Universitat Pompeu Fabra Barcelona
Department of Economics and Business

Economic Working Paper Series
Working Paper No. 1458

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Updated version: July 2016
(November 2014)

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Abstract

In this paper we compute analytically the at-the-money second derivative of the implied volatility curve as a function of the strike price, for correlated stochastic volatility models. We obtain an expression for the short-time limit of this second derivative in terms of the first and second Malliavin derivatives of the volatility process and the correlation parameter.

Keywords: Anticipating Itô's formula, Malliavin calculus, Hull and White formula, stochastic volatility models

AMS subject classification: 91G99

1 Introduction

It is well-known that stochastic volatility models capture some important features of the implied volatility. For example, its variation with respect to the strike price, described graphically as a *smile* or *skew*. Although these properties of the implied volatility surface are well-known in the literature, there are only some few papers devoted to their analytical proof. Among them, we remark the paper by Renault and Touzi (1996), where the authors have figured out that,

*Supported by grants ECO2014-59885-P and MTM2013-40782-P

†Supported by CONACyT grant 220303

in the uncorrelated case, the implied volatility is, as a function of the strike, a locally convex function with a stationary point at the forward stock price. More recently, Alòs, León and Vives (2007) have figured out, in the correlated case, an explicit expression for the short-time limit of the at-the-money skew slope in terms of the Malliavin derivative operator of the volatility process. This result, together with this article, establishes that the Malliavin calculus is a powerful tool to analyze the properties of the implied volatility.

This paper deals with the analytical study of the second derivative of the implied volatility curve as a function of the strike price. Our method uses implicit differentiation and Malliavin calculus techniques, and gives explicit expressions for this second derivative, for both correlated and uncorrelated stochastic volatility models (see (3) and Theorem 13 below). The representation (3) (i.e., the expression for the second derivative in the uncorrelated case) is the main tool in our analysis and allows us to analyze the at-the-money short-time behaviour of the mentioned second derivative in terms of the Malliavin derivative of the volatility process. The obtained formulas recover, in particular, the convexity results by Renault and Touzi (1996). Moreover, in Theorem 13 we consider the correlated case and we prove that this short-time limit can be written in terms of the Malliavin derivatives of the volatility process and the correlation parameter, as it was proved for the skew (see Alòs, León and Vives (2007)). This analysis allows us to establish a condition for the at-the-money local convexity of the implied volatility, in terms of the correlation parameter and the Malliavin derivatives of the volatility process.

The paper is organized as follows. In Section 2 we introduce the framework and the notation that we utilize in this paper. Section 3 is devoted to the study of the uncorrelated case. We prove that in this case the implied volatility has a stationary point at the forward stock price. Moreover, we obtain an expression for the second derivative of the implied volatility that allows us to prove its local convexity, as well as to compute its short-time at-the-money limit. In Section 4 we extend our results to the correlated case, and we prove an expression for the short-time limit of the at-the-money second derivative. Finally, a particular example of our results is given in Section 5, namely the case of classical diffusion volatilities.

2 Statement of the problem and notation

In this paper we consider the following model for the log-price of a stock under a risk-neutral probability measure P:

$$X_t = x + \tilde{r}t - \frac{1}{2}\int_0^t \sigma_s^2 ds + \int_0^t \sigma_s \left(\rho dW_s + \sqrt{1-\rho^2} dB_s\right), \quad t \in [0,T]. \quad (1)$$

Here, x is the current log-price, $\tilde{r}$ is the instantaneous interest rate, W and B are standard Brownian motions defined on a complete probability space $(\Omega, \mathcal{G}, P)$, and σ is a square-integrable and right-continuous stochastic process adapted to the filtration generated by W. In the following we denote by $\mathcal{F}^W$ and $\mathcal{F}^B$ the filtrations generated by W and B. Moreover we define $\mathcal{F} := \mathcal{F}^W \vee \mathcal{F}^B$.

It is well-known that there is no arbitrage opportunity if we price an European call with strike price K by the formula

$$V_t = e^{-\tilde{r}(T-t)} E_t[(e^{X_T} - K)_+],$$

where E_t is the $\mathcal{F}_t-$conditional expectation with respect to P (i.e., $E_t(Z) = E(Z|\mathcal{F}_t)$). In the sequel, we make use of the following notation:

- $v_t^2 = \frac{1}{T-t} \int_t^T \sigma_u^2 du$. That is, v represents the future average volatility.
- $M_t = E_t \left(\int_0^T \sigma_u^2 du \right), \ t \in [0, T]$.
- $BS(t, x, k, \sigma)$ denotes the price of an European call option under the classical Black-Scholes model with constant volatility σ, current log stock price x, time to maturity $T - t$, strike price $K = \exp(k)$ and interest rate $\tilde{r}$. Remember that in this case

  $$BS(t, x, k, \sigma) = e^x N(d_+) - e^{k - \tilde{r}(T-t)} N(d_-),$$

  where N denotes the cumulative probability function of the standard normal law and

  $$d_\pm := \frac{k_t^* - k}{\sigma\sqrt{T-t}} \pm \frac{\sigma}{2}\sqrt{T-t},$$

  with $k_t^* := x + \tilde{r}(T - t)$.

3 The uncorrelated case

In this section we first study the uncorrelated case $\rho = 0$.

Let us define the implied volatility $I = I(t, X_t, k)$ as the adapted stochastic process such that $V_t = BS(t, X_t, k, I)$. Notice that, as σ is independent to the filtration generated by B, option prices are given by the so-called *Hull and White formula* (see for example Hull and White (1987))

$$V_t = E_t (BS(t, X_t, k, v_t)), \quad t \in [0, T]. \tag{2}$$

Sometimes we use the convention $I = I(t, k)$ in order to simplify the notation. We will need the following result, that can be deduced directly from the proof of Proposition 5.1 in Alòs, León and Vives (2007) and (2).

Proposition 1 *(The implied volatility skew) Consider the model (1) with $\rho = 0$. Then, for all $t \in [0, T]$, $\frac{\partial I}{\partial k}(t, k_t^*) = 0$.*

Remark 2 *The above result proves that, fixed $t \in [0, T]$, the implied volatility $I(t, k)$ has, in the uncorrelated case, a stationary point at $k = k_t^*$. Notice that this result is independent of the stochastic volatility model and agrees with Theorem 4.2 in Renault and Touzi (1996), where is established that the implied volatility, as a function of the strike, is continuous differentiable, decreasing for in-the-money options and increasing for out-of-the-money options (see also Proposition 5 in Renault (1997)).*

3.1 The at-the-money implied volatility smile

Now our purpose in this section is to study the at-the-money second derivative $\frac{\partial^2 I}{\partial k^2}(t, k_t^*)$ in the uncorrelated case (i.e., $\rho = 0$). We prove that this is positive. Consequently, for every fixed $t \in [0, T]$, the implied volatility $I(t, X_t, k)$ is a locally convex function of k. Moreover, we prove that $\lim_{T \to t} \frac{\partial^2 I}{\partial k^2}(t, k_t^*)$ is well-defined and finite, which is figured out explicitly (see Theorem 5 below).

We assume that the reader is familiar with the elementary results of the Malliavin calculus, as given for instance in Nualart (2006). In the remaining of this paper $\mathbb{D}_W^{1,2}$ denotes the domain of the Malliavin derivative operator D^W with respect to the Brownian motion W. It is well-known that $\mathbb{D}_W^{1,2}$ is a dense subset of $L^2(\Omega)$ and that D^W is a closed and unbounded operator from $\mathbb{D}_W^{1,2}$ to $L^2([0, T] \times \Omega)$. We also consider the iterated derivatives $D^{n,W}$, for $n > 1$, whose domains will be denoted by $\mathbb{D}_W^{n,2}$. We will use the notation $\mathbb{L}_W^{n,2} = L^2([0, T]; \mathbb{D}_W^{n,2})$.

For our purpose, we introduce the following hypotheses:

(H1) σ^2 belongs to $\mathbb{L}_W^{1,2}$, and there exists an adapted process $Y = \{Y_r, r \in [0, T]\} \in L^4 (\Omega \times [0, T])$ such that $\left|E_r \left(D_r^W \sigma_u^2\right)\right| \leq Y_r$, for all $t \leq r \leq u \leq T$.

(H2) For every $t \in [0, T]$, there exists an $\mathcal{F}_t^W$–measurable random variable $D_t^+ \sigma_t^2$ such that

$$\lim_{T \to t} \frac{\int_t^T E_t \left(\sup_{r \leq u \leq T} \left|E_r \left(D_r^W \sigma_u^2 - D_t^+ \sigma_t^2\right)\right|^4\right) dr}{T - t} = 0.$$

(H3) There exist two deterministic, integrable and right continuous functions $\sigma_1, \sigma_2 : [0, T] \to (0, \infty)$ such that

$$\sigma_1(t) \leq \sigma_t \leq \sigma_2(t), \quad t \in [0, T].$$

Remark 3 *Notice that under (H1), the Clark-Ocone formula gives us that (see, for instance, Nualart (2006)),*

$$M_t = M_0 + \int_0^t \left(\int_s^T E_s \left(D_s^W \sigma_r^2\right) dr \right) dW_s, \quad t \in [0, T],$$

with $M_0 = E\left(\int_0^T \sigma_s^2 ds\right)$.

Before stating the main result of this section, we establish the following auxiliary result, whose proof is in Section 7.

Lemma 4 *Let* $r \in [t,T]$ *and* $\Lambda_r = E_r\left(BS(t,X_t,k_t^*,v_t)\right)$*, then*

$$\exp\left(\frac{BS^{-1}(t,X_t,k_t^*,\Lambda_r)^2(T-t)}{8}\right) \leq E_r\left(\exp\left(\frac{v_t^2(T-t)}{8}\right)\right)$$

and, for $k=2,3$

$$\left(BS^{-1}(t,X_t,k_t^*,\Lambda_r)\right)^{-k} \leq E_r\left(\frac{\exp\left(\frac{kv_t^2(T-t)}{8}\right)}{v_t^k}\right).$$

Theorem 5 *Assume that* $\rho = 0$ *in model (1), and that Hypotheses (H1) and (H3) are satisfied. Then, for all* $t \in [0,T]$,

$$\frac{\partial^2 I}{\partial k^2}(t,k_t^*) = \frac{1}{2}\frac{E_t\left[\int_t^T \Psi''\left(E_u\left(BS\left(t,X_t,k_t^*,v_t\right)\right)\right)U_u^2 du\right]}{\frac{\partial BS}{\partial \sigma}\left(t,X_t,k_t^*,I(t,k_t^*)\right)}, \tag{3}$$

where

$$\Psi\left(a\right) := \frac{\partial^2 BS}{\partial k^2}\left(t,X_t,k_t^*,BS^{-1}\left(t,X_t,k_t^*,a\right)\right)$$

and

$$U_u := E_u\left(D_u^W\left(BS\left(t,X_t,k_t^*,v_t\right)\right)\right),\ u \in [t,T].$$

Moreover, if Hypothesis (H2) also holds,

$$\lim_{T\to t}\frac{\partial^2 I}{\partial k^2}(t,k_t^*) = \frac{(D_t^+\sigma_t^2)^2}{12\sigma_t^5}.$$

Proof. From the definition of the implied volatility I, we have

$$\begin{aligned}\frac{\partial^2}{\partial k^2}V_t &= \frac{\partial^2 BS}{\partial k^2}\left(t,X_t,k;I\right) + 2\frac{\partial^2 BS}{\partial k\partial\sigma}\left(t,X_t,k;I\right)\frac{\partial I}{\partial k} \\ &+ \frac{\partial^2 BS}{\partial \sigma^2}\left(t,X_t,k;I\right)\left(\frac{\partial I}{\partial k}\right)^2 + \frac{\partial BS}{\partial\sigma}\left(t,X_t,k;I\right)\frac{\partial^2 I}{\partial k^2}.\end{aligned}$$

By Proposition 1, last equality becomes

$$\frac{\partial BS}{\partial \sigma}\left(t,X_t,k_t^*,I(t,k_t^*)\right)\frac{\partial^2 I}{\partial k^2}\left(t,k_t^*\right) = \frac{\partial^2}{\partial k^2}V_t|_{k=k_t^*} - \frac{\partial^2 BS}{\partial k^2}\left(t,X_t,k_t^*,I(t,k_t^*)\right).$$

Thus (2) gives

$$\begin{aligned}&\frac{\partial BS}{\partial \sigma}\left(t,X_t,k_t^*,I(t,k_t^*)\right)\frac{\partial^2 I}{\partial k^2}\left(t,k_t^*\right) \\ &= E_t\left[\frac{\partial^2 BS}{\partial k^2}\left(t,X_t,k_t^*,v_t\right) - \frac{\partial^2 BS}{\partial k^2}\left(t,X_t,k_t^*,I(t,k_t^*)\right)\right]. \end{aligned} \tag{4}$$

But the last term on the right-hand side of (4) can be written as

$$\begin{aligned}
&\frac{\partial^2 BS}{\partial k^2}\left(t, X_t, k_t^*, I(t, k_t^*)\right) \\
&= \frac{\partial^2 BS}{\partial k^2}\left(t, X_t, k_t^*, BS^{-1}\left(V_t\right)\right) \\
&= \frac{\partial^2 BS}{\partial k^2}\left(t, X_t, k_t^*, BS^{-1}\left(E_t\left(BS\left(t, X_t, k_t^*, v_t\right)\right)\right)\right),
\end{aligned}$$

where, in this case, we denote $BS^{-1}(t, X_t, k_t^*, \cdot)$ by $BS^{-1}(\cdot)$ in order to simplify the notation. Consequently, using (4), we can establish

$$\begin{aligned}
&\frac{\partial BS}{\partial \sigma}\left(t, X_t, k_t^*, I(t, k_t^*)\right)\frac{\partial^2 I}{\partial k^2}\left(t, k_t^*\right) \\
&= E_t\left(\frac{\partial^2 BS}{\partial k^2}\left(t, X_t, k_t^*, v_t\right)\right) \\
&\quad - E_t\left(\frac{\partial^2 BS}{\partial k^2}\left(t, X_t, k_t^*, BS^{-1}\left(E_t\left(BS\left(t, X_t, k_t^*, v_t\right)\right)\right)\right)\right) \\
&= E_t\left[\frac{\partial^2 BS}{\partial k^2}\left(t, X_t, k_t^*, BS^{-1}\left(BS\left(t, X_t, k_t^*, v_t\right)\right)\right)\right. \\
&\quad \left. - \frac{\partial^2 BS}{\partial k^2}\left(t, X_t, k_t^*, BS^{-1}\left(E_t\left(BS\left(t, X_t, k_t^*, v_t\right)\right)\right)\right)\right]. \qquad (5)
\end{aligned}$$

Now the proof is decomposed into several steps.

Step 1. Let us first prove (3). The Clark-Ocone formula (see Nualart (2006)), together with Hypotheses (H1) and (H3), leads to

$$BS\left(t, X_t, k_t^*, v_t\right) = E_t\left(BS\left(t, X_t, k_t^*, v_t\right)\right) + \int_t^T U_r dW_r,$$

where

$$\begin{aligned}
U_r &= E_r\left(D_r^W\left(BS\left(t, X_t, k_t^*, v_t\right)\right)\right) \\
&= E_r\left(\left(\frac{\partial BS}{\partial \sigma}\left(t, X_t, k_t^*, v_t\right)\right)\frac{D_r^W M_T}{2(T-t)v_t}\right), \quad r > t. \qquad (6)
\end{aligned}$$

Hence, using equality (5), we get

$$
\begin{aligned}
&\frac{\partial BS}{\partial \sigma}\left(t,X_t,k_t^*,I(t,k_t^*)\right)\frac{\partial^2 I}{\partial k^2}\left(t,k_t^*\right)\\
&= E_t\left[\frac{\partial^2 BS}{\partial k^2}\left(t,X_t,k_t^*,BS^{-1}\left(E_t\left(BS\left(t,X_t,k_t^*,v_t\right)\right)+\int_t^T U_r dW_r\right)\right)\right.\\
&\quad\left.-\frac{\partial^2 BS}{\partial k^2}(t,X_t,k_t^*,BS^{-1}\left(E_t\left(BS\left(t,X_t,k_t^*,v_t\right)\right)\right))\right]\\
&= E_t\left[\int_t^T \Psi'\left(E_t\left(BS\left(t,X_t,k_t^*,v_t\right)\right)+\int_t^u U_r dW_r\right)U_u dW_u\right.\\
&\quad\left.+\frac{1}{2}\int_t^T \Psi''\left(E_t\left(BS\left(t,X_t,k_t^*,v_t\right)\right)+\int_t^u U_r dW_r\right)U_u^2 du\right]\\
&= E_t\left[\int_t^T \Psi'\left(E_u\left(BS\left(t,X_t,k_t^*,v_t\right)\right)\right)U_u dW_u\right.\\
&\quad\left.+\frac{1}{2}\int_t^T \Psi''\left(E_u\left(BS\left(t,X_t,k_t^*,v_t\right)\right)\right)U_u^2 du\right]\\
&= \frac{1}{2}E_t\left[\int_t^T \Psi''\left(E_u\left(BS\left(t,X_t,k_t^*,v_t\right)\right)\right)U_u^2 du\right],
\end{aligned}
$$

where, in the last equality, we use the fact that

$$
\Psi'(a)=\frac{1}{4}-\frac{1}{\left(BS^{-1}\left(a\right)\right)^2\left(T-t\right)},
$$

Lemma 4 and Hypothesis (H3). This proves (3).

Step 2. Here we show that

$$
\frac{E_t\left[\int_t^T \Psi''(\Lambda_r)\left(U_r^2-\left(E_r\left(\partial_\sigma BS(t,X_t,k_t^*,v_t)\frac{(T-r)D^+\sigma_t^2}{2(T-t)v_t}\right)\right)^2\right)dr\right]}{\exp(X_t)(T-t)^{1/2}},
$$

where Λ_r is defined in Lemma 4, converges to 0 as $T\to t$.

By Schwarz inequality, we can write, for $r > t$,

$$\begin{aligned}
&\left| U_r^2 - \left(E_r \left(\partial_\sigma BS(t, X_t, k_t^*, v_t) \frac{(T-r)D^+\sigma_t^2}{2(T-t)v_t} \right) \right)^2 \right| \\
&= \left| E_r \left(\partial_\sigma BS(t, X_t, k_t^*, v_t) \frac{1}{2(T-t)v_t} \int_r^T (D_r^W \sigma_u^2 + D_t^+ \sigma_t^2) du \right) \right| \\
&\quad \times \left| E_r \left(\partial_\sigma BS(t, X_t, k_t^*, v_t) \frac{1}{2(T-t)v_t} \int_r^T (D_r^W \sigma_u^2 - D_t^+ \sigma_t^2) du \right) \right| \\
&\leq \frac{Ce^{2X_t}}{(T-t)} \left| E_r \left(\frac{\exp\left(-\frac{v_t^2(T-t)}{8}\right)}{v_t} \int_r^T (D_r^W \sigma_u^2 + D_t^+ \sigma_t^2) du \right) \right| \\
&\quad \times \left| E_r \left(\frac{\exp\left(-\frac{v_t^2(T-t)}{8}\right)}{v_t} \int_r^T (D_r^W \sigma_u^2 - D_t^+ \sigma_t^2) du \right) \right| \\
&\leq \frac{Ce^{2X_t}}{(T-t)} E_r \left(\frac{\exp\left(-\frac{v_t^2(T-t)}{4}\right)}{v_t^2} \right) \left(E_r \left(\left[\int_r^T (D_r^W \sigma_u^2 + D_t^+ \sigma_t^2) du \right]^2 \right) \right)^{1/2} \\
&\quad \times \left(E_r \left(\left[\int_r^T (D_r^W \sigma_u^2 - D_t^+ \sigma_t^2) du \right]^2 \right) \right)^{1/2} \\
&\leq \frac{C(T-r)^2 e^{2X_t}}{(T-t)} E_r \left(\frac{\exp\left(-\frac{v_t^2(T-t)}{4}\right)}{v_t^2} \right) \left(\sup_{r \leq u \leq T} E_r \left((D_r^W \sigma_u^2 + D_t^+ \sigma_t^2)^2 \right) \right)^{1/2} \\
&\quad \times \left(\sup_{r \leq u \leq T} E_r \left((D_r^W \sigma_u^2 - D_t^+ \sigma_t^2)^2 \right) \right)^{1/2}.
\end{aligned}$$

Hence, Lemma 4 gives

$$
\begin{aligned}
&\frac{E_t\left[\int_t^T \Psi''(\Lambda_r)\left(U_r^2 - \left(E_r\left(\partial_\sigma BS(t,X_t,k_t^*,v_t)\frac{(T-r)D_t^+\sigma_t^2}{2(T-t)v_t}\right)\right)^2\right)dr\right]}{\exp(X_t)(T-t)^{1/2}} \\
&\leq \frac{C}{(T-t)}E_t\left[\int_t^T E_r\left(\exp\left(\frac{v_t^2(T-t)}{8}\right)\right)E_r\left(\frac{\exp\left(\frac{3v_t^2(T-t)}{8}\right)}{v_t^3}\right)E_r\left(\frac{\exp\left(-\frac{v_t^2(T-t)}{4}\right)}{v_t^2}\right)\right. \\
&\quad \left.\times\left(\sup_{r\leq u\leq T}E_r\left((D_r^W\sigma_u^2 + D_t^+\sigma_t^2)^2\right)\right)^{1/2}\left(\sup_{r\leq u\leq T}E_r\left((D_r^W\sigma_u^2 - D_t^+\sigma_t^2)^2\right)\right)^{1/2}dr\right] \\
&\leq \frac{C}{(T-t)}\left(E_t\int_t^T\left(E_r\left(\exp\left(\frac{v_t^2(T-t)}{8}\right)\right)E_r\left(\frac{\exp\left(\frac{3v_t^2(T-t)}{8}\right)}{v_t^3}\right)\right.\right. \\
&\quad \left.\left.\times E_r\left(\frac{\exp\left(-\frac{v_t^2(T-t)}{4}\right)}{v_t^2}\right)\right)^2 dr\right)^{1/2} \\
&\quad \times\left(E_t\int_t^T\sup_{r\leq u\leq T}\left|E_r\left((D_r^W\sigma_u^2 + D_t^+\sigma_t^2)\right)\right|^4 dr\right)^{1/4} \\
&\quad \times\left(E_t\int_t^T\sup_{r\leq u\leq T}\left|E_r\left((D_r^W\sigma_u^2 - D_t^+\sigma_t^2)\right)\right|^4 dr\right)^{1/4} \\
&\to 0, \quad \text{as } T\to t,
\end{aligned}
$$

due to Hypotheses (H2) and (H3). Thus the claim of this part of the proof is true.

Step 3. Finally we prove that $\lim_{T\to t}\frac{\partial^2 I}{\partial k^2}(t,k_t^*) = \frac{(D_t^+\sigma_t^2)^2}{12\sigma_t^5}$.

From Step 2, we obtain

$$
\begin{aligned}
&\lim_{T\to t}\frac{\partial^2 I}{\partial k^2}(t,k_t^*) \\
&= \frac{1}{2}\lim_{T\to t}\frac{E_t\left[\int_t^T\Psi''(\Lambda_r)\left(E_r\left(\partial_\sigma BS(t,X_t,k_t^*,v_t)\frac{(T-r)D^+\sigma_t^2}{2(T-t)v_t}\right)\right)^2 dr\right]}{\frac{1}{\sqrt{2\pi}}(T-t)^{1/2}e^{X_t}} \\
&= \frac{1}{2}\lim_{T\to t}\frac{E_t\left[(D^+\sigma_t^2)^2\int_t^T\Psi''(\Lambda_r)\left(E_r\left(\partial_\sigma BS(t,X_t,k_t^*,v_t)\frac{(T-r)}{2v_t}\right)\right)^2 dr\right]}{\frac{1}{\sqrt{2\pi}}(T-t)^{5/2}e^{X_t}}.
\end{aligned}
$$

Note that the right continuity of σ and (H3) imply

$$\frac{1}{2}\frac{(D^+\sigma_t^2)^2\int_t^T \Psi''(\Lambda_r)\left(E_r\left(\partial_\sigma BS(t,X_t,k_t^*,v_t)\frac{(T-r)}{2v_t}\right)\right)^2 dr}{\frac{1}{\sqrt{2\pi}}(T-t)^{5/2}e^{X_t}}$$
$$\rightarrow \frac{(D^+\sigma_t^2)^2}{12\sigma_t^5} \quad \text{as } T\rightarrow t, \quad \text{w.p.1},$$

and

$$\frac{1}{2}\frac{(D^+\sigma_t^2)^2\int_t^T \Psi''(\Lambda_r)\left(E_r\left(\partial_\sigma BS(t,X_t,k_t^*,v_t)\frac{(T-r)}{2v_t}\right)\right)^2 dr}{\frac{1}{\sqrt{2\pi}}(T-t)^{5/2}e^{X_t}}$$
$$\leq \frac{1}{2}\frac{(D^+\sigma_t^2)^2}{(T-t)}\int_t^T E_r\left(\exp\left(\frac{v_r^2(T-t)}{8}\right)\right)E_r\left(\frac{\exp\left(\frac{3v_r^2(T-t)}{8}\right)}{v_t^3}\right)E_r\left(\frac{\exp\left(-\frac{v_t^2(T-t)}{4}\right)}{v_t^2}\right)dr.$$

Therefore, the result follows from the dominated convergence theorem and the fact that

$$\frac{1}{(T-t)}\int_t^T E_t\left(\left(E_r\left(\exp\left(\frac{v_r^2(T-t)}{8}\right)\right)E_r\left(\frac{\exp\left(\frac{3v_r^2(T-t)}{8}\right)}{v_t^3}\right)\right.\right.$$
$$\left.\left.\times E_r\left(\frac{\exp\left(-\frac{v_t^2(T-t)}{4}\right)}{v_t^2}\right)\right)^2\right)dr\rightarrow E_t\left(\frac{1}{\sigma_t^{10}}\right), \quad \text{as } T\rightarrow t,$$

which follows from (H3). Now the proof is finished. ■

Remark 6 *Notice that the arguments in the proof of the above theorem can be adapted to the study of volatility models that do not satisfy (H1), (H2) and (H3), as we can see in the following example, where (H2) and the last part of (H1) are not satisfied.*

Example 7 *Fractional noises with $H<1/2$ were introduced in Alòs, León and Vives (2007) to describe the empirical skew slope of the implied volatililty. This idea has been further developed in Gatheral, Jaisson and Rosembaum (2014), and Bayer, Friz and Gatheral (2016), where the authors have proved these models to be very efficient in the description of real market data. Following the ideas in these papers, we consider a function f in $\mathcal{C}_b^2$ such that is lower bounded by a positive constant and the volatility process $\sigma_t^2=f(Y_t)$. Here*

$$dY_t=\nu dW_t^H-\alpha(Y_t-m)dt.$$

with

$$W_t^H:=\int_0^t(t-s)^{H-1/2}dW_t,$$

and ν, α and m positive constants. It is easy to see that $\sigma \in \mathbb{L}_W^{1,2}$ and that Hypothesis (H3) holds. Moreoer, a standard computation gives us that

$$\begin{aligned} D_r Y_t &= \nu(t-r)^{H-1/2} - \alpha\nu \int_r^t e^{-\alpha(t-s)}(s-r)^{H-1/2} ds \\ &=: h(t,r) \end{aligned}$$

and then

$$D_r \sigma_t^2 = f'(Y_t) h(t,r).$$

Notice that, as $H < 1/2$, the above Malliavin derivative satisfies neither the last part of (H1), nor (H2). However, we can make use of a similar procedure to study its short-time behaviour. In fact, Clark-Ocone formula gives us that

$$BS\left(t, X_t, k_t^*, v_t\right) = E_t\left(BS\left(t, X_t, k_t^*, v_t\right)\right) + \int_t^T U_r dW_r,$$

where U_r is defined as in (6). Then, the arguments of the proof of Theorem 5 give us that (3) holds. On the other hand,

$$\begin{aligned} &\left| U_r^2 - \left(E_r \left(\partial_\sigma BS(t, X_t, k_t^*, v_t) \frac{(\int_r^T h(s,r)ds) f'(Y_t)^2}{2(T-t)v_t} \right) \right)^2 \right| \\ &= \left| E_r \left(\partial_\sigma BS(t, X_t, k_t^*, v_t) \frac{1}{2(T-t)v_t} \int_r^T h(s,r)(f'(Y_u) + f'(Y_t)) ds \right) \right| \\ &\quad \times \left| E_r \left(\partial_\sigma BS(t, X_t, k_t^*, v_t) \frac{1}{2(T-t)v_t} \int_r^T \int_r^T h(s,r)(f'(Y_u) + f'(Y_t)) ds \right) \right| \\ &\leq \frac{Ce^{2X_t}}{(T-t)} \left(\int_r^T h(s,r) ds \right)^2 \sup_{t \leq r \leq u \leq T} |E_r(f'(Y_u) - f'(Y_t))| \end{aligned}$$

Hence, Lemma 4 gives that

$$\begin{aligned} &(T-t)^{1-2H} \frac{E_t\left[\int_t^T \Psi''(\Lambda_r)\left(U_r^2 - \left(E_r\left(\partial_\sigma BS(t, X_t, k_t^*, v_t)\frac{(\int_r^T h(s,r)ds) f'(Y_t)}{2(T-t)v_t}\right)\right)^2\right) dr\right]}{\exp(X_t)(T-t)^{1/2}} \\ &\leq \frac{C}{(T-t)^{2+2H}} \sup_{r \leq u \leq T} |E_r(f'(Y_u) - f'(Y_t))| \int_t^t \left(\int_r^T h(s,r) ds \right)^2 dr \\ &\leq C \left(\sup_{t \leq r \leq u \leq T} |E_r(f'(Y_u) - f'(Y_t))| \right) \\ &\to 0, \quad \text{as } T \to t. \end{aligned}$$

Then, the same arguments as in the proof of Theorem 5 give us that

$$\lim_{T \to t} (T-t)^{1-2H} \frac{\partial^2 I}{\partial k^2}(t, k^*) = \frac{(\nu f'(Y_t))^2}{12\sigma_t^5}.$$

Remark 8 *The above result gives us an explicit expression for the at-the-money second derivative that allows us to study its main properties. In particular, (3) implies that*

$$\begin{aligned}
&\frac{\partial^2 I}{\partial k^2}(t,k_t^*) \\
&= \frac{1}{\frac{\partial BS}{\partial \sigma}(t,x_t,k_t^*,I(t,k_t^*))} \\
&\quad \times E_t\left[2e^{-X_t}\int_t^T \frac{\sqrt{2\pi}\exp\left(\frac{\left(BS^{-1}(E_u(BS(t,X_t,k_t^*,v_t)))\right)^2(T-s)}{8}\right)}{\left(BS^{-1}\left(E_u\left(BS\left(t,X_t,k_t^*,v_t\right)\right)\right)\right)^3(T-t)^{3/2}}U_u^2 du\right].
\end{aligned}$$

Thus $\frac{\partial^2 I}{\partial k^2}(t,k_t^*) > 0$ *w.p.1. This, jointly with Proposition 1, proves that, fixed* $t \in [0,T]$, *the implied volatility* $I(t,k)$ *is, in the uncorrelated case, a locally convex function of the strike with a minimum at* $k = k_t^*$. *This agrees with the previous results by Renault and Touzi (1996) and by Renault (1997).*

4 The correlated case

This section is devoted to extend the above results to the correlated case. We will need the following hypotheses:

(H1') σ^2 belongs to $\mathbb{L}_W^{2,4}$, and there exists a positive and adapted process $Y = \{Y_r, r \in [0,T]\}$ such that, for all $r > t$ $E_t(Y_r) \le C$, for some positive constant C, and such that, for all $t < \theta < r < u < T$

$$\left|E_\theta\left((D_\theta^W \sigma_r)^2\right)\right| + \left|E_\theta((D_\theta^W D_r^W \sigma_u)^2)\right| \le Y_\theta.$$

(H2') For every $t \in [0,T]$, there exists a $\mathcal{F}_t^W-$measurable random variable $D_t^+\sigma_t^2$ and a positive constant $\varepsilon > 0$ such that, if $T - t < \varepsilon$

$$E_t\left(\sup_{t\le r\le u\le T}\left|E_r\left(D_r^W\sigma_u^2 - D_t^+\sigma_t^2\right)\right|^4\right) \le C(T-t)^\delta,$$

for some positive constants C and δ. Moreover, there exists a $\mathcal{F}_t^W-$measurable random variable $\left(D_t^+\right)^2\sigma_t^2$ such that

$$\lim_{T\to t}\frac{\int_t^T E_t\left(\sup_{s\le r\le u\le T}\left|E_r\left(D_s^W D_r^W \sigma_u^2 - \left(D_t^+\right)^2\sigma_t^2\right)\right|\right)ds}{T-t} = 0.$$

(H3') Condition (H3) holds and there exists a positive constant a such that $\sigma_1(t) > a$, for all $t > 0$.

(H4) For every fixed $t > 0$, $\sup_{s,r,\theta\in[t,T]} E_t\left(\left(\sigma_s\sigma_r - \sigma_\theta^2\right)^2\right) \to 0$ as $T \to t$.

Henceforth we use the notation

$$G(t,x,k,\sigma) := \left(\partial_{xx}^2 - \partial_x\right) BS(t,x,k,\sigma)$$

and

$$\Gamma_s := \sigma_s \int_s^T \left(D_s^W \sigma_r^2 \right) dr.$$

In order to prove our results on the implied volatility smile, we will make use of the following results on correlated stochastic volatility models proved in Alòs, León and Vives (2007). Although Lemma 10 is well-known, we state it for the convenience of the reader.

Lemma 9 *(Lemma 4.1 in Alòs, León and Vives (2007)) Let* $0 \le t \le s < T$, $\rho \in (-1,1)$ *and* $\mathcal{G}_t := \mathcal{F}_t \vee \mathcal{F}_T^W$. *Then for every* $n \ge 0$, *there exists* $C = C(n,\rho)$ *such that*

$$\left| E\left(\partial_x^n G\left(s, X_s, k_t^*, v_s\right) \right| \mathcal{G}_t) \right| \le C e^{X_t} \left(\int_t^T \sigma_s^2 ds \right)^{-\frac{1}{2}(n+1)}.$$

Lemma 10 *(Lemma 6.1 in Alòs, León and Vives (2007)) Assume the model (1) is satisfied. Then,* $I(t,k_t^*)\sqrt{T-t}$ *tends to zero a.s., as* $T \to t$.

Theorem 11 *(Theorem 4.2 in Alòs, León and Vives (2007)) Consider the model (1) and assume that* $\sigma \in \mathbb{L}_W^{1,2}$. *Then we have that, for* $0 \le t \le T$,

$$V_t = E_t\left(BS\left(t, X_t, k, v_t\right)\right) + \frac{\rho}{2} E_t \left(\int_t^T e^{-\tilde{r}(s-t)} \partial_x G(s, X_s, k, v_s) \Gamma_s ds \right). \quad (7)$$

Theorem 12 *(Adaptation of Theorem 6.3 in Alòs, León and Vives (2007)) Assume the model (1) is satisfied and that Hypotheses (H1'), (H2), (H3) and (H4) hold. Then*

$$\lim_{T\to t} \frac{\partial I}{\partial k}(t,k_t^*) = \lim_{T\to t} \frac{\rho}{2\sigma_t^3} \frac{1}{(T-t)^2} E_t \left(\int_t^T \Gamma_s ds \right) = \rho \frac{D_t^+ \sigma_t}{2\sigma_t}, \quad (8)$$

where $D_t^+ \sigma_t := \frac{D_t^+ \sigma_t^2}{2\sigma_t}$.

Proof. Let us denote $V_t(X_t,k)$ the option price with log-stock price X_t and log-strike k. Notice that, as

$$BS(t,x,k,\sigma) = e^k BS(t, x-k, 0, \sigma)$$

we get

$$V_t(X_t,k) = BS(t,X_t,k,I(t,X_t,k)) = e^k BS(t, X_t - k, 0, I(t,X_t,k)). \quad (9)$$

On the other hand,

$$\begin{aligned} V_t(X_t,k) &= E_t(e^{X_T} - e^k)_+ \\ &= E_t(e^{X_T - X_t} e^{X_t} - e^k)_+ \\ &= e^k E_t(e^{X_T - X_t} e^{X_t - k} - 1)_+ \\ &= e^k V_t(X_t - k, 0), \end{aligned} \quad (10)$$

and then, (9) and (10) imply that

$$e^k BS(t, X_t - k, 0, I(t, X_t, k)) = e^k V_t(X_t - k, 0)$$

That is

$$BS(t, X_t - k, 0, I(t, X_t, k)) = BS(t, X_t - k, 0, I(t, X_t - k, 0)),$$

which implies that

$$I(t, X_t, k) = I(t, X_t - k, 0).$$

In particular, this proves that

$$\partial_k I(t, X_t, k) = -\partial_x I(t, X_t - k, 0).$$

Now the result follows directly from Theorem 6.3 in Alòs, León and Vives (2007)). ■

Now we are in a position to prove the main result of this Section.

Theorem 13 *Assume that the model (1), and Hypotheses (H1'), (H2), (H3') and (H4) are satisfied. Then,*

$$\lim_{T \to t} \frac{\partial^2 I}{\partial k^2}(t, k_t^*) = \left(\frac{1}{12} - \frac{7}{24}\rho^2 \right) \frac{(D_t^+ \sigma_t^2)^2}{\sigma_t^5} + \frac{\rho^2}{6\sigma_t^3} (D_t^+)^2 \sigma_t^2.$$

Proof. From the definition of the implied volatility I, we have

$$\begin{aligned}
& \frac{\partial^2}{\partial k^2} V_t \\
= \; & \frac{\partial^2 BS}{\partial k^2} (t, X_t, k_t^*, I(t, k_t^*)) \\
& + 2 \frac{\partial^2 BS}{\partial k \partial \sigma} (t, X_t, k_t^*, I(t, k_t^*)) \frac{\partial I(t, k_t^*)}{\partial k} \\
& + \frac{\partial^2 BS}{\partial \sigma^2} (t, X_t, k_t^*, I(t, k_t^*)) \left(\frac{\partial I(t, k_t^*)}{\partial k} \right)^2 \\
& + \frac{\partial BS}{\partial \sigma} (t, X_t, k_t^*, I(t, k_t^*)) \frac{\partial^2 I(t, k_t^*)}{\partial k^2}. \qquad (11)
\end{aligned}$$

Now, Equality (7) gives us that

$$\begin{aligned}
& \frac{\partial^2}{\partial k^2} V_t \\
= \; & \frac{\partial^2}{\partial k^2} E_t (BS(t, X_t, k_t^*, v_t)) + \frac{\rho}{2} E_t \left(\int_t^T e^{-\tilde{r}(s-t)} \frac{\partial^3 G}{\partial k^2 \partial x} (s, X_s, k_t^*, v_s) \Gamma_s ds \right) \\
= \; & \frac{\partial^2}{\partial k^2} (V_t(0)) + \frac{\rho}{2} E_t \left(\int_t^T e^{-\tilde{r}(s-t)} \frac{\partial^3 G}{\partial k^2 \partial x} (s, X_s, k_t^*, v_s) \Gamma_s ds \right) \\
= \; & \frac{\partial^2}{\partial k^2} \left(BS(t, X_t, k_t^*, I^0(t, k_t^*)) \right) + \frac{\rho}{2} E_t \left(\int_t^T e^{-\tilde{r}(s-t)} \frac{\partial^3 G}{\partial k^2 \partial x} (s, X_s, k_t^*, v_s) \Gamma_s ds \right),
\end{aligned}$$

where $V_t(0)$ denotes the option price in the case $\rho = 0$ and $I^0(t, k_t^*)$ the corresponding implied volatiltiy.

On the other hand, we can write

$$\begin{aligned}
& \frac{\partial^2}{\partial k^2}\left(BS(t, X_t, k_t^*, I^0(t, k_t^*))\right) \\
= & \frac{\partial^2 BS}{\partial k^2}(t, X_t, k_t^*, I^0(t, k_t^*)) \\
& +2\frac{\partial^2 BS}{\partial k \partial \sigma}(t, X_t, k_t^*, I^0(t, k_t^*))\frac{\partial I^0}{\partial k}(t, k_t^*) \\
& +\frac{\partial^2 BS}{\partial \sigma^2}(t, X_t, k_t^*, I^0(t, k_t^*))\left(\frac{\partial I^0}{\partial k}(t, k_t^*)\right)^2 \\
& +\frac{\partial BS}{\partial \sigma}(t, X_t, k_t^*, I^0(t, k_t^*))\frac{\partial^2 I^0}{\partial k^2}(t, k_t^*).
\end{aligned}$$

Then, from Proposition 1, $\frac{\partial I^0}{\partial k}(t, k_t^*) = 0$ and we get

$$\begin{aligned}
& \frac{\partial^2}{\partial k^2}\left(BS(t, X_t, k_t^*, I^0(t, k_t^*))\right) \\
= & \frac{\partial^2 BS}{\partial k^2}(t, X_t, k_t^*, I^0(t, k_t^*)) \\
& +\frac{\partial BS}{\partial \sigma}(t, X_t, k_t^*, I^0(t, k_t^*))\frac{\partial^2 I^0}{\partial k^2}(t, k_t^*),
\end{aligned}$$

which gives us that

$$\begin{aligned}
& \frac{\partial^2}{\partial k^2}V_t \\
= & \frac{\partial^2 BS}{\partial k^2}(t, X_t, k_t^*, I^0(t, k_t^*)) \\
& +\frac{\partial BS}{\partial \sigma}(t, X_t, k_t^*, I^0(t, k_t^*))\frac{\partial^2 I^0}{\partial k^2}(t, k_t^*) \\
& +\frac{\rho}{2}E_t\left(\int_t^T e^{-\tilde{r}(s-t)}\frac{\partial^3 G}{\partial k^2 \partial x}(s, X_s, k_t^*, v_s)\Gamma_s ds\right).
\end{aligned}$$

This, jointly with (11), allows us to write

$$
\begin{aligned}
&\frac{\partial^2 I}{\partial k^2}(t,k_t^*) \\
=\ & \frac{\partial^2 I^0}{\partial k^2}(t,k_t^*)\frac{\frac{\partial BS}{\partial\sigma}\left(t,X_t,k_t^*;I^0(t,k_t^*)\right)}{\frac{\partial BS}{\partial\sigma}(t,X_t,k_t^*,I(t,k_t^*))} \\
&-\frac{\frac{\partial^2 BS}{\partial\sigma^2}(t,X_t,k_t^*;I(t,k_t^*))}{\frac{\partial BS}{\partial\sigma}(t,X_t,k_t^*,I(t,k_t^*))}\left(\frac{\partial I(t,k_t^*)}{\partial k}\right)^2 \\
&-2\frac{\frac{\partial^2 BS}{\partial k\partial\sigma}(t,X_t,k_t^*;I(t,k_t^*))}{\frac{\partial BS}{\partial\sigma}(t,X_t,k_t^*,I(t,k_t^*))}\frac{\partial I(t,k_t^*)}{\partial k} \\
&+\frac{\rho}{2}\frac{E_t\left(\int_t^T e^{-\tilde{r}(s-t)}\frac{\partial^3 G}{\partial k^2\partial x}(s,X_s,k_t^*,v_s)\Gamma_s ds\right)}{\frac{\partial BS}{\partial\sigma}(t,X_t,k_t^*,I(t,k_t^*))} \\
&+\frac{\frac{\partial^2 BS}{\partial k^2}(t,X_t,k_t^*,I^0(t,k_t^*))-\frac{\partial^2 BS}{\partial k^2}(t,X_t,k_t^*;I(t,k_t^*))}{\frac{\partial BS}{\partial\sigma}(t,X_t,k_t^*,I(t,k_t^*))} \\
:\ =\ & \frac{\partial^2 I^0}{\partial k^2}(t,k_t^*)\frac{\frac{\partial BS}{\partial\sigma}\left(t,X_t,k_t^*;I^0(t,k_t^*)\right)}{\frac{\partial BS}{\partial\sigma}(t,X_t,k_t^*,I(t,k_t^*))}+T_1+T_2+T_3+T_4. \qquad (12)
\end{aligned}
$$

Note that, by Lemma 10, it is easy to check that $\frac{\frac{\partial BS}{\partial\sigma}\left(t,X_t,k_t^*;I^0(t,k_t^*)\right)}{\frac{\partial BS}{\partial\sigma}(t,X_t,k_t^*,I(t,k_t^*))}\to 1$ as $T\to t$. Now the proof is decomposed into several steps.

Step 1. Let us see that $T_1\to 0$ as $T\to t$. We can write

$$
\begin{aligned}
T_1 &= -\frac{\frac{\partial^2 BS}{\partial\sigma^2}(t,X_t,k_t^*;I(t,k_t^*))}{\frac{\partial BS}{\partial\sigma}(t,X_t,k_t^*,I(t,k_t^*))}\left(\frac{\partial I(t,k_t^*)}{\partial k}\right)^2 \\
&= -\frac{(T-t)I(t,k_t^*)}{4}\left(\frac{\partial I(t,k_t^*)}{\partial k}\right)^2.
\end{aligned}
$$

From Lemma 10 we know that $(T-t)I(t,k_t^*)\to 0$ as $T\to t$. Therefore Theorem 11 implies that $T_1\to 0$ as $T\to t$.

Step 2. We claim $T_2=-\frac{\partial I(t,k_t^*)}{\partial k}$. Indeed, this follows directly from the fact that

$$
\frac{\frac{\partial^2 BS}{\partial k\partial\sigma}(t,X_t,k_t^*;I(t,k_t^*))}{\frac{\partial BS}{\partial\sigma}(t,X_t,k_t^*,I(t,k_t^*))}=\frac{1}{2}.
$$

Step 3. To deal with T_3, we apply the anticipating Itô's formula (see for example Nualart (2006)) to the process

$$
e^{-\tilde{r}(s-t)}\frac{\partial^3 G}{\partial k^2\partial x}(s,X_s,k_t^*,v_s)\int_s^T\Gamma_r dr
$$

and taking conditional expectations it follows that

$$\begin{aligned}
&\frac{\rho}{2}E_t\left(\int_t^T e^{-\tilde{r}(s-t)}\frac{\partial^3 G}{\partial k^2\partial x}(s,X_s,k_t^*,v_s)\Gamma_s ds\right)\\
=\ &\frac{\rho}{2}E_t\left(\frac{\partial^3 G}{\partial k^2\partial x}(t,X_t,k_t^*,v_t)\int_t^T\Gamma_s ds\right)\\
&+\frac{\rho^2}{4}E_t\left(\int_t^T e^{-\tilde{r}(s-t)}\left(\frac{\partial^3}{\partial x^3}-\frac{\partial^2}{\partial x^2}\right)\frac{\partial^3 G}{\partial k^2\partial x}(s,X_s,k_t^*,v_s)(\int_s^T\Gamma_r dr)\Gamma_s ds\right)\\
&+\frac{\rho^2}{2}E_t\left(\int_t^T e^{-\tilde{r}(s-t)}\frac{\partial^4 G}{\partial k^2\partial x^2}(s,X_s,k_t^*,v_s)(\int_s^T D_s^W\Gamma_r dr)\sigma_s ds\right).
\end{aligned}$$

Then, applying the anticipating Itô's formula again,we get

$$\begin{aligned}
T_3 &= \frac{\frac{\rho}{2}E_t\left(\int_t^T e^{-\tilde{r}(s-t)}\frac{\partial^3 G}{\partial k^2\partial x}(s,X_s,k_t^*,v_s)\Gamma_s ds\right)}{\frac{\partial BS}{\partial\sigma}(t,X_t,k_t^*,I(t,k_t^*))}\\
&= \frac{1}{\frac{\partial BS}{\partial\sigma}(t,X_t,k_t^*,I(t,k_t^*))}\left[\frac{\rho}{2}E_t\left(\frac{\partial^3 G}{\partial k^2\partial x}(t,X_t,k_t^*,v_t)\int_t^T\Gamma_s ds\right)\right.\\
&\quad+\frac{\rho^2}{4}E_t\left(\left(\frac{\partial^3}{\partial x^3}-\frac{\partial^2}{\partial x^2}\right)\frac{\partial^3 G}{\partial k^2\partial x}(t,X_t,k_t^*,v_t)\int_t^T(\int_s^T\Gamma_r dr)\Gamma_s ds\right)\\
&\quad+\frac{\rho^2}{2}E_t\left(\frac{\partial^4 G}{\partial k^2\partial x^2}(t,X_t,k_t^*,v_t)\int_t^T(\int_s^T D_s^W\Gamma_r dr)\sigma_s ds\right)\\
&\quad+\frac{\rho^3}{8}E_t\left(\int_t^T e^{-\tilde{r}(s-t)}\left(\frac{\partial^3}{\partial x^3}-\frac{\partial^2}{\partial x^2}\right)^2\frac{\partial^3 G}{\partial k^2\partial x}(s,X_s,k_t^*,v_s)\left(\int_s^T(\int_r^T\Gamma_\theta d\theta)\Gamma_r dr\right)\Gamma_s ds\right)\\
&\quad+\frac{\rho^3}{4}E_t\left(\int_t^T e^{-\tilde{r}(s-t)}\left(\frac{\partial^3}{\partial x^3}-\frac{\partial^2}{\partial x^2}\right)\frac{\partial^4 G}{\partial k^2\partial x^2}(s,X_s,k_t^*,v_s)\left(\int_s^T D_s^W(\int_r^T\Gamma_\theta d\theta)\Gamma_r dr\right)\sigma_s ds\right)\\
&\quad+\frac{\rho^3}{4}E_t\left(\int_t^T e^{-\tilde{r}(s-t)}\left(\frac{\partial^3}{\partial x^3}-\frac{\partial^2}{\partial x^2}\right)\frac{\partial^4 G}{\partial k^2\partial x^2}(s,X_s,k_t^*,v_s)\left(\int_s^T(\int_r^T D_r^W\Gamma_\theta d\theta)\sigma_r dr\right)\Gamma_s ds\right)\\
&\quad\left.+\ \frac{\rho^3}{2}E_t\left(\int_t^T e^{-\tilde{r}(s-t)}\frac{\partial^5 G}{\partial k^2\partial x^3}(s,X_s,k_t^*,v_s)\left(\int_s^T D_s^W(\int_r^T D_r^W\Gamma_\theta d\theta)\sigma_r dr\right)\sigma_s ds\right)\right]\\
&= T_3^1+T_3^2+T_3^3+T_3^4+T_3^5+T_3^6+T_3^7.
\end{aligned}$$

Notice that, from Lemma 8 and Hypotheses (H1') and (H3'), $T_3^4+T_3^5+T_3^6+T_3^7$

tends to zero as $T \to t$. On the other hand, (8) implies that

$$\begin{aligned}
\lim_{T\to t} T_3^1 &= \frac{\rho}{2}\lim_{T\to t}\frac{E_t\left(\frac{\partial^3 G}{\partial k^2\partial x}(t,X_t,k_t^*,v_t)\int_t^T \Gamma_s ds\right)}{\frac{\partial BS}{\partial\sigma}(t,X_t,k_t^*,I(t,k_t^*))} \\
&= \lim_{T\to t}\frac{\rho}{4}\frac{1}{v_t^3(T-t)^2}E_t\left(\int_t^T \Gamma_s ds\right) \\
&= \frac{1}{2}\lim_{T\to t}\frac{\partial I(t,k_t^*)}{\partial k}. \qquad (13)
\end{aligned}$$

In a similar way we can see that

$$\begin{aligned}
\lim_{T\to t} T_3^2 &= \frac{\frac{\rho^2}{4}E_t\left(\left(\frac{\partial^3}{\partial x^3}-\frac{\partial^2}{\partial x^2}\right)\frac{\partial^3 G}{\partial k^2\partial x}(t,X_t,k_t^*,v_t)\int_t^T(\int_s^T \Gamma_r dr)\Gamma_s ds\right)}{\frac{\partial BS}{\partial\sigma}(t,X_t,k_t^*,I(t,k_t^*))} \\
&= \lim_{T\to t}\frac{\rho^2}{4}E_t\left(-\frac{15}{v_t^7(T-t)^4}\int_t^T(\int_s^T \Gamma_r dr)\Gamma_s ds\right) \\
&= \lim_{T\to t}\frac{\rho^2}{8}E_t\left(-\frac{15}{v_t^7(T-t)^4}\left(\int_t^T \Gamma_s ds\right)^2\right) \\
&= -\frac{15\rho^2}{32\sigma_t^5}\left(D_t^+\sigma_t^2\right)^2. \qquad (14)
\end{aligned}$$

Finally, we can write

$$\lim_{T\to t} T_3^3 = \frac{\rho^2}{2}\lim_{T\to t}E_t\left(\frac{3}{\sigma_t^5(T-t)^3}\int_t^T(\int_s^T D_s^W\Gamma_r dr)\sigma_s ds\right).$$

Since

$$\begin{aligned}
D_s^W\Gamma_r &= D_s^W\sigma_r\left(\int_r^T D_r^W\sigma_\theta^2 d\theta\right)+\sigma_r\left(\int_r^T D_s^W D_r^W\sigma_\theta^2 d\theta\right) \\
&= \frac{D_s^W\sigma_r^2}{2\sigma_r}\left(\int_r^T D_r^W\sigma_\theta^2 d\theta\right)+\sigma_r\left(\int_r^T D_s^W D_r^W\sigma_\theta^2 d\theta\right),
\end{aligned}$$

and taking into account (H2'), it follows that

$$\begin{aligned}
\lim_{T\to t} T_3^3 &= \frac{\rho^2}{4}\lim_{T\to t}E_t\left(\frac{3}{\sigma_t^5(T-t)^3}\int_t^T\left(\int_s^T D_s^W\sigma_r^2\left(\int_r^T D_r^W\sigma_\theta^2 d\theta\right)dr\right)ds\right) \\
&\quad+\frac{\rho^2}{4}\lim_{T\to t}E_t\left(\frac{3}{\sigma_t^3(T-t)^3}\int_t^T\left(\int_s^T\left(\int_r^T D_s^W D_r^W\sigma_\theta^2 d\theta\right)dr\right)ds\right) \\
&= \frac{\rho^2}{8}\left(\frac{1}{\sigma_t^5}\left(D_t^+\sigma_t^2\right)^2\right)+\frac{\rho^2}{4}\left(\frac{1}{\sigma_t^3}(D_t^+)^2\sigma_t^2\right).
\end{aligned}$$

This, jointly with (13) and (14) implies that

$$\lim_{T\to t} T_3 = \rho^2 \left(-\frac{11}{32\sigma_t^5} \left(D_t^+ \sigma_t^2 \right)^2 + \frac{1}{4\sigma_t^3} (D_t^+)^2 \sigma_t^2 \right) + \frac{1}{2} \lim_{T\to t} \frac{\partial I(t,k_t^*)}{\partial k}.$$

- *Step 4*. Let us study the term T_4. We can write, from Theorem 10,

$$\begin{aligned}
& \frac{\partial^2 BS}{\partial k^2}(t, X_t, k_t^*, I^0(t,k_t^*)) - \frac{\partial^2 BS}{\partial k^2}(t, X_t, k_t^*, I(t,k_t^*)) \\
= \; & \frac{\partial^2 BS}{\partial k^2}(t, X_t, k_t^*, BS^{-1}\left(E_t\left(BS\left(t, X_t, k_t^*, v_t\right)\right)\right)) \\
& - \frac{\partial^2 BS}{\partial k^2}\left(t, X_t, k_t^*, BS^{-1}\left(E_t\left(BS\left(t, X_t, k_t^*, v_t\right)\right)\right.\right. \\
& \left.\left.\left. + \frac{\rho}{2} E_t \left(\int_t^T e^{-\tilde{r}(s-t)} \frac{\partial G}{\partial x}(s, X_s, k_t^*, v_s) \Gamma_s ds \right)\right)\right) \\
= \; & \Psi\left(E_t\left(BS\left(t, X_t, k_t^*, v_t\right)\right)\right) \\
& - \Psi\left(E_t \left(BS\left(t, X_t, k_t^*, v_t\right) + \frac{\rho}{2} E_t \left(\int_t^T e^{-\tilde{r}(s-t)} \frac{\partial G}{\partial x}(s, X_s, k_t^*, v_s) \Gamma_s ds \right)\right)\right) \\
= \; & -\Psi'(\mu\left(T,t\right)) \left(\frac{\rho}{2} E_t \left(\int_t^T e^{-\tilde{r}(s-t)} \frac{\partial G}{\partial x}(s, X_s, k_t^*, v_s) \Gamma_s ds \right)\right),
\end{aligned}$$

  where Ψ is defined in Theorem 5 and $\mu\left(T,t\right)$ is a positive value between $E_t\left(BS\left(t, X_t, k_t^*, v_t\right)\right)$ and

$$E_t \left(BS\left(t, X_t, k_t^*, v_t\right) + \frac{\rho}{2} E_t \left(\int_t^T e^{-\tilde{r}(s-t)} \frac{\partial G}{\partial x}(s, X_s, k_t^*, v_s) \Gamma_s ds \right)\right).$$

As in the proof of *Step 3,* it follows that

$$
\begin{aligned}
T_4 &= -\frac{\rho}{2}\frac{\Psi'(\mu(T,t))E_t\left(\int_t^T e^{-\tilde{r}(s-t)}\frac{\partial G}{\partial x}(s,X_s,k_t^*,v_s)\Gamma_s ds\right)}{\frac{\partial BS}{\partial\sigma}(t,X_t,k_t^*,I(t,k_t^*))} \\
&= -\frac{\rho}{2}\frac{\Psi'(\mu(T,t))}{\frac{\partial BS}{\partial\sigma}(t,X_t,k_t^*,I(t,k_t^*))}\left[E_t\left(\frac{\partial G}{\partial x}(t,X_t,k_t^*,v_t)\int_t^T\Gamma_s ds\right)\right. \\
&\quad +\frac{\rho}{2}E_t\left(\left(\frac{\partial^3}{\partial x^3}-\frac{\partial^2}{\partial x^2}\right)\frac{\partial G}{\partial x}(t,X_t,k_t^*,v_t)\int_t^T(\int_s^T\Gamma_r dr)\Gamma_s ds\right) \\
&\quad +\rho E_t\left(\frac{\partial^2 G}{\partial x^2}(t,X_t,k_t^*,v_t)\int_t^T(\int_s^T D_s^W\Gamma_r dr)\sigma_s ds\right) \\
&\quad +\frac{\rho^2}{4}E_t\left(\int_t^T e^{-\tilde{r}(s-t)}\left(\frac{\partial^3}{\partial x^3}-\frac{\partial^2}{\partial x^2}\right)^2\frac{\partial G}{\partial x}(s,X_s,k_t^*,v_s)\left(\int_s^T(\int_r^T\Gamma_\theta d\theta)\Gamma_r dr\right)\Gamma_s ds\right) \\
&\quad +\frac{\rho^2}{2}E_t\left(\int_t^T e^{-\tilde{r}(s-t)}\left(\frac{\partial^3}{\partial x^3}-\frac{\partial^2}{\partial x^2}\right)\frac{\partial^2 G}{\partial x^2}(s,X_s,k_t^*,v_s)\left(\int_s^T D_s^W(\int_r^T\Gamma_\theta d\theta\Gamma_r)dr\right)\sigma_s ds\right) \\
&\quad +\frac{\rho^2}{2}E_t\left(\int_t^T e^{-\tilde{r}(s-t)}\left(\frac{\partial^3}{\partial x^3}-\frac{\partial^2}{\partial x^2}\right)\frac{\partial^2 G}{\partial x^2}(s,X_s,k_t^*,v_s)\left(\int_s^T(\int_r^T D_r^W\Gamma_\theta d\theta)\sigma_r dr\right)\Gamma_s ds\right) \\
&\quad \left.+\rho^2 E_t\left(\int_t^T e^{-\tilde{r}(s-t)}\frac{\partial^3 G}{\partial x^3}(s,X_s,k_t^*,v_s)\left(\int_s^T D_s^W(\int_r^T D_r^W\Gamma_\theta d\theta\sigma_r)dr\right)\sigma_s ds\right)\right] \\
&= T_4^1+T_4^2+T_4^3+T_4^4+T_4^5+T_4^6+T_4^7.
\end{aligned}
$$

It is easy to see that, from Lemma 9, $T_4^4+T_4^5+T_4^6+T_4^7\to 0$ as $T\to t$.
Now,

$$
\begin{aligned}
\lim_{T\to t}T_4^1 &= -\frac{\rho}{2}\lim_{T\to t}\frac{\Psi'(\mu(T,t))}{\frac{\partial BS}{\partial\sigma}(t,X_t,k_t^*,I(t,k_t^*))}E_t\left(\frac{\partial G}{\partial x}(t,X_t,k_t^*,v_t)\int_t^T\Gamma_s ds\right) \\
&= \frac{\rho}{2}\lim_{T\to t}\frac{e^{-X_t}}{BS^{-1}(\mu(T,t))^2(T-t)^{\frac{3}{2}}}E_t\left(\frac{\partial G}{\partial x}(t,X_t,k_t^*,v_t)\int_t^T\Gamma_s ds\right) \\
&= \frac{\rho}{4}\lim_{T\to t}\frac{1}{BS^{-1}(\mu(T,t))^2 v_t(T-t)^2}E_t\left(\int_t^T\Gamma_s ds\right).
\end{aligned}
$$

Notice that $BS^{-1}(\mu(T,t))$ is an intermediate value between $I^0(t,k_t^*)$ and $I(t,k_t^*)$. Then, Theorem 3.1 in Durrleman (2007) gives us that $BS^{-1}(\mu(T,t))\to\sigma_t$ as $T\to t$, and this implies that

$$\lim_{T\to t}T_4^1=\frac{1}{2}\lim_{T\to t}\frac{\partial I_t}{\partial k}(t,k_t^*).$$

On the other hand,

$$\begin{aligned}
\lim_{T\to t} T_4^2 &= -\frac{\rho^2}{4}\lim_{T\to t}\frac{\Psi'(\mu(T,t))}{\frac{\partial BS}{\partial\sigma}(t,X_t,k_t^*,I(t,k_t^*))} \\
&\quad \times E_t\left(\left(\frac{\partial^3}{\partial x^3}-\frac{\partial^2}{\partial x^2}\right)\frac{\partial G}{\partial x}(t,X_t,k_t^*,v_t)\int_t^T(\int_s^T \Gamma_r dr)\Gamma_s ds\right) \\
&= \frac{\rho^2}{4}\lim_{T\to t}\frac{e^{-X_t}}{BS^{-1}(\mu(T,t))^2(T-t)^{\frac{3}{2}}} \\
&\quad \times E_t\left(\left(\frac{\partial^3}{\partial x^3}-\frac{\partial^2}{\partial x^2}\right)\frac{\partial G}{\partial x}(t,X_t,k_t^*,v_t)\int_t^T(\int_s^T \Gamma_r dr)\Gamma_s ds\right) \\
&= \frac{\rho^2}{4\sigma_t^7}\lim_{T\to t}\frac{3}{(T-t)^4}\lim_{T\to t}\left(\int_t^T(\int_s^T \Gamma_r dr)\Gamma_s ds\right) \\
&= \frac{\rho^2}{8\sigma_t^7}\lim_{T\to t}\frac{3}{(T-t)^4}\lim_{T\to t}\left(\int_t^T \Gamma_s ds\right)^2 \\
&= \frac{3\rho^2}{32\sigma_t^5}\left(D_t^+\sigma_t^2\right)^2.
\end{aligned}$$

Finally,

$$\begin{aligned}
T_4^3 &= -\frac{\rho^2}{2}\lim_{T\to t}\frac{1}{BS^{-1}(\mu(T,t))^2(T-t)} \\
&\quad \times E_t\left(\frac{1}{v_t^3(T-t)^2}\int_t^T(\int_s^T D_s^W\Gamma_r dr)\sigma_s ds\right) \\
&= -\frac{\rho^2}{2\sigma_t^5}\lim_{T\to t}\frac{1}{(T-t)^3}E_t\left(\int_t^T(\int_s^T D_s^W\Gamma_r dr)\sigma_s ds\right) \\
&= -\frac{1}{3}\lim_{T\to t} T_3^3.
\end{aligned}$$

Thus the proof is complete.

■

Remark 14 *The hypotheses of the above theorem can be substituted by other adequate integrability conditions. That is, as in Example 7, we can change Hypotheses (H1) and (H2) by suitable conditions to deal with fractional noises.*

Remark 15 *The above results prove that both the short-time at-the-money skew and smile depend directly on the short-time behaviour of the Malliavin derivatives of the volatility process.*

Remark 16 *Notice that the short-time limit implied volatility is a locally convex function around k_t^* if and only if*

$$\left(\frac{1}{12}-\frac{7}{24}\rho^2\right)\frac{(D_t^+\sigma_t^2)^2}{\sigma_t^5}+\frac{\rho^2}{6\sigma_t^3}(D_t^+)^2\sigma_t^2\geq 0.$$

If $\frac{(D_t^+)^2\sigma_t^2}{6\sigma_t^3}-\frac{7}{24}\frac{(D_t^+\sigma_t^2)^2}{\sigma_t^5}\geq 0$ *this is satisfied independently of the correlation parameter* ρ*. If* $\frac{(D_t^+)^2\sigma_t^2}{6\sigma_t^3}-\frac{7}{24}\frac{(D_t^+\sigma_t^2)^2}{\sigma_t^5}<0$ *this condition holds if*

$$\rho^2\leq\frac{\frac{1}{12\sigma_t^5}(D_t^+\sigma_t^2)^2}{\frac{7}{24\sigma_t^5}(D_t^+\sigma_t^2)^2-\frac{1}{6\sigma_t^3}(D_t^+)^2\sigma_t^2\sigma_t^2}.$$

5 Examples

5.1 Diffusion stochastic volatilities

In this subsection we assume that $\sigma=f(Y)$, for some positive function f, and where Y is the solution of a stochastic differential equation:

$$dY_r=a\left(r,Y_r\right)dr+b\left(r,Y_r\right)dW_r,\ r\in[0,T] \tag{15}$$

for some real functions $a,b\in\mathcal{C}_b^2$. Then we can prove the following result

Proposition 17 *Let us consider the model (1) with $\sigma=f(Y)$, where $f\in\mathcal{C}_b^2$ is such that $f(x)>c$, for some positive constant c, and Y is the solution of (15). Then*

$$\begin{aligned}&\lim_{T\to t}\frac{\partial^2 I}{\partial k^2}(t,k_t^*)\\ =&\ \frac{1}{3\sigma_t^3}\left(f'(Y_t)b(t,Y_t)\right)^2\\ +&\ \frac{\rho^2}{3\sigma_t^3}\left(b^2(t,Y_t)\left(-\frac{5}{2}\left(f'(Y_t)\right)^2+\sigma_t f''(Y_t)\right)+\sigma_t f'(Y_t)\frac{\partial b}{\partial x}(t,Y_t)b(t,Y_t)\right)\end{aligned}$$

Proof. Note that (H3') and (H4) are true in this case. Then, classical arguments (see for example Nualart (2006)) give us that $Y\in\mathbb{L}_W^{2,2}$ and that, for all $s<r$

$$D_s^W Y_r=\int_s^r\frac{\partial a}{\partial x}(u,Y_u)D_s^W Y_u du+b(s,Y_s)+\int_s^r\frac{\partial b}{\partial x}(u,Y_u)D_s^W Y_u dW_u \tag{16}$$

and for all $\tau<s<r$

$$\begin{aligned}&D_\tau^W D_s^W Y_r\\ =&\int_s^r\frac{\partial^2 a}{\partial x^2}(u,Y_u)\left(D_\tau^W Y_u\right)\left(D_s^W Y_u\right)du+\int_s^r\frac{\partial a}{\partial x}(u,Y_u)D_\tau^W D_s^W Y_u du\\ &+\frac{\partial b}{\partial x}(s,Y_s)D_\tau^W Y_s+\int_s^r\frac{\partial^2 b}{\partial x^2}(u,Y_u)\left(D_\tau^W Y_u\right)\left(D_s^W Y_u\right)dW_u\\ &+\int_s^r\frac{\partial b}{\partial x}(u,Y_u)D_\tau^W D_s^W Y_u dW_u\end{aligned} \tag{17}$$

Taking now into account that

$$D_s^W \sigma_u^2 = 2\sigma_u f'(Y_u) D_s^W Y_u$$

and

$$\begin{aligned} & D_\tau^W D_s^W \sigma_u^2 \\ = \;& 2\left(f'(Y_u)\right)^2 \left(D_r^W Y_u\right)\left(D_s^W Y_u\right) + 2\sigma_u f''(Y_u)\left(D_r^W Y_u\right)\left(D_s^W Y_u\right) + 2\sigma_u f'(Y_u) D_\tau^W D_s^W Y_u \\ = \;& 2\left(\left(f'(Y_u)\right)^2 + \sigma_u f''(Y_u)\right)\left(D_r^W Y_u\right)\left(D_s^W Y_u\right) + 2\sigma_u f'(Y_u) D_\tau^W D_s^W Y_u, \end{aligned}$$

together with (16) and (17), it can be deduced that (H1') and (H2') are satisfied with

$$D_t^+ \sigma_t^2 = 2\sigma_t f'(Y_t) b(t, Y_t),$$

and

$$\left(D_t^+\right)^2 \sigma_t^2 := 2\left[\left(\left(f'(Y_t)\right)^2 + \sigma_t f''(Y_t)\right) b^2(t, Y_t) + \sigma_t f'(Y_t) \frac{\partial b}{\partial x}(t, Y_t) b(t, Y_t)\right].$$

Then, Theorem 13 yields

$$\begin{aligned} & \lim_{T\to t} \frac{\partial^2 I}{\partial k^2}(t, k_t^*) \\ = \;& \left(\frac{1}{12} - \frac{7}{24}\rho^2\right) \frac{(D_t^+ \sigma_t^2)^2}{\sigma_t^5} + \frac{\rho^2}{6\sigma_t^3}\left(\left(D_t^+\right)^2 \sigma_t^2\right) \\ = \;& \left(\frac{1}{12} - \frac{7}{24}\rho^2\right) \frac{\left(2\sigma_t f'(Y_t) b(t, Y_t)\right)^2}{\sigma_t^5} \\ + \;& \frac{\rho^2}{3\sigma_t^3}\left(\left(\left(f'(Y_t)\right)^2 + \sigma_t f''(Y_t)\right) b^2(t, Y_t) + \sigma_t f'(Y_t) \frac{\partial b}{\partial x}(t, Y_t) b(t, Y_t)\right) \\ = \;& \frac{1}{3\sigma_t^3}\left(f'(Y_t) b(t, Y_t)\right)^2 \\ + \;& \frac{\rho^2}{3\sigma_t^3}\left(b^2(t, Y_t)\left(-\frac{5}{2}\left(f'(Y_t)\right)^2 + \sigma_t f''(Y_t)\right) + \sigma_t f'(Y_t) \frac{\partial b}{\partial x}(t, Y_t) b(t, Y_t)\right), \end{aligned}$$

and now the proof is complete. ■

Remark 18 *Notice that the obtained expression for* $\lim_{T\to t} \frac{\partial^2 I}{\partial k^2}(t, k_t^*)$ *does not depend on the function* a.

Remark 19 *The short-time implied volatility is convex either when*

$$b^2(t, Y_t)\left(-\frac{5}{2}\left(f'(Y_t)\right)^2 + \sigma_t f''(Y_t)\right) + \sigma_t f'(Y_t) \frac{\partial b}{\partial x}(t, Y_t) b(t, Y_t) \geq 0,$$

or when

$$b^2(t, Y_t)\left(-\frac{5}{2}\left(f'(Y_t)\right)^2 + \sigma_t f''(Y_t)\right) + \sigma_t f'(Y_t) \frac{\partial b}{\partial x}(t, Y_t) b(t, Y_t) < 0$$

and

$$\rho^2 \leq \frac{\left(f'(Y_t)b(t,Y_t)\right)^2}{b^2(t,Y_t)\left(\frac{5}{2}\left(f'(Y_t)\right)^2 - \sigma_t f''(Y_t)\right) - \sigma_t f'(Y_t)\frac{\partial b}{\partial x}(t,Y_t)b(t,Y_t)}.$$

Example 20 *Let us suppose that* $\sigma_t = c + Y_t$*, with a positive constant* c *and where* $Y_t = \sqrt{Z_t}$*, for a CIR process* Z *of the form*

$$dZ_t = -\kappa(Z_t - \theta) + \nu\sqrt{Z_t}dW_s.$$

Here κ*,* ν *and* θ *are positive constants such that* $\frac{2\kappa\theta}{\nu^2} \geq 0$*. Then, we have (see for example Alòs and Ewald (2008)) that* Y *satisfies (15) with*

$$a(t,Y_t) = \left(\frac{\kappa\theta}{2} - \frac{\nu^2}{8}\right)\frac{1}{Y_t} - \frac{\kappa}{2}Y_t$$

and $b(t,Y_t) = \frac{\nu}{2}$*. Even when* a *is not bounded, a limit argument (see Alòs and Ewald (2008)) gives us that that* $D_t^+\sigma_t^2 = \nu\sigma_t$ *and* $(D_t^+)^2\sigma_t^2 = \frac{\nu^2}{2}$*. Then*

$$\lim_{t\to T}\frac{\partial^2 I}{\partial k^2}(t,k_t^*) = \frac{\nu^2}{12\sigma_t^3}\left(1 - \rho^2\frac{5}{2}\right),$$

that gives us that this short-time limit volatility is locally convex around k_t^* *when* $\rho^2 \leq 2/5$.

5.2 Fractional noises with $H > 1/2$

Here we analyze the model (18) below driven by a fractional noise.

Example 21 *Assume that the squared volatility* σ^2 *can be written as* $\sigma^2 = f(Y)$*, where* $f \in \mathcal{C}_b^2$ *and* Y *is a process of the form*

$$Y_r = m + (Y_t - m)\,e^{-\alpha(r-t)} + c\sqrt{2\alpha}\int_t^r \exp\left(-\alpha\left(r-s\right)\right)dW_s^H, \qquad (18)$$

for some positive constants m, c *and* α *and where* $W_s^H := \int_0^s (s-u)^{H-\frac{1}{2}}dW_u$*, for some* $H > 1/2$*. This class of models have been introduced in Comte and Renault (1998) to capture the long-time behaviour of the implied volatility. Notice that (see for example Alòs, Mazet and Nualart (2000))* $\int_t^r \exp\left(-\alpha\left(r-s\right)\right)dW_s^H$ *can be written as*

$$\left(H - \frac{1}{2}\right)\int_0^r \left(\int_s^r \mathbf{1}_{[t,r]}(u)\exp\left(-\alpha\left(r-u\right)\right)(u-s)^{H-\frac{3}{2}}du\right)dW_s,$$

from where it follows easily that hipotheses (H1'), (H2') (H3') and (H4) hold, with $\delta = 4H - 2$ *and* $D_t^+\sigma_t = (D_t^+)^2\sigma_t = 0$*. Then, independently on the correlation parameter,* $\lim_{t\to T}\frac{\partial^2 I}{\partial k^2}(t,k_t^*) = 0$*. This means that the introduction of fractional noises with* $H > 1/2$ *does not give a contribution to the short-time smile.*

6 Conclusions

By means of Malliavin calculus we have studied the second derivative of the implied volatility as a function of the strike price, both in the uncorrelated and in the correlated cases. Moreover, we explicitly compute its at-the-money short-time limit in terms of the first and second Malliavin derivatives of the volatility process and the correlation parameter. As a particular example, we study this limit for classical diffusion volatility models as well as for fractional volatilities. This methodology allows us to derive a condition for the at-the-money local convexity of the implied volatility, in terms of the correlation parameter and the Malliavin derivatives of the volatility process.

7 Proof of Lemma 4

This section is devoted to the proof of Lemma 4. We first observe that $BS^{-1}(t, X_t, k_t^*, \cdot)$ and $\exp(\cdot)$ are two convex function on R^+. Therefore, Jensen inequality implies

$$\begin{aligned}
&\exp\left(\frac{BS^{-1}(t, X_t, k_t^*, \Lambda_r)^2(T-t)}{8}\right) \\
&= \exp\left(\frac{BS^{-1}(t, X_t, k_t^*, E_r(BS(t, X_t, k_t^*, v_t)))^2(T-t)}{8}\right) \\
&\leq \exp\left(E_r\left(\frac{BS^{-1}(t, X_t, k_t^*, BS(t, X_t, k_t^*, v_t))^2(T-t)}{8}\right)\right) \\
&\leq E_r\left(\exp\left(\frac{v_t^2(T-t)}{8}\right)\right).
\end{aligned}$$

Similarly, using that $x \mapsto x^{-k}$, $k = 2, 3$, is a convex function on R^+ and the Taylor expansion for $BS^{-1}(t, X_t, k_t^*, \cdot)$, we have

$$\begin{aligned}
&\left(BS^{-1}(t, X_t, k_t^*, \Lambda_r)\right)^{-k} \\
&\leq \left(\frac{\sqrt{2\pi}}{\sqrt{T-t}} E_r\left(BS(t, X_t, k_t^*, v_t)\right) e^{-X_t}\right)^{-k} \\
&\leq E_r\left(\left(\frac{\sqrt{2\pi}}{\sqrt{T-t}} BS(t, X_t, k_t^*, v_t) e^{-X_t}\right)^{-k}\right),
\end{aligned}$$

which implies the result due to the mean value theorem.

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