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On the Impact of Leverage Constraints on Asset Prices and Trading Volume[†]

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Abstract

Researchers have used stylized facts on asset prices and trading volume in stock markets, in particular correlations between trading volume and price changes, to support theories where agents are not rational expected utility maximizers. This paper shows that this empirical evidence is in fact consistent with a standard infinite horizon - perfect information - expected utility economy where some agents face leverage constraints similar to those found in today's financial markets. In addition, and in sharp contrast to the theories above, we explain the qualitative differences that are observed between trading volume on stock and on futures markets.

We consider a continuous-time economy where agents maximize the integral of discounted utility from consumption under both budget and leverage constraints. We find a closed form solution, up to a negative constant, for the equilibrium prices and demands by assuming a constant risk-free rate and market price for risk. We show that, at the equilibrium, the ratio of stock holdings volatility to stock price volatility is an increasing function of the stock price. We also show that leverage constraints can generate mean reversion. We then argue that those features are consistent with the above empirical evidence. Finally, as a side result, we find that leverage constraints can decrease the initial stock price as well as return volatility. We briefly discuss some regulatory implications.

I. INTRODUCTION

The price-volume relationship observed in stock and futures markets has been a subject of interest in empirical finance for the last twenty years. Researchers in this area have identified some well-known stylized facts among which we highlight the following:

- S1. i) Stock market trading volume is increasing with the absolute change in prices.
 - ii) Bullish stock markets generate more trading volume per unit of price change than bearish markets (see for instance Harris (1986), Jain and Joh (1988), Karpoff (1987)).
 - iii) Current stock market trading volume is positively correlated with past returns (Lakonishok and Smidt (1986 & 1989)).
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- S2. i) Futures market trading volume is also increasing in the absolute value of the price change.
 - ii) In sharp contrast with stock markets, futures trading volume is not significantly different in bullish and bearish markets. (see Karpoff (1987 & 1988)).

The evidence regarding stock markets has been widely used to support various theories on agents' beliefs and preferences. For instance, Epps' (1975) model assumes agents have heterogeneous beliefs, but systematically ignore news arriving to the market. The main prediction of the model is that there will be greater trading volume for positive versus negative price changes. According to Epps, the presence of this asymmetry in the data validates his assumptions on agents' beliefs. Second, Chen (1992) argues that the stock market trading volume evidence supports a non-expected utility theory, called prospect theory, where individuals tend to lock in their profits as fast as possible but delay their losses. Finally, Copeland (1976) argues that the trading volume evidence supports the assumption of sequential, rather than simultaneous information arrival. These

interpretations of trading volume evidence share a common critique: since these models focus on primitives such as agents' preferences, beliefs or information, none of them can explain why the price-volume relation should have different qualitative properties in stock and in futures markets¹.

Complementary to these models, there are frictions-based models trying to explain the empirical evidence. For instance, Karpoff (1988) offers a verbal argument suggesting that the different volume patterns in stock and futures could be due to the lower cost of short selling on futures markets; however, we are not aware of any work formalizing his insight. On the contrary, Lakonishok and Smidt (1986 & 1989) show that tax-related trading motives predict exactly the opposite asymmetry from that actually observed in stock markets. As far as we know, there is not a theoretical model that can reconcile empirical facts S1 and S2. The main objective of this paper is to propose an explanation based on the existence of potentially binding leverage constraints. We will show constraints similar to those faced by agents in actual financial markets can generate the type of price-volume relation that is observed in both stock and futures markets.

An even larger debate on agent's rationality has been launched by the finding that expected stock returns are mean-reverting. In their two seminal papers, DeBondt and Thaler(1985 & 1987) interpret mean reversion as evidence that agents are poor Bayesian updaters and "over-react" to news. Since then, it has been demonstrated that this evidence is also consistent with many other hypotheses. For instance, Ball and Kothari (1989) argue that changes in expected returns are due to changes in the leverage, and therefore the risk, of the companies affected by the news. He and Leland (1993) show that mean reversion can be simply implied by decreasing relative risk aversion and constant volatility of equilibrium prices². A second objective of this paper is to demonstrate that leverage constraints can exacerbate the mean reversion of excess returns.

¹Moreover, Copeland's model is not consistent with the well-known fact that trading volume is an increasing function of the heterogeneity of earnings forecasts.

²Evidence about mean reversion is about both indices and individual stocks, whereas He and Leland's model, like ours, contains only one single risky asset. We believe however that this type of model can

This paper constructs a simple stylized economy where the above properties of asset prices and trading volume arise in equilibrium, without having to make any strong assumptions about preferences, beliefs or information. Instead, we assume that some agents are facing leverage constraints that can potentially bind. This is a fairly realistic assumption since these constraints exist in current financial markets, either as a consequence of explicit regulations or of normal business practices (e.g. commercial banks collateral policies). For instance, on the US stock market, individual investors must satisfy a 50% margin requirement, whereas brokers frequently purchase stocks at a 20 or 25% margin³. Another example of leverage-constrained agents is institutional investors (mutual and pension funds), which not only have to keep liquid assets in order to pay redemptions, but also often do not have the authority to trade on futures markets in order to increase their risk exposure with negligible down payments⁴. Furthermore, there is some indirect empirical evidence that leverage constraints can actually be binding and have explanatory power as far as trading volume is concerned. For instance, Hardouvelis and Peristiani (1992) found that trading volume in the Japanese stock market is significantly correlated with changes in margin requirements⁵.

The portfolio choice problem of an investor facing leverage constraints was first studied by Grossman and Vila (1992), for the case of an agent maximizing his expected utility from terminal wealth, and by Vila and Zariphopoulou (1994), for the case of an investor who has an infinite horizon and consumes a continuous flow of goods. These papers use a very general specification of the leverage constraint that encompasses the case of the individual investor purchasing stocks on margins as well as that of an institutional investor facing liquidity constraints. Both papers find that the leverage constraint binds for high

provide a powerful insight on individual stock data if one is willing to accept that most investors trade only in a reduced number of securities. Then, indeed, the wealth effect of a change in the price of, say, security A will mostly affect the demand for security A rather than being 'diluted' across the demands for all other securities in the market.

³Source: Brady Commission Report (1988).

⁴According to the Brady commission report, 60% of mutual funds are not allowed to trade on index futures markets.

⁵The evidence about margin requirements and volatility is more ambiguous: Hardouvelis (1990) and Pruitt (1993) find a negative correlation, whereas Kumar et al. (1991) and Hsieh and Miller (1990) find none. We provide a possible explanation for these results later in the paper.

levels of wealth, but that these agents already adopt portfolio choices different from those of unconstrained agents at lower levels of wealth. Even though the constraint is not binding when the agent is “poor”, he realizes the constraint he faces will prevent him from taking full advantage of future price increases and, therefore, prefers to invest less in the risky asset than an unconstrained agent with the same wealth.

In this paper we look at the equilibrium of an economy where some agents face leverage constraints similar to the ones used in the above papers and are allowed to trade with unconstrained traders. Trading volume between the two types of agents has some striking properties. As in Vila and Zariphopoulou, the constrained agents behave like unconstrained agents when they are “poor”, but become significantly more risk averse as soon as they start making profits on the stock market and therefore see their wealth getting closer to the binding region. This implies that upward movements in prices exacerbate the differences between the two types of agents, thereby increasing trading volume; whereas, downward movements have the opposite effect. Moreover, when good news arrives on the market, and the stock price increases, the equilibrium expected return from the stock must decrease since constrained agents have become more risk-averse. Thus, a stylized 2-agent economy can capture both the mean reversion of expected returns and the asymmetry of trading volume on stock markets. On the other hand, this type of trading due to leverage constraints should be absent on index futures markets, since all agents on those markets are able to take risky positions with small margins. An argument based on leverage constraints can therefore simultaneously explain the evidence on stock and on futures markets.

From a methodological point of view the research closest to ours addresses the equilibrium effects of portfolio insurance. Grossman and Zhou (1993) derive the qualitative implications of the introduction of some agents with wealth constraints (portfolio insurers) on equilibrium prices. Unfortunately, their method is not applicable to our problem since it requires a finite horizon, for which there does not exist any closed form solution to most consumption portfolio problems with portfolio constraints,

including ours. Basak (1994) studies the same problem as Grossman and Zhou but allows for intermediate consumption. Our construction of equilibrium is similar to one of the examples in his paper. On the other hand, no equilibrium analysis of the leverage constraint problem has been done before. The reason is mainly technical: the consumption/portfolio choice problem of an agent facing leverage constraints is very complex, even when the coefficients of the stock return process are assumed to be constant. Grossman and Vila (1992) find a closed form solution for the limit of a terminal wealth problem when the investor's horizon goes to infinity. Vila and Zariphopoulou (1994) are able to derive some qualitative properties of the optimal policies for the problem with intermediate consumption. By using a slightly different change of variable from theirs, we find a closed form solution to the latter problem when the constraint is not initially binding. It is important to point out that this change of variable is not specific to the leverage constraint problem. It can be used to solve the Bellman equations of a variety of constrained optimization problems. This provides a simpler alternative to the martingale methods that are currently used, as long as one is willing to make analytical assumptions such as CRRA utility functions and constant market price of risk. Using this result, we solve for an equilibrium of an economy with a fixed number of securities and a linear storage technology. We then derive the qualitative properties of equilibrium variables and find that they match the existing empirical evidence

The remainder of this paper is organized as follows. In section II.A we present the basic economy without leverage constraints and solve for the equilibrium price process. We then introduce the leverage constraint in section II.B and solve for both equilibrium demands and price processes. In section III we compare the equilibria of both economies. We discuss the possible extension of our results to an economy where the bond is in zero net supply in section IV. Finally, we discuss our various results and conclude in section V. Proofs are in the Appendix.

III. THE MODEL

2.A. The Economy without leverage constraints

In this section we define a benchmark economy where agents do not face leverage constraints. First, we describe the economy and set the main assumptions. Then, we solve for the consumption/investment problem of the representative agent and finally compute and describe the equilibrium price process.

We assume the existence of a representative agent with an infinite horizon who, at any time t , can allocate his wealth in two assets. The first asset is a bond with a constant rate of return $r > 0$, the other a stock. Each share of stock pays a continuous flow of stochastic dividends, $\delta(t)$ (in units of the unique consumption good):

$$\frac{d\delta(t)}{\delta(t)} = \mu_{\delta}(t) dt + \sigma_{\delta}(t) dB(t)$$

We define $P(t)$ to be the ex-dividend price of the risky asset at time t and assume that

$$dP(t) + \delta(t) dt = (\mu(t) + r) P(t) dt + \sigma(t) P(t) dB(t)$$

where $B(t)$ is a standard Brownian motion on the underlying probability space $(\Omega, \mathcal{F}, \mathcal{P})$.

The agent is initially endowed with N shares of the risky asset and the net supply of shares of this asset is constant (and equal to N).

The net supply of bonds is assumed to be infinitely elastic at the constant rate of interest, r . Provided the non-negativity condition is satisfied, this is equivalent to assuming a linear storage technology⁶. We prove in this paper that it is possible to choose parameters such that the non-negativity condition holds in equilibrium. We also discuss the possible

⁶Linear technologies were first used in the Finance literature by Cox, Ingersoll and Ross (1985). Contrary to us, however, they assume that the risky assets are in infinitely elastic supply, whereas the bond is in constant net supply.

extension of our results to an economy where the bond is in zero net supply in section IV of the paper.

Furthermore, we conjecture that the market price of risk is constant at the equilibrium:

$$K(t) \equiv \frac{\mu(t)}{\sigma(t)} = K > 0 \quad a.e. \quad (1)$$

Condition (1) is clearly weaker than the typical partial equilibrium assumption of constant coefficients and gives us the necessary additional degree of freedom to compute an equilibrium price process while keeping the consumption/investment problem of the agent tractable. We verify later that, given sufficient assumptions on agents' preferences and the dividend process, there always exists a stock price process such that (1) holds and markets clear, even when leverage constraints are introduced in the economy.

The representative agent maximizes his expected utility from the flow of consumption $c(t)$. The utility function is assumed to exhibit constant relative risk aversion (CRRA),

$$u(c) \equiv \frac{c^{1-\alpha}}{1-\alpha}, \quad \alpha > 1$$

Finally, we assume that there are no transaction costs involved in trading any of the assets and normalize the size of the total population to 1.

2.A.1. The Consumption/Investment Problem

Given the assumptions above, our agent solves the following program at any time t :

$$\left. \begin{aligned} V(W(t)) &\equiv \underset{\{c,\pi\}}{\text{Max}} E_t \int_t^\infty e^{-\rho s} u(c(s)) ds \\ s.t. \quad dW(s) &= (rW(s) - c(s)) ds + \pi(s) (\mu(s) ds + \sigma(s) dB(s)) \\ W(s) &\geq 0 \quad a.e. \\ W(t) &> 0 \quad \text{given} \end{aligned} \right\} (2)$$

where $\pi(s)$ is the (dollar) amount invested in the risky asset, $W(s)$ is the wealth of the investor at time s and E_t denotes the conditional expectation given the available information up to time t , \mathcal{F}_t , where \mathcal{F}_t is the extension of the σ -algebra generated by the Brownian Motion $B(t)$.

The Bellman equation associated with problem (2) is:

$$-\rho V(W) + u(c(W)) + (rW - c(W)) V'(W) - \frac{K^2}{2} \frac{(V'(W))^2}{V''(W)} = 0 \quad (3)$$

where c is such that $u'(c) = V'(W)$

To solve the second order differential equation (3) we need two boundary conditions. The first condition follows from the fact that the only feasible policy when $W=0$ is $c=\pi=0$. The second condition is the transversality condition.

$$\lim_{W \rightarrow 0} V(W) = -\infty \quad (3a)$$

$$\lim_{t \rightarrow \infty} E e^{-\rho t} V(W(t)) = 0 \quad (3b)$$

Assumption (1) implies that neither $\mu(t)$ nor $\sigma(t)$ enter in the system $\{(3),(3a),(3b)\}$. Therefore, our problem is identical to Merton's (1971). Its solution is:

$$V(W) = k \frac{W^{1-\alpha}}{1-\alpha}; \quad c(t) = k^{-\frac{1}{\alpha}} W(t); \quad \pi(t) \sigma(t) = \frac{K}{\alpha} W(t) \quad (4)$$

where $k^{-\frac{1}{\alpha}} \equiv \frac{1}{\alpha} (\rho - (1-\alpha)r - \frac{(1-\alpha)K^2}{2\alpha}) > 0$.

The equations in (4) describe the optimal consumption and portfolio policy given r , K and the processes $\mu(t)$ and $\sigma(t)$. The next step is to find $(r, K, \mu(t), \sigma(t))$ such that markets clear and assumption (1) holds.

2.A.2. Equilibrium

There are three markets in our economy: the stock, the bond and the consumption good market. Since the net supply of bonds is assumed to be infinitely elastic, and since the consumption good market clears by Walras law, we only need to clear the stock market. The resulting restrictions on the price process will be seen to be fairly weak; therefore, we shall call an equilibrium of the economy with an infinitely elastic bond supply a *restricted equilibrium*, and reserve the word *equilibrium* for the set-up discussed in the last section of the paper where we assume the risk free asset is in zero net supply. More formally, we define:

Definition 1: (restricted equilibrium in the benchmark economy)

$\{r, K, P(t); c(t), \pi(t)\}$ is a restricted equilibrium iff:

- i) $\{c(t), \pi(t)\}$ is optimal given r, K and $P(t)$, and
- ii) $\pi(t) = NP(t)$ a.e.

We can solve for a *restricted equilibrium* of the benchmark economy in a variety of ways. For clarity of exposition, the following mimics the method we will need to use in the next section, where leverage constraints are introduced.

We want to use (4) to find a closed form for the equilibrium demands and prices. It is helpful to define a new variable $z(t)$, equal to the marginal value of wealth

$$z(t) \equiv V'(W(t)) = k W(t)^{-\alpha} = U'(c(t)) \quad (5)$$

and write the optimal controls as functions of this new variable, rather than wealth. The reason why this change of variable is helpful is that the law of motion of $z(t)$ can easily be written in terms of exogenous variables. This implies that we will be able to write the optimal controls as functions of exogenous variables and stock price only, rather than in a feedback form like (4). It will then be easy to use the stock market clearing condition to find the law of motion of equilibrium stock prices.

To write the optimal demand for the risky asset as a function of $z(t)$ we just need to substitute (5) in (4) and get:

$$n(t) \equiv \frac{\pi(t)}{P(t)} = \frac{K}{\alpha \sigma(t) P(t)} k^{\frac{1}{\alpha}} z(t)^{-\frac{1}{\alpha}} \quad (6)$$

We now use the stock market clearing condition $n(t) = N$ to obtain a closed form solution for the equilibrium excess expected return and volatility in terms of $z(t)$:

$$\sigma(t) P(t) = \frac{K}{N \alpha} k^{\frac{1}{\alpha}} z(t)^{-\frac{1}{\alpha}} \quad (7)$$

$$\mu(t) = \frac{K^2}{N \alpha P(t)} k^{\frac{1}{\alpha}} z(t)^{-\frac{1}{\alpha}} \quad (8)$$

Equations (7) and (8) give us the value of the coefficients of the restricted equilibrium price process as functions of the process $z(t)$. This is not so useful yet since $z(t)$ has so far been defined only as a function of another optimal control (wealth). We therefore need to find the law of motion of the process $z(t)$ in terms of state variables. This can be readily obtained from the law of motion of wealth, as shown in the following lemma:

Lemma 1: $\frac{dz}{z} = (\rho - r) dt - K dB$ (9)

Proof: see Appendix A.

Expressions (7)-(9) characterize the restricted equilibrium price process. This means that we have uniquely determined the price process from its initial conditions provided the stochastic differential equation $dP(t) + \delta(t) dt = (\mu(t) + r) P(t) dt + \sigma(t) P(t) dB(t)$ satisfies the usual regularity properties (it is locally Lipschitz and satisfies the polynomial growth condition). We now derive some interesting properties of this price process. In particular, we are interested in the change of the expected excess returns when prices change, or, in other words, whether and when mean reversion occurs:

$$\begin{aligned}
d\mu(t) &= \text{drift terms} + \left(-\frac{1}{\alpha} \mu(t) \frac{dz(t)}{z(t)} - \mu(t) \frac{dP(t)}{P(t)}\right) \\
\Leftrightarrow \frac{d\mu(t)}{\mu(t)} &= \text{drift terms} + \left(\frac{K}{\alpha} - \sigma(t)\right) dB(t)
\end{aligned} \tag{10}$$

Thus, mean reversion occurs if and only if $\sigma(t) > K/\alpha$, i.e. when volatility is high enough. This is consistent with the evidence of DeBondt and Thaler (1985), who found that stocks exhibiting mean reversion had an average beta greater than one.

The results obtained so far are summarized in *Theorem 1* and formally proved in the Appendix B.

Theorem 1: Consider an economy with a representative agent solving program (2) and a total supply of N shares of the risky asset satisfying assumption (1), then:

i) A price process $P(t)$ is a restricted equilibrium price process iff

$$dP(t) + \delta(t) dt = \left(\frac{K^2}{N\alpha} k^{\frac{1}{\alpha}} z^{-\frac{1}{\alpha}} + r P(t)\right) dt + \frac{K}{N\alpha} k^{\frac{1}{\alpha}} z^{-\frac{1}{\alpha}} dB(t)$$

ii) The returns on the risky asset are mean-reverting iff $\sigma(t) > K/\alpha$

Corollary 1: Given the initial price $P(0)$, the interest rate r and the Sharpe ratio K , there exists an unique restricted equilibrium price process of the above economy.

Proof: see Appendix B.

Theorem 1 gives a closed form solution for the restricted equilibrium price process. Note that even though we are able to fully characterize the law of motion of $P(t)$, variables such as r , K , and $P(0)$ are left indeterminate. This is the sense under which we said earlier that market clearing in an economy with an infinitely elastic bond supply imposes only weak restrictions on the price process. These restrictions are nevertheless quite sufficient for our purposes since we only aim at constructing a price process that clears the stock market so as to be able to analyze the intensity of trading on that market. On the other hand, this

model would obviously be inappropriate if we were interested in looking at the impact of leverage constraints on, say, the risk-free interest rate. For that purpose, one would need a completely different specification of the bond supply. Such an extension is discussed in section IV of the paper.

Let's now ensure ourselves that the restricted equilibrium we constructed is reasonable, in the sense that it is always possible to choose parameters such that the aggregate capital stock is always non-negative. The following theorem proves that this is indeed feasible:

Theorem 2 (non-negativity for the benchmark economy):

$$\text{Assume that: } \frac{d\delta(t)}{\delta(t)} = \mu_\delta dt + \sigma_\delta dB(t)$$

$$\text{Let } r \equiv \rho + \alpha \left(\mu_\delta - \frac{1}{2} \alpha \sigma_\delta^2 \left(\frac{1}{\alpha} + 1 \right) \right) \text{ and } K \equiv \sigma_\delta \alpha$$

$$\text{Then } P(0) \leq k^{\frac{1}{\alpha}} \delta(0) \Rightarrow W(t) - \pi(t) \geq 0 \text{ a.e.}$$

Proof: see Appendix C.

2.B. The Economy with Leverage Constraints

We introduce leverage constraints by assuming that a fraction $\lambda \in (0, 1)$ of the agents are limited in their ability to borrow at the risk-free rate, r . Specifically, their optimal controls (c, π) must satisfy an additional constraint:

$$\pi(t) \leq L (W(t)^+, \sigma(t)^-) \quad (11)$$

Condition (11) is a fairly general type of constraint which says that the wealthier an agent is, the more he will be allowed to invest in the risky asset, but the more risky his investment is, the less money he will be able to borrow. This type of constraints is observed in actual financial markets. For instance, individual investors in US stock markets have to respect a 50% margin requirement, which means that their stock holdings, π , must be less than twice their total wealth W . Mutual funds are another example of agents who

are limited in the amount of risk they can take. Not only do they face voluntary restrictions depending on their investment objective (aggressive growth / growth / growth and income)⁷, but they must also keep liquid assets in order to pay redemptions. In our stylized 2-asset economy the amount of risk an agent takes is given by his exposure to the Brownian motion, $B(t)$, and is equal to $\pi(t)\sigma(t)$. (11) puts an upward limit on this risk exposure.

In order to solve the consumption/investment problem of an investor explicitly facing (11), we need to adopt a special functional form for $L(\cdot, \cdot)$. The following constraint is both easy to interpret economically and very tractable mathematically:

$$\pi(t) \sigma(t) \leq a (W(t) + L); \quad a < \frac{K}{\alpha}, \quad L > 0 \quad (12)$$

First, note that if $\sigma(t)$ is constant, (12) is identical to the constraint used in Grossman and Vila (1992). The constant “ L ” can then be interpreted as a fixed credit line and σ/a as the fraction of the purchases of risky asset that needs to be put down by the investor. Equation (12) states that the total exposure to the Brownian motion $B(t)$, which is also equal to the standard deviation of wealth, is constrained to be less than some affine function of wealth. Thus, the variable which is effectively constrained by (12) is the amount of risk that an individual can take as a function of his wealth. Note that, given our choice of utility functions, $a < K/a$ would imply that the leverage constraint is never binding and $L > 0$ that the constraint is always binding.

This closed form is also very tractable analytically since the maximum exposure to the Brownian motion is independent of $\sigma(t)$. This happens to be exactly the condition we need to have wealth as the only state variable of the constrained agent’s maximization problem, provided the market price of risk, K , is constant. The value function of the constrained agent will then be implicitly defined as the solution of an ordinary differential equation, which we will be able to solve explicitly by using the same change of variable as we used

⁷ For instance, during the eighties, 60% of US mutual funds were not allowed to trade on index futures markets (source: Brady commission report).

for the benchmark case. We will finally solve for an equilibrium price process of an economy where a fraction, λ , of the agents face the constraint (12) and a fraction, $1-\lambda$, does not⁸ and check that our conjecture of constant market price of risk is satisfied.

2.B.1 Consumption/Investment Problem of the Constrained Trader.

At time t , the representative constrained trader solves the following program:

$$\left. \begin{aligned}
 V^c(W(t), K(t)) &= \text{Max}_{\{c, \pi\}} E_t \int_t^\infty e^{-\rho s} u(c(s)) ds \\
 \text{s.t. } dW(s) &= (rW(s) - c(s)) ds + \pi(s) (\mu(s) ds + \sigma(s) dB(s)) \\
 W(s) &\geq 0 \text{ a.e.} \\
 W(t) &> 0 \text{ given.} \\
 \pi(t)\sigma(t) &\leq a(W(t) + L)
 \end{aligned} \right\} (13)$$

where again $K(t) \equiv \frac{\mu(t)}{\sigma(t)}$.

Since for any particular state of the world the leverage constraint might or might not be binding, we split the state space in two possible regions. By NB we denote the Non Binding region, i.e. the set of states of the world where the constrained traders' leverage constraint is not binding. In a similar fashion, by B we denote the Binding region, or set of states of the world where the leverage constraint binds. These two regions are formally defined below.

We conjecture that the following holds at the equilibrium:

a) NB is of the form $\{W \in (0, W^*)\}$.

This conjecture states that the leverage constraint binds at high levels of wealth, which Vila and Zariphopoulou (1994) prove to hold in the case of constant coefficients. We show that the same result holds in our economy even though the leverage constraint we

⁸It is easy to prove that most of the results we derive below still hold if all agents are facing leverage constraints but that a fraction λ of the agents faces a tighter constraint, in the sense of the parameters 'a' and 'L' being strictly smaller, than the rest of agents.

use has been slightly modified to take into account the stochastic nature of volatility in our framework.

$$b) \quad K(t) \equiv \frac{\mu(t)}{\sigma(t)} = K > 0 \quad a.e. \quad (1bis)$$

Assumption (1bis) means that the same assumption we used in the benchmark is still consistent with equilibrium in the economy with leverage constraints. It implies that the market price of risk is the same for both economies. Just as in the benchmark economy, (1bis) will also imply that the maximization problem of the agents will be the same as in the case of constant coefficients. However, compared to the constant coefficient case, we have an additional degree of freedom that is essential in order to clear the stock market.

By standard techniques one can then prove that if V^c is twice continuously differentiable, then it must satisfy the following Bellman equation,

$$-\rho V^c(W) + u(c) + (rW - c) V^{c'}(W) + \text{Max}_{\sigma \pi(W) \leq a(W+L)} \left\{ \mu \pi V^{c'}(W) + \frac{1}{2} \sigma^2 \pi^2 V^{c''}(W) \right\} = 0 \quad (14)$$

subject to the following two boundary conditions:

$$\begin{aligned} \lim_{W \rightarrow 0} V^c(W) &= -\infty \\ \lim_{t \rightarrow \infty} E e^{-\rho t} V^c(W(t)) &= 0 \quad (\text{transversality condition}) \end{aligned} \quad (15)$$

where c is defined by $u'(c) = V'(W)$.

In the interior of NB , we have:

$$-\rho V^c(W) + u(c) + (rW - c) V^{c'}(W) - \frac{K^2}{2} \frac{(V^{c'}(W))^2}{V^{c''}(W)} = 0 \quad (16a)$$

In the interior of B , (14) can be written as:

$$-\rho V^c + u(c) + (rW - c) V^{c'} + aK(W+L) V^{c'} + \frac{1}{2} a^2 (W+L)^2 V^{c''} = 0 \quad (16b)$$

The B and NB regions are formally defined as:

$$B \equiv \left\{ \omega \in \Omega \left| -\frac{K V^{c'}(W)}{V^{c''}(W)} \geq a (W + L) \right. \right\}$$

$$NB \equiv \left\{ \omega \in \Omega \left| -\frac{K V^{c'}(W)}{V^{c''}(W)} \leq a (W + L) \right. \right\}$$

We would like to work with (16a) and (16b) instead of the initial maximization problem (13). Standard verification theorems tell us that, under some regularity conditions, if (16a) and (16b) admit a twice continuously differentiable solution, then this solution is the value function associated with (13). Vila and Zariphopoulou (1994) prove the existence of such a solution to {(16a), (16b)}⁹. Their result is stated in the following lemma:

Lemma 2 (Vila and Zariphopoulou - 1994): There exists an unique twice continuously differentiable function V^c solving the differentiable equation (14) subject to the boundary condition (15). This solution is the value function defined in (13).

The next stage is to explicitly solve for the value function and the optimal controls. Unfortunately, to the best of our knowledge, there does not exist a closed form solution to (16b). Vila and Zariphopoulou (1994) were able to exhibit some qualitative properties of the optimal policy in the NB region in the case of μ and σ constant. By using a different change of variable from theirs', we can go an step further and find a closed form solution up to a constant of integration for these controls. We are then able to characterize the equilibrium price process by using the same approach as in the benchmark economy.

We now solve for the value function in the non-binding region, NB . Even though we are using a different change of variable, the techniques we are using are similar to those in Grossman and Vila (1992) and Vila and Zariphopoulou (1994). We make a change of variable which consists in looking at the function V not as a function of current wealth but as a function of the current marginal utility. This change of variable is motivated by

⁹The problem they study is slightly different from ours. However, it is easy to check that our problem is equivalent to theirs whenever $K(t)=K$ a.e. .

Cvitanic and Karatzas' (1993) work on optimization under cone constraints and He and Pages' (1993) work on constrained optimization with labor income. The difference between their approach and ours is that they use martingale methods to prove the equivalence between the original problem and its convex dual, and then derive the Bellman equation associated with the dual problem. Instead, we directly transform the primal's Bellman equation¹⁰. This allows us to sidestep the construction of the convex dual program and simplifies the derivation of both the optimal controls and the value function. This short-cut is a direct payoff from the assumptions we have made: CRRA utility functions, constant interest rate and Sharpe ratio. As long as one is willing to make similar assumptions, the change of variable described below can be used for a variety of different constrained optimization problems, such as consumption/savings choice under borrowing constraints (with or without labor income) or short sales constraints. This technique is therefore not specific to the leverage constraint problem.

As in the benchmark economy, we define a new variable equal to the marginal value of wealth:

$$y(t) \equiv V^{c'}(W(t)) \quad (20)$$

We also define a transformed value function $Q(t)$:

$$Q(y(t)) \equiv V^c(W(t)) - y(t)W(t) \quad (21)$$

$Q(y)$ can be interpreted as the value function associated with the convex dual of (13) in the non-binding region¹¹.

Note that (21) implies that:

$$Q'(y(t)) = -W(t) \quad (22)$$

¹⁰Note also that leverage constraints do not immediately fit in Cvitanic and Karatzas' framework since it is the dollar amount, rather than the fraction of wealth, invested in the risky asset that enters our constraint.

¹¹We refer the reader to Cvitanic and Karatzas (1993) and He and Pages (1993) for a thorough description of convex duality methods for constrained optimization problems under continuous time.

$$Q''(y(t)) = - \frac{1}{(V^c)''(W(t))} \quad (23)$$

With this transformation, the optimal controls are given by:

$$c(t)^{-\alpha} = y(t) \quad (24)$$

$$\pi(t) \sigma(t) = y(t) Q''(y(t)) \mathbb{K} \quad (25)$$

Substituting (20)-(25) into (17) yields the desired differential equation:

$$-\rho Q(y) - \frac{y^b}{b} + (\rho - r)y Q'(y) + \frac{1}{2} K^2 y^2 Q''(y) = 0 \quad (26)$$

where $b \equiv \frac{\alpha-1}{\alpha} \in (0,1)$. Note that $b-1 = -1/\alpha$.

Equation (26) is a linear differential equation in Q . Now, standard calculus yields the following:

$$Q(y) \text{ solves (26)} \Leftrightarrow Q(y) = M \frac{y^b}{b} + K_1^c y^{\alpha_1} + K_2^c y^{\alpha_2} \quad (27)$$

where,

$$\text{i) } M \equiv \frac{1}{b \left(\frac{\rho}{1-\alpha} - r - \frac{K^2}{2\alpha} \right)} = -k^{1/\alpha} < 0 \quad (28)$$

$$\text{ii) } \alpha_i \text{ is a root of } -\rho + (\rho-r)\alpha_i + \frac{1}{2} K^2 \alpha_i (\alpha_i - 1) = 0 \quad (29)$$

(by convention $\alpha_1 > 1$ and $\alpha_2 < 0$)

$$\text{iii) } K_1^c \text{ and } K_2^c \text{ are the constants of integration.}$$

The usual method to compute the constants of integration is to use the boundary conditions and the smoothness properties of the value function, V , and therefore Q . However, since the Bellman equation does not have a closed form solution in the binding region, we cannot use the value-matching and the smooth pasting condition to solve explicitly for the constants of integration, K_1^c and K_2^c ; Nonetheless, we are able to sign these constants which is sufficient in our model to determine the qualitative properties of

equilibrium demands and prices. *Theorem 3* gives the closed form solution for Q and the signs of the constants of integration. *Corollary 2* contains the closed form solution for the optimal wealth and portfolio as functions of the Lagrange multiplier, $y(t)$, and the constant K_2^c .

Theorem 3: Let V^c be a twice continuously differentiable function that solves (14), then:

- a) The constraint (12) is not binding on $y \in (y^*, \infty) \Leftrightarrow \hat{W} \in (0, W^*)$, where W^* is endogenously determined by the value matching and smooth pasting conditions.
- b) Furthermore, if $r - \rho - 1/2 K^2 < 0$, then (12) is binding on $(0, y^*)$.
- c) $Q(y) = My^b / b + K_2^c y^{\alpha_2}$ on (y^*, ∞) .
- d) $K_2^c < 0$.

Proof: see Appendix D.

Corollary 2: In NB, the optimal wealth and portfolio of the constrained trader are given

by:

$$W^c(t) = -My^{b-1} - K_2^c \alpha_2 y^{\alpha_2-1} \quad (30)$$

$$\pi^c(t) \sigma(t) = (M(b-1)y^{b-1} + K_2^c \alpha_2 (\alpha_2 - 1)y^{\alpha_2-1}) K ; \quad c^c(t) = y(t)^{-\frac{1}{\alpha}} \quad (31)$$

There are two points to note. First, parts a) and b) of *Theorem 3* have already been proved by Vila and Zariphopoulou (1994). Appendix D contains a slightly different proof, consistent with the change of variables we are using throughout the paper. Second, the condition $r - \rho - 1/2 K^2 < 0$ is not improbable. In fact, it is likely to be satisfied in actual financial markets: for instance, let $K = 0.25$, then (31) is implied by $r < 12\%$.

We finish this section on optimization by pointing out that, since we still maintain the assumption of $K(t)=K$, there is no difference between the unconstrained trader's optimization problem in this economy and that of the representative agent in the

benchmark economy; hence, his optimal demands and wealth and the law of motion of $z(t)$ are already known.

2.B.2. Equilibrium Analysis

We solve for the equilibrium in the same fashion as the benchmark economy: namely, we find $(r, K, \mu(t), \sigma(t))$ such that markets clear. We first find the laws of motion of the marginal utilities of both traders, then prove the existence of a *restricted equilibrium* in this economy, which is formally defined as follows:

Definition 2: (restricted equilibrium in the leverage constrained economy)

$\{r, K, P(t); c^c(t), \pi^c(t), c^u(t), \pi^u(t)\}$ is a restricted equilibrium iff:

- i) $\{c^c(t), \pi^c(t), c^u(t), \pi^u(t)\}$ is optimal given $r, K(t)$ and $P(t)$, and
- ii) $\lambda \pi^c(t) + (1 - \lambda) \pi^u(t) = NP(t)$ a.e. (32)

Superscript “c” denotes the demand of an agent facing leverage constraints, as described in this section, and superscript “u” denotes the demand of an unconstrained agent as described in section 2.A.

To obtain the law of motion for the process $y(t)$ we follow the same procedure as in the benchmark economy. The law of motion of the lagrangian multiplier of the unconstrained trader, as previously mentioned, is given in *Lemma 1*¹². *Lemma 3* describes the law of motions for the two types of traders. As in the benchmark economy, this lemma gives all we need to express the optimal controls as functions of the stock price and the model’s primitives and derive the qualitative properties of asset prices and trading volume at equilibrium.

¹²Note that we are using $z(t)$ both for the marginal utility of the representative agent of the benchmark economy and the unconstrained trader of the leverage constrained economy. This is an abuse of notations as $z(0)$ will be the same across economies only if the initial price, and therefore initial wealth, is constant across economies, which is not the case in general.

Lemma 3: Suppose $y \in (y^*, \infty)$, then:

$$\frac{d y(t)}{y(t)} = (\rho - r) dt - K dB(t) \quad (33)$$

$$\frac{d z(t)}{z(t)} = (\rho - r) dt - K dB(t) \quad (34)$$

where $z(t) \equiv V^{u_1}(t)$

Proof: see Appendix E.

Note that (32) and (33) imply that:

$$\frac{z(t)}{y(t)} = \frac{z(0)}{y(0)} = \text{constant.}$$

We show later that the marginal value of wealth is larger for the unconstrained trader, $y(0) < z(0)$, which implies that the unconstrained trader will consume less than a constrained agent with the same wealth since $c = (V'(W))^{-1/\alpha}$.

We are now able to solve for the equilibrium variables of the restricted equilibrium in the economy with leverage constraints.

Theorem 4: A price process $P(t)$ is a restricted equilibrium price process iff:

$$\begin{aligned} dP(t) + \delta(t) dt = & \\ & \left(\frac{K^2}{N} [(1-\lambda) \frac{1}{\alpha} k^{\frac{1}{\alpha}} z(t)^{\frac{1}{\alpha}} + \lambda \left(\frac{y(0)z(t)}{z(0)k} \right)^{\frac{1}{\alpha}} + K_2^c \alpha_2 (\alpha_2 - 1) \left(\frac{y(0)z(t)}{z(0)} \right)^{\alpha_2 - 1}] + r P(t) \right) dt \\ & + \frac{K}{N} [(1-\lambda) \frac{1}{\alpha} k^{\frac{1}{\alpha}} z(t)^{\frac{1}{\alpha}} + \lambda \left(\frac{y(0)z(t)}{z(0)k} \right)^{\frac{1}{\alpha}} + K_2^c \alpha_2 (\alpha_2 - 1) \left(\frac{y(0)z(t)}{z(0)} \right)^{\alpha_2 - 1}] dB(t) \end{aligned}$$

a.e. in NB, and:

$$\begin{aligned} dP(t) + \delta(t) dt = & \left(\frac{K}{N} [(1-\lambda) \frac{K}{\alpha} k^{\frac{1}{\alpha}} z(t)^{\frac{1}{\alpha}} + \lambda (a(W^c(t) + L))] + r P(t) \right) dt \\ & + \frac{1}{N} [(1-\lambda) \frac{K}{\alpha} k^{\frac{1}{\alpha}} z(t)^{\frac{1}{\alpha}} + \lambda (a(W^c(t) + L))] dB(t) \end{aligned}$$

a.e. in B.

Proof: see Appendix F.

Corollary 3: Given the initial price $P(0)$, the interest rate r and the Sharpe ratio K , there exists an unique restricted equilibrium price process.

Proof: see Appendix G.

Theorem 4 and *Corollary 3* prove the existence of and provide a characterization for the restricted equilibrium price process. These results follow directly from our assumption of a constant market price of risk, rather than constant coefficients of the return process, which yields an extra degree of freedom necessary to clear the stock market. The need for this extra degree of freedom in the economy with leverage constraints is apparent once we realize that, contrary to the benchmark case, there does not exist any r and K such that the restricted equilibrium price process associated with these variables has constant coefficients.

As in the benchmark case, we check whether it is possible to select parameters such that the economy is not a net borrower at the risk free rate. The next theorem proves this to hold in the non-binding region, NB ; however, the lack of a closed form solution for optimal consumption in the binding region implies that we cannot prove or disprove this result in B .

Theorem 5 (non-negativity in the NB region):

Suppose that current wealth of the constrained agents lies in the non-binding region, NB , and that the dividend process is given by:

$$\frac{d\delta(t)}{\delta(t)} = \mu_\delta dt + \sigma_\delta dB(t)$$

Let $r \equiv \rho + \alpha (\mu_\delta - \frac{1}{2} \alpha \sigma_\delta^2 (\frac{1}{\alpha} + 1))$ and $K \equiv \sigma_\delta \alpha$

Then $P(0) \leq P^(0) \Rightarrow W(t) - \pi(t) \geq 0$ a.e. in NB .*

$P^(0)$ is s.t. $N\delta(0) = \lambda c^c(0, P^*(0)) + (1 - \lambda) c^u(0, P^*(0))$*

Proof: See Appendix H.

III. BENCHMARK ECONOMY Vs. LEVERAGE CONSTRAINED ECONOMY

We are now ready to discuss the main point of this paper: how the presence of leverage constrained agents affects asset prices and trading volume. Here, we compare the values of the endogenous variables in the leverage constrained economy and benchmark economy, while keeping the sample path of our exogenous state variables constant¹³. All derivations are made assuming that the leverage constraint is non-binding in the current state.

The first equilibrium result concerns the impact of the introduction of leverage constraints on the mean reversion of the equilibrium return process.

Theorem 6: Suppose that current wealth of the constrained agents lies in the non-binding region NB and that $P(t) \neq 0$, then the restricted equilibrium price process is mean reverting iff

$$\sigma(t) > \frac{K}{\alpha} - K(\alpha_2 - 1 + \frac{1}{\alpha}) D(t)$$

where $D(t)$ is a strictly negative function.

Proof: see Appendix I.

A comparison of *Theorem 6* and *Theorem 1* shows that the condition on volatility required to get mean-reversion is weaker in the leverage constrained economy than in the benchmark economy. This is a fairly weak result as we can not express this condition in terms of exogenous variables due to the indeterminacy already mentioned. We discuss how to obtain a sharper condition in section IV of the paper.

¹³This means it is incorrect to estimate the effect of leverage constraints by running a regression that includes asset prices among the independent variables since the price process is changed when new constraints are introduced. This endogeneity problem has often been ignored by empirical researchers and can therefore potentially explain the seemingly contradictory results on volatility and margin requirements mentioned in the introduction.

The intuition for why mean reversion should be easier to obtain in the presence of leverage constraints is straightforward. Suppose that good news arrives to the market, moving the price up. The constrained agent sees his wealth getting closer to the switching point W^* , becomes more risk averse and, therefore, does not react as much to future news as he would if he were not facing the leverage constraint. This implies a sharper decline in the return process volatility. Since we assume that expected excess return and volatility are proportional, this also implies a sharper decline of the expected excess return. Note that the same intuition applies to the case where expected returns and volatility are not proportional but simply positively correlated.

Even though our leverage constrained agents behave as if they had increasing relative risk aversion, there is no contradiction between this result and He and Leland's'. He and Leland proved that decreasing relative risk aversion is associated with mean reversion *when volatility is constant* (whereas equilibrium volatility is stochastic in our environment). The (casual) intuition for their result is that when prices go up, agents want to invest more in the risky asset. Consequently, expected returns must decrease in order for the stock market to clear. In our model, when prices rise some agents become more risk averse and, consequently, they do not react to future news as much as they would if their risk aversion had remained low. This implies that volatility is going down when prices are moving up. Since we conjecture a constant Sharpe ratio equilibrium, expected returns are mean-reverting. Unfortunately, it also suggests that an assumption like "1bis" that we imposed for purely technical reasons may be crucial for some of our results. It is therefore a relevant empirical question to decide which assumption, constant volatility or constant Sharpe ratio, is the most plausible (if any of the two is).

We now move to the most important part of this paper, namely the analysis of trading volume. Since we only have two types of traders, we can examine the variation in one agent's demand to compute the volume of trading. As mentioned in the introduction, the fact that equilibrium demands are functions of the Brownian motion, $B(t)$, which is an unbounded variation process, means that the volume of trading during any finite time

interval is infinite. We can, however, define an instantaneous volume equal to the volatility of stock holdings, $\sigma_n(t)$. In order to prove that volume of trading per absolute unit of price change is increasing in the stock price, we need to show that the ratio of demand volatility to price volatility is increasing in the stock price.

The first step to prove this statement is to express the equilibrium demand of one of the agents, say the unconstrained agent, as a function of only exogenous variables. This is readily done by recalling expression (6) which gives us the unconstrained trader's demand for $P(t)$ different from zero :

$$n^u(t) = \frac{K}{\alpha \sigma(t) P(t)} \left(k^{\frac{1}{\alpha}} z(t)^{-\frac{1}{\alpha}} \right)$$

Now, we use the characterization of a restricted equilibrium price process in NB to get:

$$n^u(t) = \frac{N}{\alpha} \frac{k^{\frac{1}{\alpha}} z(t)^{-\frac{1}{\alpha}}}{(1-\lambda) \frac{1}{\alpha} k^{\frac{1}{\alpha}} z(t)^{-\frac{1}{\alpha}} + \lambda \left(\frac{1}{\alpha} \left(\frac{y(0) z(t)}{z(0) k} \right)^{-\frac{1}{\alpha}} + K_2^c \alpha_2 (\alpha_2 - 1) \left(\frac{y(0) z(t)}{z(0)} \right)^{\alpha_2 - 1} \right)}$$

A little bit of algebra allows us to express $n^u(t)$ as

$$n^u(t) = \frac{1}{C_1 - C_2 z(t)^{\alpha_2 - b}} \quad (35)$$

where C_1 and C_2 are positive constants.

We can now easily compute the demand volatility by applying Ito's lemma to the above expression. We get:

$$dn^u(t) = \text{drift terms} + \frac{C_2 (\alpha_2 - b) z(t)^{\alpha_2 - b}}{(C_1 - C_2 z(t)^{\alpha_2 - b})^2} \frac{dz(t)}{z(t)}$$

or,

$$dn^u(t) = \text{drift terms} + \frac{K C_2 (b - \alpha_2) z(t)^{\alpha_2 - b}}{(C_1 - C_2 z(t)^{\alpha_2 - b})^2} dB(t) \quad (36)$$

The last expression implies that the equilibrium stock holdings of the unconstrained traders is increasing with the value of the Brownian motion. This should not come as a surprise since when prices go up, the constrained trader sees himself getting closer to the point where the leverage constraint binds and therefore finds investing in the risky asset less profitable. In other words, the better the news, the more risk averse the constrained trader becomes compared to the unconstrained trader. This naturally implies that in order for the stock market to clear, we need to have the constrained agent selling some of his shares to the unconstrained agent.

Notice that (36) implies that the volatility of equilibrium demand, σ_n , is an increasing function of $B(t)$. We interpret this as saying that trading volume is larger when the stock price is high, or, in other words, when past returns have been high, which is exactly what Lakonishok and Smidt (1986 and 1989) report. This should be intuitive as well since the wealthier the constrained agent becomes, the closer he gets to the point where the constraint binds, the more his behavior differs from that of an unconstrained agent; hence, more trading occurs at the equilibrium.

However, in order to explain the entire evidence on trading volume, we have yet to prove that trading volume per absolute unit of price change is also increasing in the stock price. This is easily done by looking at the ratio of stock holdings volatility over price volatility, which we can easily obtain from (36) and *Theorem 4*:

$$\left| \frac{\sigma_n(t)}{\sigma(t) P(t)} \right| = \frac{1}{(C_1 - C_2 z(t)^{\alpha_2 - b})^2} \times \frac{N C_2 (b - \alpha_2) z(t)^{\alpha_2 - b}}{(1 - \lambda) \frac{1}{\alpha} k^{\frac{1}{\alpha}} z(t)^{-\frac{1}{\alpha}} + \lambda \left(\frac{1}{\alpha} \left(\frac{y(0) z(t)}{z(0) k} \right)^{-\frac{1}{\alpha}} + K_2^c \alpha_2 (\alpha_2 - 1) \left(\frac{y(0) z(t)}{z(0)} \right)^{\alpha_2 - 1} \right)} \quad (37)$$

It can easily be checked that both terms in the RHS of (37) are decreasing in $z(t)$.

We can then collect our three results on equilibrium demand to obtain the following theorem which is the most important result of this paper:

Theorem 7: Suppose that current wealth of the constrained agents lies in the non-binding region NB and that $P(t) \neq 0$, then the equilibrium demand of the unconstrained traders, the volatility of equilibrium demands of both traders, as well as the ratio of demand volatility to price volatility are increasing in the stock price.

Notice that no such liquidity-motivated trading would occur if the risky asset had no margin requirements or could be traded without restrictions. In fact, given our interpretation of the parameter ' α ' in the leverage constraint as being inversely proportional to the margin requirement, it is easy to prove that the smaller is ' α ', the larger are the effects described above. This explains why we observe a significantly asymmetric behavior of trading volume in stock markets but not in futures markets. In that sense, our leverage constraints-based model is a more satisfactory explanation of the empirical evidence on trading volume than models based on non-expected utility theory or special structure of beliefs across investors since these models do not explain why trading volume on stock and futures markets have different qualitative features. Moreover, in sharp contrast to the above theories, our hypothesis is empirically testable. The fraction of large agents facing leverage constraints, such as mutual funds, has considerably increased over the years (the fraction of US equity owned by mutual funds has more than doubled between 1981 and 1987). If our model is to be believed, so must have the degree of asymmetry between trading volume in bullish versus bearish markets. This prediction could easily be tested in future research.

IV. DISCUSSION AND EXTENSION OF THE RESULTS.

As already discussed earlier, the set-up used so far is perfectly appropriate for studying trading volume, but much less so for studying asset prices. We have not been able to address many questions of interest, for instance how the presence of leverage constraints affects the price level or return volatility, and have only been able to obtain some weak results concerning mean reversion. The next logical step is to extend our results to an economy with a constant net supply of bonds so as to put more restrictions on the equilibrium prices, while keeping the assumption of constant market price of risk in at least part of the state space. The model would then be a continuous time analog of the celebrated Lucas (1978) asset pricing model.

We review in this section the technical difficulties associated with such a strategy. The main issue is that, while it is easy to characterize an equilibrium price process, we have not been able to get a proof of existence. We show here that, in the NB region, an equilibrium price process of the leverage constrained economy must necessarily exhibit mean reversion and that both the volatility and the initial price will be lower than in the benchmark economy. We also show that the market price of risk has to be non-constant in the B region. Finally, we also point out that all the results on trading volume from the previous section straightforwardly hold in the economy with a constant supply of bonds.

As we did in the case of the bond in infinitely elastic supply, let us start by studying a benchmark economy, where no agent faces leverage constraints.

We first define the concept of *equilibrium*. Note that every time we use the word *equilibrium* we refer to an economy where the bond is in constant (zero) net supply, while we keep the term *restricted equilibrium* for the economy where the bond is in infinitely elastic supply.

Definition 3: (equilibrium in the benchmark economy)

$\{r, K, P(t); c(t), \pi(t)\}$ is an equilibrium of the benchmark economy iff:

- i) $\{c(t), \pi(t)\}$ is optimal given r, K and $P(t)$, and
- ii) $\pi(t) = NP(t)$ a.e.
- iii) $c(t) = N\delta(t)$ a.e.

We then prove the existence of and characterize the unique equilibrium price process of the benchmark economy in the case where dividends follow a geometric Brownian motion:

Theorem 8: Assume that: $\frac{d\delta(t)}{\delta(t)} = \mu_\delta dt + \sigma_\delta dB(t)$

Then $\{r \equiv \rho + \alpha(\mu_\delta - \frac{1}{2}\alpha\sigma_\delta^2(\frac{1}{\alpha} + 1)), K \equiv \sigma_\delta\alpha; c(t), \pi(t)\}$ is the unique equilibrium price process of the benchmark economy, where:

- i) $P(0) = k^{\frac{1}{\alpha}} \delta(0)$
- ii) $dP(t) + \delta(t) dt = (\frac{K^2}{\alpha} + r) P(t) dt + \frac{K}{\alpha} P(t) dB(t)$
- iii) $c(t)$ and $\pi(t)$ are given by (4).

Proof: Identical to that of Theorem 2 and Corollary 1 and therefore omitted.

Corollary 4: The equilibrium return process of the benchmark economy is neither mean-reverting nor mean-averting.

Now that we have fully described the (unique) equilibrium price process, let's move to the leverage constrained economy. We first extend our definition of equilibrium to the leverage constrained economy:

Definition 4: (equilibrium in the leverage constrained economy)

- $\{r, K, P(t); c^c(t), \pi^c(t), c^u(t), \pi^u(t)\}$ is an equilibrium in NB (B) iff:
 - i) $\{c^c(t), \pi^c(t), c^u(t), \pi^u(t)\}$ is optimal given $r, K(t)$ and $P(t)$,

$$ii) \lambda \pi^c(t) + (1-\lambda) \pi^u(t) = NP(t) \text{ a.e. in } NB (B)$$

$$iii) \lambda c^c(t) + (1-\lambda) c^u(t) = N\delta(t) \text{ a.e. in } NB (B)$$

- o $\{r, K, P(t); c^c(t), \pi^c(t), c^u(t), \pi^u(t)\}$ is an equilibrium iff it is an equilibrium in NB and B .

Remark: One can immediately conclude from this definition than in the benchmark economy, an equilibrium is also trivially a restricted equilibrium. This obviously implies that all the results we obtained in the case of an infinitely elastic supply of bonds also apply in this new setting.

As a first step, we will impose a constant net supply of bonds in the NB region. The reason why this is easier than to directly impose constant bond supply in the entire state space is that we have explicit expressions for the optimal controls in that region. Once we have characterized price processes that clear markets in the NB region, we will look at the B region, for which analytical methods are not available.

Theorem 9: Suppose that the current wealth of the constrained agents lies in the non-binding region NB and that the dividend process is given by:

$$\frac{d\delta(t)}{\delta(t)} = \mu_\delta dt + \sigma_\delta dB(t)$$

Then $\{r \equiv \rho + \alpha(\mu_\delta - \frac{1}{2}\alpha\sigma_\delta^2(\frac{1}{\alpha} + 1)), K \equiv \sigma_\delta\alpha, P(t); c(t), \pi(t)\}$ is the unique

equilibrium in the NB region, where:

$$i) P(0) \text{ is s.t. } N\delta(0) = \lambda c^c(0, P(0)) + (1-\lambda) c^u(0, P(0))$$

$$ii) dP(t) + \delta(t) dt =$$

$$\begin{aligned} & \left(\frac{K^2}{N} [(1-\lambda) \frac{1}{\alpha} k^{\frac{1}{\alpha}} z(t)^{-\frac{1}{\alpha}} + \lambda \frac{1}{\alpha} \left(\frac{y(0)z(t)}{z(0)k} \right)^{-\frac{1}{\alpha}} + K_2 \alpha_2 (\alpha_2 - 1) \left(\frac{y(0)z(t)}{z(0)} \right)^{\alpha_2 - 1}] + r P(t) \right) dt \\ & + \frac{K}{N} [(1-\lambda) \frac{1}{\alpha} k^{\frac{1}{\alpha}} z(t)^{-\frac{1}{\alpha}} + \lambda \left(\frac{1}{\alpha} \left(\frac{y(0)z(t)}{z(0)k} \right)^{-\frac{1}{\alpha}} + K_2 \alpha_2 (\alpha_2 - 1) \left(\frac{y(0)z(t)}{z(0)} \right)^{\alpha_2 - 1}] \right) dB(t) \end{aligned}$$

$$iii) c^u(t) \text{ and } \pi^u(t) \text{ are as in equation (2).}$$

$$iv) c^c(t) \text{ and } \pi^c(t) \text{ are as in Corollary 2.}$$

Proof: Identical to that of *Theorem 6* and *Corollary 4* and therefore omitted.

We now compare the price processes found in *Theorems 8* and *9* to see how leverage constraints affect the equilibrium asset prices. The first result relates to mean reversion and is given in the following corollary:

Corollary 5: *The equilibrium price process in NB is mean reverting a.e.*

Proof: See Appendix J.

Comparing *Corollaries 4* and *5* gives us a much stronger result than what we obtained in the restricted equilibrium case. Instead of just having the condition for mean-reversion weakened by the introduction of leverage constraints, we go from an economy where mean reversion never occurs at the equilibrium to an economy where mean reversion always occurs. Note that since an equilibrium price process in *NB* is also a restricted equilibrium price process, this means that even if the bond is in infinitely elastic supply, leverage constraints can generate mean reversion. The intuition for this result is given in Section III of the paper.

The second result we obtain concerns equilibrium volatility:

Corollary 6: *The return volatility of the equilibrium price process in NB is a.e. lower than the return volatility of the equilibrium price of the benchmark economy.*

Proof: see Appendix K.

Corollary 6 says that the return volatility of an equilibrium price process in *NB* must necessarily be lower than that of the equilibrium price process of the benchmark economy. Moreover, since we have already proved the existence of a restricted equilibrium price process such that $K(t)=K$ a.e., *Corollary 6* implies that leverage constraints *can* generate a lower volatility even in the case of an infinitely elastic supply of riskless asset. In this sense, our model seems to back up financial regulators' belief that the more one constrains

agents in their leverage possibilities, the lower will be return volatility. However, this does not mean that leverage constraints are necessarily desirable. On the contrary, the weighted sum of the agents' expected utilities is strictly lower in the constrained than in the benchmark economy. The optimal risk sharing between agents with the same concave utility function is indeed obtained when each agent is holding the same quantity of stocks, as in the benchmark economy. This is obviously not happening here since we saw that the constrained agents sell some of their shares to the unconstrained agents whenever the price is going up. In essence, this result suggests that while the introduction of constraints on leverage can be an effective policy tool to control volatility, at the same time it might also lead to inefficient risk sharing among agents and become a welfare reducing market intervention.

Finally, we establish the impact of leverage constraints on the price level. As expected, since leverage constraints increase the effective aggregate risk aversion in the market, the initial price must be lower than in the benchmark economy.

Corollary 7: The initial value of the equilibrium price process in NB is lower than the initial equilibrium price in the benchmark economy.

Proof: see Appendix L.

We have therefore been able to obtain some very sharp results about asset prices when assuming a constant net supply of bonds in the NB region. However these results will be meaningful only if we can guarantee a constant supply of bonds in the entire state space. The following result proves that it is not possible to keep the market price of risk constant in the B region when the bond is in zero net supply.

Theorem 10: There does not exist an equilibrium s.t. $K(t) = K$ a.e..

Proof: see Appendix M.

Does *Theorem 10* mean an end to the possible extension of our results to an economy where the bond is in zero net supply? Not necessarily. It may be possible to construct an equilibrium where $K(t)$ is constant in NB , but not in B . We have unfortunately not been able to prove or disprove the existence of such a process. However, in an earlier version of this paper, we proved that all the qualitative properties of the price process constructed in this section would also hold in any equilibrium where $K(t)$ follows a Markov process in B , if such an equilibrium exists. The intuition for this result is straightforward: whatever happens in the B region affects the agents' decisions in the NB region only through the boundary conditions of the Bellman equations. This implies that only the values of the constants of integration are affected by a change in the equilibrium price process in the B region. In that sense, all the results in this section should be interpreted as necessary conditions. If an equilibrium in the economy with the bond in zero net supply exists and is such that the interest rate and the market price of risk in the NB region are unchanged by the presence of leverage constraints, then it must be the case that the price level and the return volatility will be lowered by the leverage constraints, and that mean reversion will always occur. However, the existence of such an equilibrium and the analysis of the equilibrium variables in the B region are still open areas for future research.

V. CONCLUSION

The main result of this paper is that both mean-reversion of stock prices and positive correlation between trading volume per absolute unit of price change and price level are consistent with a standard infinite horizon - perfect information - expected utility economy where some agents are facing leverage constraints comparable to those prevailing in actual financial markets. This means that, even though previous research has indicated that some of those facts were also consistent with more uncommon theoretical frameworks, this empirical evidence can not be used to prove the superiority of particular behavioral theories over standard expected utility theory. In that sense, the basic message of this paper is quite negative: it just reminds us of the dangers of drawing strong conclusions from equilibrium variables on unobservables such as preferences, beliefs or information.

More positively, we also note that our model is, as far as we know, the first one to generate dynamics of trading volume that fits the stylized facts identified by the empirical literature. The presence of some investors facing leverage constraints generates trading in our economy even though all agents are identical in every other respect (preferences, initial endowments, information...). We have shown that the volume of trading generated by the presence of constrained traders per unit of price change is an increasing function of the stock price, which is consistent with the empirical evidence presented in the introduction. Moreover, such trading generated by leverage constraints should be negligible in futures markets since margin requirements are much smaller than in stock markets. This is consistent with the empirical evidence brought forward by Karpoff (1987 and 1988) and others, which indicates that trading volume per absolute unit of price change and price level are uncorrelated in futures markets. We are not aware of any other formal model generating these two features simultaneously.

At this stage, we have to put a 'caveat' to our results: our model does not aim at explaining the entire volume of trading in stock or futures markets¹⁴. Presumably, most of

¹⁴For such a general model, see for instance Wang (1994).

the trading volume occurring in actual financial markets is generated by differences in beliefs or changes in risk aversion. However, what we have done is to focus on one particular motive for trading, namely differences in liquidity across agents, and argue that this motive could help us understand empirical phenomena that previous theoretical research could not satisfactorily explain. We also have been able to get some tentative results on how leverage constraints affect asset prices. We have shown that leverage constraints can generate mean reversion and lower both stock prices and return volatility. Unfortunately, a problem of indeterminacy of equilibrium prevented us from obtaining sharper results.

Finally, from a methodological point of view, we have built on the results of Vila and Zariphopoulou (1994) to sharpen their characterization of the optimal policies of an agent facing leverage constraints and made a first step in understanding the equilibrium implications of such constraints. The change of variable we used to solve the leverage constraint problem are applicable to a much more general class of constrained optimization problems. When applicable, our technique is much simpler than martingale methods currently used and should therefore be helpful for applied financial economists. Our results are obviously limited by our assumption of constant Sharpe ratio in the *NB* region. This assumption is very convenient analytically but clearly not general enough. Existence and characterization of the solution to the leverage constraint problem under more general assumptions, as well as equilibrium analysis of an economy with the bond in zero net supply, are questions of interest that should be solved by future research.

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APPENDIX

Appendix A.

Lemma 1:

We have:

$$W(t) = k^{\frac{1}{\alpha}} z^{-\frac{1}{\alpha}}$$

By Ito's lemma, this implies:

$$dW = -\frac{1}{\alpha} k^{\frac{1}{\alpha}} z^{-\frac{1}{\alpha}} \frac{dz}{z} + \frac{1}{2\alpha} \left(\frac{1}{\alpha} + 1\right) k^{\frac{1}{\alpha}} z^{-\frac{1}{\alpha}} \text{Var}\left(\frac{dz}{z}\right) \quad (\text{A1})$$

On the other hand, from the law of motion of W , we get:

$$\begin{aligned} dW &= (rW - c) dt + \pi (\mu dt + \sigma dB) \\ &= \left(r k^{\frac{1}{\alpha}} z^{-\frac{1}{\alpha}} - z^{-\frac{1}{\alpha}}\right) dt + \frac{1}{\alpha} k^{\frac{1}{\alpha}} z^{-\frac{1}{\alpha}} (K^2 dt + K dB) \end{aligned} \quad (\text{A2})$$

Assume that:

$$\frac{dz}{z} = \mu_z dt + \sigma_z dB \quad (\text{A3})$$

Then, equating terms of (A1) and (A2) in dB yields:

$$\sigma_z = -K \quad (\text{A4})$$

Given (A3), equating the terms in $z^{-\frac{1}{\alpha}} dt$ yields:

$$\begin{aligned} r k^{\frac{1}{\alpha}} - 1 + \frac{1}{\alpha} K^2 k^{\frac{1}{\alpha}} &= -\frac{1}{\alpha} k^{\frac{1}{\alpha}} \mu_z + \frac{1}{2\alpha} \left(\frac{1}{\alpha} + 1\right) k^{\frac{1}{\alpha}} K^2 \\ \Leftrightarrow \mu_z &= \rho - r \quad (\text{by definition of } k). \end{aligned}$$

QED.

Appendix B.

Corollary 1:

Let $X(t) \equiv z(t)^{\frac{1}{\alpha}}$.

Then Theorem 1 and (6) are equivalent to the system of equations (B1a) and (B1b), where:

$$\frac{dX}{X} = \left(-\frac{1}{\alpha} (\rho - r) + \frac{1}{2} \frac{1}{\alpha} \left(\frac{1}{\alpha} + 1\right) K^2\right) dt + \frac{1}{\alpha} K dB \quad (\text{B1a})$$

$$dP + \delta dt = \left(\frac{K^2}{N\alpha} k^{\frac{1}{\alpha}} X + rP\right) dt + \left(\frac{K}{N\alpha} k^{\frac{1}{\alpha}} X\right) dB \quad (\text{B1b})$$

It can easily be seen that (B1a) and (B1b) satisfy the Lipschitz and growth conditions of Theorem 6.2.2. of Arnold (1992) and therefore uniquely determine the processes $X(t)$ and $P(t)$ from their initial conditions.

QED.

Appendix C.

Theorem 2:

At time 0, we have:

$$c(0) = k^{-\frac{1}{\alpha}} W(0) = k^{-\frac{1}{\alpha}} NP(0) \leq N \delta(0) \quad (\text{C1})$$

where the first equality comes from the solution of the agent problem and the second from the assumption on initial endowments.

Note also that:

$$c(t) = z(t)^{-\frac{1}{\alpha}} \quad (C2)$$

and that:

$$\frac{dz}{z} = (\rho - r) dt - K dB \quad (C3)$$

Applying Ito's lemma on (C2) and substituting for the values of r and K given in *Theorem 2* yields:

$$\frac{dc}{c} = \mu_\delta dt + \sigma_\delta dB = \frac{d(N \delta(t))}{N \delta(t)} \quad (C4)$$

(C4) and (C1) yields $c(t) \leq N \delta(t)$ a.e. .

This implies $W(t) - \pi(t) \geq 0$ since $W(0) - \pi(0) = 0$ by assumption.

Q.E.D.

Appendix D.

Theorem 3:

We prove this theorem in four steps.

1) NB contains an open interval $W \in (0, W^*) \Leftrightarrow y \in (y^*, \infty)$.

We have to prove that on a neighborhood of zero wealth,

$$K(t) = K \text{ a.e.} \Rightarrow -\frac{V^{c'}(W)}{V^{c''}(W)} K \leq a(W+L).$$

Suppose not. Then we must have:

$$\begin{aligned} -\rho V^c(W) + \frac{[(V^c)'(W)]^{\frac{\alpha-1}{\alpha}}}{1-\alpha} + \{rW - [V^{c'}(W)]^{\frac{1}{\alpha}}\} V^{c'}(W) \\ + aK(W+L) V^{c'}(W) + \frac{1}{2} a^2 (W+L)^2 V^{c''}(W) = 0 \end{aligned}$$

$$\begin{aligned} \Leftrightarrow \rho \frac{V^c(W)}{(V^c)'(W)} - \frac{(V^{c'}(W))^{\frac{1}{\alpha}}}{1-\alpha} - (rW - [V^{c'}(W)]^{\frac{1}{\alpha}}) = \\ aK(W+L) + \frac{1}{2} a^2 (W+L)^2 \frac{V^{c''}(W)}{V^{c'}(W)} \end{aligned}$$

Taking the limit when $W \rightarrow 0^+ \Leftrightarrow V^{c'}(W) \rightarrow +\infty$ yields:

$$\lim_{W \rightarrow 0} aKL + \frac{a^2}{2} L^2 \frac{V^{c''}(W)}{V^{c'}(W)} = \lim_{W \rightarrow 0} \frac{\rho V^c(W)}{V^{c'}(W)}$$

$$\Rightarrow \lim_{W \rightarrow 0} aKL + \frac{a^2}{2} L^2 \frac{V^{c''}(W)}{V^{c'}(W)} \leq 0$$

$$\Rightarrow \lim_{W \rightarrow 0} -\frac{V^{c''}(W)}{V^{c'}(W)} \geq \frac{2K}{aL}$$

$$\Rightarrow \lim_{W \rightarrow 0} -\frac{V^{c'}(W)}{V^{c''}(W)} K \leq \frac{1}{2} aL \Rightarrow \text{contradiction with (12) binding. Q.E.D.}$$

$$2) y \in (y^*, \infty) \Rightarrow Q(y) = M \frac{y^b}{b} + K_2^c y^{\alpha_2}$$

From (22) and (27), we know that:

$$W(t) = -M y^{b-1} - K_1^c \alpha_1 y^{\alpha_1-1} - K_2^c \alpha_2 y^{\alpha_2-1} \quad (D1)$$

We also know that:

$$0 \leq \lim_{y \rightarrow \infty} W(y) < \infty$$

Taking the limit of (D1) when y goes to infinity yields:

$$K_1^c = 0.$$

$$3) y \in (y^*, \infty) \Rightarrow y \in NB$$

Or, in other words, there does not exist an interval (y_1, y_2) s.t.

$$\Psi(y_1) = \Psi(y_2) = 0$$

$$\Psi(y) > 0 \quad \forall y \in (y_1, y_2)$$

(D2)

$$(\Rightarrow \psi'(y_1) > 0, \psi'(y_2) < 0)$$

Where:

$$\Psi(y) \equiv a(W(y) + L) - \pi(y)\sigma(y)$$

$$= [aL + y^{b-1} M(-a + \frac{K}{\alpha}) + y^{\alpha_1-1} \alpha_1 K_1^c(-a - (\alpha_1 - 1)K) + y^{\alpha_2-1} \alpha_2 K_2^c(-a - (\alpha_2 - 1)K)] \quad (D3)$$

By contradiction; suppose that such an interval exists.

Then:

a) if $K_1^c \leq 0$ and $K_2^c \geq 0$, then $\psi'(y) > 0$.

$$\Rightarrow \text{contradiction with } \Psi(y_1) = \Psi(y_2)$$

b) if $K_1^c \leq 0$ and $K_2^c \leq 0$, then:

$$\text{Let: } A \equiv \sigma(y_1)[\Psi(y_1) + y_1 \Psi'(y_1)] - \sigma(y_2)[\Psi(y_2) + y_2 \Psi'(y_2)]$$

$$\Leftrightarrow A = (y_1^{b-1} - y_2^{b-1}) M(-a + \frac{K}{\alpha})(1 - \frac{1}{\alpha}) + (y_1^{\alpha_1-1} - y_2^{\alpha_1-1}) \alpha_1^2 K_1^c(-a - (\alpha_1 - 1)K) \\ + (y_1^{\alpha_2-1} - y_2^{\alpha_2-1}) \alpha_2^2 K_2^c(-a - (\alpha_2 - 1)K)$$

$$\Rightarrow A < 0$$

$$\Rightarrow \text{contradiction with } \Psi(y_1) = \Psi(y_2) \text{ and } \psi'(y_1) > 0, \psi'(y_2) < 0.$$

c) if $K_1^c \geq 0$ and $K_2^c \geq 0$, then:

Notice that by construction $y_2 < y^*$.

Since, by definition, $\psi(y^*) = 0$, we have:

$$M y^{*b-1} \left(\frac{K}{\alpha} - a \right) < -aL$$

$$\Rightarrow M y_2^{b-1} \left(\frac{K}{\alpha} - a \right) < -aL$$

$$\Rightarrow 0 < y_2^{\alpha_1-1} \alpha_1 K_1^c(-a - (\alpha_1 - 1)K) + y_2^{\alpha_2-1} \alpha_2 K_2^c(-a - (\alpha_2 - 1)K) \quad (D4)$$

$$\Rightarrow \text{contradiction with } K_1^c \geq 0 \text{ and } K_2^c \geq 0.$$

d) if $K_1^c \geq 0$ and $K_2^c \leq 0$, then:

Recall that α_1 and α_2 are defined by:

$$-\rho + (\rho - r) \alpha_i + \frac{1}{2} K^2 \alpha_i (\alpha_i - 1) = 0 \Leftrightarrow \alpha_i^2 - (1 + \frac{2(r - \rho)}{K^2}) \alpha_i - \frac{2r}{K^2} = 0$$

$$\Rightarrow \alpha_1 + \alpha_2 = 1 + \frac{2(r-\rho)}{K^2}$$

Suppose now that $r - \rho - K^2/2 < 0$, then $\alpha_1 + \alpha_2 < 2$ and therefore

$$\alpha_1 - 1 < 1 - \alpha_2 \quad (D5)$$

We have shown in part c) of the proof that $y_2 < y^*$ implies:

$$0 < y_2^{\alpha_1-1} \alpha_1 K_1^c (-a - (\alpha_1 - 1)K) + y_2^{\alpha_2-1} \alpha_2 K_2^c (-a - (\alpha_2 - 1)K) \quad (D6)$$

But then (D5) in (D4) implies:

$$\left| y_2^{\alpha_1-1} \alpha_1 (\alpha_1 - 1) K_1^c (-a - (\alpha_1 - 1)K) \right| < \left| y_2^{\alpha_2-1} \alpha_2 (\alpha_2 - 1) K_2^c (-a - (\alpha_2 - 1)K) \right|$$

$\Rightarrow \psi'(y_2) > 0 \Rightarrow \text{contradiction} \Rightarrow \text{QED.}$

4) Since $K(t)=K$ in both NB and B , $K_2^u = 0$ ("u" refers to the unconstrained trader)

Since the constraint (12) is binding in some states, we must have:

$$V^c(W(y(t))) < V^u(W(y(t)))$$

$$\Leftrightarrow \frac{M}{\alpha-1} y^b + K_2^c (1 - \alpha_2) y^{\alpha_2} < \frac{M}{\alpha-1} (y^{b-1} + \frac{K_2^c \alpha_2}{M} y^{\alpha_2-1})^{1-\alpha} \quad (D7)$$

Notice that:

$$\lim_{y \rightarrow \infty} \frac{V^c(W(y))}{V^u(W(y))} = 1$$

It must therefore be that for y large enough:

$$\begin{aligned} & \left| \frac{dV^c(W(y))}{dy} \right| < \left| \frac{dV^u(W(y))}{dy} \right| \\ \Rightarrow & (-M) \left| -\frac{1}{\alpha} y^{b-1} + \frac{K_2^c \alpha_2 (\alpha_2 - 1)}{M} y^{\alpha_2-1} \right| \\ & < (-M) (y^{b-1} + \frac{K_2^c \alpha_2}{M} y^{\alpha_2-1})^{-\alpha} \left| -\frac{1}{\alpha} y^{b-2} + \frac{K_2^c \alpha_2 (\alpha_2 - 1)}{M} y^{\alpha_2-2} \right| \\ \Rightarrow & \frac{(y^{-\frac{1}{\alpha}} + \frac{K_2^c \alpha_2}{M} y^{\alpha_2-1})^{-\alpha}}{y} > 1 \\ \Rightarrow & K_2^c < 0 \text{ since both } \alpha_2 \text{ and } M \text{ are negative.} \end{aligned}$$

QED.

Appendix E.

Lemma 3:

We have for the constrained trader:

$$W(t) = -M y^{b-1} - K_2^c \alpha_2 y^{\alpha_2-1}$$

By Ito's lemma, this implies:

$$\begin{aligned} dW &= (-M(b-1)y^{b-1} - K_2^c \alpha_2 (\alpha_2 - 1) y^{\alpha_2-1}) \frac{dy}{y} \\ &+ \frac{1}{2} (-M(b-1)(b-2)y^{b-1} - K_2^c \alpha_2 (\alpha_2 - 1)(\alpha_2 - 2) y^{\alpha_2-1}) \text{Var}\left(\frac{dy}{y}\right) \end{aligned} \quad (E1)$$

On the other hand, from the law of motion of W , we get:

$$\begin{aligned} dW &= (rW - c) dt + \pi (\mu dt + \sigma dB) \\ &= (r(-M y^{b-1} - K_2^c \alpha_2 y^{\alpha_2-1}) - y^{b-1}) dt \\ &+ (M(b-1)y^{b-1} + K_2^c \alpha_2 (\alpha_2 - 1) y^{\alpha_2-1}) (K^2 dt + K dB) \end{aligned} \quad (E2)$$

Assume that:

$$\frac{dy}{y} = \mu_y dt + \sigma_y dB \quad (\text{E3})$$

Then, equating terms of (E1) and (E2) in dB yields:

$$\sigma_y = -K \quad (\text{E4})$$

Given (E4), equating the terms in $y^{\alpha_2-1} dt$ yields:

$$\begin{aligned} K_2^c \alpha_2 (\alpha_2 - 1) \mu_y - \frac{1}{2} K_2^c \alpha_2 (\alpha_2 - 1) (\alpha_2 - 2) K^2 &= K_2^c \alpha_2 (\alpha_2 - 1) K^2 - K_2^c \alpha_2 r \\ \Rightarrow (\alpha_2 - 1) \mu_y &= (1 - \alpha_2) K^2 + r - \frac{1}{2} (\alpha_2 - 2) (\alpha_2 - 1) K^2 \\ \Rightarrow (\alpha_2 - 1) \mu_y &= r - \rho + (\rho - r) \alpha_2 \\ &(\text{since } -\rho + (\rho - r) \alpha_2 + \frac{1}{2} K^2 \alpha_2 (\alpha_2 - 1) = 0) \\ \Rightarrow \mu_y &= \rho - r \end{aligned} \quad (\text{E5})$$

One can easily check that equating the terms in $y^{b-1} dt$ yields (E5) as well. Substituting (E4) and (E5) in (E3) yields the desired law of motion for y .

QED.

Appendix F.

Theorem 4:

1) *NB* region:

a) If $P(t) = 0$, the stock market clearing condition is trivially satisfied.

b) assume $P(t) \neq 0$,

By (31), we have:

$$\sigma(t) n^c(t) \equiv \frac{\sigma(t) \pi^c(t)}{P(t)} = (M(b-1)y(t)^{b-1} + K_2^c \alpha_2 (\alpha_2 - 1) y(t)^{\alpha_2-1}) \frac{K}{P(t)} \quad (\text{F1})$$

Since the unconstrained trader solves the same problem as in the benchmark economy, it is easy to see that:

$$\sigma(t) n^u(t) \equiv \frac{\sigma(t) \pi^u(t)}{P(t)} = M(b-1)z(t)^{b-1} \frac{K}{P(t)} \quad (\text{F2})$$

The stock market clearing condition is:

$$\lambda n^c(t) + (1-\lambda) n^u(t) = N \quad (\text{F3})$$

Substituting (F1) and (F2) in (F3) yields:

$$\begin{aligned} \sigma(t) P(t) &= \frac{K}{N} \times \{ \lambda \{ M(b-1) \left(\frac{y(0)z(t)}{z(0)} \right)^{b-1} + K_2^c \alpha_2 (\alpha_2 - 1) \left(\frac{y(0)z(t)}{z(0)} \right)^{\alpha_2-1} \} \\ &\quad + (1-\lambda) (M(b-1)z(t)^{b-1}) \} \end{aligned} \quad (\text{F4})$$

We have:

$$dP(t) + \delta(t)dt = (K\sigma(t) + r) P(t) dt + \sigma(t) P(t) dB(t) \quad (\text{F5})$$

Substituting (F4) into (F5) yields the price process in *NB*.

2) *B* region:

Since the leverage constraint is binding,

$$\sigma(t) n^c(t) = \frac{\sigma(t) \pi^c(t)}{P(t)} = \frac{a}{P(t)} (W^c(t) + L) \quad (\text{F6})$$

Substituting (F2) and (F5) in (F3) gives the equilibrium volatility. Substituting the equilibrium volatility into (F5) yields the result.

QED.

Appendix G.

Corollary 3 :

1) NB region:

Define:

$$X(t) \equiv y(t)^{-\frac{1}{\alpha}}$$

$$Y(t) \equiv y(t)^{\alpha_2 - 1}$$

Then, Lemma 3 and Theorem 5 are equivalent to the system of equations (G1a), (G1b) and (G1c), where:

$$\frac{dX}{X} = \left(-\frac{1}{\alpha} (\rho - r) + \frac{1}{2} \frac{1}{\alpha} \left(\frac{1}{\alpha} + 1 \right) K^2 \right) dt + \frac{1}{\alpha} K dB \quad (G1a)$$

$$\frac{dY}{Y} = \left[(\alpha_2 - 1) (\rho - r) + \frac{1}{2} (\alpha_2 - 1) (\alpha_2 - 2) K^2 \right] dt + (\alpha_2 - 1) K dB \quad (G1b)$$

$$dP + \delta dt = \left\{ \frac{K^2}{N} \left[\frac{k^\alpha}{\alpha} (\lambda \left(\frac{y(0)}{z(0)} \right)^\alpha X + K_2^c \alpha_2 (\alpha_2 - 1) Y) + (1 - \lambda) X \right] + rP \right\} dt \quad (G1c)$$

$$+ \left(\frac{K}{N} \left[\frac{k^\alpha}{\alpha} (\lambda \left(\frac{y(0)}{z(0)} \right)^\alpha X + K_2^c \alpha_2 (\alpha_2 - 1) Y) + (1 - \lambda) X \right] \right) dB$$

It can easily be seen that (G1a), (G1b) and (G1c) satisfy the Lipschitz and growth conditions of Theorem 6.2.2. of Arnold (1994) and therefore uniquely determine the processes $X(t)$, $Y(t)$ and $P(t)$ from their initial conditions.

2) B region:

In this region the equivalent to the system {(G1a), (G1b), (G1c)} is given by:

$$\frac{dX}{X} = \left(-\frac{1}{\alpha} (\rho - r) + \frac{1}{2} \frac{1}{\alpha} \left(\frac{1}{\alpha} + 1 \right) K^2 \right) dt + \frac{1}{\alpha} K dB \quad (G2a)$$

$$dW^c(t) = (rW^c(t) - c(W^c(t)) + aK(W^c(t) + L)) dt + a(W^c(t) + L) dB \quad (G2b)$$

$$dP + \delta dt = \left\{ \frac{K}{N} \left[\lambda a(W^c + L) + (1 - \lambda) \frac{k^\alpha}{\alpha} K X \right] + rP \right\} dt \quad (G2c)$$

$$+ \frac{1}{N} \left[\lambda a(W^c + L) + (1 - \lambda) \frac{k^\alpha}{\alpha} K X \right] dB$$

Lemma G: $\exists \Gamma > 0$ s.t. $\forall W \in B$, $c^c(W) < \Gamma W$.

Proof: Suppose not. Then there exists W' such that $c^c(W') \geq \left(\frac{k\rho}{\alpha} \right)^{\frac{1}{1-\alpha}} W'$

$$\Rightarrow u(c^c(W')) \geq \frac{\rho k}{\alpha} \frac{W'^{1-\alpha}}{1-\alpha}$$

$$\Rightarrow \frac{\alpha}{\rho} u(c^c(W')) \geq \frac{k}{1-\alpha} W'^{1-\alpha} = V^u(W')$$

But, (16b) implies:

$$V^c(W') > \frac{\alpha}{\rho} u(c^c(W'))$$

Therefore, $V^c(W') > V^u(W')$. A contradiction. Q.E.D.

It can easily be seen that *Lemma 2* (in the text) and *Lemma G* imply that (G2a), (G2b) and (G2c) satisfy the Lipschitz and growth conditions of Theorem 6.2.2. of Arnold (1994) and therefore uniquely determine the processes $X(t)$, $W^c(t)$ and $P(t)$ from their initial conditions.

QED.

Appendix H.

Theorem 5:

a) Let $c^c(W(t))$ be the consumption of a constrained trader with wealth $W(t)$. By previous results, $c^c(W)$ is continuous, strictly increasing and s.t. $c^c(0) = 0$ and $c^c(W^*) > 0$. The same properties hold for the consumption of the unconstrained trader. Therefore the equation defining $P(0)$ has a unique solution for $\delta(0)$ small enough. By construction, initial aggregate consumption is no larger than aggregate dividends..

b) Let $C^{Ag}(t) = \lambda c^c(t) + (1-\lambda) c^u(t)$ be aggregate consumption at time t .

We know that

$$\frac{d c^c}{c^c} = \frac{d c^u}{c^u} = - \frac{1}{\alpha} ((\rho-r) dt - K dB) + \frac{1}{2 \alpha} \left(\frac{1}{\alpha} + 1\right) K^2 dt$$

We then have:

$$\frac{d C^{Ag}}{C^{Ag}} = - \frac{1}{\alpha} ((\rho-r) dt - K dB) + \frac{1}{2 \alpha} \left(\frac{1}{\alpha} + 1\right) K^2 dt$$

Substituting the values of r and K given in *Theorem 5* yields:

$$\begin{aligned} \frac{d C^{Ag}}{C^{Ag}} &= \frac{d \delta}{\delta} \\ \Rightarrow C^{Ag}(t) &\leq N \delta(t) \text{ a.e.} \end{aligned}$$

$\Rightarrow W^{Ag}(t) \geq N P(t)$ since $W^{Ag}(0) = N P(0)$ by assumption.

QED.

Appendix I.

Theorem 6:

We have:

$$\mu(t) = \frac{K^2}{N P(t)} \left[(1-\lambda) \frac{1}{\alpha} k^{\frac{1}{\alpha}} z(t)^{\frac{1}{\alpha}} + \lambda \left(\frac{y(0) z(t)}{z(0) k} \right)^{\frac{1}{\alpha}} + K_2^c \alpha_2 (\alpha_2 - 1) \left(\frac{y(0) z(t)}{z(0) k} \right)^{\alpha_2 - 1} \right] \quad (11)$$

By Ito's lemma, we get:

$$\begin{aligned} d\mu(t) &= \text{drift terms} - \mu(t) \frac{dP(t)}{P(t)} + \frac{K^2}{N P(t)} \times \\ &\times \left\{ (1-\lambda) \left(-\frac{1}{\alpha^2}\right) k^{\frac{1}{\alpha}} z(t)^{\frac{1}{\alpha}} + \lambda \left[\left(\frac{y(0) z(t)}{z(0) k} \right)^{\frac{1}{\alpha}} \left(-\frac{1}{\alpha^2}\right) + K_2^c \alpha_2 (\alpha_2 - 1)^2 \left(\frac{y(0) z(t)}{z(0) k} \right)^{\alpha_2 - 1} \right] \right\} \frac{dz}{z} \\ \Rightarrow \frac{d\mu(t)}{\mu(t)} &= \text{drift terms} + \left[\frac{K}{\alpha} - \sigma(t) - K \left(\alpha_2 - 1 + \frac{1}{\alpha} \right) D(t) \right] dB \end{aligned}$$

where,

$$D(t) \equiv \frac{\lambda K_2^c \alpha_2 (\alpha_2 - 1) \left(\frac{y(0) z(t)}{z(0) k} \right)^{\alpha_2 - 1}}{\left(1-\lambda\right) \left(-\frac{1}{\alpha^2}\right) k^{\frac{1}{\alpha}} z(t)^{\frac{1}{\alpha}} + \lambda \left[\left(\frac{y(0) z(t)}{z(0) k} \right)^{\frac{1}{\alpha}} \left(-\frac{1}{\alpha^2}\right) + K_2^c \alpha_2 (\alpha_2 - 1)^2 \left(\frac{y(0) z(t)}{z(0) k} \right)^{\alpha_2 - 1} \right]} < 0$$

The return process is therefore mean reverting iff

$$\sigma(t) > \frac{K}{\alpha} - K(\alpha_2 - 1 + \frac{1}{\alpha}) D(t) \quad (I2)$$

QED.

Appendix J.

Corollary 5:

Now note that (I1) implies:

$$\begin{aligned} \sigma(t) &= \frac{K}{\alpha} + K(\alpha_2 - 1 + \frac{1}{\alpha}) D(t) \times \\ &\quad \frac{(1-\lambda)(-\frac{1}{\alpha^2})k^{\frac{1}{\alpha}}z(t)^{-\frac{1}{\alpha}} + \lambda((\frac{y(0)z(t)}{z(0)k})^{-\frac{1}{\alpha}}(-\frac{1}{\alpha^2}) + K_2^c \alpha_2 (\alpha_2 - 1)^2 (\frac{y(0)z(t)}{z(0)k})^{\alpha_2-1})}{NP(t)} \end{aligned}$$

$$\Rightarrow \sigma(t) > \frac{K}{\alpha} - \frac{K}{\alpha^2} (\alpha_2 - 1 + \frac{1}{\alpha}) D(t)$$

$$\Rightarrow \sigma(t) > \frac{K}{\alpha} - K(\alpha_2 - 1 + \frac{1}{\alpha}) D(t)$$

The first step follows from $NP(t) = W^{ng}(t)$, the second from $\alpha > 1$.

QED.

Appendix K.

Corollary 6:

Note that since the bond is in zero net supply $P(t) = W^{ng}(t) / N > 0$.

We have:

$$\sigma(t) = \frac{K}{NP(t)} [(1-\lambda)\frac{1}{\alpha}k^{\frac{1}{\alpha}}z(t)^{-\frac{1}{\alpha}} + \lambda(\frac{1}{\alpha}(\frac{y(0)z(t)}{z(0)k})^{-\frac{1}{\alpha}} + K_2^c \alpha_2 (\alpha_2 - 1) (\frac{y(0)z(t)}{z(0)k})^{\alpha_2-1})] \quad (J1)$$

But since the bond is in zero net supply, we must have:

$$\begin{aligned} NP(t) &= \lambda W^c(t) + (1-\lambda) W^u(t) \\ \Rightarrow NP(t) &= (1-\lambda)k^{\frac{1}{\alpha}}z(t)^{-\frac{1}{\alpha}} + \lambda((\frac{y(0)z(t)}{z(0)k})^{-\frac{1}{\alpha}} - K_2^c \alpha_2 (\frac{y(0)z(t)}{z(0)k})^{\alpha_2-1}) \end{aligned} \quad (J2)$$

Since $\alpha_2 < 0$, (J2) in (J1) implies:

$$\sigma(t) < K/\alpha.$$

QED

Appendix L.

Corollary 7:

$$\lambda c^c(0, P(0)) + (1-\lambda)c^u(0, P(0)) > k^{-\frac{1}{\alpha}} W(0) = k^{-\frac{1}{\alpha}} NP(0) \text{ since } K_2 < 0.$$

Moreover, $\lambda c^c(0, P(0)) + (1-\lambda)c^u(0, P(0))$ is increasing in the second argument. Comparing the consumption good market clearing conditions for both economies for $P(0)$ equal to the initial price of the benchmark economy yields the result.

QED.

Appendix M.

Theorem 10:

(by contradiction) Suppose $K(t)$ constant. Then the value function of the unconstrained trader is identical to the value function of the benchmark economy trader. But, by *Corollary 6*, initial wealth is larger in the benchmark economy, which means that the benchmark trader has a strictly larger expected utility at time 0 than the unconstrained trader. But this is impossible since the optimal policy of the benchmark trader (keep his initial endowment of shares and consume the dividends) is still feasible => contradiction.

QED.

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