# A closed-form option pricing approximation formula for a fractional Heston model

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## Abstract

We present a method to develop simple option pricing approximation formulas for a fractional Heston model, where the volatility process is defined by means of a fractional integration of a diffusion process. This model preserves the short-time behaviour of the Heston model, at the same time it explains the slow decrease of the smile amplitude when time to maturity increases. Then, by means of classical Itô's calculus we decompose option prices as the sum of the classical Black-Scholes formula with volatility parameter equal to the root-mean-square future average volatility plus a term due to correlation and a term due to the volatility of the volatility. This decomposition procedure does not need the volatility process to be Markovian and allows us to develop easy-to-apply approximation formulas for option prices and implied volatilities, as well as to study their accuracy. Numerical examples are given.

Keywords: Stochastic volatility, Heston model, Itô's calculus, fractional Brownian motion

JEL Classification: G13

Mathematics Subject Classification (2000): 91B28, 91B70

# 1 Introduction

One of the most popular extensions of the classical Black-Scholes model is to allow the volatility to be a stochastic process (see for example Hull and White (1987), Scott (1987), Stein and Stein (1991), Heston (1993), and Ball and Roma (1994), among others). Classical stochastic volatility diffusion models, where the volatility also follows a diffusion process, are able to reproduce some important features of the implied volatility as its variation with respect to the strike price, described graphically as a *smile* or *skew* (see Renault and Touzi (1996)). Nevertheless, they do not easily explain its dependence on time to maturity (*term structure*).

For example, stochastic volatility effects appear to be still significant for very long maturities(see Bollerslev and Mikkelsen (1996)). In practice, the decrease of the smile amplitude when time to maturity increases turns out to be much slower than it goes according to the standard stochastic volatility models.

Long-memory features for the volatility process have been introduced in Comte and Renault (1998) and in Coutin, Comte and Renault (2012) by the introduction of fractional noises in the description of the stochastic volatility process. This technique allows us to endow the volatility process with high persistence in the long run in order to capture the steepness of long term volatility smiles without overincreasing the short run persistence. An extension of these models have been studied recently in Corlay, Levobits and Lévy-Vehel (2014), where the volatility process is driven by a fractional Brownian motion (fBm) where the Hurst parameter is allowed to vary in time. In this paper the authors developed numerical techniques, based on funtional quantization-based cubature methods, to get accurate approximate option prices.

Even when the introduction of fractional noises is a powerful technique to explain the term-structure of the implied volatility, fractional stochastic volatilities are not Markovian process nor semimartingales. In consequence, their mathematical structure is more complex. This becomes an important handicap in the construction of simple and easy-to-apply techniques for option pricing and hedging, as closed-forms approximations of option prices, and, up to our knowledge, only numerical methods have been presented in this framework.

Our main goal in this paper is to present a simple method to construct option pricing approximation formulas for a fractional stochastic volatility model. The presented model, based, as in Coutin, Comte and Renault (2012), on the fractional integration techniques, allows us to preserve the short-time behaviour of the Heston model, at the same time it explains the slow flattening of the implied volatility when time to maturity increases. Our approximation formula is obtained by the same procedure presented in Alòs (2012), where by using classical Itô's calculus we decompose option prices as the sum of the classical Black-Scholes formula with volatility parameter equal to the root-mean-square future average volatility plus a term due to correlation and a term due to the volatility of the volatility. This methodology does not need the volatility process to be a diffusion nor to be Markovian so it appears as a natural tool to study models with fractional volatilities, that gives us simple and easy-to-apply op-

tion pricing approximation formulas. Moreover, the obtained approximations for option prices allow us to deduce an approximation for the implied volatility. This approximation gives us a tool to study the short-time and the long-time implied volatility behaviour and to see that the proposed model explains the slow decrease of the implied volatility slope.

The paper is organized as follows: Section 2 is devoted to the introduction of an extension of the Heston model by means of fractional calculus. In Section 3 we develop adequate option pricing approximation formulas and we study their accuracy . In Section 4 we study the behaviour of the corresponding implied volatility and we compare it with the classical stochastic volatility case.

## 2    Fractional volatility models

The aim of this section is to define a fractional volatility model that will allow us to reproduce efficiently the short-time and the long-time behaviour of the implied volatility. To this aim, we will introduce the main concepts of fractional integration.

### 2.1    Fractional derivatives and integrals

We recall the basic facts on fractional derivatives and integrals that we will need along the paper. We will use the notation of Samko et al. (1993), which gives a complete survey of fractional integrals and derivatives. Let $f \in L^1([0, T])$ and $\alpha > 0$. The left-sided fractional Riemann-Liouville integral of $f$ of order $\alpha$ on $[0, T]$ is given at almost all $t$ by

$$I_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - r)^{\alpha - 1} f(r) dr.$$

The inverse operation of the fractional integration is the fractional differentation. Consider $p \geq 1$ and let $I_{0+}^{\alpha}(L^p)$ be the image of $L^p([0, T])$ by the operator $I_{0+}^{\alpha}$. If $f \in I_{0+}^{\alpha}(L^p)$ and $\alpha \in (0, 1)$, the function $\phi$ such that $f = I_{0+}^{\alpha} \phi$ is unique in $L^p([0, T])$ and it agrees with the left-sided Riemann-Liouville derivative of $f$ or order $\alpha$ defined by

$$D_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(1 - \alpha)} \frac{d}{dt} \int_0^t \frac{f(r)}{(t - r)^{\alpha}} dr.$$

This derivative has the Weyl representation

$$D_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(1 - \alpha)} \left( \frac{f(t)}{t^{\alpha}} + \alpha \int_0^t \frac{f(t) - f(r)}{(t - r)^{\alpha + 1}} dr \right),$$

where the convergence of the integrals at the singularity $t = r$ holds almost surely if $p = 1$ and in the $L^p$−sense for $p > 1$.

## 2.2  The model and notations

We will consider a stochastic model for stock prices in a time interval $[0, T]$ under a risk neutral probability $P^*$ :

$$dS_t = rS_t dt + \sigma_t S_t \left( \rho dW_t^* + \sqrt{1 - \rho^2} dB_t^* \right), \ t \in [0, T], \tag{1}$$

where $r$ is the instantaneous interest rate (supposed to be constant), $W_t^*$ and $B_t^*$ are independent standard Brownian motions defined on a probability space $(\Omega, \mathcal{F}, P)$, $\rho \in [-1, 1]$. In the following we will denote by $\mathcal{F}^{W^*}, \mathcal{F}^{B^*}$ the filtrations generated respectively by $W^*$ and $B^*$. Moreover we define $\mathcal{F} := \mathcal{F}^{W^*} \vee \mathcal{F}^{B^*}$.

We assume that the volatility process is given by the sum of a fractional integral and a fractional derivative of a of a diffusion process $\tilde{\sigma}_s^2$ adapted to the filtration generated by $W$. The fractional integral term will allow us to explain the observed long-time behaviour of the implied volatility, as in Comte, Coutin and Renault (1998), while the fractional derivative part will explain the short-time behaviour of the implied volatility. More precisely, consider a CIR process of the form

$$\tilde{\sigma}_t^2 = \theta + \left( \tilde{\sigma}_0^2 - \theta \right) e^{-\kappa t} + \nu \int_0^t \exp\left(-\kappa(t-u)\right) \sqrt{\tilde{\sigma}_u^2} dW_u \tag{2}$$

where $\tilde{\sigma}_0^2, \theta, \kappa$ and $\nu$ are positive constants satisfying the condition $\frac{2\kappa\theta}{\nu^2} \geq 1$, which implies that $\tilde{\sigma}_s^2 > 0$ a.s. We will denote $Y_t = \theta + \left( \tilde{\sigma}_0^2 - \theta \right) e^{-\kappa t}$ and $Z_t = \int_0^t \exp\left(-\kappa(t-u)\right) \sqrt{\tilde{\sigma}_u^2} dW_u$. Then we will assume the volatility process is given by

$$\sigma_t^2 = Y_t + c_1 \nu Z_t + c_2 \nu I_{0+}^\alpha Z(t), \tag{3}$$

for some $\alpha \in (0, 1/2)$ and for some positive constants $c_1, c_2$.

**Remark 1** *If $c_2 = 0$ and $c_1 = 1$, the above process coindices with $\tilde{\sigma}_t^2$. Notice also that $E\left( \sigma_t^2 \right) = E\left( \tilde{\sigma}_t^2 \right) = \theta + \left( \tilde{\sigma}_0^2 - \theta \right) e^{-\kappa t}$.*

**Proposition 2** *Take $\alpha \in (0, 1/2)$ and $T \geq 0$. Assume that $\frac{2\kappa\theta}{\nu^2} \geq 1$ and $\left( 1 - c_1 - c_2 \frac{T^\alpha}{\alpha \Gamma(\alpha)} \right) \geq 0$. Then*

$$\sigma_t^2 \geq \tilde{\sigma}_0^2 e^{-\kappa t} + \theta \left( 1 - e^{-\kappa t} \right) \left( 1 - c_1 - c_2 \frac{T^\alpha}{\alpha \Gamma(\alpha)} \right) \ a.s.$$

**Proof.** We can write

$$\begin{aligned}
\sigma_t^2 &= Y_t + c_1 \nu Z_t \\
&\quad + c_2 \nu \frac{1}{\Gamma(\alpha)} \int_0^t (t-r)^{\alpha-1} Z(r) dr,
\end{aligned}$$

We know, from the positivity property of the Heston volatility process, that, a.s., $\nu Z_r > - \left( \theta + \left( \tilde{\sigma}_0^2 - \theta \right) e^{-\kappa r} \right) = -Y_t$ for all initial condition $\tilde{\sigma}_0^2$. Then, letting

$\tilde{\sigma}_0^2 \to 0$ we have that $\nu Z_r \geq -\theta\left(1 - e^{-\kappa r}\right)$ a.s. This allows us to write

$$
\begin{aligned}
\sigma_t^2 \; &\geq \; Y_t - c_1\theta\left(1 - e^{-\kappa r}\right) \\
&\quad - c_2 \frac{\theta}{\Gamma(\alpha)} \int_0^t (t-r)^{\alpha-1}\left(1 - e^{-\kappa r}\right) dr \\
&\geq \; Y_t - c_1\theta\left(1 - e^{-\kappa r}\right) \\
&\quad - c_2 \frac{\theta\left(1 - e^{-\kappa t}\right)}{\Gamma(\alpha)} \int_0^t (t-r)^{\alpha-1} dr \\
&= \; Y_t - c_1\theta\left(1 - e^{-\kappa t}\right) - c_2 \frac{\theta\left(1 - e^{-\kappa t}\right) t^\alpha}{\alpha\Gamma(\alpha)} \\
&= \; \theta + \left(\tilde{\sigma}_0^2 - \theta\right) e^{-\kappa t} - c_1\theta\left(1 - e^{-\kappa t}\right) \\
&\quad - c_2 \frac{\theta\left(1 - e^{-\kappa t}\right) t^\alpha}{\alpha\Gamma(\alpha)} \\
&= \; \tilde{\sigma}_0^2 e^{-\kappa t} + \theta\left(1 - e^{-\kappa t}\right)\left(1 - c_1 - c_2 \frac{t^\alpha}{\alpha\Gamma(\alpha)}\right) \\
&\geq \; \tilde{\sigma}_0^2 e^{-\kappa t} + \theta\left(1 - e^{-\kappa t}\right)\left(1 - c_1 - c_2 \frac{T^\alpha}{\alpha\Gamma(\alpha)}\right)
\end{aligned}
$$

which is a positive quantity because

$$
\left(1 - c_1 - c_2 \frac{T^\alpha}{\alpha\Gamma(\alpha)}\right) \geq 0
$$

∎

**Remark 3** *If $c_2 = 0$ $\sigma_t^2$ is a CIR process with vol-vol equal to $c_1\nu$ and the condition $\left(1 - c_1 - c_2 \frac{T^\alpha}{\alpha\Gamma(\alpha)}\right) \geq 0$ reduces to $1 \geq c_1$. If $\alpha = 0$ $\sigma_t^2$ is a CIR process with vol-vol equal to $(c_1 + c_2)\nu$ and this condition reduces to $1 \geq c_1 + c_2$.*

**Remark 4** *Notice that the above result implies that*

$$
\sigma_t^2 \geq \min\left(\tilde{\sigma}_0^2, \theta\left(1 - c_1 - c_2 \frac{T^\alpha}{\alpha\Gamma(\alpha)}\right)\right).
$$

*Then, if $\left(1 - c_1 - c_2 \frac{T^\alpha}{\alpha\Gamma(\alpha)}\right) > 0$, the volatility process $\sigma$ is uniformly lower bounded by a positive constant.*

It is well-known that the price of an European call option at time $t$ is given by

$$
V_t = e^{-r(T-t)} E^*\left(X_T - K\right), \tag{4}
$$

where $K$ is the strike price and $E^*$ denotes the expectation with respect to $P^*$.

In the sequel we will make use of the following notation

- $BS(t, x, \sigma)$ will denote the price of a European call option under the classical Black-Scholes model with constant volatility $\sigma$, current log stock price $x$, time to maturity $T - t$, strike price $K$ and interest rate $r$. Remember that in this case

$$BS(t, x, \sigma) = e^x N(d_+) - K e^{-r(T-t)} N(d_-),$$

  where $N$ denotes the cumulative probability function of the standard normal law and

$$d_\pm := \frac{x - x_t^*}{\sigma\sqrt{T - t}} \pm \frac{\sigma}{2}\sqrt{T - t},$$

  with $x_t^* := \ln K - r(T - t)$.

- $\mathcal{L}_{BS}\left(\sigma^2\right)$ stands for the Black-Scholes differential operator, in the log variable, with volatility $\sigma$ :

$$\mathcal{L}_{BS}\left(\sigma^2\right) = \partial_t + \frac{1}{2}\sigma^2\partial_{xx}^2 + (r - \frac{1}{2}\sigma^2)\partial_x - r\cdot$$

  It is well known that $\mathcal{L}_{BS}\left(\sigma^2\right)BS(\cdot, \cdot, \sigma) = 0$.

- $v_t^2 = \frac{1}{T-t}\int_t^T E^*\left(\sigma_s^2 \middle| \mathcal{F}_t\right)ds$. That is, $v_t^2$ denotes the square time future average volatility.

- $M_t = \int_0^T E^*\left(\sigma_s^2 \middle| \mathcal{F}_t\right)ds$. Notice that $v_t^2 = \frac{1}{T-t}\left(M_t - \int_0^t \sigma_s^2 ds\right)$.

## 2.3 Martingale representation of the future expected volatility

In our study we will need a explicit expression for $dM_t$ by means of Clark-Ocone formula. For this, we assume that the reader is familiar with the elementary results of the Malliavin calculus, as given for instance in Nualart (1995). The set $\mathbb{D}_W^{1,2}$ will denote the domain of the Malliavin derivative operator $D^W$. It is well-known that $\mathbb{D}_W^{1,2}$ is a dense subset of $L^2(\Omega)$ and that $D^W$ is a closed and unbounded operator from $L^2(\Omega)$ to $L^2([0, T] \times \Omega)$.

The next result is proved in Alòs and Ewald (2008):

**Proposition 5** *Assume the condition $2\kappa\theta > \nu^2$. Then , for all $0 < s < t < T$ the random variable $\tilde{\sigma}_t^2 \in \mathbb{D}_W^{1,2}$ and*

$$D_s\tilde{\sigma}_t^2 = \nu\sqrt{\tilde{\sigma}_t^2}f(t, s),$$

*where $f(t, s) := \exp\left(\int_s^t \left(-\frac{\kappa}{2} - \left(\frac{\kappa\theta}{2} - \frac{\nu^2}{8}\right)\frac{1}{\tilde{\sigma}_u^2}\right)du\right)$*

**Remark 6** *Notice that the condition $2\kappa\theta > \nu^2$ implies that $\frac{\kappa\theta}{2} - \frac{\nu^2}{8}$ is positive. Then it follows that $\left|D_s\tilde{\sigma}_t^2\right| \leq \nu\sqrt{\tilde{\sigma}_t^2}$.*

**Remark 7** *From the above result can easily check that, for all $0 < s < t < T$ the random variable $\sigma_t^2 \in \mathbb{D}_W^{1,2}$ and that*

$$D_s^W\sigma_t^2 = c_1 D_s^W\tilde{\sigma}_t^2 + \frac{c_2}{\Gamma(\alpha)}\int_s^t (t-r)^{\alpha-1}D_s\tilde{\sigma}_r^2 dr \tag{5}$$

The above result allows us to proof the following martingale representation for the future expected volatility:

**Proposition 8** *Assume the condition $2\kappa\theta > \nu^2$ and that $\left(1 - c_1 - c_2\frac{T^\alpha}{\alpha\Gamma(\alpha)}\right) \geq 0$. Then, for every fixed $t \in [0,T]$*

$$dM_t = \nu A(T,t)\sqrt{\tilde{\sigma}_t^2}dW_t,$$

*where $A(T,t) := \int_t^T \left(\frac{c_2}{\alpha\Gamma(\alpha)}(T-u)^\alpha + c_1\right)\exp(-\kappa(u-t))du$.*

**Proof.** From the Clark-Ocone formula and (5) we deduce that

$$
\begin{aligned}
\sigma_t^2 &= E\left(\sigma_t^2\right) + \int_0^t E\left[D_s^W\sigma_t^2\middle|\mathcal{F}_s\right]dW_s \\
&= E\left(\sigma_t^2\right) + \int_0^t \left[c_1\nu\exp(-\kappa(t-s))\right.\\
&\quad + \left.\left(\frac{c_2}{\Gamma(\alpha)}\int_s^t (t-r)^{\alpha-1}\exp(-\kappa(r-s))dr\right)\sqrt{\tilde{\sigma}_s^2}\right]dW_s \\
&= : E\left(\sigma_t^2\right) + \int_0^t a(t,s)dW_s.
\end{aligned}
$$

It is easy to see that $dM_t = \left(\int_t^T a(r,t)dr\right)dW_t$. Then some algebra gives us that

$$
\begin{aligned}
dM_t &= \nu\left(\int_t^T (c_1\exp(-\kappa(r-t))\right.\\
&\quad + \left.\frac{c_2}{\Gamma(\alpha)}\int_t^r (r-u)^{\alpha-1}\exp(-\kappa(u-t))du dr\sqrt{\tilde{\sigma}_t^2}\right)dW_t\right)\\
&= \nu\left(\int_t^T c_1\exp(-\kappa(r-t)) + \frac{c_2}{\alpha\Gamma(\alpha)}(T-u)^\alpha\exp(-\kappa(r-t))dr\right)\\
&= \nu\left(\int_t^T \left(\frac{c_2}{\alpha\Gamma(\alpha)}(T-u)^\alpha + c_1\right)\exp(-\kappa(u-t))du\right)\sqrt{\tilde{\sigma}_t^2}dW_t,
\end{aligned}
$$

and now the proof is complete.

**Corollary 9** *There exists a positive constant $C\left(\sigma_0^2, \alpha, T\right)$ such that, for every $t > 0$ and $p > 1$*

$$E\left(\sigma_t^p\right) \leq C\left(\sigma_0^2, \alpha, T\right).$$

∎

**Proof.** We can write

$$
\begin{aligned}
E\left(\sigma_t^p\right) &\leq C\left(\sigma_0^2\right)\left(1 + E\left(\int_0^t |a(t,s)|^2\right)^{\frac{p}{2}} ds\right) \\
&\leq C\left(\sigma_0^2, \alpha\right)\left(1 + E\left(\int_0^t (1 + (t-s)^\alpha) \tilde{\sigma}_s^2 ds\right)^{\frac{p}{2}}\right) \\
&\leq C\left(\sigma_0^2, \alpha\right)\left(1 + \left(\int_0^t (t-s)^\alpha E\left(\tilde{\sigma}_s^p\right) ds\right)\right),
\end{aligned}
$$

and now the results follows directly from the fact that the CIR process (2) has uniformly bounded moments (see for example Alfonsi (2010)). ∎

## 3    Option pricing approximation

In this section we will develope an approximation method following the same ideas as in Alòs (2012). We will need the following lemma:

**Lemma 10** *Let $0 \leq t \leq s \leq T$ and $\mathcal{G}_t := \mathcal{F}_t \vee \mathcal{F}_T^{W^*}$. Then for every $n \geq 0$, there exists $C = C(n, \rho)$ such that*

$$
\left| E^*\left( \partial_x^n \left( \frac{\partial^2}{\partial x^3} - \frac{\partial}{\partial x^2} \right)(s, X_s, v_s) \Big| \mathcal{G}_t \right) \right| \leq C \left( \int_s^T E\left(\sigma_\theta^2 \big| \mathcal{F}_s\right) d\theta \right)^{-\frac{1}{2}(n+1)}.
$$

Our price approximation method is based in the following decomposition result that can be proved following the same arguments as in Alòs (2012). In the following we take $t = 0$ for the sake of simplicity.

**Proposition 11** *Assume the model (1), where the volatility process $\sigma = \{\sigma_s, s \in [0,T]\}$ satisfies the conditions $2\kappa\theta > \nu^2$ and $\left(1 - c_1 - c_2 \frac{T^\alpha}{\alpha\Gamma(\alpha)}\right) > 0$. Then*

$$
\begin{aligned}
V_0 = {}& BS\left(0, X_0; v_0\right) \\
&+ \frac{1}{2} E^*\left( \int_0^T e^{-rs} H\left(s, X_s, v_s\right) \sigma_s d\langle M, X\rangle_s \right) \\
&+ \frac{1}{8} E^*\left( \int_0^T e^{-rs} K\left(s, X_s, v_s\right) d\langle M, M\rangle_s \right),
\end{aligned}
\tag{6}
$$

*where*

$$
H\left(s, X_s, v_s\right) := \left( \frac{\partial^3}{\partial x^3} - \frac{\partial^2}{\partial x^2} \right) BS\left(s, X_s, v_s\right)
$$

*and*

$$K\left(s, X_s, v_s\right) := \left(\frac{\partial^4}{\partial x^4} - 2\frac{\partial^3}{\partial x^3} + \frac{\partial^2}{\partial x^2}\right) BS\left(s, X_s, v_s\right).$$

**Theorem 12** *Fix $T > 0$. Assume the model (1), where the volatility process $\sigma = \{\sigma_s, s \in [0, T]\}$ satisfies the conditions $2\kappa\theta > \nu^2$ and $\left(1 - c_1 - c_2\frac{T^\alpha}{\alpha\Gamma(\alpha)}\right) > C$ for some positive constant $C$. Then*

$$\left| V_0 - BS\left(0, X_0; v_0\right) - \frac{1}{2}H\left(0, X_0; v_0\right) E^*\left(\int_t^T \sigma_s d\left\langle M, W^*\right\rangle_s\right) \right.$$
$$\left. -\frac{1}{8}K\left(0, X_0; v_0\right) E^*\left(\int_0^T d\left\langle M, M\right\rangle_s\right) \right|$$
$$\leq \quad C\left(T, \sigma_0, \alpha\right)\left(\rho^2\nu^2 + \rho\nu^3 + \nu^4\right) \tag{7}$$

*for some positive constant $C\left(T, \sigma_0, \alpha\right)$.*

**Proof.** Consider the processes $e^{-rt}H\left(t, X_t; v_t\right) U_t$ and $e^{-rt}K\left(t, X_t; v_t\right) R_t$, where

$$U_t := \frac{1}{2}E^*\left(\int_t^T \sigma_s d\left\langle M, W^*\right\rangle_s \middle| \mathcal{F}_t\right)$$

*and*

$$R_t := \frac{1}{8}E^*\left(\int_t^T d\left\langle M, M\right\rangle_s \middle| \mathcal{F}_t\right)$$

It is easy to check that

$$e^{-rT}H\left(T, X_T; v_T\right) U_T = 0, e^{-rT}K\left(T, X_T; v_T\right) R_T = 0.$$

Then, the same arguments as in Alòs (2012) we can write

$$0 = H\left(0, X_0; v_0\right) U_0$$
$$-\frac{1}{2}E^*\left(\int_0^T e^{-rs}H\left(s, X_s, v_s\right)\sigma_s d\left\langle M, W^*\right\rangle_s\right)$$
$$+\frac{1}{2}E^*\left(\int_0^T e^{-rs}\left(\frac{\partial^3}{\partial x^3} - \frac{\partial^2}{\partial x^2}\right) H\left(s, X_s, v_s\right) U_s\sigma_s d\left\langle M, W^*\right\rangle_s\right)$$
$$+\frac{1}{2}E^*\left(\int_0^T e^{-rs}\left(\frac{\partial^4}{\partial x^4} - 2\frac{\partial^3}{\partial x^3} + \frac{\partial^2}{\partial x^2}\right) H\left(s, X_s, v_s\right) U_s d\left\langle M, M\right\rangle_s\right).$$

9

and

$$
\begin{aligned}
0 \;=\; & K\left(0, X_0; v_0\right) R_0 \\
& -\frac{1}{8} E^*\left(\int_0^T e^{-rs} K\left(s, X_s, v_s\right) d\left\langle M, M\right\rangle_s\right) \\
& +\frac{\rho}{2} E^*\left(\int_0^T e^{-rs}\left(\frac{\partial^3}{\partial x^3} - \frac{\partial^2}{\partial x^2}\right) K\left(s, X_s, v_s\right) R_s \sigma_s d\left\langle M, W^*\right\rangle_s\right) \\
& +\frac{1}{8} E^*\left(\int_0^T e^{-rs}\left(\frac{\partial^4}{\partial x^4} - 2\frac{\partial^3}{\partial x^3} + \frac{\partial^2}{\partial x^2}\right) K\left(s, X_s, v_s^\delta\right) R_s d\left\langle M, M\right\rangle_s\right).
\end{aligned}
$$

This, together with (6), allows us to write

$$
\begin{aligned}
V_t \;=\; & BS\left(0, X_0; v_0\right) + H\left(0, X_0; v_0\right) U_0 + K\left(0, X_0; v_0\right) R_0 \\
& +\frac{\rho}{2} E^*\left(\int_0^T e^{-r(s-t)}\left(\frac{\partial^3}{\partial x^3} - \frac{\partial^2}{\partial x^2}\right) H\left(s, X_s, v_s\right) U_s \sigma_s d\left\langle M, W^*\right\rangle_s\right) \\
& +\frac{1}{8} E^*\left(\int_0^T e^{-rs}\left(\frac{\partial^4}{\partial x^4} - 2\frac{\partial^3}{\partial x^3} + \frac{\partial^2}{\partial x^2}\right) H\left(s, X_s, v_s\right) U_s d\left\langle M, M\right\rangle_s\right) \\
& +\frac{\rho}{2} E^*\left(\int_0^T e^{-r(s-t)}\left(\frac{\partial^3}{\partial x^3} - \frac{\partial^2}{\partial x^2}\right) K\left(s, X_s, v_s\right) R_s \sigma_s d\left\langle M, W^*\right\rangle_s\right) \\
& +\frac{1}{8} E^*\left(\int_0^T e^{-rs}\left(\frac{\partial^4}{\partial x^4} - 2\frac{\partial^3}{\partial x^3} + \frac{\partial^2}{\partial x^2}\right) K\left(s, X_s, v_s\right) R_s d\left\langle M, M\right\rangle_s\right) \\
\;=\; & BS\left(t, X_t; v_t\right) + H\left(t, X_t; v_t\right) U_t + K\left(t, X_t; v_t\right) R_t + T_1 + T_2 + T_3 + T_4.
\end{aligned}
$$

Notice that

$$
\begin{aligned}
|U_s| \;\leq\; & \nu\rho E^*\left(\left.\int_s^T \sqrt{\sigma_r^2}\sqrt{\tilde{\sigma}_r^2} A(T, r) dr\right| \mathcal{F}_s\right) \\
\;\leq\; & \nu\rho \int_s^T E^*\left(\left.\sqrt{\sigma_r^2}\sqrt{\tilde{\sigma}_r^2}\right| \mathcal{F}_s\right) A(T, r) dr \\
\;\leq\; & C\nu\rho \int_s^T \sqrt{E^*\left(\sigma_r^2 | \mathcal{F}_s\right) E^*\left(\tilde{\sigma}_r^2 | \mathcal{F}_s\right)} A(T, r) dr \\
\;\leq\; & C\nu\rho \int_s^T \sqrt{E^*\left(\sigma_r^2 | \mathcal{F}_s\right) E^*\left(\tilde{\sigma}_r^2 | \mathcal{F}_s\right)} A(T, r) dr
\end{aligned}
$$

and

$$
|R_s| \leq \nu^2 E^*\left(\left.\int_s^T \tilde{\sigma}_t^2 A^2(T, r) dr\right| \mathcal{F}_s\right).
$$

Then, Lemma 10 and Remark 4 give us that

$$
\begin{aligned}
T_1 \;\leq\; & C\nu^2\rho^2 E\left(\left(\int_0^T e^{-r(s-t)}\left(\int_s^T E\left(\sigma_\theta^2\big|\mathcal{F}_s\right)d\theta\right)^{-\frac{5}{2}}\sqrt{\sigma_s^2}\right.\right. \\
& \left.\left.\times\left(\int_s^T \sqrt{E^*\left(\sigma_r^2\big|\mathcal{F}_s\right)E^*\left(\tilde{\sigma}_r^2\big|\mathcal{F}_s\right)}A(T,r)dr\right)ds\right)\right) \\
\;\leq\; & C\nu^2\rho^2 E\left(\int_0^T (T-s)^{-\frac{5}{2}}\sqrt{\sigma_s^2}\left(\int_s^T \sqrt{E^*\left(\sigma_r^2\big|\mathcal{F}_s\right)E^*\left(\tilde{\sigma}_r^2\big|\mathcal{F}_s\right)}dr\right)ds\right),
\end{aligned}
$$

and then, Corollary 9 gives us that

$$
T_1 \leq C\left(T,\sigma_0,\alpha\right)
$$

In a similar way

$$
\begin{aligned}
T_2 \;\leq\; & C\nu^3\rho\int_t^T e^{-r(s-t)}\left(\int_s^T E\left(\sigma_\theta^2\big|\mathcal{F}_s\right)d\theta\right)^{-3} \\
& \times(T-s)\left(\int_s^T\left(\frac{c_2}{\alpha\Gamma(\alpha)}(T-u)^\alpha+c_1\right)e^{-\kappa(u-s)}du\right)^3 ds \\
\;\leq\; & C\nu^3\rho,
\end{aligned}
$$

$$
\begin{aligned}
T_3 \;\leq\; & C\nu^3\rho\int_t^T e^{-r(s-t)}\left(\int_s^T E\left(\sigma_\theta^2\big|\mathcal{F}_s\right)d\theta\right)^{-3} \\
& \times(T-s)\left(\int_s^T\left(\frac{c_2}{\alpha\Gamma(\alpha)}(T-u)^\alpha+c_1\right)e^{-\kappa(u-s)}du\right)^3 ds \\
\;\leq\; & C\nu^3\rho,
\end{aligned}
$$

and

$$
\begin{aligned}
T_4 \;\leq\; & C\left(\alpha,\lambda\right)\nu^4\int_t^T e^{-r(s-t)}\left(\int_s^T E\left(\sigma_\theta^2\big|\mathcal{F}_s\right)d\theta\right)^{-\frac{7}{2}} \\
& \times(T-s)\left(\int_s^T\left(\frac{c_2}{\alpha\Gamma(\alpha)}(T-u)^\alpha+c_1\right)e^{-\kappa(u-s)}du\right)^4 ds \\
\;\leq\; & C\nu^4,
\end{aligned}
$$

and now the proof is complete. ∎

## 3.1 An explicit form for the approximation

For an European call option, it is easy to check that

$$H(0, x, \sigma) := \frac{e^x}{\sigma\sqrt{2\pi T}} \exp\left(-\frac{d_+^2}{2}\right)\left(1 - \frac{d_+}{\sigma\sqrt{T}}\right)$$

and

$$K(0, x, \sigma) = \frac{e^x}{\sigma\sqrt{2\pi T}} \exp\left(-\frac{d_+^2}{2}\right)\left[\left(-\frac{d_+}{\sigma\sqrt{T}} + \frac{d_+^2}{\sigma^2 T}\right) - \frac{1}{\sigma^2 T}\right].$$

Moreover,

$$E^*\left(\int_0^T \sigma_s^2 ds\right) = \theta + \frac{(\sigma_0^2 - \theta)}{\kappa}\left(1 - e^{-\kappa T}\right),$$

$$E^*\left(\int_0^T \sigma_s d\langle M, W^*\rangle_s\right)$$

$$= \nu\rho\int_0^T\left(\int_s^T\left(\frac{c_2}{\alpha\Gamma(\alpha)}(T-u)^\alpha + c_1\right)\exp(-\kappa(u-s))du\right)E^*\left(\sqrt{\sigma_s^2\tilde{\sigma}_s^2}\right)ds$$

$$= : \nu\rho\int_0^T Z(T,s)E^*\left(\sqrt{\sigma_s^2\tilde{\sigma}_s^2}\right)ds$$

and

$$E^*\left(\int_0^T d\langle M, M\rangle_s\right) = \nu^2\int_0^T Z^2(T,s)E^*\left(\tilde{\sigma}_s^2\right)ds$$

Then we can easily obtain the following explicit approximation formulas by substituting the above quantities in the approximation expressions proposed in Theorem 10. Moreover, notice that $\sigma_s\sqrt{\tilde{\sigma}_s^2} = \theta + \left(\tilde{\sigma}_0^2 - \theta\right)e^{-\kappa t} + O\left(\nu^2\right)$, which allows us to obtain the explicit closed-form approximation formula

$$BS(0, X_0; v_0) + \frac{\nu\rho}{2}H(0, X_0, v_0)\int_0^T E^*\left(\tilde{\sigma}_s^2\right)Z(T,s)ds$$

$$+ \frac{\nu^2}{8}K(0, X_0, v_0)\int_0^T E^*\left(\tilde{\sigma}_s^2\right)Z^2(T,s)ds$$

$$= BS(0, X_0; v_0) + \frac{\nu\rho}{2}H(0, X_0, v_0)\int_0^T\left[\theta + \left(\tilde{\sigma}_0^2 - \theta\right)e^{-\kappa s}\right]Z(T,s)ds$$

$$+ \frac{\nu^2}{8}K(0, X_0, v_0)\int_0^T\left[\theta + \left(\tilde{\sigma}_0^2 - \theta\right)e^{-\kappa s}\right]Z^2(T,s)ds. \tag{8}$$

**Example 13** *Let us consider $S_0 = 90, r = 0.05, \theta = 0.09, \sigma_0 = 0.04, \kappa = 3, \nu = 0.3, \rho = -0.5, c_1 = 0, c_2 = 0.1$ and $\alpha = 0.2$. In the following figure we can see the corresponding error of approximation (in %) relative to the option price obtained by a 1000,000 Monte Carlo simulation, for times to maturity $T = 0.5, T = 1$ and $T = 3$. We can see the observed errors are lesser than 0.6%.*
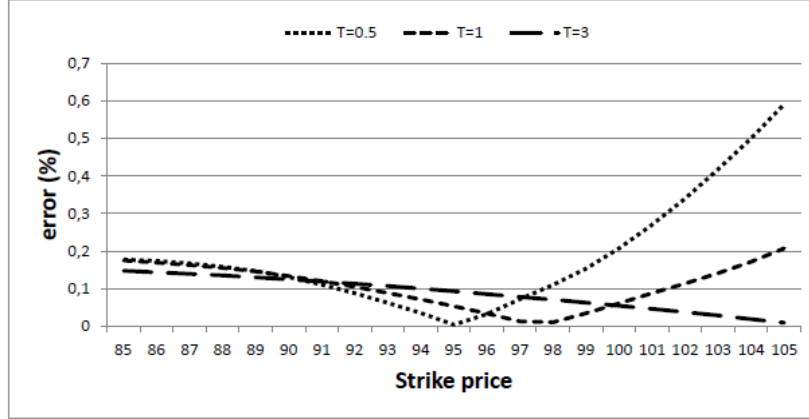
Figure 1: Fig. 1: Approximation error (%) for $T = 0.5, T = 1$ and $T = 3$

# 4   Implied volatility behaviour

In this section we study the flattening of the implied volatility for our model. It is easy to deduce from (8) by using Taylor expansions as in Fouque, Papanicolaou and Sircar (2000), the following approximation for the implied volatility

$$
\hat{I} \quad : \quad = v_0 + \frac{\nu\rho}{2v_0 T}\left(1 - \frac{d_+}{v_0\sqrt{T}}\right)E^*\left(\int_0^T Y_s Z(T,s)ds\right)
$$
$$
+ \frac{\nu^2}{8v_t\,(T-t)}\left[\left(-\frac{d_+}{v_0\sqrt{T}} + \frac{d_+^2}{v_0^2 T}\right) - \frac{1}{v_0^2 T}\right]E^*\left(\int_0^T Y_s Z^2(T,s)ds\right)
$$

Notice that, as

$$
d_+ = \frac{x - x_t^*}{v_0\sqrt{T}} + \frac{v_0\sqrt{T}}{2}
$$

the first expression is linear in the initial log-stock price $x$, and the second one is quadratic in $x$. Then we deduce that the second term in the right-hand side of this expression allows us to describe the skew effect, while the last one is necessary to describe a smile.

It is easy to check that

$$
\lim_{T\to t}\frac{\partial\hat{I}}{\partial X_t}\left(x_t^*\right) = \frac{\nu\rho c_1}{2\sigma_0}
$$

which coincides with the short-time skew slope of a classical Heston model with volatility of the volatility equal to $\nu c_1$. On the other hand, as

$$
Z(T,s) = \left(\int_s^T\left[\frac{c_2}{\alpha\Gamma\,(\alpha)}(T-u)^\alpha + c_1\right]\exp(-\kappa(u-s))du\right)
$$

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the flattening of the implied volatility skew is slower than in the Heston case, as we can see in the following example.

**Example 14** *Let us consider $\theta = \sigma_0 = 0.04, \kappa = 3, \nu = 0.3, \rho = -0.5$ and $\alpha = 0.4$. In the following figure we plot the derivative (in absolute value)$\left|\frac{\partial \hat{I}}{\partial X_t}(x_t^*)\right|$ as a function of time to maturity and for different values of $c_1$ and $c_2$. Notice that in the case $c_2 = 0$ there is not the fractional integral term in the definition of the volatility process. Then this implied volatility skew tends to a constant as time to maturity tends to zero, while it decays strongly as time to maturity increases, as in the classical Heston case. In the case $c_1 = 0$ the implied volatility slope tends to zero as time to maturity tends to zero, but it flattens slowly as time to maturity increases. Then it explains the slow flattening of the implied volatility when time to maturity increases. Finally, taking $c_1 \neq 0$ and $c_2 \neq 0$ the implied volatility skew tends to a constant when time to maturity is near zero (as in the classical Heston case) and at the same time we capture the slow decrease of the implied volatility slope.*
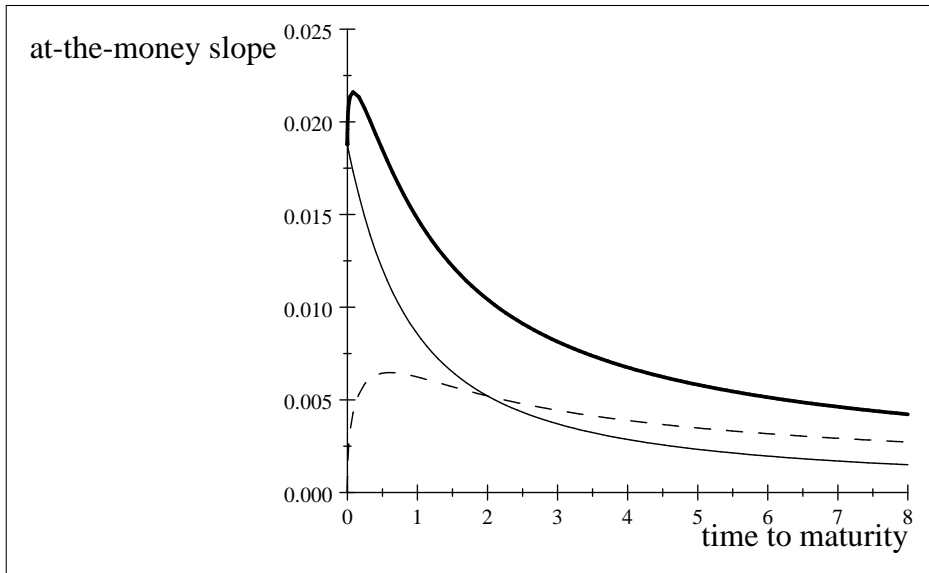


Fig. 2: $\left|\frac{\partial \hat{I}}{\partial X_t}(x_t^*)\right|$ *as a function of time to maturity, for $c_1 = 0.1, c_2 = 0$ (solid); $c_1 = 0, c_2 = 0.1$ (dash) and $c_1 = 0.1, c_2 = 0.1$ (thick)*

**Conclusion 15** *We have presented a method to construct simple option pricing approximation formulas for a fractional volatility model and we have studied its accuracy. Moreover, we have seen the corresponding approximation of the implied volatility captures the slow flattening when time to maturity increases. This ability to explain the implied volatility smile and the simplicity of the option pricing approximation formulas makes the presented model potentially interesting in finance.*

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