infty} \|N_i\| |\varphi|^{-i}$ for $|\varphi| < 1$. If, moreover, $|\varphi| > R$ then φ lies in the convergence region of the Laurent series for $N(z)$ and so $\sum_{i=0}^{\infty} \|N_i\| |\varphi|^{-i} < \infty$. It follows that $E(\sum_{i=0}^{\infty} \|N_i\| \|Y_{t+i}\|) < \infty$ for all $t \in \mathbb{Z}$. This implies that $\sum_{i=0}^{\infty} \|N_i\| \|Y_{t+i}\|$ converges almost surely for all $t \in \mathbb{Z}$ (Williams, 1991, Result 6.5.(c)). And this in turn implies that $\sum_{i=0}^{\infty} N_i Y_{t+i}$ converges almost surely for all $t \in \mathbb{Z}$. Since for $t \in \mathbb{Z}$ and $m \geq 0$, $\|\sum_{i=0}^m N_i Y_{t+i}\| \leq \sum_{i=0}^{\infty} \|N_i\| \|Y_{t+i}\| \in \mathcal{L}^1$ and $\sum_{i=0}^{\infty} N_i Y_{t+i}$ converges a.s., it follows that

$\sum_{i=0}^{\infty} N_i E(Y_{t+i} | \mathcal{F}_t) = E(N(L)Y_t | \mathcal{F}_t)$ a.s. by the conditional version of the Dominated Convergence Theorem (Williams, 1991, Property 9.7.(g)).

(ii) Since $E\|E(N(L)Y_t | \mathcal{F}_t)\| \leq E(E(\|N(L)Y_t\| | \mathcal{F}_t)) = E\|N(L)Y_t\|$, the result follows if $N(L)Y \in \mathcal{S}^n$. To that end, $\|E(\sum_{i=0}^{\infty} N_i Y_{t+i})\| \leq E\|\sum_{i=0}^{\infty} N_i Y_{t+i}\| \leq E(\sum_{i=0}^{\infty} \|N_i\| \|Y_{t+i}\|) = \sum_{i=0}^{\infty} \|N_i\| E(\|Y_{t+i}\|) < \infty$. Thus $N(L)Y_t \in \mathcal{L}^1$ for all $t \in \mathbb{Z}$. For $|\theta| < 1$, let $\max\{R, |\theta|\} < |\varphi| < 1$, then inequality (5) again implies that $|\theta|^t E\|\sum_{i=0}^{\infty} N_i Y_{t+i}\| \leq |\theta|^t \sum_{i=0}^{\infty} \|N_i\| E\|Y_{t+i}\| \leq |\theta|^t \sum_{i=0}^{\infty} |\varphi|^{-t-i} C(Y, 0, |\varphi|) \|N_i\| = C(Y, 0, |\varphi|) |\theta/\varphi|^t \sum_{i=0}^{\infty} |\varphi|^{-i} \|N_i\|$ for all $t \geq 0$. Since $\sum_{i=0}^{\infty} |\varphi|^{-i} \|N_i\| < \infty$, the last term tends to zero as $t \rightarrow \infty$ so $N(L)Y \in \mathcal{S}^n$.

(iii) Absolute summability implies that the order of summation of a series is irrelevant (Rudin, 1976, Theorem 3.55). A simple extension of that result implies that for $t \in \mathbb{Z}$, if $\sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \|M_j\| \|N_i\| \|Y_{t+i+j}\| < \infty$ a.s., then $M(L)(N(L)Y_t) = \sum_{j=0}^{\infty} M_j \sum_{i=0}^{\infty} N_i Y_{t+i+j} = \sum_{k=0}^{\infty} (\sum_{i+j=k} M_j N_i) Y_{t+k} = (M(L)N(L))Y_t$ a.s. By the Monotone Convergence Theorem and inequality (5) again, $E\left(\sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \|M_j\| \|N_i\| \|Y_{t+i+j}\|\right) = \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \|M_j\| \|N_i\| E\|Y_{t+i+j}\| \leq \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \|M_j\| \|N_i\| |\varphi|^{-t-i-j} C(Y, t, \varphi) \leq |\varphi|^{-t} C(Y, t, \varphi) \sum_{j=0}^{\infty} |\varphi|^{-j} \|M_j\| \sum_{i=0}^{\infty} |\varphi|^{-i} \|N_i\|$ for all $t \in \mathbb{Z}$. This last term is finite if $R < |\varphi| < 1$. Thus $\sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \|M_j\| \|N_i\| \|Y_{t+i+j}\| < \infty$ a.s. by Result 6.5.(c) of Williams (1991). Finally, the fact that $M(L)N(L)Y \in \mathcal{S}^n$ follows from (ii). \square

Proof of Lemma 3.2. (i) Let $N(z) = \sum_{i=0}^p N_i z^i$. Then $\det(N_0) = \det(N(0)) \neq 0$ because $0 \in \mathbb{D}$. Therefore, the process X can be obtained recursively as $X_t = -N_0^{-1} N_1 X_{t-1} - \dots - N_0^{-1} N_p X_{t-p} + N_0^{-1} Y_t$ for all $t \geq 0$, with $X_t = \tilde{X}_t$ for $t < 0$. To prove that it is in \mathcal{S}^n , first note that $X_t \in \mathcal{L}^1$ for all $t \geq 0$. Now let $0 < |\theta| < 1$ and define $Q(z) = N(\theta z)$. Then $\det(Q(z)) \neq 0$ for all $|z| < |\theta|^{-1}$ and $Q(L)(\theta^t X_t) = \theta^t N(L)X_t = \theta^t Y_t$ for all $t \geq 0$. Thus $Q^{-1}(z) = \sum_{i=0}^{\infty} Q^i z^i$ converges in $|\theta|^{-1}\mathbb{D}$ and $\theta^t X_t = G_{-1t} \tilde{X}_{-1} + \dots + G_{-pt} \tilde{X}_{-p} + \sum_{i=0}^t Q^i \theta^{t-i} Y_{t-i}$ for $t \geq 0$, where the matrices G_{it} are exponentially decaying. It follows that $E\|\theta^t X_t\| \leq \|G_{-1t}\| E\|\tilde{X}_{-1}\| + \dots + \|G_{-pt}\| E\|\tilde{X}_{-p}\| + \sum_{i=0}^t \|Q^i\| |\theta|^{t-i} E\|Y_{t-i}\|$ for $t \geq 0$. Since $Y \in \mathcal{S}^n$, inequality (5) implies that $\sum_{i=0}^t \|Q^i\| |\theta|^{t-i} E\|Y_{t-i}\| \leq \sum_{i=0}^t \|Q^i\| |\theta/\varphi|^{i-t} C(Z, 0, |\varphi|) = C(Z, 0, |\varphi|) |\theta/\varphi|^{-t} \sum_{i=0}^t \|Q^i\| |\theta/\varphi|^i$ for $|\varphi| < 1$ and $t \geq 0$. If we further require that $|\theta|^2 < |\varphi| < |\theta|$, then $\sum_{i=0}^{\infty} \|Q^i\| |\theta/\varphi|^i < \infty$ and $|\theta/\varphi|^{-t} \rightarrow 0$, so $E\|\theta^t X_t\| \rightarrow 0$ and therefore $X \in \mathcal{S}^n$.

(ii) If \hat{X} is another solution, then $N(L)(X_t - \hat{X}_t) = 0$ a.s. for $t \geq 0$. Since $\det(N_0) \neq 0$, $X_{\bar{t}+1} = \hat{X}_{\bar{t}+1}$ a.s. whenever $X_t = \hat{X}_t$ a.s. for $t \leq \bar{t}$. But $X_t = \hat{X}_t$ a.s. for $t < 0$, therefore $X_t = \hat{X}_t$ a.s. for all $t \in \mathbb{Z}$. \square

Proof of Theorem 3.1. First note that by Definition 2.2 (i), $M_f^{-1}(z)$ has a Laurent series expansion that converges for all $|z| \geq R$ for some $R < 1$. It follows from Lemma 3.1 (ii) that $\{E(M_f^{-1}(L)\varepsilon_t|\mathcal{I}_t) : t \in \mathbb{Z}\} \in \mathcal{S}^n$.

(i) By Lemma 3.2, the observation above and (2) imply that $X \in \mathcal{S}^n$. Since the right hand side of (2) is \mathcal{I}_t -measurable and $M_b(0)$ is invertible, X is adapted to \mathcal{I} . Thus Definition 3.2 (ii) is satisfied. To see that Definition 3.2 (iv) is satisfied, apply the operator $E(M_f(L)(\cdot)|\mathcal{I}_t)$ to both sides of (2) and use Lemma 3.1 (iii). Finally, let \hat{X} be another solution so that

$$E(M(L)(X_t - \hat{X}_t)|\mathcal{I}_t) = E(M(L)X_t|\mathcal{I}_t) - E(M(L)\hat{X}_t|\mathcal{I}_t) = 0 \quad \text{a.s.} \quad t \geq 0. \quad (6)$$

Since $X - \hat{X} \in \mathcal{S}^n$, $M(L)(X - \hat{X}) \in \mathcal{S}^n$. Applying the operator $E(M_f^{-1}(L)(\cdot)|\mathcal{I}_t)$ to both sides of (6) and using Lemma 3.1 (iii), we have that $E(M_b(L)(X_t - \hat{X}_t)|\mathcal{I}_t) = 0$ a.s. for all $t \geq 0$. But since X and \hat{X} are adapted to \mathcal{I} , $M_b(L)(X_t - \hat{X}_t) = 0$ a.s. for all $t \geq 0$. Finally, Lemma 3.2 (ii) implies that \hat{X} is indistinguishable from X .

(ii) Take the expected value of (3) with respect to \mathcal{I}_t , the result then follows from exactly the same argument as used in (i).

(iii) Suppose a solution (X, \mathcal{I}) exists. Then applying the operator $E(M_f(L)(\cdot)|\mathcal{I}_t)$ to both sides of Definition 3.2 (iv) we obtain

$$E(M_0(L)M_b(L)X_t|\mathcal{I}_t) = E(M_f^{-1}(L)\varepsilon_t|\mathcal{I}_t) \quad \text{a.s.} \quad t \geq 0. \quad (7)$$

If any partial index is positive then $\kappa_1 > 0$ and the first equation of (7) can be written as $e_1' M_b(L)X_{t-\kappa_1} = E(e_1' M_f^{-1}(L)\varepsilon_t|\mathcal{I}_t)$ a.s. for all $t \geq 0$, where $e_1 = (1, 0, \dots, 0)' \in \mathbb{R}^n$. For $t = 0$ in particular, $e_1' M_b(L)X_{-\kappa_1} = E(e_1' M_f^{-1}(L)\varepsilon_0|\mathcal{I}_0)$ a.s. Since $M_b(0)$ and $M_f(\infty)$ are invertible, it is always possible to choose ε and/or initial conditions of X that violate this last equation and therefore equation (7) as well. \square

Proof of Corollary 3.1. Follows from (2). \square

Proof of Theorem 3.2. The result is proven by repeated application of the Argument Principle (Ahlfors, 1979, Section 5.2). By Theorem 2.3 (i), $\det(M_f(z))$ can have zeros only in \mathbb{D} and since it is also a polynomial in z^{-1} with $\det(M_f(\infty)) \neq 0$, it has an equal number of zeros and poles inside $r\mathbb{T}$. Thus, the winding number of $\det(M_f(z))$ about the origin is zero. By Theorem 2.3 (iii), $\det(M_b(z))$ is analytic and non-zero inside and on $r\mathbb{T}$, thus it winds zero times around the origin. Finally, the winding number of $\det(M_0(z)) = z^{\sum_{i=1}^n \kappa_i}$ around the

origin is $\sum_{i=1}^n \kappa_i$. Adding it all up, the total number of times that $\det(M(z))$ winds around the origin is $\sum_{i=1}^n \kappa_i$. By Theorem 2.4, the set of $M(z)$ with $\kappa_1 \leq \kappa_n + 1$ is generic. Thus, for a generic LREM, $\sum_{i=1}^n \kappa_i$ is zero, negative, or positive, according to whether the partial indices are all zero, all non-positive with some negative, or all non-negative with some positive respectively. \square

Proof of Corollary 3.2. By the Argument Principle again, if r is as in Proposition 2.1, the number of times that $\det(M(z))$ winds around the origin as $r\mathbb{T}$ is traversed counter clockwise is the number of zeros of $\det(M(z))$ in $r\mathbb{D}$ minus the number of poles of $\det(M(z))$ in $r\mathbb{D}$. \square

Proof of Lemma 4.1. The “only if” part. If $M_f(z)$ and $M_b(z)$ are lower triangular, then the i -th diagonal element of $M(z)$ is given by $M_{f,ii}(z)M_{0,ii}(z)M_{b,ii}(z)$ and this is an ILWHF of $M_{ii}(z)$ relative to $\rho\mathbb{T}$. Thus $M_{ii,0}(z) = M_{0,ii}(z)$.

The “if” part. The construction of the ILWHF relative to $\rho\mathbb{T}$ proceeds along the same lines of reasoning as in Theorem 2.1. We will factorize $z^q M(z)$ into $F(z)B(z)$, where $F(z) \in \mathbb{R}^{n \times n}[z]$ has all the zeros of $z^q M(z)$ in $\rho\mathbb{D}$ and $B(z) \in \mathbb{R}^{n \times n}[z]$ has all the zeros of $z^q M(z)$ in $\rho\mathbb{D}^c$ and both factors are lower triangular. Then we will obtain the column reduced form of $F(z)$ while maintaining the lower triangular structure of the factors.

Consider the $n = 1$ case first. Let $z^q M(z) = M_f(z)(z^q M_0(z))M_b(z)$ be an ILWHF with respect to $\rho\mathbb{T}$. Then following the proof of Theorem 2.1, we can set $F(z) = M_f(z)z^q M_0(z) \in \mathbb{R}[z]$ and $B(z) = M_b(z) \in \mathbb{R}[z]$. Now suppose that the result is true for all $n = 1, \dots, k-1$ and let $M(z) \in \mathbb{R}_{pq}^{k \times k}(z)$ be non-singular and lower triangular. Then all that remains is to obtain the k -th row of $F(z)$ and $B(z)$. It also follows from the induction hypothesis that we may set $F_{kk}(z) = M_{kk,f}(z)z^q M_{kk,0}(z)$ and $B_{kk}(z) = M_{kk,b}(z)$. Next, we solve $M_{k,k-1}(z) = F_{k,k-1}(z)B_{k-1,k-1}(z) + F_{kk}(z)B_{k,k-1}(z)$ for $F_{k,k-1}(z)$ and $B_{k,k-1}(z)$. Substituting in what is known, $M_{k,k-1}(z) = F_{k,k-1}(z)M_{k-1,k-1,b}(z) + M_{kk,f}(z)z^q M_{kk,0}(z)B_{k,k-1}(z)$ and rearranging

$$\frac{M_{k,k-1}(z)}{M_{k-1,k-1,b}(z)M_{kk,f}(z)z^q M_{kk,0}(z)} = \frac{F_{k,k-1}(z)}{M_{kk,f}(z)z^q M_{kk,0}(z)} + \frac{B_{k,k-1}(z)}{M_{k-1,k-1,b}(z)}.$$

Now apply the operator $[\cdot]_{\rho\mathbb{D}}$ to both sides and rearrange to obtain

$$B_{k,k-1}(z) = M_{k-1,k-1,b}(z) \left[\frac{M_{k,k-1}(z)}{M_{k-1,k-1,b}(z)M_{kk,f}(z)z^q M_{kk,0}(z)} \right]_{\rho\mathbb{D}},$$

which must be a polynomial of degree at most $p + q$ because $\deg(M_{k-1,k-1,b}(z)) \leq p + q$ by Theorem 2.3 and the denominators in $\left[\frac{M_{k,k-1}(z)}{M_{k-1,k-1,b}(z)M_{kk,f}(z)z^q M_{kk,0}(z)} \right]_{\rho\mathbb{D}}$ are all factors of

$M_{k-1,k-1,b}(z)$. Similarly applying the operator $[\cdot]_{\rho\mathbb{D}^c}$ and rearrange we obtain

$$F_{k,k-1}(z) = M_{kk,f}(z)z^q M_{kk,0}(z) \left[\frac{M_{k,k-1}(z)}{M_{k-1,k-1,b}(z)M_{kk,f}(z)z^q M_{kk,0}(z)} \right]_{\rho\mathbb{D}^c},$$

which, following the same logic as before, must be a polynomial of degree at most $p + q$.

Likewise, $B_{k,k-2}(z), F_{k,k-2}(z), \dots, B_{k,1}(z), F_{k,1}(z)$ are solved sequentially.

Next, if $F(z)$ is column reduced, we simply factor out the highest power of z from each column to obtain the ILWHF. If not, the key to obtaining the column reduced form without destroying the lower triangular structure of $F(z)$ and $B(z)$ is to notice that the degrees of the diagonal elements of $F(z)$ are in descending order. If $\delta_{21} = \deg(F_{21}(z)) > \deg(F_{11}(z)) = \delta_{11}$, then $\delta_{21} > \deg(F_{22}(z)) = \delta_{22}$ and if the leading coefficient of $F_{21}(z)$ is K , then the unimodular transformation

$$U(z) = \left[\begin{array}{cc|c} 1 & 0 & 0 \\ -Kz^{\delta_{21}-\delta_{22}} & 1 & 0 \\ \hline 0 & 0 & I_{n-2} \end{array} \right]$$

has the property that $\deg([F(z)U(z)]_{21}) < \deg(F_{21}(z))$. Thus, we can apply similar transformations until we have reduced the degree of the $(2, 1)$ element to at most $\deg(F_{11}(z))$. Next, we use the $(3, 3)$ element of $F(z)$ to reduce the degree of the $(3, 1)$ element. We continue in this fashion until the degree of the first column is $\deg(F_{11}(z))$. Then we proceed to the second column and so on until the degree of the i -th column is $\deg(F_{ii}(z))$. Clearly, at the conclusion of this algorithm we will have obtained a column reduced matrix polynomial. \square

Proof of Theorem 4.1. The “if” part. Suppose $M_f(z) = \begin{bmatrix} M_{f,11}(z) & 0 \\ M_{f,21}(z) & M_{f,22}(z) \end{bmatrix}$ and $M_b(z) = \begin{bmatrix} M_{b,11}(z) & 0 \\ M_{b,21}(z) & M_{b,22}(z) \end{bmatrix}$. Then, just as in Lemma 4.1, the result follows from the fact that $M_{11,0}(z) = M_{0,11}(z)$ and $M_{22,0}(z) = M_{0,22}(z)$.

The “only if” part. Let $M_{11,b}(z)M_{11,0}(z)M_{11,f}(z)$ and $M_{22,b}(z)M_{22,0}(z)M_{22,f}(z)$ be ILWHFs relative to $\rho\mathbb{T}$ of $M_{11}(z)$ and $M_{22}(z)$ respectively. Let $U_1(z), U_2(z) \in \mathbb{R}^{n \times n}[z]$ be unimodular matrices such that $M_{11,b}(z)U_1(z)$ and $M_{22,b}(z)U_2(z)$ are in lower triangular Hermite form respectively. Likewise, let $V_1(z), V_2(z) \in \mathbb{R}^{n \times n}[z]$ be unimodular matrices such that $V_1(z)M_{11,f}(z^{-1})$ and $V_2(z)M_{22,f}(z^{-1})$ are in lower triangular Hermite form respectively. The existence of these reduced forms follows from Theorem 6.3.2 of Kailath (1980). Then $\begin{bmatrix} V_1(z^{-1}) & 0 \\ 0 & V_2(z^{-1}) \end{bmatrix} M(z) \begin{bmatrix} U_1(z) & 0 \\ 0 & U_2(z) \end{bmatrix}$ is lower triangular and each diagonal block has partial indices in descending order by Lemma 4.1. The result then follows from Lemma 4.1. \square

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