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**Pricing and hedging Margrabe options
with stochastic volatilities**

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Abstract

A Margrabe or exchange option is an option to exchange one asset for another. In a general stochastic volatility framework, by taking the second asset as a numeraire, we derive pricing as well as approximate pricing formulae for Margrabe options. The correlated Stein & Stein and the 3/2 model are studied as particular examples. Moreover, we derive the general mean-variance optimal hedging strategy and show that it is a delta-hedge only in case of zero correlation between asset prices and volatility.

Key words. Stochastic volatility; Margrabe options; change of numeraire; mean-variance hedging; Malliavin calculus

AMS subject classification. 60G44, 60H07, 91G20

1 Introduction

Consider two risky assets, S^1 and S^2 . A Margrabe option, see Margrabe [18], gives the buyer the right, but not the obligation, to exchange the second asset for the first at maturity T . Its payoff thus is

$$\max(S_T^1 - S_T^2, 0) = (S_T^1 - S_T^2)_+.$$

The main pricing method for Margrabe options is to switch to a new measure by taking S^2 as a numeraire. This allows a reduction of the problem to pricing a European call on asset S^1 , expressed relative to numeraire S^2 , with strike equal to one. In particular, the pricing formula does not depend on the risk-free interest rate, given that it is the same for both assets. For reviews of the classical proof and interesting discussions, see Carmona & Durrleman [9] as well as Poulsen [21]. One major application is in FX markets where the two assets represent currencies, see Davies [11].

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As discussed in Wüthrich, Bühlmann & Furrer (2010) [24], an insurance company can achieve solvency by investing into Margrabe options enabling them to exchange their asset portfolio for a valuation portfolio (VaPo). The latter consists of financial instruments which replicate the insurance liabilities. The swapping will be done each time the value of the VaPo exceeds that one of the available assets. This allows for a dynamic hedge by continuously observing the financial market.

Our main contribution is to extend Margrabe's option pricing formula to stochastic volatility models. Option pricing in this context has been studied in Antonelli & Scarlatti [6] by using power series in the correlation parameter as well as in Alòs [1], Alòs & Ewald [3] by Malliavin Calculus, and in Alòs [2] by classical Itô calculus, amongst others. Regarding exchange options in particular, Antonelli, Ramponi & Scarlatti [7] extend the method of their earlier paper by first deriving a pricing PDE and then providing a power series expansion in all three possible correlation coefficients. Carmona and Su ([10]) present a linear approximation that allows that authors to study the corresponding implied and local correlations. Alòs and Léon [4] focus on random strike basket options and derive a linear approximation formula for the implied volatility, based in the study of the short-time implied volatility skew.

We study Margrabe options in a general stochastic volatility framework by taking, as in the classical case, the second asset as a numeraire for a measure change. In other words, we switch to the so-called dual market measure, and derive a general option pricing formula. Hereby we encounter typical problems when working with stochastic volatility models: stochastic exponentials which are candidates for density processes to be used for a measure change need not be true martingales; under a Girsanov transform, the filtration of the new Brownian motion can be strictly smaller than the filtration of the original Brownian motion; some integrability issues turn out to be rather delicate; and, last but not least, the resulting financial markets are in general incomplete. Therefore, typically there does not exist a replicating strategy, and we discuss mean-variance hedging under the martingale measure for Margrabe options. It results that the optimal hedging strategy is a delta-hedge only in the (unrealistic) case that the asset prices are uncorrelated to the volatility, in the correlated case one has to incorporate a correction term. For the corresponding mean-variance price, we provide a general decomposition formula in terms of the quadratic variation of the expected future modified volatility and its covariation with the log-return of the asset expressed in the new numeraire. Based on this general decomposition formula, we develop an approximation result and provide the corresponding error bounds in terms of higher Greeks. For an Ornstein-Uhlenbeck stochastic volatility model we can get the approximation in closed form, while for the 3/2 model we obtain its short time limit.

The paper is organised as follows: In the next section, we state the Margrabe option pricing problem and introduce the dual change of measure. In the third section, we derive a general Margrabe option pricing formula for stochastic volatility models, and provide our approximations. Mean-variance hedging of Margrabe options is then studied in the fourth section. The necessary explicit computations for the 3/2-model are performed in an appendix.

2 Statement of the problem and notations

Let $(\Omega, \mathcal{F}, \mathbb{F}, P)$ be a filtered probability space where the filtration satisfies the usual conditions with \mathcal{F}_0 being trivial up to P -null sets, and fix a finite but arbitrary time horizon $T > 0$. All stochastic processes are RCLL and defined on $\Omega \times [0, T]$. We assume that $(\Omega, \mathcal{F}, \mathbb{F}, P)$ supports at least three independent Brownian motions B, W and Z . A process X resp. a stochastic integral process $\int \vartheta dX$ is distinguished in the notation from the random variables X_t and $\int_0^t \vartheta_s dX_s$ we get by evaluating them at time t . (In-)equalities between stochastic processes are in the sense of indistinguishability, whereas between random variables they are to be understood in the a.s. sense (if the dependency on the measure can be dropped). Let E_t^P denote the \mathcal{F}_t -conditional P -expectation. A martingale measure, sometimes called risk-neutral measure, for a (possibly vector-valued) process X is a probability measure Q such that X is a local Q -martingale. A continuous martingale M is called square-integrable if its bracket process $\langle M \rangle$ is integrable, i.e. $E[\langle M \rangle_T] < \infty$. We denote by $L^2(M)$ the space of all M -integrable processes ϑ such that $\int \vartheta dM$ is a square-integrable martingale.

The financial market consists of two tradable assets S^1 and S^2 . In particular, in the exchange option context considered agents do not trade in the bank account. It will henceforth be assumed that P is a martingale measure for the pair (S^1, S^2) . We consider two correlated stochastic volatility models for the asset prices S^1, S^2 and we will assume that the corresponding volatility processes $\sigma^X, \sigma^Y > 0$ depend on the Brownian motion Z . Moreover, σ^Y is negatively correlated (or uncorrelated) with S^2 . More precisely, we will assume that the risk-neutral dynamics under P for the pair (S^1, S^2) are given as

$$\begin{aligned} dS_t^2 &= \sigma_t^Y S_t^2 \left(\rho_{23} dZ_t + \sqrt{1 - \rho_{23}^2} dW_t \right), \quad t \in [0, T], \\ dS_t^1 &= \sigma_t^X S_t^1 \left(\rho_{12} \left(\rho_{23} dZ_t + \sqrt{1 - \rho_{23}^2} dW_t \right) + \sqrt{1 - \rho_{12}^2} dB_t \right), \quad t \in [0, T], \end{aligned} \quad (1)$$

where $\rho_{12} \in (-1, 1)$, $\rho_{23} \in (-1, 0]$, and both $(\sigma_t^X)^2, (\sigma_t^Y)^2$ are \mathbb{F}^Z -adapted processes (with σ^X, σ^Y being their respective positive roots).

As in the classical case, if the same risk-free interest rate is assumed for both assets, we may w.l.o.g. assume it to be zero for the purpose of pricing Margrabe options. See in particular [8] for the case of stochastic interest rates which may depend on the assets.

Example 1 *We will illustrate the general results with an example, namely the 3/2-model (see [16] for a survey) where the volatility y is given as strong solution to*

$$dy_t = \kappa y_t (\theta - y_t) dt + \nu y_t^{3/2} dZ_t,$$

for some positive real constants κ, θ and ν .

It is well-known (and follows by Itô's formula) that the Heston and the 3/2-model are reciprocal in the sense that the inverse $1/y$ of the 3/2 volatility follows a Heston dynamics,

albeit with different coefficients. More precisely,

$$d\left(\frac{1}{y_t}\right) = k' \left(\theta' - \frac{1}{y_t}\right) dt + \nu' \left(\frac{1}{\sqrt{y_t}}\right) dZ_t,$$

with $k' = \kappa\theta$, $\theta' = \frac{\kappa+\nu^2}{\kappa\theta}$ and $\nu' = -\nu$. Notice that $\frac{2k'\theta'}{(\nu')^2} = 2\frac{\kappa+\nu^2}{\nu^2} > 2$, which implies that the Heston process $1/y$ is positive.

Our goal is to evaluate a Margrabe option with payoff

$$(S_T^1 - S_T^2)_+.$$

As valuation concept we consider conditional expectation under the chosen risk-neutral measure,

$$\begin{aligned} V_t &= E_t \left[S_T^2 \left(\frac{S_T^1}{S_T^2} - 1 \right)_+ \right] \\ &=: E_t [S_T^2 (Y_T - 1)_+] \end{aligned}$$

where we denote by $Y = S^1/S^2$ the asset price S^1 expressed in the new numeraire S^2 . Notice that the situation is symmetric, since it is easy to see that there exists two Brownian motions \widetilde{W} , \widetilde{B} such that

$$dS_t^2 = \sigma_t^Y S_t^2 d\widetilde{W}_t, \quad dS_t^1 = \sigma_t^X S_t^1 d\widetilde{B}_t.$$

So we could have equally well expressed S^2 in terms of S^1 . However, we have chosen the notation such that calculations are facilitated by choosing S^2 as numeraire.

Remark on valuation in incomplete markets. The dynamics in (1) induce an incomplete market, and are formulated under one particular martingale measure chosen by the agent. As there is, by the second fundamental theorem of asset pricing, no unique martingale measure, it can be selected by several considerations, depending on the agent's risk preference. Possible choices are the minimal martingale measure, the minimal entropy martingale measure, or a risk-neutral measure obtained by some calibration procedure. There are also alternative valuation concepts like utility indifference pricing which do not result in taking the expectation with respect to some martingale measure (which, however, will not be considered in this paper). So there will not be a unique price valid for all agents, as in the complete market setting studied by [18], but rather some valuation concept used by an individual agent. For this agent, we do however take this risk-neutral measure P as fixed for the remainder of the paper.

We have that S^2 can be written as a Doléans-Dade stochastic exponential,

$$\begin{aligned} S^2 &= S_0^2 \mathcal{E} \left(\int \sigma^Y \left(\rho_{23} dZ + \sqrt{1 - \rho_{23}^2} dW \right) \right) \\ &= S_0^2 \exp \left(\int \sigma^Y \left(\rho_{23} dZ + \sqrt{1 - \rho_{23}^2} dW \right) - \frac{1}{2} \int (\sigma^Y)^2 ds \right). \end{aligned} \tag{2}$$

Assumption (A.1) S^2 is a martingale (and not a strict local martingale).

Under (A.1), we have that by the abstract Bayes' formula,

$$E_t [S_T^2 (Y_T - 1)_+] = S_t^2 \widehat{E}_t [(Y_T - 1)_+] \quad (3)$$

where \widehat{E}_t denotes the conditional expectation under the probability measure \widehat{P} which has Radon-Nikodym density process with respect to P as

$$\left. \frac{d\widehat{P}}{dP} \right|_{\mathcal{F}_t} = \frac{S_t^2}{S_0^2}.$$

The measure change from P to \widehat{P} comprises a *dual market transform* with respect to asset S^2 .

Example 2 *One well-known criterion for a stochastic exponential as in (2) to be a martingale is that there is an $\varepsilon > 0$ such that*

$$E \left[\exp \left(\varepsilon \int_0^T (\sigma_t^Y)^2 dt \right) \right] < \infty,$$

this follows e.g. from [17], Section 6.2, Example 3 (note that there is an expectation sign missing). This criterion is fulfilled for both the Heston as well as the 3/2-model which in turn follows from [12], p. 18 and Theorem 4.1, respectively. Here it is important to note that by our assumption $\rho_{23} \leq 0$; in case of a positive ρ_{23} there is a counterexample in [23].
□

Now we compute

$$\begin{aligned} dY_t &= \frac{dS_t^1}{S_t^2} - \frac{S_t^1}{(S_t^2)^2} dS_t^2 + \frac{S_t^1}{(S_t^2)^3} d\langle S^2, S^2 \rangle_t - \frac{1}{(S_t^2)^2} d\langle S^1, S^2 \rangle_t \\ &= Y_t \sigma_t^X \left(\rho_{12} \left(\rho_{23} dZ_t + \sqrt{1 - \rho_{23}^2} dW_t \right) + \sqrt{1 - \rho_{12}^2} dB_t \right) \\ &\quad - Y_t \sigma_t^Y \left(\rho_{23} dZ_t + \sqrt{1 - \rho_{23}^2} dW_t \right) \\ &\quad + Y_t (\sigma_t^Y)^2 dt - Y_t \rho_{12} \sigma_t^X \sigma_t^Y dt \end{aligned}$$

The processes $\widehat{Z} = Z - \rho_{23} \int \sigma^Y dt$ and $\widehat{W} = W - \sqrt{1 - \rho_{23}^2} \int \sigma^Y dt$ are Brownian motions under the probability measure \widehat{P} . By a straightforward computation,

$$dY_t = Y_t \left[(\sigma_t^X \rho_{12} - \sigma_t^Y) \left(\rho_{23} d\widehat{Z}_t + \sqrt{1 - \rho_{23}^2} d\widehat{W}_t \right) + \sigma_t^X \sqrt{1 - \rho_{12}^2} dB_t \right], \quad (4)$$

so in particular Y is a local \widehat{P} -martingale. Moreover, if we define $U := \mathcal{L}(Y)$ (the stochastic logarithm of Y , so $\mathcal{E}(U) = Y$), by Itô's formula we get

$$dU_t = (\sigma_t^X \rho_{12} - \sigma_t^Y) \left(\rho_{23} d\widehat{Z}_t + \sqrt{1 - \rho_{23}^2} d\widehat{W}_t \right) + \sigma_t^X \sqrt{1 - \rho_{12}^2} dB_t.$$

Note that we have $\mathbb{F}^{\widehat{Z}} \subset \mathbb{F}^Z$, but in general \mathbb{F}^Z need not be included into $\mathbb{F}^{\widehat{Z}}$.

Assumption (A.2) It holds that $\mathbb{F}^{\widehat{Z}} = \mathbb{F}^Z$.

Example 3 (A.2) is fulfilled e.g. in the 3/2-model (with respect to P) because, as we shall see later, $(\sigma^Y)^2$ is again a 3/2-model under \widehat{P} , albeit with different constants (see Appendix 1), hence as strong solution to an SDE driven by \widehat{Z} is in particular $\mathbb{F}^{\widehat{Z}}$ -adapted. It follows that $Z = \widehat{Z} + \rho_{23} \int \sigma^Y dt$ is then also $\mathbb{F}^{\widehat{Z}}$ -adapted, hence $\mathbb{F}^{\widehat{Z}} = \mathbb{F}^Z$. An analogous argument also gives that (A.2) is fulfilled in the Heston model as well.

Remark on the classical Margrabe formula. Notice that by straightforward computations based on Itô's formula, for every deterministic process $a(t)$

$$\begin{aligned} C_{BS}(T, \ln Y_T, a(T)) &= C_{BS}(t, \ln Y_t, a(t)) \\ &+ \int_t^T \mathcal{L}_{BS} \left((\sigma_s^X)^2 + (\sigma_s^Y)^2 - 2\rho_{12}\sigma_s^X\sigma_s^Y \right) C_{BS}(s, \ln Y_s, a(s)) ds \\ &+ \int_t^T \left(\frac{\partial C_{BS}}{\partial \sigma} \right) (s, \ln Y_s, a(s)) da(s) \\ &+ \text{local martingale terms,} \end{aligned}$$

where $C_{BS}(T, \ln y, \sigma)$ denotes the Black-Scholes price for a European call option with strike one maturing at T , on an asset with log-price $\ln y$ and volatility σ , and $\mathcal{L}_{BS}(\sigma)$ denotes the Black-Scholes operator with volatility σ . Obviously, if σ_t^X and σ_t^Y are constants, $\rho_{23} = 0$ and the resulting Feynman-Kac equation is the Black-Scholes equation with variance given by $(\sigma^X)^2 + (\sigma^Y)^2 - 2\rho_{12}\sigma^X\sigma^Y$, from which we deduce the classical Margrabe formula by choosing

$$a^2(s) = (\sigma_s^X)^2 + (\sigma_s^Y)^2 - 2\rho_{12}\sigma_s^X\sigma_s^Y,$$

see [18]. If they are deterministic, the exchange option price is given by the Black-Scholes price with variance

$$\frac{1}{T} \int_0^T \left((\sigma_s^X)^2 + (\sigma_s^Y)^2 - 2\rho_{12}\sigma_s^X\sigma_s^Y \right) ds;$$

then it is easy to check that it suffices to take

$$a^2(s) = \frac{1}{T-s} \int_s^T \left((\sigma_s^X)^2 + (\sigma_s^Y)^2 - 2\rho_{12}\sigma_s^X\sigma_s^Y \right) ds. \quad \square$$

3 An extension of the Margrabe formula to the stochastic volatility case

3.1 A decomposition formula for the option price

We denote by \widehat{E}_s the conditional expectation with respect to $\mathcal{F}_s^{\widehat{Z}}$, and let the modified squared volatility be given as

$$a^2(r) := (\sigma_r^X)^2 + (\sigma_r^Y)^2 - 2\rho_{12}\sigma_r^X\sigma_r^Y.$$

Let us also consider the averaged modified squared volatility

$$\begin{aligned}\widehat{a}^2(s) &:= \frac{1}{T-s}\widehat{E}_s\left[\int_s^T a^2(r) dr\right] \\ &= \frac{1}{T-s}\widehat{E}_s\left[\int_0^T a^2(r) dr - \int_0^s a^2(r) dr\right] \\ &= \frac{1}{T-s}\left(M_s - \int_0^s a^2(r) dr\right),\end{aligned}$$

where

$$\widehat{M}_s := \widehat{E}_s\left[\int_0^T a^2(r) dr\right]. \quad (5)$$

Then it follows via Itô's formula by straightforward computations that

$$\begin{aligned}C_{BS}(T, \ln Y_T, \widehat{a}(T)) &= C_{BS}(t, \ln Y_t, \widehat{a}(t)) \\ &+ \int_t^T \frac{\partial}{\partial x} \left(\frac{\partial^2 C_{BS}}{\partial x^2} - \frac{\partial C_{BS}}{\partial x} \right) (s, \ln Y_s, \widehat{a}(s)) d\langle U, \widehat{M} \rangle_s \\ &+ \int_t^T \left(\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x} \right) \left(\frac{\partial^2 C_{BS}}{\partial x^2} - \frac{\partial C_{BS}}{\partial x} \right) (s, \ln Y_s, \widehat{a}(s)) d\langle \widehat{M}, \widehat{M} \rangle_s \\ &+ \text{local martingale terms.}\end{aligned} \quad (6)$$

Here x refers to the second argument of C_{BS} . The local martingale terms are

1.

$$\int_t^T \frac{\partial C_{BS}}{\partial x} (s, \ln Y_s, \widehat{a}(s)) \left((\sigma_t^X \rho_{12} - \sigma_t^Y) \left(\rho_{23} d\widehat{Z}_t + \sqrt{1 - \rho_{23}^2} d\widehat{W}_t \right) + \sigma_t^X \sqrt{1 - \rho_{12}^2} dB_t \right); \quad (7)$$

2.

$$\int_t^T \left(\frac{\partial^2 C_{BS}}{\partial x^2} - \frac{\partial C_{BS}}{\partial x} \right) (s, \ln Y_s, \widehat{a}(s)) dM_s. \quad (8)$$

Assumption (A.3) We will henceforth assume that M and U are square-integrable \widehat{P} -martingales.

Remark 4 (A.3) is satisfied for most of the classical stochastic volatility models. An explicit proof in the context of the 3/2 model is given in the Appendix.

For the sake of simplicity we will assume the next assumption. One could replace it by model-dependent integrability assumptions which in the interest of readability we omit here since they would render the proofs quite tedious.

Assumption (A.4) There exists a strictly positive constant ε such that $\min(\sigma^X, \sigma^Y) > \varepsilon$.

We will make use of Lemma 4 in [7]:

Lemma 5 Assume that (A.4) holds. Then, for all $n \geq 2$ and $0 \leq t \leq s \leq T$ there exists a positive constant C such that

$$\widehat{E} \left[\left| \frac{\partial^n C_{BS}}{\partial x^n}(s, \ln Y_s, \widehat{a}(s)) \right| \middle| a(u), u \in [t, s] \right] \leq C(T-s)^{1-n/2}.$$

Theorem 6 Consider our basic model (1). In the case that (A.1) – (A.3) hold, we have the following decomposition formula for the option price:

$$\begin{aligned} V_t = S_t^2 & \left\{ C_{BS}(t, \ln Y_t, \widehat{a}(t)) + \widehat{E}_t \left[\frac{1}{2} \int_t^T \frac{\partial}{\partial x} \left(\frac{\partial^2 C_{BS}}{\partial x^2} - \frac{\partial C_{BS}}{\partial x} \right) (s, \ln Y_s, \widehat{a}(s)) d \langle U, \widehat{M} \rangle_s \right. \right. \\ & \left. \left. + \frac{1}{8} \int_t^T \left(\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x} \right) \left(\frac{\partial^2 C_{BS}}{\partial x^2} - \frac{\partial C_{BS}}{\partial x} \right) (s, \ln Y_s, \widehat{a}(s)) d \langle \widehat{M}, \widehat{M} \rangle_s \right] \right\}. \end{aligned} \quad (9)$$

Proof. By Lemma 5, the expectation of the integrals in (9) are finite. Since (A.3) holds, the \widehat{P} -expectations of the local martingale terms are zero, and the formula follows directly from (6). ■

Assumption (A.5) The $a^2(r)$ associated martingale is square-integrable, so that by the martingale representation formula there exists a process $\Lambda(r, u)$ such that

$$a^2(r) = \widehat{E} [a^2(r)] + \int_0^r \Lambda(r, u) d\widehat{Z}_u,$$

where

$$|\Lambda(r, u)| \leq \nu \lambda(r, u),$$

where $\nu \in (0, 1)$ and λ is a square integrable (wrt. \widehat{P}) process whose moments are uniformly bounded by one in r and u . Noticing that

$$d\widehat{M}_s = \nu \left(\int_s^T \lambda(r, s) dr \right) d\widehat{Z}_s,$$

we moreover have that

$$\widehat{E}_t \left[\int_0^T d \langle U, \widehat{M} \rangle_s \right] = \int_0^t \left(\int_s^T \int_r^T \Phi(t, r, s) dt dr \right) d\widehat{Z}_s$$

and

$$\widehat{E}_t \left[\int_0^T d \langle \widehat{M}, \widehat{M} \rangle_s \right] = \int_0^t \left(\int_s^T \int_r^T \int_u^T \Psi(t, u, r, s) dt du dr \right) d\widehat{Z}_s,$$

where

$$|\Phi(t, r, s)| \leq \rho_{23} \nu \phi(t, r, s)$$

and

$$|\Psi(t, u, r, s)| \leq \nu^2 \psi(t, u, r, s),$$

for some square integrable (with respect to \widehat{P}) processes ϕ and ψ whose moments are uniformly bounded by one in t, r, s and u .

Now we are in a position to prove the main result of this section.

Theorem 7 *Assume (A.1) - (A.5). Then an approximation result can be given as*

$$\begin{aligned} S_t^{-1} V_t &= C_{BS}(t, \ln Y_t, \widehat{a}(t)) \\ &+ \frac{1}{2} H(t, \ln Y_t, \widehat{a}_t) \widehat{E}_t \left[\int_t^T d \langle U, \widehat{M} \rangle_s \right] \\ &+ \frac{1}{8} K(t, \ln Y_t, \widehat{a}_t) \widehat{E}_t \left[\int_t^T d \langle \widehat{M}, \widehat{M} \rangle_s \right] \\ &+ O(\rho_{23} \nu + \nu^2)^2, \end{aligned}$$

$$\text{where } H := \frac{\partial}{\partial x} \left(\frac{\partial^2 C_{BS}}{\partial x^2} - \frac{\partial C_{BS}}{\partial x} \right), K := \left(\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x} \right) \left(\frac{\partial^2 C_{BS}}{\partial x^2} - \frac{\partial C_{BS}}{\partial x} \right).$$

Proof. We follow the notation $Q_t := \widehat{E}_t \left[\int_t^T d \langle U, \widehat{M} \rangle_s \right]$, $R_t := \widehat{E}_t \left[\int_t^T d \langle \widehat{M}, \widehat{M} \rangle_s \right]$. Then by applying Itô's formula to the process

$$C_{BS}(t, \ln Y_t, \widehat{a}(t)) + \frac{1}{2} H(t, \ln Y_t, \widehat{a}(t)) Q_t + \frac{1}{8} K(t, \ln Y_t, \widehat{a}(t)) R_t,$$

and taking into account that $Q_T = R_T = 0$, it follows that

$$\begin{aligned}
& C_{BS}(T, \ln Y_T, \widehat{a}(T)) \\
&= C_{BS}(t, \ln Y_t, \widehat{a}(t)) + \frac{1}{2} H(t, \ln Y_t, \widehat{a}(t)) Q_t + \frac{1}{8} K(t, \ln Y_t, \widehat{a}(t)) R_t \\
&+ \frac{1}{2} \int_t^T \partial_x H(s, \ln Y_s, \widehat{a}(s)) d\langle U, Q \rangle_s \\
&+ \frac{1}{2} \int_t^T (\partial_x^2 - \partial_x) H(s, \ln Y_s, \widehat{a}(s)) d\langle \widehat{M}, Q \rangle_s \\
&+ \frac{1}{2} \int_t^T (\partial_x^3 - \partial_x^2) H(s, \ln Y_s, \widehat{a}(s)) Q_s d\langle U, \widehat{M} \rangle_s \\
&+ \frac{1}{8} \int_t^T (\partial_x^2 - \partial_x)^2 H(s, \ln Y_s, \widehat{a}(s)) Q_s d\langle \widehat{M}, \widehat{M} \rangle_s \\
&+ \frac{1}{2} \int_t^T \partial_x K(s, \ln Y_s, \widehat{a}(s)) d\langle U, R \rangle_s \\
&+ \frac{1}{2} \int_t^T (\partial_x^2 - \partial_x) K(s, \ln Y_s, \widehat{a}(s)) d\langle \widehat{M}, R \rangle_s \\
&+ \frac{1}{2} \int_t^T (\partial_x^3 - \partial_x^2) K(s, \ln Y_s, \widehat{a}(s)) R_s d\langle U, \widehat{M} \rangle_s \\
&+ \frac{1}{8} \int_t^T (\partial_x^2 - \partial_x)^2 K(s, \ln Y_s, \widehat{a}(s)) R_s d\langle \widehat{M}, \widehat{M} \rangle_s \\
&+ \text{local martingale terms.}
\end{aligned}$$

We can show, similarly as in Section A.3 that the local martingale terms are in fact true martingale terms starting at zero. Then, taking conditional expectations we get that

$$\begin{aligned}
V_t &= C_{BS}(t, \ln Y_t, \widehat{a}(t)) + \frac{1}{2}H(t, \ln Y_t, \widehat{a}(t))Q_t + \frac{1}{8}K(t, \ln Y_t, \widehat{a}(t))R_t \\
&+ E_t \left\{ \frac{1}{2} \int_t^T \partial_x H(s, \ln Y_s, \widehat{a}(s)) d\langle U, Q \rangle_s \right. \\
&+ \frac{1}{2} \int_t^T (\partial_x^2 - \partial_x) H(s, \ln Y_s, \widehat{a}(s)) d\langle \widehat{M}, Q \rangle_s \\
&+ \frac{1}{2} \int_t^T (\partial_x^3 - \partial_x^2) H(s, \ln Y_s, \widehat{a}(s)) Q_s d\langle U, \widehat{M} \rangle_s \\
&+ \frac{1}{8} \int_t^T (\partial_x^2 - \partial_x)^2 H(s, \ln Y_s, \widehat{a}(s)) Q_s d\langle \widehat{M}, \widehat{M} \rangle_s \\
&+ \frac{1}{2} \int_t^T \partial_x K(s, \ln Y_s, \widehat{a}(s)) d\langle U, R \rangle_s \\
&+ \frac{1}{2} \int_t^T (\partial_x^2 - \partial_x) K(s, \ln Y_s, \widehat{a}(s)) d\langle \widehat{M}, R \rangle_s \\
&+ \frac{1}{2} \int_t^T (\partial_x^3 - \partial_x^2) K(s, \ln Y_s, \widehat{a}(s)) R_s d\langle U, \widehat{M} \rangle_s \\
&\left. + \frac{1}{8} \int_t^T (\partial_x^2 - \partial_x)^2 K(s, \ln Y_s, \widehat{a}(s)) R_s d\langle \widehat{M}, \widehat{M} \rangle_s \right\}.
\end{aligned}$$

Now, by Lemma 5 and assumption (A.5), it follows that

$$V_t - C_{BS}(t, \ln Y_s, \widehat{a}(t)) - \frac{1}{2}H(t, \ln Y_s, \widehat{a}(t))Q_t - \frac{1}{8}K(t, \ln Y_s, \widehat{a}(t))R_t$$

is less or equal than

$$\begin{aligned}
& CE_t \left\{ \frac{1}{2} \int_t^T \left[(T-s)^{-1} + (T-s)^{-\frac{1}{2}} \right] d\langle U, Q \rangle_s \right. \\
& + \frac{1}{2} \int_t^T \left[(T-s)^{-\frac{3}{2}} + (T-s)^{-1} + (T-s)^{-\frac{1}{2}} \right] d\langle \widehat{M}, Q \rangle_s \\
& + \frac{1}{2} \int_t^T \left[(T-s)^{-2} + (T-s)^{-\frac{3}{2}} + (T-s)^{-1} \right] Q_s d\langle U, \widehat{M} \rangle_s \\
& + \frac{1}{8} \int_t^T \left[(T-s)^{-\frac{5}{2}} + (T-s)^{-2} + (T-s)^{-\frac{3}{2}} + (T-s)^{-1} \right] Q_s d\langle \widehat{M}, \widehat{M} \rangle_s \\
& + \frac{1}{2} \int_t^T \left[(T-s)^{-\frac{3}{2}} + (T-s)^{-1} + (T-s)^{-\frac{1}{2}} \right] d\langle U, R \rangle_s \\
& + \frac{1}{2} \int_t^T \left[(T-s)^{-3} + (T-s)^{-\frac{5}{2}} + (T-s)^{-2} + (T-s)^{-\frac{3}{2}} + (T-s)^{-1} \right] d\langle \widehat{M}, R \rangle_s \\
& + \frac{1}{2} \int_t^T \left[(T-s)^{-\frac{7}{2}} + (T-s)^{-3} + (T-s)^{-\frac{5}{2}} + (T-s)^{-2} + (T-s)^{-\frac{3}{2}} \right] R_s d\langle U, \widehat{M} \rangle_s \\
& + \left. \frac{1}{8} \int_t^T \left[(T-s)^{-4} + (T-s)^{-\frac{7}{2}} + (T-s)^{-3} + (T-s)^{-\frac{5}{2}} + (T-s)^{-2} \right] R_s d\langle \widehat{M}, \widehat{M} \rangle_s \right\} \\
& = T_1 + \dots + T_8,
\end{aligned}$$

where C is a positive constant. Assumption 5 allows us then to estimate

$$\begin{aligned}
T_1 &\leq C\rho_{23}^2\nu^2, & T_2 &\leq C\rho_{23}\nu^3, & T_3 &\leq C\rho_{23}^2\nu^2, & T_4 &\leq C\rho_{23}\nu^3 \\
T_5 &\leq C\rho_{23}\nu^3, & T_6 &\leq C\nu^4, & T_7 &\leq C\rho_{23}\nu^3, & T_8 &\leq C\nu^4
\end{aligned}$$

and the proof is complete. ■

Definition 8 The implied volatility $I(T, t)$ for exchange options is the process such that

$$S_t^2 C_{BS}(t, \ln Y_t, I(T, t)) = V_t.$$

Remark 9 Define

$$\begin{aligned}
\widehat{I}(T, t) &:= \sqrt{\widehat{a}^2(t)} + \frac{1}{2} \left(1 - \frac{d}{\sqrt{\widehat{a}^2(t)}(T-t)} \right) \frac{\rho_{23}}{\sqrt{\widehat{a}^2(t)}(T-t)} \widehat{E}_t \left[\int_t^T (\sigma_s^x \rho_{12} - \sigma_s^y) d\langle Z, \widehat{M} \rangle_s \right] \\
&+ \frac{1}{8} \left(-\frac{d}{\sqrt{\widehat{a}^2(t)}(T-t)} + \frac{d^2}{\widehat{a}^2(t)(T-t)} - \frac{1}{\widehat{a}^2(t)(T-t)} \right) \frac{1}{\sqrt{\widehat{a}^2(t)}(T-t)} \widehat{E}_t \left[\int_t^T d\langle \widehat{M}, \widehat{M} \rangle_s \right],
\end{aligned}$$

where

$$d := \frac{\ln Y_t}{\sqrt{\widehat{a}^2(t)}(T-t)} + \frac{\sqrt{\widehat{a}^2(t)}(T-t)}{2}.$$

Then it is easy to see by means of a Taylor expansion as in [5] that

$$\widehat{I}(T, t) - I(T, t) = o(\rho_{23}\nu + \nu^2)^2.$$

3.2 An explicit expression for the approximation formula

Let us denote by $\mathbb{D}_{\hat{Z}}^{1,2}$ the domain of the derivative operator $D^{\hat{Z}}$ in the Malliavin calculus sense. $\mathbb{D}_{\hat{Z}}^{1,2}$ is a dense subset of $L_{\hat{P}}^2(\Omega)$ and $D^{\hat{Z}}$ is a closed and unbounded operator from $L_{\hat{P}}^2(\Omega)$ into $L_{\hat{P}}^2([0, T] \times \Omega)$ (see for example [19] for a detailed introduction to these notions). The martingale representation theorem states that for a square-integrable martingale M adapted to the Brownian filtration $\mathbb{F}^{\hat{Z}}$ there exists $\theta \in L^2(\hat{Z})$ such that

$$M_t = M_0 + \int_0^t \theta_u d\hat{Z}_u.$$

The integrand θ is called a *martingale kernel*. It can be represented by the Clark-Ocone formula, in case $M \in \mathbb{D}_{\hat{Z}}^{1,2}$, as a conditional expectation of the Malliavin derivative with respect to the Brownian filtration,

$$\theta_t = E_t \left[D_u^{\hat{Z}} (M_T) \right].$$

Recall that

$$a^2(r) := (\sigma_r^X)^2 + (\sigma_r^Y)^2 - 2\rho_{12}\sigma_r^X\sigma_r^Y.$$

Given that the to $a^2(r)$ associated martingale is square-integrable, by the martingale representation formula there exists a process $\Lambda(r, u) \in L^2(\hat{Z})$ such that

$$a^2(r) = \hat{E} [a^2(r)] + \int_0^r \Lambda(r, u) d\hat{Z}_u.$$

Then we may apply stochastic Fubini to get

$$\widehat{M}_s = M_0 + \int_0^s \left(\int_u^T \Lambda(r, u) dr \right) d\hat{Z}_u, \quad (10)$$

$$\hat{E}_t \left[\int_t^T d \langle U, \widehat{M} \rangle_s \right] = \rho_{23} \hat{E}_t \left[\int_t^T (\sigma_s^X \rho_{12} - \sigma_s^Y) \left(\int_s^T \Lambda(r, u) dr \right) \right],$$

and

$$\hat{E}_t \left[\int_t^T d \langle \widehat{M}, \widehat{M} \rangle_s \right] = \hat{E}_t \left[\int_t^T \left(\int_s^T \Lambda(r, u) dr \right)^2 ds \right].$$

Now the problem reduces to compute – or approximate – \widehat{M}_0 and $\Lambda(r, u)$.

Example 10 *Let us assume that both σ^X and σ^Y are Ornstein-Uhlenbeck processes under the martingale measure P . More precisely, we will assume that*

$$d\sigma_t^X = \kappa^X (m^X - \sigma_t^X) dt + \nu^X dZ_t$$

and

$$d\sigma_t^Y = \kappa^Y (m^Y - \sigma_t^Y) dt + \nu^Y dZ_t$$

for some positive constants $\kappa^X, m^X, \nu^X, \kappa^Y, m^Y$ and ν^Y . Then it follows that

$$d\sigma_t^Y = \kappa (m - \sigma_t^Y) dt + \nu^Y d\widehat{Z}_t$$

where $\kappa := \kappa^Y - \rho_{23}\nu^Y$ and $m := \frac{m^Y}{\kappa^Y - \rho_{23}\nu^Y}$. Then, some straightforward computations give us the following martingale representations

$$\sigma_t^X = \widehat{E}(\sigma_t^X) + \int_0^t f(t, s) d\widehat{Z}_s$$

and

$$\sigma_t^Y = \widehat{E}(\sigma_t^Y) + \int_0^t g(t, s) d\widehat{Z}_s,$$

where

$$\begin{aligned} \widehat{E}(\sigma_t^Y) &= m + (\sigma_0^Y - m) \exp(-\kappa(t - s)), \\ \widehat{E}(\sigma_t^X) &= E(\sigma_t^X) + \rho_{23}\nu^Y \int_0^t \exp(-\kappa^X(t - s)) \widehat{E}(\sigma_s^Y) ds \end{aligned}$$

and

$$\begin{aligned} g(t, s) &= \nu^Y \exp(-\kappa(t - s)), \\ f(t, s) &= \nu^X \nu^Y \int_s^t \exp(-\kappa^X(t - r)) \exp(-\kappa(r - s)) dr + \nu^X \exp(-\kappa^X(t - s)). \end{aligned}$$

Then, it is easy to check that (10) holds with

$$\Lambda(t, s) = 2\widehat{E}_s \left[\int_s^T \left((\sigma_t^X \rho_{12} - \sigma_t^Y) (\rho_{12} f(t, s) - g(t, s)) + \sigma_t^X \sqrt{1 - \rho_{12}^2} f(t, s) \right) dt \right].$$

Example 11 Let us assume that both σ^X and σ^Y are 3/2 models under the martingale measure P . More precisely, we will assume that

$$d(\sigma_t^X)^2 = \kappa^X (\sigma_t^X)^2 \left(m^X - (\sigma_t^X)^2 \right) dt + \nu^X \left((\sigma_t^X)^2 \right)^{\frac{3}{2}} dZ_t$$

and

$$d(\sigma_t^Y)^2 = \kappa^Y (\sigma_t^Y)^2 \left(m^Y - (\sigma_t^Y)^2 \right) dt + \nu^Y \left((\sigma_t^Y)^2 \right)^{\frac{3}{2}} dZ_t.$$

Then, it is not direct to obtain an explicit expression for $\Lambda(t, s)$, but from the results in the Appendix it is easy to see that

$$\lim_{t \rightarrow T} \left(\widehat{E}_t \left[\int_t^T d \langle U, \widehat{M} \rangle_s \right] - \frac{1}{2} \rho_{23} (\sigma_t^X \rho_{12} - \sigma_t^Y) \Lambda(t, t) (T - t)^2 \right) = 0$$

and

$$\lim_{t \rightarrow T} \left(\widehat{E}_t \left[\int_t^T d \langle M, M \rangle_s \right] - \frac{1}{3} \Lambda^2(t, t) (T - t)^3 \right) = 0,$$

where

$$\Lambda(t, t) := \nu^y \left((\sigma_t^Y)^2 \right)^{\frac{3}{2}} + \nu^x \left((\sigma_t^X)^2 \right)^{\frac{3}{2}} - \rho_{12} \left(\nu^x (\sigma_t^x)^2 \sigma_t^y + \nu^y (\sigma_t^y)^2 \sigma_t^x \right).$$

Then, the following short-time approximation formula is easily deduced:

$$\begin{aligned} & S_t^2 \{ C_{BS}(t, \ln Y_t, a(t)) \\ & + \frac{1}{2} H(t, \ln Y_t, a(t)) \rho_{23} (\sigma_t^X \rho_{12} - \sigma_t^Y) \Lambda(t, t) (T-t)^2 \\ & \times \frac{1}{8} K(t, \ln Y_t, a(t)) \Lambda(t, t)^2 (T-t)^3 \}. \end{aligned}$$

Moreover, we can check that

$$\lim_{T \rightarrow t} \widehat{I}(T, t) = \sqrt{a^2(t)} + \frac{\rho_{23} (\sigma_t^x \rho_{12} - \sigma_t^y)}{4 \left(\sqrt{a^2(t)} \right)^3} \Lambda(t, t) \ln Y_t + \frac{\Lambda^2(t, t)}{24 \left(\sqrt{a^2(t)} \right)^5} (\ln Y_t)^2,$$

which gives us the short-time limit of the implied volatility approximation, as a quadratic function of $\ln Y_t$.

4 Mean-variance hedging of Margrabe options

The stochastic volatility model considered induces an incomplete market, so not every claim is replicable by trading with the underlying asset. Here we choose as hedging instrument the process Y , that is the asset price S^1 expressed in the new numeraire S^2 . In particular, one of the main applications of Margrabe options is in FX markets, so in that case we would hedge with asset S^1 in terms of the new currency as given by S^2 . In this section, we aim to minimize the remaining risk using a quadratic criterion, formulated under the martingale measure \widehat{P} . A quadratic criterion can be considered under the statistical measure as well, but the then resulting theory has conceptual drawbacks. A more comprehensive discussion of mean-variance hedging can be found e.g. in [22].

For the purpose of mean-variance hedging, the following strategy set is appropriate:

Definition 12 *A strategy ϑ is called admissible if $\vartheta \in L^2(Y)$, i.e. $\int \vartheta dY$ is a square-integrable \widehat{P} -martingale.*

Note that although Y might be a strict local martingale, the gains process $\int \vartheta dY$ is a square-integrable martingale for every admissible strategy ϑ . Given a claim $H \in L^2(\mathcal{F}_T, \widehat{P})$, define a square-integrable martingale V via

$$V_t := \widehat{E}[H | \mathcal{F}_t], \quad 0 \leq t \leq T. \quad (11)$$

For an initial capital c and a self-financing strategy ϑ , the associated value process is $c + \int \vartheta dY$. Our goal is to minimize

$$\widehat{E} \left[\left(H - c - \int_0^T \vartheta_t dY_t \right)^2 \right] \quad (12)$$

over all constants c and all $\vartheta \in L^2(Y)$. For the following result see [22] and the references therein.

Theorem 13 *Optimal mean-variance hedging strategy.* Consider the Kunita-Watanabe decomposition of V ,

$$V = \widehat{E}[H] + \int \vartheta^H dY + L, \quad (13)$$

with $\vartheta^H \in L^2(Y)$ and a square-integrable martingale L with $L_0 = 0$, strongly orthogonal to Y . The optimal initial capital c^* and optimal strategy ϑ^* minimizing the quadratic functional (12) are $c^* = \widehat{E}[H]$, $\vartheta^* = \vartheta^H$. The optimal strategy is unique in the sense that for two optimal strategies ϑ^*, ψ^* the resulting stochastic integral processes are indistinguishable, or equivalently, $\int (\vartheta^* - \psi^*)^2 d[Y] = 0$.

One can interpret $c^* + \int \vartheta^H dY$ as the part of the risk which is attainable, so can be perfectly replicated by means of the hedging strategy ϑ^H , whereas, L is the part of the risk that is totally unhedgeable. Thus L_T is the risk-component of the claim H that cannot be accessed by trading in the underlying. To quantify this inaccessible risk, we are often interested in calculating the variance of the remaining hedging error,

$$\mathcal{R}_T(\vartheta^H) := \widehat{E}[L_T^2].$$

Since by strong orthogonality (13) implies that $\langle V, Y \rangle = \int \vartheta^H d\langle Y, Y \rangle$, we can determine the optimal mean-variance hedging strategy ϑ^H by calculating the formal derivative

$$\vartheta^H = d\langle V, Y \rangle / d\langle Y, Y \rangle.$$

To sum up, the mean-variance hedging approach is a method yielding both a fair price c^* and an optimal hedging strategy ϑ^* . In particular, the fair price is the expectation of the claim under the chosen martingale measure which seems to be a very suitable extension of the pricing rule for complete markets.

For the remainder of this section, we assume that (A.1) and (A.2) are in place. We have in our case, see (3), that $H = (Y_T - 1)_+$ which we assume to be in $L^2(\widehat{P})$. By the put-call parity, this is equivalent to saying that $Y_T \in L^2(\widehat{P})$ which e.g. is satisfied in the 3/2-model. Hence

$$V_t = \widehat{E}_t[(Y_T - 1)_+].$$

Here, see (4),

$$dY/Y = \xi^{1/2} dR,$$

where $\xi := \omega^2 + \mu^2 + \nu^2$ with

$$\begin{aligned}\omega_t &= (\sigma_t^X \rho_{12} - \sigma_t^Y) \rho_{23}, \\ \mu_t &= (\sigma_t^X \rho_{12} - \sigma_t^Y) \sqrt{1 - \rho_{23}^2}, \\ \nu_t &= \sigma_t^X \sqrt{1 - \rho_{12}^2}.\end{aligned}$$

The Brownian motion R is given as

$$R = \left(\omega \widehat{Z} + \mu \widehat{W} + \nu B \right) / \sqrt{(\omega^2 + \mu^2 + \nu^2)}.$$

Hence the two Brownian motions \widehat{Z} and R have quadratic covariation

$$d \langle \widehat{Z}, R \rangle = \omega / \sqrt{(\omega^2 + \mu^2 + \nu^2)} dt =: \omega' dt.$$

There exists a Brownian motion R^\perp in \mathbb{F} which is uncorrelated to R (the choice is not unique) which we fix.

Moreover, we denote the various martingale kernels as follows, up to finite variation (FV) terms (which play no role here since they will drop out when forming brackets):

$$d(\sigma^X)^2 = m^X d\widehat{Z}, \quad d(\sigma^Y)^2 = m^Y d\widehat{Z}, \quad d\sigma^X \sigma^Y = m^{XY} d\widehat{Z}. \quad (\text{all plus FV terms})$$

It results that

$$\begin{aligned}d\xi &= d \left\{ (\sigma^X)^2 - 2\rho_{12} \sigma^X \sigma^Y + (\sigma^Y)^2 \right\} \\ &=: \zeta d\widehat{Z} + \text{FV terms},\end{aligned}$$

with

$$\zeta = m^X - 2\rho_{12} m^{XY} + m^Y.$$

While Y is a Markov process conditional on \widehat{Z} , it is not a Markov process *per se*, but the pair (Y, \widehat{Z}) is. Hence, and since we are in a Brownian framework, there exists a smooth function $v(t, y, z)$ such that

$$V_t = v \left(t, Y_t, \widehat{Z}_t \right).$$

As V is a martingale, there are no FV terms in its canonical decomposition. It results by Itô's formula that

$$\begin{aligned}dV &= \frac{\partial v}{\partial y} dY + \frac{\partial v}{\partial z} \zeta d\widehat{Z} \\ &= \left(\frac{\partial v}{\partial y} \xi^{1/2} Y + \frac{\partial v}{\partial z} \zeta \omega' \right) dR + \frac{\partial v}{\partial z} \zeta \sqrt{1 - \omega'} dR^\perp,\end{aligned}$$

so the Kunita-Watanabe decomposition is given as

$$dV = \left(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z} \omega' \zeta \xi^{-1/2} Y^{-1} \right) dY + \frac{\partial v}{\partial z} \zeta \sqrt{1 - \omega'} dR^\perp.$$

Therefore, the optimal mean-variance hedging strategy is

$$\vartheta^* = \frac{d \langle V, Y \rangle}{d \langle Y, Y \rangle} = \frac{\partial v}{\partial y} + \frac{\partial v}{\partial z} \frac{\omega' \zeta}{\xi^{1/2} Y}.$$

In particular, ϑ^* is a Delta-hedge if and only if $\omega = 0$ which is the case if and only if $\rho_{23} = 0$. To show that ϑ^* is admissible, we will have to show that

$$\widehat{E} \left[\int_0^T \left(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z} \frac{\omega' \zeta}{\xi^{1/2} Y} \right)^2 dt \right] < \infty.$$

Notice first that the Delta $\partial v / \partial y$ and the Vega $\partial v / \partial z$ are bounded. On the other hand, if $1/Y$ has finite moments of all orders, it suffices to check that, for some $p > 1$,

$$\widehat{E} \left[\int_0^T \left(\frac{\omega' \zeta}{\xi^{1/2}} \right)^{2p} dt \right] < \infty.$$

Remark 14 *In the 3/2-model, the moments of $1/\xi$ are bounded by the moments of a CIR process which are finite because of the results in [12]. Then, in this case we only need to check that*

$$\widehat{E} \left[\int_0^T (\omega \zeta)^{2q} dt \right] < \infty,$$

for some $q > 1$. This is satisfied if $\kappa > \frac{3}{2}\nu^2$. In fact, notice that $\widehat{E} (\omega \zeta)^{2q}$ is bounded by the moments of order $4q$ of the inverse of a CIR, which is finite if $4q < \frac{2k'\theta'}{\nu^2} - 1$. Then, taking $q > 1$ in such a way that $4q \in \left(4, \frac{2k'\theta'}{\nu^2} - 1 \right)$ (here $\frac{2k'\theta'}{\nu^2} - 1 = 2\frac{\kappa + \nu^2}{\nu^2} - 1 > 4$ because $\kappa > \frac{3}{2}\nu^2$), the admissibility is proved.

To find the minimal risk, we proceed as follows: Recall that

$$dV = \left(\frac{\partial v}{\partial y} \xi^{1/2} Y + \frac{\partial v}{\partial z} \zeta \omega' \right) dR + \frac{\partial v}{\partial z} \zeta \sqrt{1 - \omega'} dR^\perp.$$

The Kunita-Watanabe decomposition can be written as

$$V = \widehat{E} [(Y_T - 1)_+] + \int \vartheta^* dY + \int \psi dR^\perp,$$

for some $\psi \in L^2(\widehat{Z})$. We can calculate ψ as formal derivative

$$\psi = \frac{d\langle V, R^\perp \rangle}{dt} = \frac{\partial v}{\partial z} \zeta \sqrt{1 - \omega'},$$

which is in $L^2(\widehat{Z})$ for the 3/2-model, see Section 4.

The variance of the hedging error is then given by

$$\mathcal{R}_T(\vartheta^*) = \widehat{E} \left[\int_0^T \psi^2 dt \right] = \int_0^T \widehat{E} \left[\left(\frac{\partial v}{\partial z} \zeta \sqrt{1 - \omega'} \right)^2 \right] dt.$$

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A Computations for the 3/2 model

In this section we will assume that both $(\sigma_t^X)^2$ and $(\sigma_t^Y)^2$ follow a 3/2 model. More precisely, we will assume that

$$d(\sigma_t^X)^2 = k^x (\sigma_t^X)^2 \left(\theta^x - (\sigma_t^X)^2 \right) dt + \nu^x \left((\sigma_t^X)^2 \right)^{\frac{3}{2}} dZ_t \quad (14)$$

and

$$d(\sigma_t^Y)^2 = k^y (\sigma_t^Y)^2 \left(\theta^y - (\sigma_t^Y)^2 \right) dt + \nu^y \left((\sigma_t^Y)^2 \right)^{\frac{3}{2}} dZ_t. \quad (15)$$

A.1 Malliavin differentiability of the 3/2 model volatility

Consider a 3/2 model σ^2 given by an equation of the form

$$d\sigma_t^2 = k\sigma_t^2 (\theta - \sigma_t^2) dt + \nu (\sigma_t^2)^{\frac{3}{2}} dW_t, \quad t \in [0, T], \quad (16)$$

where W is a standard Brownian motion and k, θ and ν are non-negative constants. It is well-known (and follows by Itô's formula) that the process $z := 1/(\sigma)^2$ is a CIR process given by

$$dz_t = k' (\theta' - z_t) dt + \nu' \sqrt{z_t} dW_t, \quad t \in [0, T], \quad (17)$$

where $k' = k\theta$, $\nu' = -\nu$ and $\theta' = \frac{k+\nu^2}{k\theta}$.

Notice that, if $k > 0$, then $\frac{2k'\theta'}{\nu'^2} = \frac{2(k+\nu^2)}{\nu^2} \geq 2$ and the dimension of the underlying Bessel process for z is greater or equal than 2. Then Corollary 4.2 in [3] gives us that, for all $t \in [0, T]$, $z_t \in \mathbb{D}_W^{1,2}$ and hence

$$D_r^W z_t = \nu' \sqrt{z_t} \exp \left(\int_r^t \left(-\frac{k'}{2} - \left(\frac{k'\theta'}{2} - \frac{\nu^2}{8} \right) \frac{1}{z_u} \right) du \right), \quad t \in [0, T]. \quad (18)$$

Now we are in a position to prove the following Lemma.

Lemma 15 *Assume the model (16). Then, for all $t \in [0, T]$, $\sigma_t^2 \in \mathbb{D}_W^{1,2}$ and*

$$D_r^W \sigma_t^2 = \nu (\sigma_t^2)^{\frac{3}{2}} \exp \left(\int_r^t \left(-\frac{k\theta}{2} - \left(\frac{k}{2} + \frac{3\nu^2}{8} \right) \sigma_u^2 \right) du \right), \quad t \in [0, T]. \quad (19)$$

Proof. This proof is based on similar approximation arguments as presented in Section 2 in [3]. Let $\varepsilon > 0$ and $\Phi_\varepsilon(x)$ be a continuously differentiable function satisfying $\Phi_\varepsilon(x) = 1$ if $x \geq 2\varepsilon$ and $\Phi_\varepsilon(x) = 0$ if $x < \varepsilon$, while $\Phi_\varepsilon(x) \leq 1$ for all $x \in \mathbb{R}$. Notice that in this case $\Phi'_\varepsilon(x) = 0$ if $x < \varepsilon$ or $x \geq 2\varepsilon$. Furthermore we define the function $\Lambda_\varepsilon(x) = \Phi_\varepsilon(x) \frac{1}{x}$ with $\Lambda_\varepsilon(0) = 0$. The function $\Lambda_\varepsilon(x)$ is bounded and continuously differentiable satisfying $\Lambda'_\varepsilon(x) = \Phi'_\varepsilon(x) \frac{1}{x} - \Phi_\varepsilon(x) \frac{1}{x^2}$. In particular $\Lambda'_\varepsilon(x) = -\frac{1}{x^2}$ if $x \geq 2\varepsilon$ and $\Lambda'_\varepsilon(x) = 0$ if $x < \varepsilon$. It is easy to see that, for all $t \in [0, T]$, $\Lambda_\varepsilon(z_t) \rightarrow (\sigma_t)^2$ in $L^2(P)$. On the other hand,

$$\begin{aligned} |D_r^W \Lambda_\varepsilon(z_t)| &= |\Lambda'_\varepsilon(z_t) D_r^W z_t| \\ &= \left| \Lambda'_\varepsilon(z_t) \nu' \sqrt{z_t} \exp \left(\int_r^t \left(-\frac{k'}{2} - \left(\frac{k'\theta'}{2} - \frac{\nu^2}{8} \right) \frac{1}{z_u} \right) du \right) \right|. \end{aligned}$$

Notice that, as $\frac{2k'\theta'}{\nu'^2} \geq 2$, $\frac{k'\theta'}{2} - \frac{\nu^2}{8} \geq \frac{\nu^2}{2} - \frac{\nu^2}{8} > 0$ and then

$$|D_r^W \Lambda_\varepsilon(z_t)| \leq \frac{C}{(z_t)^{\frac{3}{2}}},$$

for some positive constant C . Then, as $E[1/z_t] < \infty, \forall t \geq 0$, Lemma 1.2.3 in [19] gives us that $(\sigma_t^Y)^2 \in \mathbb{D}_W^{1,2}$ and that

$$D_r^W (\sigma_t^Y)^2 = \nu (\sigma_t^Y)^{\frac{3}{2}} \exp \left(\int_r^t \left(-\frac{k\theta}{2} - \left(\frac{k}{2} + \frac{3\nu^2}{8} \right) \sigma_u^2 \right) du \right).$$

■

A.2 Martingale representation for $(\sigma_t^Y)^2$, $(\sigma_t^X)^2$ and $\sigma_t^X \sigma_t^Y$.

In this section we will apply the Clark-Ocone formula (see for example [19]) to find a martingale representation for $(\sigma_t^Y)^2$, $(\sigma_t^X)^2$ and $\sigma_t^X \sigma_t^Y$. Our first step will be the computation of the corresponding Malliavin derivatives.

Notice that under the change of numeraire the process $(\sigma_t^Y)^2$ is again a 3/2 model. In fact,

$$\begin{aligned} d(\sigma_t^Y)^2 &= k^y (\sigma_t^Y)^2 (\theta^y - (\sigma_t^Y)^2) dt + \nu^y \left((\sigma_t^Y)^2 \right)^{\frac{3}{2}} d\widehat{Z}_t + \nu^y \rho_{23} \left((\sigma_t^Y)^2 \right)^2 dt \\ &= (k^y - \nu^y \rho_{23}) (\sigma_t^Y)^2 \left(\frac{k^y \theta^y}{k^y - \nu^y \rho_{23}} - (\sigma_t^Y)^2 \right) dt + \nu^y \left((\sigma_t^Y)^2 \right)^{\frac{3}{2}} d\widehat{Z}_t. \end{aligned}$$

Then, Lemma 6 implies that, for all $t \in [0, T]$, $(\sigma_t^Y)^2 \in \mathbb{D}_{\widehat{Z}}^{1,2}$ and that

$$D_r^{\widehat{Z}} (\sigma_t^Y)^2 = \nu^y \left((\sigma_t^Y)^2 \right)^{\frac{3}{2}} \exp \left(\int_r^t \left(-\frac{k\theta}{2} - \left(\frac{k^y - \nu^y \rho_{23}}{2} + \frac{3\nu^2}{8} \right) (\sigma_u^Y)^2 \right) du \right),$$

which satisfies

$$\left| D_r^{\widehat{Z}} (\sigma_t^Y)^2 \right| \leq \nu^y \left((\sigma_t^Y)^2 \right)^{\frac{3}{2}},$$

provided $k^y - \nu^y \rho_{23} \geq 0$.

On the other hand, even when $(\sigma^X)^2$ is not a 3/2 model under the new measure, Lemma 6.3.1 in [19], jointly with (19) gives us the following martingale representation

$$(\sigma_t^X)^2 = \widehat{E} (\sigma_t^X)^2 + \int_0^t \widehat{E}_s \left(D_s^Z (\sigma_t^X)^2 - \rho_{23} (\sigma_t^X)^2 \int_s^t D_s^Z \sigma_r^Y d\widehat{Z}_r \right) d\widehat{Z}_s.$$

In a similar way, for the product $\sigma^X \sigma^Y$, the kernel of the martingale representation is

$$\widehat{E}_s \left[D_s^Z (\sigma_t^X \sigma_t^Y) - \sigma_t^X \sigma_t^Y \int_s^t D_s^Z (\sigma_u^Y) d\widehat{Z}_u \right].$$

Corollary 16 *The above results prove that we get for the martingale kernel*

$$\begin{aligned} \Lambda(s, t) &= \widehat{E}_s \left(D_r^{\widehat{Z}} (\sigma_t^Y)^2 + D_s^Z (\sigma_t^X)^2 - 2\rho_{12} D_s^Z (\sigma_t^X \sigma_t^Y) + 2\rho_{12} \left(\rho_{23} (\sigma_t^X)^2 + \sigma_t^X \sigma_t^Y \right) \int_s^t D_s^Z \sigma_r^Y d\widehat{Z}_r \right). \end{aligned}$$

A.3 On the martingale condition

To show that the stochastic integrals with respect to M and U are square-integrable martingales, and not strict local ones, we will use the criterion that a local martingale L whose square bracket $[L]$ is integrable is a square-integrable martingale.

As for the first term in (7), note first that the Delta $\partial C_{BS}/\partial x$ is bounded, so what remains is to show that

$$\widehat{E} \left[\int_0^T \left((\sigma_t^X)^2 + (\sigma_t^Y)^2 - 2\rho_{12}\sigma_t^X\sigma_t^Y \right) dt \right] < \infty. \quad (20)$$

Regarding the second term (8), in concrete examples one has to show that

$$\widehat{E} \left[\int_0^T \left(\left(\frac{\partial^2 C_{BS}}{\partial x^2} - \frac{\partial C_{BS}}{\partial x} \right) (t, \ln Y_t, \widehat{a}(t)) \right)^2 d\langle M, M \rangle_t \right] < \infty. \quad (21)$$

We will now carry out the required steps for the 3/2 model. As $(\sigma^Y)^2$ is again a 3/2 model under \widehat{P} , it follows by Theorem 4.1 of [12] that $\widehat{E} \left[\int_0^T (\sigma_t^Y)^2 dt \right]$ is finite. With a measure change and Hölder's and Jensen's inequality we get

$$\widehat{E} \left[\int_0^T (\sigma_t^X)^2 dt \right] = E \left[\frac{S_T^2}{S_0^2} \int_0^T (\sigma_t^X)^2 dt \right] \leq \sqrt{E \left[\left(\frac{S_T^2}{S_0^2} \right)^2 \right]} \sqrt{TE \left[\int_0^T (\sigma_t^X)^4 dt \right]} < \infty$$

which holds again by the results in [12]. By the elementary inequality

$$x^2 + y^2 - 2\rho xy \leq 4(x^2 + y^2) \text{ for } \rho \in [-1, +1], \quad (22)$$

it follows that (20) is fulfilled.

Regarding (21), notice first that we can bound the squared difference of the Gamma and Delta for the Black-Scholes price in the log-stock price x as

$$\left| \left(\frac{\partial^2 C_{BS}}{\partial x^2} - \frac{\partial C_{BS}}{\partial x} \right) (t, x, \sigma) \right|^2 \leq C\sigma^{-2}(T-t)^{-1},$$

hence

$$\widehat{E} \left[\int_0^T \left| \frac{\partial^2 C_{BS}}{\partial x^2} - \frac{\partial C_{BS}}{\partial x} \right|^2 d\langle M, M \rangle_t \right] \leq C\widehat{E} \left[\int_0^T (\sigma_s^Y)^{-2} (T-t)^{-1} d\langle M, M \rangle_t \right].$$

Further, we can evaluate $\langle M, M \rangle$ by the martingale representation (10) of M as

$$d\langle M, M \rangle_t = \left(\int_t^T \Lambda(r, t) dr \right)^2 dt,$$

Notice that the moments of $\Lambda(r, t)$ are bounded by the moments of S_T^2/S_t^2 and the moments of $(\sigma_t^X)^2, (\sigma_t^Y)^2$ up to order $2p$, for some $p > 1$, which are finite by the results in [12] since $(\sigma^X)^2$ is given as a 3/2 model under the measure P , and we can proceed by similar arguments as in the $(\sigma^Y)^2$ -case. We conclude that (21) is fulfilled. Summing up, in the 3/2 model, Assumption (A.3) is valid.