On the tractability of the piecewise-linear approximation for general discrete-choice network revenue management

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Abstract

The choice network revenue management (RM) model incorporates customer purchase behavior as customers purchasing products with certain probabilities that are a function of the offered assortment of products, and is the appropriate model for airline and hotel network revenue management, dynamic sales of bundles, and dynamic assortment optimization. The underlying stochastic dynamic program is intractable and even its certainty-equivalence approximation, in the form of a linear program called Choice Deterministic Linear Program ($CDLP$) is difficult to solve in most cases. The separation problem for $CDLP$ is NP-complete for MNL with just two segments when their consideration sets overlap; the affine approximation of the dynamic program is NP-complete for even a single-segment MNL. This is in contrast to the independent-class (perfect-segmentation) case where even the piecewise-linear approximation has been shown to be tractable. In this paper we investigate the piecewise-linear approximation for network RM under a general discrete-choice model of demand. We show that the gap between the $CDLP$ and the piecewise-linear bounds is within a factor of at most 2. We then show that the piecewise-linear approximation is polynomially-time solvable for a fixed consideration set size, bringing it into the realm of tractability for small consideration sets; small consideration sets are a reasonable modeling tradeoff in many practical applications. Our solution relies on showing that for any discrete-choice model the separation problem for the linear program of the piecewise-linear approximation can be solved exactly by a Lagrangian relaxation. We give modeling extensions and show by numerical experiments the improvements from using piecewise-linear approximation functions.

1 Introduction and literature review

Revenue management is the control of the sale of a limited quantity of a resource (hotel rooms for a night, airline seats, advertising slots etc.) to a heterogenous population with different valuations for a unit of the resource. The resource is perishable, and for simplicity sake, we assume that it perishes at a fixed point of time in the future. The firm has to decide what products to offer (at a given price for each product), the tradeoff being selling too much at too low a price early and running out of capacity, or, rejecting too many low-valuation customers and ending up with excess unsold inventory.

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In industries such as hotels, advertising and airlines, the products consume bundles of different resources (multi-night stays, bundles of ad slots, multi-leg itineraries) and the decision to accept or reject a particular product at a certain price depends on the future demands and revenues for all the resources used by the product and indirectly, on all the resources in the network. Network revenue management (network RM) is control based on the demands for the entire network. Chapter 3 of Talluri and van Ryzin [21] contains the necessary background on network RM.

RM incorporating more realistic models of customer behavior as customers choosing from set of offered products have recently become popular, initiated in Talluri and van Ryzin [20] for the single-resource problem. Bodea, Ferguson, and Garrow [3] for instance use choice data from a large hotel chain and empirically study the suitability of choice models. Vulcano, van Ryzin, and Chaar [24] and Newman, Ferguson, Garrow, and Jacobs [15] and Talluri [19] study estimation of choice models from sales data.

The choice network RM problem can be formulated as a stochastic dynamic program with exponentially large state and action spaces (exponential in the number of products). Since the dynamic programming formulation is computationally intractable, many approximation methods have been proposed starting with Gallego, Iyengar, Phillips, and Dubey [5] and Liu and van Ryzin [12], who formulate the choice deterministic linear program ($CDLP$) and show that it gives an upper bound on the value function. Since $CDLP$ has an exponential number of decision variables it has to be solved using column generation and Liu and van Ryzin [12] show that this can be done in polynomial time for the MNL model of choice when the consideration sets of the different customer segments are disjoint. However, generating the columns is NP-complete when the segment consideration sets overlap even under the MNL model with just two segments (Bront, Méndez-Díaz, and Vulcano [4], Rusmevichientong, Shmoys, Tong, and Topaloglu [16]). Kunnumkal and Talluri [10] show that the affine approximation of the dynamic program is NP-hard even for a single-segment MNL model. These negative results show us the limits of tractability just for the MNL model, leaving little hope for general discrete choice models.

Kunnumkal and Topaloglu [11] and Zhang and Adelman [25] study decomposition procedures and an affine relaxation of the dynamic program. In the same vein, Meissner and Strauss [14] look at time-sensitive bid-price controls based on a decomposition procedure. All these methods yield upper bounds on the value function that are provably tighter than the $CDLP$ upper bound but intractable even for a single-segment MNL model of choice ([10]). Recently Vossen and Zhang [23] study the affine approximation using Dantzig-Wolfe decomposition ideas.

The difficulty of approximations for choice network RM is in contrast to the perfect segmentation case (also called the independent-class assumption) where both the affine and piecewise-linear approximations to the dynamic program are tractable (Tong and Topaloglu [22], Kunnumkal and Talluri [9]). So choice network RM appears to be significantly more difficult, even for restricted MNL choice models.

In this paper we study the piecewise-linear approximation of the dynamic program for general discrete choice models. In this approximation, the value functions are replaced by separable piecewise-linear functions that are more flexible and general than the affine approximations considered previously. We show that the gap between the $CDLP$ and the piecewise-linear bounds is within a factor of at most 2. We note that this is not an asymptotic-type bound, relying on demand and capacity scaling, and that it holds for any choice model; as far as are aware, it is the first known constant-factor bound for this problem.

Now, since we work with general discrete-choice models, we expect all the negative complexity results for the MNL model and affine approximation to carry over for the piecewise-linear approximation. So naturally we have to make some assumptions or restrict our realm of applicability: we assume that the choice set is not very large so that one could possibly enumerate all the subsets of the choice set.

The main theoretical result of this paper is the following: If r_i^1 is the initial capacity of resource i, and τ the number of time periods, and $|\mathcal{J}|$ the size of the consideration set, then the linear program representing the piecewise-linear approximation has $\mathcal{O}(2^{|\mathcal{J}|}\tau \prod_i r_i^1)$ constraints (and the separation problem of this linear program is NP-complete, even for MNL). We show that the piecewise-linear approximation linear program is equivalent to a linear program with $\mathcal{O}(2^{|\mathcal{J}|}\tau \sum_i r_i^1)$ constraints. So the complexity is reduced from $\prod_i r_i^1$ to a $\sum_i r_i^1$ factor, a significant reduction and for a fixed $|\mathcal{J}|$ gives a polynomial-time complexity to the approximation. This result holds for any general discrete-choice model.

We show that the reduced linear program can be alternatively viewed as a Lagrangian relaxation of the network RM dynamic program that uses $\mathcal{O}(2^{|\mathcal{J}|}\tau)$ multipliers. The Lagrangian relaxation method we propose is new and is attractive since it decomposes the network problem into a number of single resource problems which are computationally much more tractable. Furthermore, we show that the optimal set of Lagrange multipliers can be obtained by solving a convex optimization problem.

The result has practical implications whenever the size of the consideration sets are small (say no more than 20). Small consideration sets can be justified in the airline setting where a segment's consideration set consists of choices (on one airline) for travel between an origin and destination, and typically there are only a few alternatives on a given date (Talluri [18]). For hotels, as the product consists of a multi-night stay and most customers arrive with a fixed duration of stay in mind, the consideration set consists of the types of rooms and products, usually not a very large number. Empirical studies in the marketing literature also motivate our assumption of small consideration sets; Hauser and Wernerfelt [7] report average consideration set sizes of 3 brands for deodorants, 4 brands for shampoos, 2.2 brands for air fresheners, 4 brands for laundry detergents and 4 brands for coffees. (Note that the study is for brands rather than choices of sizes or colors.) Furthermore, another line of marketing research finds great value in deliberately limiting customer choices to a small number (Iyengar and Lepper [8]).

To obtain tractability in optimization one can potentially model demand as consisting of multiple segments, each with a small consideration set. We therefore extend the relaxation to multiple segments with distinct small consideration sets. Our numerical results show significant improvements over the benchmark *CDLP* method and even the affine relaxation—for some examples, we tighten the upper bound on the dynamic program by nearly 20% compared to the *CDLP* and 10% compared to the affine relaxation. Our theoretical results also provide a basis for the Lagrangian relaxation and its relation to the dynamic program, along the lines first proposed in Meissner and Strauss [14]. Going back further, decomposing the problem to obtain tractability is quite classical (§6.4.2 of Bertsekas [2]; Maglaras and Meissner [13] in a related context) and this paper bridges the decomposition and approximation approaches to difficult stochastic dynamic programs.

From a technical viewpoint, while many possible approximations (polynomials, trignometric functions, Lyapunov functions) to difficult stochastic dynamic programs can be considered (and indeed, been proposed), few of them can be shown to be tractable. This is especially true for the choice network revenue management where even with a simple MNL model, the affine relaxation is NP-hard. Our research sheds light on the modeling tradeoffs involved in obtaining solvability of good approximations, such as with piecewise-linear functions.

The remainder of the paper is organized as follows: In §2 we describe the network choice RM model, the notation, and the stochastic dynamic program and the linear-programming based approximations we consider in this paper, the $CDLP$, affine and piecewise-linear relaxations. In §3 we describe the Lagrangian relaxation approach and in §4, we show the connection between piecewise-linear and Lagrangian relaxation approaches. In §5 we perform numerical experiments to determine the performance of the piecewise-linear approximation compared to the $CDLP$ and the affine relaxation.

2 Model and notation

A product is a specification of a price and the set of resources that it consumes. For example, a product could be an itinerary-fare class combination for an airline network, where an itinerary is a combination of flight legs; in a hotel network, a product would be a multi-night stay for a particular room type at a certain price point.

Time is discrete and assumed to consist of τ intervals, indexed by t. The booking horizon begins at time $t = 1$ and ends at $t = \tau$; all the resources perish instantaneously at time $\tau + 1$. We make the standard assumption that the time intervals are fine enough so that the probability of more than one customer arriving in any single time period is negligible. The underlying network has m resources which are indexed by i, and n products which are indexed by j. We index resources by i or l, and products by j, and time periods by t. We refer to the set of all resources as $\mathcal{I}$ and the set of all products as $\mathcal{J}$. A product j uses a subset of resources $\mathcal{I}_j \subseteq \mathcal{I}$ and similarly, a resource i is used by a subset $\mathcal{J}_i \subseteq \mathcal{J}$ of products.

We use superscripts on vectors to index the vectors. For example, we write the resource capacity vector associated with time period t as $\boldsymbol{r}^t$. We use subscripts to indicate components. For example, r_i^t represents the capacity on resource i in time period t. We use $\mathbb{1}_{[\cdot]}$ as the indicator function, 1 if true and 0 if false.

We let $\boldsymbol{r}^1 = \left[r_i^1\right]$ represent the initial capacity on the resources and $\boldsymbol{r}^t = [r_i^t]$ denote the remaining capacity on resource i at beginning of time period t. The remaining capacity r_i^t takes values in the set $\mathcal{R}_i = \left\{0, \ldots, r_i^1\right\}$ and $\mathcal{R} = \prod_i \mathcal{R}_i$ represents the state space.

2.1 Demand model

In each time period the firm offers a subset S of its products for sale, called the *offer set*. A customer arrives with probability α and given an offer set S, an arriving customer purchases a product j in the set S or decides not to purchase. The no-purchase option is indexed by 0 and is always present for the customer. We let $P_j(S)$ denote the probability that the firm sells product j given that a customer arrives and the offer set is S. Clearly, $P_j(S) = 0$ if $j \notin S$. The probability of no sale given a customer arrival is $P_0(S) = 1 - \sum_{j \in S} P_j(S)$. We assume that the choice probabilities are given by an oracle, as the model represents a general discrete-choice model; they could be conceivably be calculated by a simple formula as in the case of the Multinomial Logit (MNL) model.

2.2 Choice dynamic program

The dynamic program (DP) to determine optimal controls is as follows. Let $V_t(\boldsymbol{r}^t)$ denote the maximum expected revenue to go, given remaining capacity $\boldsymbol{r}^t$ at the beginning of period t. Then

$V_t(\boldsymbol{r}^t)$ must satisfy the Bellman equation

$$V_t(\boldsymbol{r}^t) = \max_{S \subset \mathcal{Q}(\boldsymbol{r}^t)} \left\{ \sum_{j \in S} \alpha P_j(S) \left[f_j + V_{t+1} \left(\boldsymbol{r}^t - \sum_{i \in \mathcal{I}_j} \boldsymbol{e}^i \right) \right] + \left[\alpha P_0(S) + 1 - \alpha \right] V_{t+1} \left(\boldsymbol{r}^t \right) \right\}, \quad (1)$$

where

$$\mathcal{Q}(\boldsymbol{r}) = \left\{ j \mid \mathbb{1}_{[j \in \mathcal{J}_i]} \leq r_i, \ \forall i \right\}$$

represents the set of products that can be offered given the capacity vector $\boldsymbol{r}$ and $\boldsymbol{e}^i$ is a vector with a 1 in the ith position and 0 elsewhere. The boundary conditions are $V_{\tau+1}(\boldsymbol{r}) = V_t(\mathbf{0}) = 0$ for all $\boldsymbol{r}$ and for all t, where $\mathbf{0}$ is a vector of all zeroes. $V^{DP} = V_1(\boldsymbol{r}^1)$ denotes the optimal expected total revenue over the booking horizon, given the initial capacity vector $\boldsymbol{r}^1$.

For brevity of notation, we assume that $\alpha = 1$ in the remaining part of the paper. We note that this is without loss of generality since this is equivalent to letting $\tilde{P}_j(S) = \alpha P_j(S)$ and $\tilde{P}_0(S) = \alpha P_0(S) + 1 - \alpha$, and working with the choice probabilities $\left\{ \tilde{P}_j(S) \mid \forall j, S \right\}$.

2.3 Linear programming formulation of the dynamic program

The value functions can, alternatively, be obtained by solving a linear program, following Schweitzer and Seidmann [17]. The linear programming formulation of the network choice RM dynamic program has a decision variable $V_t(\boldsymbol{r})$ for each state vector $\boldsymbol{r}$ in each period t and is as follows:

$$
\begin{aligned}
V^{DP} = \quad & \min_V V_1(\boldsymbol{r}^1) \\
\text{s.t.} \quad & \\
(DP) \quad & V_t(\boldsymbol{r}) \geq \sum_j P_j(S) \left[f_j + V_{t+1} \left(\boldsymbol{r} - \sum_{i \in \mathcal{I}_j} \boldsymbol{e}^i \right) - V_{t+1}(\boldsymbol{r}) \right] + V_{t+1}(\boldsymbol{r}) \qquad (2) \\
& \qquad \forall \boldsymbol{r} \in \mathcal{R}, S \subset \mathcal{Q}(\boldsymbol{r}), t \\
& V_t(\boldsymbol{r}) \geq 0
\end{aligned}
$$

with the boundary condition that $V_{\tau+1}(\cdot) = 0$. Both the dynamic program (1) and linear program DP are computationally intractable, but the linear program DP turns out to be useful in developing value function approximation methods. In the following sections, we describe methods to approximate the value function.

2.4 Choice-based deterministic linear program

The choice-based deterministic linear program proposed in Gallego et al. [5] and Liu and van Ryzin [12] is given by

$$
\begin{aligned}
V^{CDLP} = \quad & \max_h \sum_t \sum_S R(S) h_{S,t} \\
\text{s.t} \quad & \\
(CDLP) \quad & \sum_t \sum_S \left[\sum_{j \in \mathcal{J}_i} P_j(S) \right] h_{S,t} \le r_i^1 \quad \forall i \qquad (3) \\
& \sum_S h_{S,t} = 1 \quad \forall t \qquad (4) \\
& h_{S,t} \ge 0 \quad \forall t, S,
\end{aligned}
$$

where

$$
R(S) = \sum_j P_j(S) f_j \qquad (5)
$$

is the expected revenue associated with offer set S. In the above linear program, we interpret the decision variable $h_{S,t}$ as the frequency with which set S is offered at time period t. The objective function measures the total expected revenues, while the first set of constraints ensure that the total expected capacity consumed on each resource does not exceed its available capacity. The second set of constraints ensures that the total frequencies add up to 1.

Liu and van Ryzin [12] show that the optimal objective function value of the $CDLP$ gives an upper bound on the optimal expected revenue. That is, $V_1(\boldsymbol{r}^1) \le V^{CDLP}$. Since $CDLP$ has an exponential number of decision variables it has to be solved using column generation. The column generation procedure is intractable in general, NP-complete for MNL with just two segments (Bront et al. [4] and Rusmevichientong et al. [16]). We use this method as our benchmark method for numerical comparisons in §5.

2.5 Affine Approximation

The affine approximation replaces the value function $V_t(\boldsymbol{r})$ by the affine function $\theta_t + \sum_i V_{i,t} r_i$ in the linear program DP to obtain the following linear program

$$
\begin{aligned}
V^{AF} = \quad & \min_{\theta, V} \theta_1 + \sum_i V_{i,1} r_i^1 \\
\text{s.t} \quad & \\
(AF) \quad & \theta_t + \sum_i V_{i,t} r_i \ge \sum_j P_j(S) \left[f_j - \sum_{i \in \mathcal{I}_j} V_{i,t+1} \right] + \theta_{t+1} + \sum_i V_{i,t+1} r_i \\
& \qquad \forall \boldsymbol{r} \in \mathcal{R}, S \subset \mathcal{Q}(\boldsymbol{r}), t \\
& \theta_t, V_{i,t} \ge 0.
\end{aligned}
$$

Zhang and Adelman [25] show that the optimal objective function value of AF gives an upper bound on the optimal expected revenue and this bound is tighter than the $CDLP$ upper bound. While the number of decision variables in AF is manageable, the number of constraints is exponential both in the number of resources as well as the products. Vossen and Zhang [23] show that AF has a reduced formulation where the number of constraints is exponential only in the number of products. Still, the problem has to be solved by constraint generation and the separation problem is intractable even for the MNL choice model with a single segment (Kunnumkal and Talluri [10]).

2.6 Piecewise-linear approximation

The main object of study in this paper is the piecewise-linear approximation which approximates the value function $V_t(\boldsymbol{r})$ by $\sum_i v_{i,t}(r_i)$ in DP. The resulting linear program is

$$
\begin{aligned}
V^{PL} = \quad & \min_{v} \sum_i v_{i,1}(r_i^1) \\
\text{s.t} \quad & \\
(PL) \quad & \sum_i v_{i,t}(r_i) \geq \sum_j P_j(S) \left[f_j + \sum_{i \in \mathcal{I}_j} \left(v_{i,t+1}(r_i - 1) - v_{i,t+1}(r_i) \right) \right] + \sum_i v_{i,t+1}(r_i) \qquad (6) \\
& \qquad \forall \boldsymbol{r} \in \mathcal{R}, S \subset \mathcal{Q}(\boldsymbol{r}), t \\
& v_{i,\tau+1}(\cdot) = 0.
\end{aligned}
$$

Since the piecewise-linear approximation uses a more refined approximation architecture than the affine approximation, it is natural to expect that it obtains a tighter upper bound on the value function. Meissner and Strauss [14] show $V^{DP} \leq V^{PL} \leq V^{AF} \leq V^{CDLP}$.

Lemma 1 below shows that an optimal solution to PL satisfies certain monotonicity properties. If we interpret $v_{i,t}(r_i)$ as the value of having r_i units of resource i at time period t, then $v_{i,t}(r_i) - v_{i,t}(r_i - 1)$ can be interpreted as the marginal value of the r_ith unit of the resource at time period t. Part (i) of the lemma shows that the marginal value of capacity is decreasing in t keeping r_i constant; part (ii) of the lemma shows that the marginal value of capacity is decreasing in r_i for a fixed t; parts (iii) and (iv) show that the value of capacity is increasing in r_i and decreasing in t.

Lemma 1. *There exists an optimal solution $\hat{v} = \{\hat{v}_{i,t}(r_i) \,|\, \forall t, i, r_i \in \mathcal{R}_i\}$ to PL such that*
(i) $\hat{v}_{i,t}(r_i) - \hat{v}_{i,t}(r_i - 1) \geq \hat{v}_{i,t+1}(r_i) - \hat{v}_{i,t+1}(r_i - 1)$ for all t, i and $r_i \in \mathcal{R}_i \backslash \{0\}$.
(ii) $\hat{v}_{i,t}(r_i) - \hat{v}_{i,t}(r_i - 1) \geq \hat{v}_{i,t}(r_i + 1) - \hat{v}_{i,t}(r_i)$ for all t, i and $r_i \in \mathcal{R}_i \backslash \left\{0, r_i^1\right\}$.
(iii) $\hat{v}_{i,t}(r_i) \geq \hat{v}_{i,t}(r_i - 1)$ for all t, i and $r_i \in \mathcal{R}_i \backslash \{0\}$.
(iv) $\hat{v}_{i,t}(r_i) \geq \hat{v}_{i,t+1}(r_i)$ for all t, i and $r_i \in \mathcal{R}_i$.

Proof. Appendix. □

The monotonicity properties described in Lemma 1 are intuitive and satisfied as well by the single resource RM problem and the network RM problem with independent demands (Talluri and van Ryzin [21] and Kunnumkal and Talluri [9]). So it is reassuring that the monotonicity properties continue to hold for an approximation to the choice network RM. Also, by Lemma 1, we can add the constraints described in parts (i)-(iv) of the lemma without affecting the optimal objective function value of PL. This can potentially speed up its solution time (Zhang and Adelman [25]). The monotonicity properties turn out to be useful in showing the equivalence between the piecewise-linear approximation and a Lagrangian relaxation of the choice network RM problem, as we shall see shortly. Finally, Lemma 1 is helpful in showing that the $CDLP$ bound is no more than twice the PL upper bound. We note that this is not an asymptotic-type relation, and does not require any demand or capacity scaling. Moreover, it holds for a general discrete-choice model.

Proposition 1. $V^{CDLP} \leq 2V^{PL}$.

Proof. By associating dual variables $\gamma = \{\gamma_i \,|\, \forall i\}$ and $\beta = \{\beta_t \,|\, \forall t\}$ with constraints (3) and (4) in $CDLP$, respectively, we write the dual of $CDLP$ as

$$\begin{aligned} V^{CDLP} = \quad & \min_{\beta,\gamma} \sum_t \beta_t + \sum_i \gamma_i r_i^1 \\ \text{s.t} \quad & \\ (dCDLP) \quad & \sum_t \beta_t \geq R(S) - \sum_i \varrho_i(S)\gamma_i \quad \forall i \qquad (7) \\ & \gamma_i \geq 0 \quad \forall i, \end{aligned}$$

where $\varrho_i(S) = \sum_{j\in\mathcal{J}_i} P_j(S)$.

Let $\hat{v} = \{\hat{v}_{i,t}(r_i) \,|\, \forall t, i, r_i \in \mathcal{R}_i\}$ denote an optimal solution to PL, and consider $\hat{\beta}_t = \sum_i[\hat{v}_{i,t}(r_i^1) - \hat{v}_{i,t+1}(r_i^1)]$ and $\hat{\gamma}_i = \hat{v}_{i,1}(r_i^1) - \hat{v}_{i,1}(r_i^1 - 1)$ (≥ 0 by Lemma 1, part (i)). We assume that $r_i^1 > 0$ for all i so that $\mathcal{Q}(\boldsymbol{r}^1) = \mathcal{J}$. We have

$$\begin{aligned} \hat{\beta}_t &= \sum_i [\hat{v}_{i,t}(r_i^1) - \hat{v}_{i,t+1}(r_i^1)] \\ &\geq R(S) - \sum_i \varrho_i(S)[\hat{v}_{i,t+1}(r_i^1) - \hat{v}_{i,t+1}(r_i^1 - 1)] \\ &\geq R(S) - \sum_i \varrho_i(S)[\hat{v}_{i,1}(r_i^1) - \hat{v}_{i,1}(r_i^1 - 1)] \\ &= R(S) - \sum_i \varrho_i(S)\hat{\gamma}_i \end{aligned}$$

where the first inequality uses the fact that $\hat{v}$ satisfies constraint (6) and the last inequality follows from part (i) of Lemma 1, which implies that $\hat{v}_{i,1}(r_i^1) - \hat{v}_{i,1}(r_i^1 - 1) \geq \hat{v}_{i,t+1}(r_i^1) - \hat{v}_{i,t+1}(r_i^1 - 1)$.

Therefore, $(\hat{\beta}, \hat{\gamma})$ is feasible to $dCDLP$ and $V^{CDLP} \leq \sum_t \hat{\beta}_t + \sum_i \hat{\gamma}_i r_i^1 = \sum_i \hat{v}_{i,1}(r_i^1) + \sum_i r_i^1[\hat{v}_{i,1}(r_i^1) - \hat{v}_{i,1}(r_i^1 - 1)]$. On the other hand, part (ii) of Lemma 1 implies that $\hat{v}_{i,t}(\cdot)$ is concave. As a result

$$\hat{v}_{i,1}(r_i^1 - 1) \geq \frac{1}{r_i^1}\hat{v}_{i,1}(0) + \frac{r_i^1 - 1}{r_i^1}\hat{v}_{i,1}(r_i^1) \geq \hat{v}_{i,1}(r_i^1) - \frac{1}{r_i^1}\hat{v}_{i,1}(r_i^1),$$

where the last inequality holds since $\hat{v}_{i,t}(0) \geq 0$ (Lemma 1, part (iv)). Therefore, $\hat{v}_{i,1}(r_i^1) \geq r_i^1[\hat{v}_{i,1}(r_i^1) - \hat{v}_{i,1}(r_i^1 - 1)]$, which implies that $V^{CDLP} \leq 2\sum_i \hat{v}_{i,1}(r_i^1) = 2V^{PL}$. □

3 Piecewise-linear and the Lagrangian relaxation

The number of constraints in PL is exponential in the number of resources as well as the number of products. So, it faces the same tractability issues as the affine approximation. In this section, we consider the Lagrangian relaxation approach to approximate dynamic programming as a more tractable alternative. We first show that a Lagrangian relaxation approach using product and time-specific Lagrange multipliers is weaker than the piecewise-linear approximation. We then consider a Lagrangian relaxation approach that uses Lagrange multipliers for every time period and offer set and show that it obtains the same upper bound as the piecewise-linear approximation.

3.1 Lagrangian relaxation using product-specific multipliers

For the network RM problem with independent demands, it is known that the piecewise-linear approximation is equivalent to a Lagrangian relaxation approach that decomposes the network problem into a number of single resource problems by allocating the revenue of each product across the resources that it uses (Kunnumkal and Talluri [9]). A natural extension of that Lagrangian relaxation approach to choice network RM is to use product and time-specific Lagrange multipliers $\lambda = \{\lambda_{i,j,t} \mid \sum_{i \in \mathcal{I}_j} \lambda_{i,j,t} = f_j, \forall t, j\}$ to decompose the network problem into a number of single resource problems. We interpret $\lambda_{i,j,t}$ as the portion of fare of product j that is allocated to resource $i \in \mathcal{I}_j$, and the constraint $\sum_{i \in \mathcal{I}_j} \lambda_{i,j,t} = f_j$ ensures that the Lagrange multipliers correspond to a valid fare allocation. Letting

$$\mathcal{Q}_i(r_i) = \left\{ j \mid \mathbb{1}_{[j \in \mathcal{J}_i]} \leq r_i \right\},$$

we solve the optimality equation

$$\nu^{\lambda}_{i,t}(r_i) = \max_{S \subset \mathcal{Q}_i(r_i)} \sum_{j \in \mathcal{J}_i} P_j(S) \left[\lambda_{i,j,t} + \nu^{\lambda}_{i,t+1}(r_i - 1) - \nu^{\lambda}_{i,t+1}(r_i) \right] + \nu^{\lambda}_{i,t+1}(r_i) \tag{8}$$

for resource i. It is possible to show that $\sum_i \nu^{\lambda}_{i,t}(r_i)$ is an upper bound on $V_t(\boldsymbol{r})$ (Kunnumkal and Topaloglu [11]). So, we can find the tightest upper bound on the optimal expected revenue by solving

$$V^{LRp} = \min_{\left\{ \lambda \mid \sum_{i \in \mathcal{I}_j} \lambda_{i,j,t} = f_j, \, \forall j,t \right\}} \sum_i \nu^{\lambda}_{i,1}(r^1_i).$$

In contrast to the independent demands setting, as the following example illustrates, we can have $V^{PL} < V^{LRp}$.

Example 1: Consider a network revenue management problem with two products, two resources and a single time period in the booking horizon. The first product uses only the first resource, while the second product uses only the second resource, and we have a single unit of capacity on each resource. Note that in the airline context, this example corresponds to a parallel flights network. The revenues associated with the products are $f_1 = 10$ and $f_2 = 1$. The choice probabilities are given in Table 1. In the Lagrangian relaxation, since we have Lagrange multipliers only for $j \in \mathcal{J}_i$, we have only two multipliers $\lambda_{1,1,1}$ and $\lambda_{2,2,1}$. Moreover, the constraint $\sum_{i \in \mathcal{I}_j} \lambda_{i,j,t} = f_j$ implies that $\lambda_{1,1,1} = f_1$ and $\lambda_{2,2,1} = f_2$. Noting that there is only a single time period in the booking horizon and that $\mathcal{Q}_i(1) = \mathcal{J}$ for $i = 1, 2$,

$$\nu^{\lambda}_{1,1}(1) = \max_{S \subset \mathcal{J}} \{P_1(S) f_1\} = 5$$

and

$$\nu^{\lambda}_{2,1}(1) = \max_{S \subset \mathcal{J}} \{P_2(S) f_2\} = 10/11$$

so that $V^{LRp} = 65/11$.

On the other hand, the linear program associated with the piecewise-linear approximation is

$$\begin{aligned} V^{PL} = \min_v \quad & v_{1,1}(1) + v_{2,1}(1) \\ \text{s.t} \quad & \\ & v_{1,1}(1) + v_{2,1}(1) \geq \max\{P_1(\{1,2\}) f_1 + P_2(\{1,2\}) f_2, P_1(\{1\}) f_1, P_2(\{2\}) f_2, 0\} \\ & v_{1,1}(1) + v_{2,1}(0) \geq \max\{P_1(\{1\}) f_1, 0\} \\ & v_{1,1}(0) + v_{2,1}(1) \geq \max\{P_2(\{2\}) f_2, 0\} \\ & v_{1,1}(0) + v_{2,1}(0) \geq 0. \end{aligned}$$

S	$P_1(S)$	$P_2(S)$
$\{1\}$	1/2	0
$\{2\}$	0	10/11
$\{1,2\}$	1/12	10/12

Table 1: Choice probabilities

Note that the first, second, third and fourth constraints correspond to the states vectors $[1,1], [1,0], [0,1]$ and $[0,0]$, respectively. It is easy to verify that an optimal solution to PL is $v_{1,1}(1) = 5, v_{1,1}(0) = 10/11, v_{2,1}(1) = 0, v_{2,1}(0) = 0$ and we have $V^{PL} = 5 < V^{LRp}$.

3.2 Lagrangian relaxation using offer set-specific multipliers

Next, we consider a Lagrangian relaxation approach with an expanded set of multipliers. We introduce some notation first.

For a resource i and offer set S, we let $\mathcal{I}_S = \{i \mid \exists j \in S \text{ with } j \in \mathcal{J}_i\}$. Therefore, $\mathcal{I}_S$ consists of the resources used by the products in S. We follow the convention that the empty set does not use any resource. We also let $\mathcal{C}_i = \{S \mid i \in \mathcal{I}_S\}$ and note that $i \in \mathcal{I}_S$ if and only if $S \in \mathcal{C}_i$. Let $\lambda_{i,S,t}$ be the portion of the revenue associated with offer set S allocated to resource $i \in \mathcal{I}_S$ and $\lambda_{\phi,S,t} = R(S) - \sum_{i \in \mathcal{I}_S} \lambda_{i,S,t}$ denote the difference between the revenue associated with the offer set and the allocations across the resources. Fixing the multipliers $\lambda = \{\lambda_{\phi,S,t}, \lambda_{i,S,t} \mid \forall t, S, i \in \mathcal{I}_S\}$ we solve the optimality equation

$$\vartheta^{\lambda}_{i,t}(r_i) = \max_{S \subset \mathcal{Q}_i(r_i)} \left\{ \mathbb{1}_{[S \in \mathcal{C}_i]} \lambda_{i,S,t} + \sum_{j \in \mathcal{J}_i} P_j(S) \left[\vartheta^{\lambda}_{i,t+1}(r_i - 1) - \vartheta^{\lambda}_{i,t+1}(r_i) \right] + \vartheta^{\lambda}_{i,t+1}(r_i) \right\} \tag{9}$$

for resource i, with the boundary condition that $\vartheta^{\lambda}_{i,\tau+1}(\cdot) = 0$.

Define recursively

$$\vartheta^{\lambda}_{\phi,t} = \max_{S \subset \mathcal{J}} \{\lambda_{\phi,S,t}\} + \vartheta^{\lambda}_{\phi,t+1}, \tag{10}$$

with the boundary condition that $\vartheta^{\lambda}_{\phi,\tau+1} = 0$. Also define

$$\Lambda = \left\{ \lambda \mid \lambda_{\phi,S,t} + \sum_{i \in \mathcal{I}_S} \lambda_{i,S,t} = R(S), \, \forall S \subset \mathcal{J}, t \right\} \tag{11}$$

and note that Λ is a convex set. Letting $V^{\lambda}_t(\boldsymbol{r}) = \sum_i \vartheta^{\lambda}_{i,t}(r_i) + \vartheta^{\lambda}_{\phi,t}$, Lemma 2 below shows that the Lagrangian relaxation obtains an upper bound on the optimal expected revenue provided the multipliers lie in the set Λ, and that this bound is potentially weaker than the piecewise-linear bound.

Lemma 2.

$$V^{DP} \le V^{PL} \le V^{\lambda}_1(\boldsymbol{r}^1) \textit{ for } \lambda \in \Lambda.$$

Proof. Appendix. □

We find the tightest upper bound on the optimal expected revenue by solving

$$V^{LRo} = \min_{\lambda \in \Lambda} V_1^{\lambda}\left(\boldsymbol{r}^1\right).$$

The following proposition gives the subgradients of $\vartheta_{i,t}^{\lambda}(r_i)$ and $\vartheta_{\phi,t}^{\lambda}$ (thereby also showing they are convex functions of λ). It follows that $V_t^{\lambda}(\boldsymbol{r})$ is also a convex function of λ and hence the above optimization problem can be efficiently solved as a convex program.

We introduce some notation for this purpose. Let $S_{i,t}^{\lambda}(r_i)$ denote an optimal solution to problem (9) where the arguments emphasize the dependence of the optimal offer set on the Lagrange multipliers and the remaining capacity on the resource. We define $S_{\phi,t}^{\lambda}$ in a similar manner for (10). Also, for a given λ let $X_{i,k}^{\lambda}$ denote the random variable which represents the remaining capacity on resource i at time period k when we offer sets that maximize the right hand side of (9) at each time period. We have the following result.

Proposition 2. *Let λ and $\hat{\lambda}$ be two sets of Lagrange multipliers. Then,*

1. $\vartheta_{i,t}^{\lambda}(r_i) \geq \vartheta_{i,t}^{\hat{\lambda}}(r_i) + \sum_{k=t}^{\tau} \sum_S \mathbb{1}_{[S \in \mathcal{C}_i]} \Pr\{S_{i,t}^{\lambda}(X_{i,k}^{\lambda}) = S \mid X_{i,t}^{\lambda} = r_i\} \left[\hat{\lambda}_{i,S,k} - \lambda_{i,S,k}\right].$

2. $\vartheta_{\phi,t}^{\lambda} \geq \vartheta_{\phi,t}^{\hat{\lambda}} + \sum_{k=t}^{\tau} \sum_S \mathbb{1}_{\left[S_{\phi,k}^{\lambda} = S\right]} [\hat{\lambda}_{\phi,S,k} - \lambda_{\phi,S,k}].$

Proof. Appendix. □

We note that besides showing that $V_1^{\lambda}(\boldsymbol{r})$ is a convex function of λ, Proposition 2 also gives an explicit expression for its subgradient. This allows us to use subgradient search to find the optimal set of Lagrange multipliers (Bertsekas [2]). We show in the following section, that by doing this, we in fact obtain the piecewise-linear bound.

4 Solving piecewise-linear approximation by Lagrangian relaxation

In this section we show that the upper bounds obtained by the piecewise-linear relaxation and the Lagrangian relaxation using offer-set specific multipliers coincide. Hence we can solve the latter instead of the former. The Lagrangian relaxation approach certain advantages: given a set of Lagrange multipliers, the problem decomposes into a number of single resource dynamic programs, which are relatively more tractable compared to the network DP. Furthermore, we can find the optimal set of Lagrange multipliers by subgradient search.

The Lagrangian bound can alternatively be obtained by solving the linear program

$$
\begin{aligned}
V^{LRo} = \min_{\lambda,v} \quad & \sum_i v_{i,1}(r_i^1) + v_{\phi,1} \\
\text{s.t.} \quad & \\
(LRo) \quad & v_{i,t}(r_i) \geq \mathbb{1}_{[S\in\mathcal{C}_i]}\lambda_{i,S,t} + \sum_{j\in\mathcal{J}_i} P_j(S)\left[v_{i,t+1}(r_i-1) - v_{i,t+1}(r_i)\right] \\
& \qquad + v_{i,t+1}(r_i) \quad \forall t, i, r_i \in \mathcal{R}_i, S \subset \mathcal{Q}_i(r_i) \\
& v_{\phi,t} \geq \lambda_{\phi,S,t} + v_{\phi,t+1} \quad \forall t, S \\
& \sum_{i\in\mathcal{I}_S} \lambda_{i,S,t} + \lambda_{\phi,S,t} = R(S) \quad \forall t, S \\
& v_{i,\tau+1}(\cdot) = 0, v_{\phi,\tau+1} = 0.
\end{aligned}
$$

Note that while the linear program LRo resembles PL, the number of constraints is reduced from a factor of $\prod_i r_i^1$ to $\sum_i r_i^1$.

While computing V^{LRo} using the subgradient search algorithm is typically more efficient than solving the linear program LRo, the linear programming formulation LRo is helpful when we compare the Lagrangian bound V^{LRo} with the piecewise-linear approximation bound V^{PL}.

Now we state the main result of the paper that makes the connection between LRo and PL.

Proposition 3.

$$V^{PL} = V^{LRo}.$$

The proof of this proposition is quite intricate so we first give a simpler (and non-rigorous) explanation using only calculus in §4.1. In the subsequent §4.2 we follow up with the rigorous proof.

4.1 An intuitive explanation for tractability of the piecewise-linear relaxation

In this section we give a simple calculus-based explanation for the equivalence of the Lagrangian relaxation and the piecewise-linear approximation. Besides providing intuition, it also gives a heuristic method for initializing the Lagrange multipliers, that turns out to be useful from a computational standpoint.

Since PL has an exponential number of constraints, we have to solve it by generating constraints on the fly. The separation problem for PL is, for each period t, given values of $\{v_{i,t}(r_i) \mid \forall t, i, r_i \in \mathcal{R}_i\}$, to check if

$$\Phi_t(v) = \max_{\boldsymbol{r}\in\mathcal{R}, S\subset\mathcal{Q}(\boldsymbol{r})} \sum_{j\in S} P_j(S)\left[f_j + \sum_{i\in\mathcal{I}_j}\left[v_{i,t+1}(r_i-1) - v_{i,t+1}(r_i)\right]\right] + \sum_i \left[v_{i,t+1}(r_i) - v_{i,t}(r_i)\right] \quad (12)$$

is greater than or equal to zero. If yes, constraint (6) is satisfied at time period t for all $\boldsymbol{r}$ and $S \subset \mathcal{Q}(\boldsymbol{r})$. Otherwise, we find a violated constraint and add it to the linear program. Recall that $R(S) = \sum_j P_j(S) f_j$ is the expected revenue from offering set S. Letting $\varrho_i(S) = \sum_{j\in\mathcal{J}_i} P_j(S)$,

$\psi_{i,t+1}(r_i) = v_{i,t+1}(r_i) - v_{i,t+1}(r_i - 1)$ and $\Delta_{i,t}(r_i) = v_{i,t+1}(r_i) - v_{i,t}(r_i)$, $\Phi_t(v)$ can be equivalently written as

$$\Phi_t(v) = \max_{\boldsymbol{r}\in\mathcal{R}, S\subset\mathcal{Q}(\boldsymbol{r})} R(S) - \sum_i \psi_i(r_i)\varrho_i(S) + \sum_i \Delta_{i,t}(r_i).$$

Note that $\varrho_i(S)$ is the expected capacity consumed on resource i when set S is offered, while $\psi_{i,t+1}(r_i)$ is the marginal value of capacity at time period $t+1$. The key part of the proof relies on showing that $\Phi_t(v)$ is equivalent to the optimization problem

$$\Pi_t(v) = \min_{\lambda\in\Lambda}\left\{\sum_i \Pi_{i,t}(v,\lambda) + \Pi_{\phi,t}(v,\lambda)\right\}.$$

where $\Pi_{i,t}(v,\lambda) = \max_{r_i\in\mathcal{R}_i, S\subset\mathcal{Q}_i(r_i)}\{\mathbb{1}_{[S\in\mathcal{C}_i]}\lambda_{i,S,t} - \psi_{i,t+1}(r_i)\varrho_i(S) + \Delta_{i,t}(r_i)\}$ and $\Pi_{\phi,t}(v,\lambda) = \max_{S\subset\mathcal{J}}\{\lambda_{\phi,S,t}\}$. That is, $\Phi_t(v)$ can be decomposed into a number of single resource problems by allocating the revenues associated with each offer set across the resources using the Lagrange multipliers $\{\lambda_{i,S,t}, \lambda_{\phi,S,t} \,|\, \forall i \in \mathcal{I}_S, S\}$. Once we establish this result, the rest of the proof proceeds by noting the correspondence between $\Pi_t(v)$ and the constraints in *LRo*.

Here we give a heuristic argument to provide intuition behind the result. Using binary variables $\{h_S \,|\, \forall S\}$, $\Phi_t(v)$ can be written as

$$\begin{aligned} \Phi_t(v) \quad &= \quad \max_{\boldsymbol{r}}\max_{h}\sum_S\left[R(S) - \sum_i \psi_{i,t+1}(r_i)\varrho_i(S)\right]h_S + \sum_i \Delta_{i,t}(r_i) \\ \text{s.t} \quad & \quad \mathbb{1}_{[S\in\mathcal{C}_i]}h_S \le r_i, \quad \forall i \\ & \quad \sum_S h_S = 1 \\ & \quad h_S \in \{0,1\}. \end{aligned} \tag{13}$$

Now, for a fixed integer $\boldsymbol{r}$ the inner maximization problem (over the h_S variables) is an integer linear programming problem. However, note that as the constraint set has a totally unimodular structure (if we write the second set of constraints as $-\sum_S h_S = -1$, each column has exactly one $+1$ and one -1). Therefore, for any integer $\boldsymbol{r}$ we can ignore the integrality requirements on h_S without affecting the optimal objective function value. For any fixed integer integer $\boldsymbol{r}$, let σ_t^* denote the optimal dual variable associated with the constraint $\sum_S h_S = 1$ (σ_t^* are hence functions of $\boldsymbol{r}$, but we suppress the dependence for brevity) and we can write the problem as

$$\Phi_t(v) \quad = \quad \max_{\boldsymbol{r}\in\mathcal{R}}\sum_S\left[R(S) - \sum_i \psi_{i,t+1}(r_i)\varrho_i(S) - \sigma_t^*\right]^+ + \sum_i \Delta_{i,t}(r_i) + \sigma_t^*$$

where $[x]^+ = \max\{x, 0\}$ and we use the facts that $\psi_{i,t+1}(0) = \infty$, and $h_S = 1$ if $R(S) - \sum_i \psi_i(r_i)\varrho_i(S) - \sigma_t^* > 0$ and $h_S = 0$ otherwise.

Moreover, from our argument on total modularity, the dual variable σ_t^* is such that exactly one set S has $R(S) - \sum_i \psi_i(r_i)\varrho_i(S) - \sigma_t^* > 0$ and this is a necessary and sufficient condition for optimality for the inner maximization problem of (13).

By Lemma 1 $\psi_{i,t+1}(\cdot)$ is non-increasing. Assuming (this is non-rigorous) it to be strictly decreasing and hence an invertible function of r_i, we can optimize over $\psi_i^d := \psi_{i,t+1}(r_i)$ instead of r_i, where the superscript indicates that ψ_i^d takes on discrete values since r_i is integer. We can write $\Phi_t(v)$ equivalently as

$$\Phi_t(v) \quad = \quad \max_{\{\psi^d \,|\, \psi_i^d \geq \psi_{i,t+1}(r_i^1),\, \forall i\}} \sum_S \left[R(S) - \sum_i \psi_i^d \varrho_i(S) - \sigma_t^* \right]^+ + \sum_i \Delta_{i,t}(\psi_i^d) + \sigma_t^*.$$

Note that σ_t^* are now functions of ψ_i^d via the inverse map.

We interpolate ψ_i^d with a continuous piecewise-linear function ψ_i and obtain a *relaxation* of $\Phi_t(v)$ in the following manner. Letting $\bar{\Delta}_{i,t}(\cdot)$ and $\bar{\sigma}_t^*$ denote linear interpolations of $\Delta_{i,t}(\cdot)$ and σ_t^*, we write a *relaxed* problem as

$$\begin{aligned} \bar{\Phi}_t(v) \quad &= \quad \max_{\{\psi \,|\, \psi_i \geq \psi_{i,t+1}(r_i^1),\, \forall i\}} \sum_S \left[R(S) - \sum_i \psi_i \varrho_i(S) - \bar{\sigma}_t^* \right]^+ + \sum_i \bar{\Delta}_{i,t}(\psi_i) + \bar{\sigma}_t^* \\ &= \quad \sum_S \left[R(S) - \sum_i \psi_i^* \varrho_i(S) - \bar{\sigma}_t^* \right]^+ + \sum_i \bar{\Delta}_{i,t}(\psi_i^*) + \bar{\sigma}_t^*, \end{aligned}$$

where $\psi^* = \{\psi_i^* \,|\, \forall i\}$ is an optimal solution to the above maximization problem. $\bar{\Phi}_t(v)$ is a relaxation since we optimize over a continuous space instead of a discrete grid.

Now assume that ψ^* is an interior optimal solution (again non-rigorous), so that it satisfies the first order set of conditions

$$-\sum_{S \in \mathcal{C}_i} \mathbb{1}_{\left[R(S) - \sum_i \psi_i^* \varrho_i(S) - \bar{\sigma}_t^* > 0\right]} \varrho_i(S) + \bar{\Delta}_{i,t}^{'}(\psi_i^*) = 0 \quad \forall i \in \mathcal{I} \tag{14}$$

where $\bar{\Delta}_{i,t}^{'}(\cdot)$ represents the first derivative of $\bar{\Delta}_{i,t}(\cdot)$, we use the fact that $\varrho_i(S) = 0$ for $S \notin \mathcal{C}_i$, and we invoke the envelope theorem to treat the optimal multipliers $\bar{\sigma}_t^*$ as constants,.

Next, we look at $\Pi_t(v)$. Introducing variables $\{h_{i,S} \,|\, \forall S\}$ we can write $\Pi_{i,t}(v, \lambda)$ equivalently as

$$\begin{aligned} \Pi_{i,t}(v,\lambda) = \quad & \max\nolimits_{r_i, h_i} \sum_{S \in \mathcal{C}_i} [\mathbb{1}_{[S \in \mathcal{C}_i]} \lambda_{i,S,t} - \psi_{i,t+1}(r_i) \varrho_i(S)] h_{i,S} + \Delta_{i,t}(r_i) \\ \text{s.t} \quad & \mathbb{1}_{[S \in \mathcal{C}_i]} h_{i,S} \leq r_i \\ & \textstyle\sum_S h_{i,S} = 1 \\ & r_i \in \mathcal{R}_i, h_{i,S} \in \{0,1\}. \end{aligned}$$

Relaxing the constraint $\sum_S h_{i,S} = 1$, we get

$$\begin{aligned} \bar{\Pi}_{i,t}(v,\lambda) = \quad & \max\nolimits_{r_i, h_i} \sum_{S \in \mathcal{C}_i} [\mathbb{1}_{[S \in \mathcal{C}_i]} \lambda_{i,S,t} - \psi_{i,t+1}(r_i) \varrho_i(S)] h_{i,S} + \Delta_{i,t}(r_i) \\ \text{s.t} \quad & \mathbb{1}_{[S \in \mathcal{C}_i]} h_{i,S} \leq r_i \\ & r_i \in \mathcal{R}_i, h_{i,S} \in \{0,1\}, \end{aligned}$$

and $\Pi_{i,t}(v,\lambda) \leq \bar{\Pi}_{i,t}(v,\lambda)$. Similarly, introducing variables $\{h_{\phi,S} \,|\, \forall S\}$ we can write $\Pi_{\phi,t}(v,\lambda)$ as

$$\begin{aligned} \Pi_{\phi,t}(v,\lambda) \quad & = \max\nolimits_{h_\phi} \textstyle\sum_S \lambda_{\phi,S,t} h_{\phi,S} \\ \text{s.t} \quad & \textstyle\sum_S h_{\phi,S} = 1 \\ & h_{\phi,S} \in \{0,1\}. \end{aligned}$$

Relaxing the constraint $\sum_S h_{\phi,S} = 1$ by associating $\bar{\sigma}_t^*$ as the corresponding multiplier, we get $\bar{\Pi}_{\phi,t}(v,\lambda) = \max_{h_{\phi,S}\in\{0,1\}} \sum_S [\lambda_{\phi,S,t} - \bar{\sigma}_t^*]h_{\phi,S} + \bar{\sigma}_t^*$, and $\Pi_{\phi,t}(v,\lambda) \leq \bar{\Pi}_{\phi,t}(v,\lambda)$. Letting

$$\bar{\Pi}_t(v) = \quad \min_{\lambda\in\Lambda} \left\{ \sum_i \bar{\Pi}_{i,t}(v,\lambda) + \bar{\Pi}_{\phi,t}(v,\lambda) \right\},$$

we have $\Pi_t(v) \leq \bar{\Pi}_t(v)$. Applying the same heuristic arguments as before to $\bar{\Pi}_{i,t}(v,\lambda)$, we have

$$\bar{\Pi}_{i,t}(v,\lambda) = \quad \max_{\left\{\psi_i \,|\, \psi_i \geq \psi_{i,t+1}(r_i^1)\right\}} \sum_{S\in\mathcal{C}_i} \left[\mathbb{1}_{[S\in\mathcal{C}_i]}\lambda_{i,S,t} - \psi_i \varrho_i(S)\right]^+ + \bar{\Delta}_{i,t}(\psi_i), \tag{15}$$

and an interior optimal solution satisfies the first order condition

$$-\sum_{S\in\mathcal{C}_i} \mathbb{1}_{[\lambda_{i,S,t}-\psi_i\varrho_i(S)>0]}\varrho_i(S) + \bar{\Delta}_{i,t}^{'}(\psi_i) = 0. \tag{16}$$

Now, consider the Lagrange multipliers $\hat{\lambda} = \left\{\hat{\lambda}_{i,S,t} \,|\, \forall S, i\in\mathcal{I}_S\right\}$ with

$$\hat{\lambda}_{i,S,t} = \frac{\psi_i^*\varrho_i(S)[R(S)-\bar{\sigma}_t^*]}{\sum_{l\in\mathcal{I}_S}\psi_l^*\varrho_l(S)}, \quad i\in\mathcal{I}_S \text{ and } \hat{\lambda}_{\phi,S,t} = \sigma_t^*, \ \forall S. \tag{17}$$

Note that $\hat{\lambda}_{i,S,t} - \psi_i^*\varrho_i(S) = \psi_i^*\varrho_i(S)\frac{R(S)-\sum_{l\in\mathcal{I}_S}\psi_l^*\varrho_l(S)-\bar{\sigma}_t^*}{\sum_{l\in\mathcal{I}_S}\psi_l^*\varrho_l(S)}$. Therefore, $\forall i$

$$\mathbb{1}_{\left[\hat{\lambda}_{i,S,t}-\psi_i^*\varrho_i(S)>0\right]} = \mathbb{1}_{\left[R(S)-\sum_{l\in\mathcal{I}_S}\psi_l^*\varrho_l(S)-\bar{\sigma}_t^*>0\right]}.$$

Consequently ψ_i^* satisfies the first order optimality condition (16) of the inner maximization problem of the relaxed Lagrangian problem. Moreover as S is the unique set with $\mathbb{1}_{\left[\hat{\lambda}_{i,S,t}-\psi_i^*\varrho_i(S)>0\right]}$, we achieve optimality.

So we have

$$\bar{\Pi}_{i,t}(v,\hat{\lambda}) = \sum_S \left[\mathbb{1}_{[S\in\mathcal{C}_i]}\hat{\lambda}_{i,S,t} - \psi_i^*\varrho_i(S)\right]^+ + \bar{\Delta}_{i,t}(\psi_i^*).$$

From this we can conclude

$$\begin{aligned}
\bar{\Pi}_t(v) &\leq \sum_i \bar{\Pi}_{i,t}(v,\hat{\lambda}) + \bar{\Pi}_{\phi,t}(v,\hat{\lambda}) \\
&= \sum_i\sum_S [\mathbb{1}_{[S\in\mathcal{C}_i]}\hat{\lambda}_{i,S,t} - \psi_i^*\varrho_i(S)]^+ + \sum_i \bar{\Delta}_{i,t}(\psi_i^*) + \bar{\sigma}_t^* \\
&= \sum_S\sum_{i\in\mathcal{I}_S} \psi_i^*\varrho_i(S)\frac{[R(S)-\sum_{l\in\mathcal{I}_S}\psi_l^*\varrho_l(S)-\bar{\sigma}_t^*]^+}{\sum_{l\in\mathcal{I}_S}\psi_l^*\varrho_l(S)} + \sum_i \bar{\Delta}_{i,t}(\psi_i^*) + \bar{\sigma}_t^* \\
&= \sum_S [R(S) - \sum_{l\in\mathcal{I}_S}\psi_l^*\varrho_l(S) - \bar{\sigma}_t^*]^+ + \sum_i \bar{\Delta}_{i,t}(\psi_i^*) + \bar{\sigma}_t^* \\
&= \bar{\Phi}_t(v)
\end{aligned}$$

where the inequality follows since $\hat{\lambda}$ is feasible to $\bar{\Pi}_t(v)$. Putting everything together, $\Phi_t(v) \leq \Pi_t(v) \leq \bar{\Pi}_t(v) \leq \bar{\Phi}_t(v)$, where the first inequality holds since $\Pi_t(v)$ is a relaxation of $\Phi_t(v)$ (Lemma 2).

At first glance, it is a bit surprising that we are able to show a strong duality result ($\Phi_t(v) = \bar{\Phi}_t(v)$) for an optimization problem on integers. Next, we outline the reason for this.

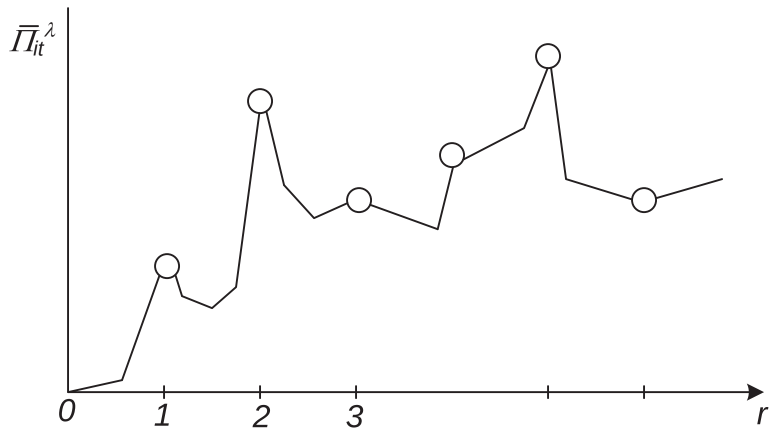

Figure 1: Shape of the relaxed (continuous) Lagrangian value function for a fixed λ for piecewise-linear functions $\psi(r)$ and $\Delta(r)$ with integer breakpoints. The function need not be convex or concave, but is piecewise-linear and convex between integer breakpoints. The maximizer therefore happens at an integer r and can be found by a simple search over r as it is a one-dimensional function.

The Lagrangian relaxation has a rather peculiar property: it so turns out that all local maxima of the Lagrangian occur at integers! To see this, consider (15) for a fixed λ and the shape of the function between two adjacent integer points: the first term in the objective is a convex function of r_i (via ψ_i which decreases as r_i increases as $\psi(r)$ is a decreasing function; hence the slope of the first term increases with increasing r_i) while the second term is a linear function ($\bar{\Delta}_{i,t}(\cdot)$ being a piece-wise linear function with breakpoints at adjacent integer values of r_i). Consequently, the objective function is *convex* between adjacent breakpoints of the piecewise-linear functions and the maximum value must occur at an integer; see Figure 1.

So the implication of the form seen in Figure 1 is that all the optima happen only at integer points so it gives us $\Phi_t(v) = \bar{\Phi}_t(v)$, which implies that $\Phi_t(v) = \Pi_t(v)$.

The simple calculus-based reasoning provides some intuition behind Proposition 3. Moreover, equation (17) also gives us a way of initializing the Lagrange multipliers. This turns out to be useful in our computational study. We emphasize that the arguments in this section constitute heuristic reasoning; in the next section we give a rigorous combinatorial proof of Proposition 3.

4.2 Proof of Proposition 3

We give a formal proof of Proposition 3. Lemma 2 implies that $V^{PL} \leq \min_{\lambda \in \Lambda} V_1^{\lambda}(\boldsymbol{r}^1) = V^{LRo}$. So it only remains to show the more difficult part: $V^{PL} \geq V^{LRo}$. The proof proceeds by considering the PL separation problem (12), which can be written as

$$\Phi_t(v) = \max_{\boldsymbol{r} \in \mathcal{R}, S \subset \mathcal{Q}(\boldsymbol{r})} \sum_j P_j(S) \left[f_j - \sum_{i \in \mathcal{I}_j} \psi_{i,t+1}(r_i) \right] + \sum_i \Delta_{i,t}(r_i),$$

where we recall that $\psi_{i,t+1}(r_i) = v_{i,t+1}(r_i) - v_{i,t+1}(r_i - 1)$ and $\Delta_{i,t}(r_i) = v_{i,t+1}(r_i) - v_{i,t}(r_i)$. Using (5) and the facts that $P_j(S) = 0$ for $j \notin S$, and $\mathbb{1}_{[S \in \mathcal{C}_i]} = 1$ for $j \in S$ and $i \in \mathcal{I}_j$, we can write $\Phi_t(v)$

as

$$\Phi_t(v) = \max_{\boldsymbol{r}\in\mathcal{R}, S\subset\mathcal{Q}(\boldsymbol{r})} R(S) - \sum_{j\in S} P_j(S) \sum_{i\in\mathcal{I}_j} \mathbb{1}_{[S\in\mathcal{C}_i]}\psi_{i,t+1}(r_i) + \sum_i \Delta_{i,t}(r_i). \tag{18}$$

Now consider the following Lagrangian relaxation of $\Phi_t(v)$, where for each time period t, we use offer set-specific multipliers to allocate the revenue associated with each offer set across the different resources and solve the optimization problem

$$\Pi_t(v) \quad = \quad \min_{\lambda\in\Lambda} \left\{ \sum_i \max_{r_i\in\mathcal{R}_i, S\subset\mathcal{Q}_i(r_i)} \left\{ \mathbb{1}_{[S\in\mathcal{C}_i]}\lambda_{i,S,t} - \sum_{j\in\mathcal{J}_i} P_j(S)\psi_{i,t+1}(r_i) + \Delta_{i,t}(r_i) \right\} + \max_{S\subset\mathcal{J}} \{\lambda_{\phi,S,t}\} \right\}$$

We show below that $\Phi_t(v) = \Pi_t(v)$ and use this result to show that $V^{PL} \geq V^{LRo}$. We begin with some preliminary results.

Lemma 3. $\Phi_t(v) \leq \Pi_t(v)$.

Proof. The proof is similar to that of Lemma 2. Let $\boldsymbol{r}^*, S^*$ be an optimal solution to $\Phi_t(v)$. Since $S^* \subset \mathcal{Q}(\boldsymbol{r}^*) \subset \mathcal{Q}_i(r_i^*)$, we have

$$\max_{r_i\in\mathcal{R}_i, S\subset\mathcal{Q}_i(r_i)} \left\{ \mathbb{1}_{[S\in\mathcal{C}_i]}\lambda_{i,S,t} - \sum_{j\in\mathcal{J}_i} P_j(S)\psi_{i,t+1}(r_i) + \Delta_{i,t}(r_i) \right\} \geq \mathbb{1}_{[S^*\in\mathcal{C}_i]}\lambda_{i,S^*,t} - \sum_{j\in\mathcal{J}_i} P_j(S^*)\psi_{i,t+1}(r_i^*) + \Delta_{i,t}(r_i^*).$$

Therefore, for any $\lambda \in \Lambda$, we have

$$\begin{aligned}
&\sum_i \max_{r_i\in\mathcal{R}_i, S\subset\mathcal{Q}_i(r_i)} \left\{ \mathbb{1}_{[S\in\mathcal{C}_i]}\lambda_{i,S,t} - \sum_{j\in\mathcal{J}_i} P_j(S)\psi_{i,t+1}(r_i) + \Delta_{i,t}(r_i) \right\} + \max_{S\subset\mathcal{J}} \{\lambda_{\phi,S,t}\} \\
&\geq \sum_i \left[\mathbb{1}_{[S^*\in\mathcal{C}_i]}\lambda_{i,S^*,t} - \sum_{j\in\mathcal{J}_i} P_j(S^*)\psi_{i,t+1}(r_i^*) + \Delta_{i,t}(r_i^*) \right] + \lambda_{\phi,S^*,t} \\
&= \sum_{i\in\mathcal{I}_{S^*}} \lambda_{i,S^*,t} + \lambda_{\phi,S^*,t} - \sum_j P_j(S^*) \sum_{i\in\mathcal{I}_j} \psi_{i,t+1}(r_i^*) + \sum_i \Delta_{i,t}(r_i^*) \\
&= R(S^*) - \sum_j P_j(S^*) \sum_{i\in\mathcal{I}_j} \psi_{i,t+1}(r_i^*) + \sum_i \Delta_{i,t}(r_i^*) = \Phi_t(v).
\end{aligned}$$

It follows that $\Phi_t(v) \leq \Pi_t(v)$. ☐

It remains to show that $\Phi_t(v) \geq \Pi_t(v)$. The following lemma shows that we can restrict ourselves to sets in $\mathcal{C}_i = \{S \,|\, i \in \mathcal{I}_S\}$ when solving the optimization problem for resource i.

Lemma 4.

$$\begin{aligned}
&\max_{r_i\in\mathcal{R}_i, S\subset Q_i(r_i)} \left\{ \mathbb{1}_{[S\in\mathcal{C}_i]}\lambda_{i,S,t} - \sum_{j\in\mathcal{J}_i} P_j(S)\psi_{i,t+1}(r_i) + \Delta_{i,t}(r_i) \right\} \\
&\qquad = \max \left\{ \Delta_{i,t}(0), \max_{r_i\in\{1,\ldots,r_i^1\}, S\in\mathcal{C}_i} \{\lambda_{i,S,t} - \sum_{j\in\mathcal{J}_i} P_j(S)\psi_{i,t+1}(r_i) + \Delta_{i,t}(r_i)\} \right\}.
\end{aligned}$$

Proof. Note that if $S \subset Q_i(0)$, then $j \notin i$ for all $j \in S$ and we have $S \notin \mathcal{C}_i$. Therefore, $\max_{S\subset Q_i(0)}\{\mathbb{1}_{[S\in\mathcal{C}_i]}\lambda_{i,S,t} - \sum_{j\in\mathcal{J}_i} P_j(S)\psi_{i,t+1}(0) + \Delta_{i,t}(0)\} = \Delta_{i,t}(0)$. For $r_i \in \{1,\ldots,r_{i,1}\}$, $Q_i(r_i) = \mathcal{J} = \mathcal{C}_i \cup \mathcal{C}_i^c$. We have

$$\begin{aligned}
&\max_{S\subset\mathcal{J}}\left\{\mathbb{1}_{[S\in\mathcal{C}_i]}\lambda_{i,S,t} - \sum_{j\in\mathcal{J}_i} P_j(S)\psi_{i,t+1}(r_i) + \Delta_{i,t}(r_i)\right\}\\
&\qquad= \max\left\{\max_{S\in\mathcal{C}_i}\left\{\lambda_{i,S,t} - \sum_{j\in\mathcal{J}_i} P_j(S)\psi_{i,t+1}(r_i) + \Delta_{i,t}(r_i)\right\}, \max_{S\in\mathcal{C}_i^c}\{\Delta_{i,t}(r_i)\}\right\}\\
&\qquad= \max\left\{\max_{S\in\mathcal{C}_i}\left\{\lambda_{i,S,t} - \sum_{j\in\mathcal{J}_i} P_j(S)\psi_{i,t+1}(r_i) + \Delta_{i,t}(r_i)\right\}, \Delta_{i,t}(r_i)\right\}.
\end{aligned}$$

Putting everything together

$$\begin{aligned}
&\max_{r_i\in\mathcal{R}_i, S\subset Q_i(r_i)}\{\mathbb{1}_{[S\in\mathcal{C}_i]}\lambda_{i,S,t} - \sum_{j\in\mathcal{J}_i} P_j(S)\psi_{i,t+1}(r_i) + \Delta_{i,t}(r_i)\}\\
&= \max\left\{\Delta_{i,t}(0), \max_{r_i\in\{1,\ldots,r_i^1\}}\left\{\max_{S\in\mathcal{C}_i}\left\{\lambda_{i,S,t} - \sum_{j\in\mathcal{J}_i} P_j(S)\psi_{i,t+1}(r_i) + \Delta_{i,t}(r_i)\right\}, \Delta_{i,t}(r_i)\right\}\right\}\\
&= \max\left\{\Delta_{i,t}(0), \max_{r_i\in\{1,\ldots,r_i^1\}, S\in\mathcal{C}_i}\left\{\lambda_{i,S,t} - \sum_{j\in\mathcal{J}_i} P_j(S)\psi_{i,t+1}(r_i) + \Delta_{i,t}(r_i)\right\}\right\}
\end{aligned}$$

where the last equality follows from $\Delta_{i,t}(\cdot)$ being a decreasing function (part (i) of Lemma 1), which implies that $\Delta_{i,t}(0)$ dominates $\Delta_{i,t}(r_i)$. □

Lemma 4 implies that we can write $\Pi_t(v)$ as the linear program

$$\begin{aligned}
\Pi_t(v) = \min_{\lambda,w} \quad & w_{\phi,t} + \sum_i w_{i,t} \\
\text{s.t.} \quad & \\
(LP_{\Pi_t(v)}) \quad & w_{i,t} \geq \Delta_{i,t}(0) \quad \forall i & (19)\\
& w_{i,t} \geq \lambda_{i,S,t} - \sum_{j\in\mathcal{J}_i} P_j(S)\psi_{i,t+1}(r) + \Delta_{i,t}(r) \quad \forall i, r\in\{1,\ldots,r_i^1\}, S\in\mathcal{C}_i & (20)\\
& w_{\phi,t} \geq \lambda_{\phi,S,t} \quad \forall S & (21)\\
& w_{\phi,t} \geq 0 & (22)\\
& \lambda_{\phi,S,t} + \sum_{i\in\mathcal{I}_S} \lambda_{i,S,t} = R(S) \quad \forall S. & (23)
\end{aligned}$$

In the following, we analyze the structure of a particular optimal solution to $LP_{\Pi_t(v)}$ that allows us to construct a feasible solution to $\Phi_t(v)$, which in turn shows that $\Phi_t(v) \geq \Pi_t(v)$. We introduce some notation that will be useful for this purpose. Given a solution (λ, w) to $LP_{\Pi_t(v)}$, let

$$\xi_{i,S,t}(r) = w_{i,t} - \left[\lambda_{i,S,t} - \sum_{j\in\mathcal{J}_i} P_j(S)\psi_{i,t+1}(r) + \Delta_{i,t}(r)\right]$$

denote the slack in constraint (20) for resource i, offer set $S \in \mathcal{C}_i$ and $r \in \{1, \ldots, r_i^1\}$. Let

$$B_{i,S}(\lambda, w) = \left\{ r \in \{1, \ldots, r_i^1\} \,|\, \xi_{i,S,t}(r) = 0 \right\}$$

denote the set of capacity levels for which constraint (20) is binding for resource i and offer set S. We use the arguments (λ, w) emphasize the dependence of the set of binding constraints on the given solution. Observe that if $B_{i,S}(\lambda, w)$ is empty, then $w_{i,t} > \lambda_{i,S,t} - \sum_{j \in \mathcal{J}_i} P_j(S)\psi_{i,t+1}(r) + \Delta_{i,t}(r)$ for all $r \in \{1, \ldots, r_{i,1}\}$.

From now on, we use a concept of optimal solutions with minimal set of binding constraints. The linear program $(LP_{\Pi_t(v)})$ has a finite optimal solution and possibly multiple ones. Naturally, any optimal solution has a set of binding constraints out of (19)–(23). Given any optimal solution, we can look for another optimal solution whose set of binding constraints is a strict subset of those of the previous optimal solution. If there is no such optimal solution, we consider that as having a minimal set of binding constraints. We have the following lemma.

Lemma 5. *Let $(\hat{\lambda}, \hat{w})$ be an optimal solution to $LP_{\Pi_t(v)}$ with a minimal number of binding constraints. Fix a set S. Either*
(i) $B_{i,S}(\hat{\lambda}, \hat{w})$ is nonempty for all $i \in \mathcal{I}_S$ and $\hat{w}_{\phi,t} = \hat{\lambda}_{\phi,S,t}$, or
(ii) $B_{i,S}(\hat{\lambda}, \hat{w})$ is empty for all $i \in \mathcal{I}_S$ and $\hat{w}_{\phi,t} > \hat{\lambda}_{\phi,S,t}$.

Proof. Suppose that the statement of the lemma is false. First consider the case where $B_{i,S}(\hat{\lambda}, \hat{w})$ is nonempty but $B_{l,S}(\hat{\lambda}, \hat{w})$ is empty for $i, l \in \mathcal{I}_S$. Let $\epsilon = \min_{r \in \{1, \ldots, r_l^1\}} \{\xi_{l,S,t}(r)\} > 0$. Let $(\widetilde{\lambda}, \widetilde{w})$ be given by $\widetilde{\lambda} = \hat{\lambda} - \delta \boldsymbol{e}^{i,S,t} + \delta \boldsymbol{e}^{l,S,t}$ and $\widetilde{w} = \hat{w}$ for some $\delta \in (0, \epsilon)$, where $\boldsymbol{e}^{i,j,k}$ is a vector with a 1 in component (i, j, k) and zeroes everywhere else. Note that $(\widetilde{\lambda}, \widetilde{w})$ is identical to $(\hat{\lambda}, \hat{w})$ except that $\widetilde{\lambda}_{i,S,t} = \hat{\lambda}_{i,S,t} - \delta$ and $\widetilde{\lambda}_{l,S,t} = \hat{\lambda}_{l,S,t} + \delta$. We show that $(\widetilde{\lambda}, \widetilde{w})$ is an optimal solution with a strictly fewer number of binding constraints which gives us a contradiction.

Notice that we only need to check constraint (20) for resources i and l and offer set S as all other resources and offer sets continue to have the same λ's and w's as before. For resource i, since $B_{i,S}(\hat{\lambda}, \hat{w})$ is nonempty, there exists $r \in \{1, \ldots, r_i^1\}$ such that $\hat{w}_{i,t} = \hat{\lambda}_{i,S,t} - \sum_{j \in \mathcal{J}_i} P_j(S)\psi_{i,t+1}(r) + \Delta_{i,t}(r)$. We have

$$\widetilde{w}_{i,t} = \hat{w}_{i,t} = \hat{\lambda}_{i,S,t} - \sum_{j \in \mathcal{J}_i} P_j(S)\psi_{i,t+1}(r) + \Delta_{i,t}(r) > \widetilde{\lambda}_{i,S,t} - \sum_{j \in \mathcal{J}_i} P_j(S)\psi_{i,t+1}(r) + \Delta_{i,t}(r),$$

which means that the number of binding constraints (20) for resource i and offer set S decreases by at least one. For resource l and offer set S, $\widetilde{w}_{l,t} - [\widetilde{\lambda}_{l,S,t} - \sum_{j \in \mathcal{J}_l} P_j(S)\psi_{l,t+1}(r) + \Delta_{l,t}(r)] = \hat{w}_{l,t} - [\hat{\lambda}_{l,S,t} - \sum_{j \in \mathcal{J}_l} P_j(S)\psi_{l,t+1}(r) + \Delta_{l,t}(r)] - \delta > 0$, for all $r \in \{1, \ldots, r_l^1\}$, where the inequality follows from the definition of δ. Therefore, all constraints (20) continue to be nonbinding for resource l and offer set S. Overall, $(\widetilde{\lambda}, \widetilde{w})$ has strictly fewer binding constraints than $(\hat{\lambda}, \hat{w})$, which gives a contradiction.

The above arguments imply that either $B_{i,S}(\hat{\lambda}, \hat{w})$ is nonempty for all $i \in \mathcal{I}_S$ or $B_{i,S}(\hat{\lambda}, \hat{w})$ is empty for all $i \in \mathcal{I}_S$. Suppose the $B_{i,S}(\hat{\lambda}, \hat{w})$ is nonempty for all $i \in \mathcal{I}_S$ but $\hat{w}_{\phi,t} > \hat{\lambda}_{\phi,St}$. In this case, pick a resource $i \in \mathcal{I}_S$ and let $\widetilde{\lambda} = \hat{\lambda} - \delta \boldsymbol{e}^{i,S,t} + \delta \boldsymbol{e}^{\phi,S,t}$ and $\widetilde{w} = \hat{w}$, where $\delta \in (0, \epsilon)$ and $\epsilon = \hat{w}_{\phi,t} - \hat{\lambda}_{\phi,S,t}$. It can be verified that the number of binding constraints (20) for resource i and offer set S strictly decreases from $(\hat{\lambda}, \hat{w})$ to $(\widetilde{\lambda}, \widetilde{w})$, while the number of binding constraints (21) remains unchanged, leading to a contradiction. This proves part (i) of the lemma.

On the other hand, if $B_{i,S}(\hat{\lambda}, \hat{w})$ is empty for all $i \in \mathcal{I}_S$ but $\hat{w}_{\phi,t} = \hat{\lambda}_{\phi,S,t}$. Pick a resource $i \in \mathcal{I}_S$. Since $B_{i,S}(\hat{\lambda}, \hat{w})$ is empty, we have $\epsilon = \min_{r\in\{1,\ldots,r_i^1\}}\{\xi_{i,S,t}(r)\} > 0$. Let $\widetilde{\lambda} = \hat{\lambda} + \delta \boldsymbol{e}^{i,S,t} - \delta \boldsymbol{e}^{\phi,S,t}$, where $\delta \in (0, \epsilon)$, and $\widetilde{w} = \hat{w}$. It can be verified that the number of binding constraints (20) for resource i and offer set S is exactly the same for $(\hat{\lambda}, \hat{w})$ and $(\widetilde{\lambda}, \widetilde{w})$, while the number of binding constraints (21) decreases by one ($\widetilde{w}_{\phi,t} = \hat{w}_{\phi,t} = \hat{\lambda}_{\phi,S,t} > \widetilde{\lambda}_{\phi,S,t}$). As a result, $(\widetilde{\lambda}, \widetilde{w})$ has strictly fewer binding constraints than $(\hat{\lambda}, \hat{w})$, which gives a contradiction. This proves part (ii) of the lemma. □

In the remainder, we focus on an optimal solution to $LP_{\Pi_t(v)}$ with a minimal number of binding constraints. Letting $(\hat{\lambda}, \hat{w})$ denote such a solution, we define

$$\hat{\mathcal{C}}_i = \left\{ S \in \mathcal{C}_i \,|\, B_{i,S}(\hat{\lambda}, \hat{w}) \text{ is nonempty} \right\},$$

$$\hat{\mathcal{C}}_\phi = \left\{ S \subset \mathcal{J} \,|\, \hat{w}_{\phi,t} = \hat{\lambda}_{\phi,S,t} \right\},$$

and

$$\hat{\mathcal{I}}^+ = \{ i \,|\, \hat{w}_{i,t} > \Delta_{i,t}(0) \},$$

where the ˆ on the sets is to remind the reader of the dependence on $(\hat{\lambda}, \hat{w})$.

Lemma 6. *Let $(\hat{\lambda}, \hat{w})$ be an optimal solution to $LP_{\Pi_t(v)}$ with a minimal number of binding constraints.*

(i) $\hat{\mathcal{C}}_i$ is nonempty for all $i \in \hat{\mathcal{I}}^+$.
(ii) If $S \in \hat{\mathcal{C}}_i$, then $S \in \hat{\mathcal{C}}_\phi$ (note that by definition the empty set does not consume any resources and so $\emptyset \notin \hat{\mathcal{C}}_i$).

Proof. For part (i), for $i \in \hat{\mathcal{I}}^+$, $\hat{w}_{i,t} > \Delta_{i,t}(0)$. Since $(\hat{\lambda}, \hat{w})$ is optimal there exists $S \in \mathcal{C}_i$ and $r \in \left\{1, \ldots, r_i^1\right\}$ such that $\hat{w}_{i,t} = \hat{\lambda}_{i,S,t} - \sum_{j\in S} P_j(S)\psi_{i,t+1}(r) + \Delta_{i,t}(r) > \Delta_{i,t}(0)$ (otherwise, we can reduce $\hat{w}_{i,t}$ contradicting optimality). Therefore $B_{i,S}(\hat{\lambda}, \hat{w})$ is nonempty and so $S \in \hat{\mathcal{C}}_i$ and $\hat{\mathcal{C}}_i$ is nonempty.

For part (ii), $S \in \hat{\mathcal{C}}_i$ implies that $i \in \mathcal{I}_S$. So we have a set S with $B_{i,S}(\hat{\lambda}, \hat{w})$ nonempty. By Lemma 5, $\hat{w}_{\phi,t} = \hat{\lambda}_{\phi,S,t}$ and so $S \in \hat{\mathcal{C}}_\phi$. □

Lemma 7. *Let $(\hat{\lambda}, \hat{w})$ be an optimal solution to $LP_{\Pi_t(v)}$ with a minimal number of binding constraints. If $\hat{\mathcal{I}}^+$ is nonempty, then $\cap_{i\in\hat{\mathcal{I}}^+}\hat{\mathcal{C}}_i$ is nonempty.*

Proof. If $|\hat{\mathcal{I}}^+| = 1$, then the statement holds trivially by part (i) of Lemma 6. Consider the case $|\hat{\mathcal{I}}^+| > 1$. If $\cap_{i\in\hat{\mathcal{I}}^+}\hat{\mathcal{C}}_i$ is empty, then this implies the following: Fix a resource $i \in \hat{\mathcal{I}}^+$. Part (i) of Lemma 6 implies that $\hat{\mathcal{C}}_i$ is nonempty. Then for every $S \in \hat{\mathcal{C}}_i$ there exists $l \in \hat{\mathcal{I}}^+$ such that $S \notin \hat{\mathcal{C}}_l$. Note that since $l \in \hat{\mathcal{I}}^+$, $\hat{w}_{l,t} > \Delta_{l,t}(0)$.

So let $i \in \hat{\mathcal{I}}^+$, $\hat{S} \in \hat{\mathcal{C}}_i$ and $l \in \hat{\mathcal{I}}^+$ with $\hat{S} \notin \hat{\mathcal{C}}_l$. If $\hat{S} \notin \hat{\mathcal{C}}_l$, there are two possibilities. First, $\hat{S} \in \mathcal{C}_l$ but $B_{l,\hat{S}}(\hat{\lambda}, \hat{w})$ is empty. But since $\hat{S} \in \hat{\mathcal{C}}_i$, $B_{i,\hat{S}}(\hat{\lambda}, \hat{w})$ is nonempty, which this contradicts part (i) of Lemma 5.

The other possibility is that $\hat{S} \notin \mathcal{C}_l$. Let

$$\epsilon = \min \left\{ \hat{w}_{l,t} - \Delta_{l,t}(0), \min_{S\in\mathcal{C}_l\setminus\hat{\mathcal{C}}_l, r\in\left\{1,\ldots,r_l^1\right\}} \{\xi_{l,S,t}(r)\} \right\} > 0$$

where we define the minimum over an empty set to be infinity (the second term could be empty). Let $\delta \in (0, \epsilon)$ and $(\widetilde{\lambda}, \widetilde{w})$ be given by $\widetilde{\lambda} = \hat{\lambda} - \delta \sum_{S \in \hat{\mathcal{C}}_l} \boldsymbol{e}^{l,S,t} + \delta \sum_{S \in \hat{\mathcal{C}}_l} \boldsymbol{e}^{\phi,S,t}$ and $\widetilde{w} = \hat{w} - \delta \boldsymbol{e}^{l,t} + \delta \boldsymbol{e}^{\phi,t}$. Therefore, we have $\widetilde{\lambda}_{l,S,t} = \hat{\lambda}_{l,S,t} - \delta$, $\widetilde{\lambda}_{\phi,S,t} = \hat{\lambda}_{\phi,S,t} + \delta$ for all $S \in \hat{\mathcal{C}}_l$. Similarly, $\widetilde{w}_{l,t} = \hat{w}_{l,t} - \delta$ and $\widetilde{w}_{\phi,t} = \hat{w}_{\phi,t} + \delta$. All other terms remain the same.

We check that $(\widetilde{\lambda}, \widetilde{w})$ is feasible and look at the set of binding constraints associated with this solution. We look at the constraints in $LP_{\Pi_t(v)}$ one by one and compare the the number of binding constraints in $(\hat{\lambda}, \hat{w})$ with the number in $(\widetilde{\lambda}, \widetilde{w})$.

Constraints (22) and (23): Since $\widetilde{w}_{\phi,t} > \hat{w}_{\phi,t}$, constraint (22) continues to hold for $(\widetilde{\lambda}, \widetilde{w})$ and the number of binding constraints do not increase. By construction $(\widetilde{\lambda}, \widetilde{w})$ satisfies constraint (23).

Constraints (19): Note that we need to check constraints (19) and (20) only for resource l. For resource l, we have $\widetilde{w}_{l,t} = \hat{w}_{l,t} - \delta > \Delta_{l,t}(0)$ and so constraint (19) continues to be nonbinding.

Constraints (20): For $S \in \mathcal{C}_l \backslash \hat{\mathcal{C}}_l$ and $r \in \{1, \ldots, r_l^1\}$, we have $\widetilde{w}_{l,t} = \hat{w}_{l,t} - \delta > \hat{w}_{l,t} - \xi_{l,S,t}(r) = \hat{\lambda}_{l,S,t} - \sum_{j \in \mathcal{J}_i} P_j(S) \psi_{l,t+1}(r) + \Delta_{l,t}(r) = \widetilde{\lambda}_{l,S,t} - \sum_{j \in \mathcal{J}_i} P_j(S) \psi_{l,t+1}(r) + \Delta_{l,t}(r)$. Note that the last equality holds by definition of $\widetilde{\lambda}$. So constraint (20) remains nonbinding.

For $S \in \hat{\mathcal{C}}_l$ and $r \in \{1, \ldots, r_l^1\} \backslash B_{l,S}(\hat{\lambda}, \hat{w})$, $\widetilde{w}_{l,t} = \hat{w}_{l,t} - \delta > \hat{\lambda}_{l,S,t} - \sum_{j \in \mathcal{J}_i} P_j(S) \psi_{l,t+1}(r) + \Delta_{l,t}(r) - \delta = \widetilde{\lambda}_{l,S,t} - \sum_{j \in \mathcal{J}_i} P_j(S) \psi_{l,t+1}(r) + \Delta_{l,t}(r)$. Therefore, constraint (20) continues to be nonbinding. For $S \in \hat{\mathcal{C}}_l$ and $r \in B_{l,S}(\hat{\lambda}, \hat{w})$, $\widetilde{w}_{l,t} = \hat{w}_{l,t} - \delta = \hat{\lambda}_{l,t} - \delta - \sum_{j \in \mathcal{J}_i} P_j(S) \psi_{l,t+1}(r) + \Delta_{l,t}(r) = \widetilde{\lambda}_{l,t} - \sum_{j \in \mathcal{J}_i} P_j(S) \psi_{l,t+1}(r) + \Delta_{l,t}(r)$. So constraints (20) are binding for all such S and r. Note that these constraints, by definition, were also binding in $(\hat{\lambda}, \hat{w})$. So, $(\widetilde{\lambda}, \widetilde{w})$ satisfies constraints (19) and (20) for resource l and the number of binding constraints is exactly the same as in $(\hat{\lambda}, \hat{w})$.

Constraint (21): For $S \in \hat{\mathcal{C}}_l$, by definition $B_{l,S}(\hat{\lambda}, \hat{w})$ is nonempty. Part (i) of Lemma 5 implies that $\hat{w}_{\phi,t} = \hat{\lambda}_{\phi,S,t}$, which means that constraint (21) is binding. We have $\widetilde{w}_{\phi,t} = \hat{w}_{\phi,t} + \delta = \hat{\lambda}_{\phi,St} + \delta = \widetilde{\lambda}_{\phi,S,t}$ and so constraint (21) holds and continues to be binding. For $S \notin \hat{\mathcal{C}}_l$, $\widetilde{\lambda}_{l,S,t} = \hat{\lambda}_{l,S,t}$. Therefore, $\widetilde{w}_{\phi,t} = \hat{w}_{\phi,t} + \delta \geq \widetilde{\lambda}_{l,S,t}$. So constraint (21) holds and the number of binding constraints do not increase.

Now we argue that the number of binding constraints (21) strictly decreases from $(\hat{\lambda}, \hat{w})$ to $(\widetilde{\lambda}, \widetilde{w})$. For the set $\hat{S}$, since $\hat{S} \in \hat{\mathcal{C}}_i$, $B_{i,\hat{S}}(\hat{\lambda}, \hat{w})$ is nonempty. By, part (i) of Lemma 5, $\hat{w}_{\phi,t} = \hat{\lambda}_{\phi,\hat{S},t}$ and so the constraint is binding in $(\hat{\lambda}, \hat{w})$. But $\widetilde{w}_{\phi,t} = \hat{w}_{\phi,t} + \delta > \hat{\lambda}_{\phi,\hat{S},t} = \widetilde{\lambda}_{\phi,\hat{S},t}$ and the constraint is nonbinding in $(\widetilde{\lambda}, \widetilde{w})$. Overall, $(\widetilde{\lambda}, \widetilde{w})$ has strictly fewer number of binding constraints (21) and they are a subset of the set of binding constraints of $(\hat{\lambda}, \hat{w})$ contradicting minimality.

Since $\hat{w}_{\phi,t} + \sum_i \hat{w}_{i,t} = \widetilde{w}_{\phi,t} + \sum_i \widetilde{w}_{i,t}$, $(\widetilde{\lambda}, \widetilde{w})$ is optimal and this gives a contradiction. □

We are now ready to show that $\Phi_t(v) \geq \Pi_t(v)$.

Proposition 4. $\Phi_t(v) \geq \Pi_t(v)$.

Proof. Let $(\hat{\lambda}, \hat{w})$ be an optimal solution to $LP_{\Pi_t(v)}$ with a minimal number of binding constraints. We consider two cases.

Case 1: Suppose that $\hat{\mathcal{C}}_i$ is empty for all i. This means that for all $S \in \mathcal{C}_i$, $B_{i,S}(\hat{\lambda}, \hat{w})$ is empty. It follows that $\hat{w}_{i,t} = \Delta_{i,t}(0)$ for all i (otherwise we can reduce $\hat{w}_{i,t}$ contradicting optimality).

Part (ii) of Lemma 5 implies that $\hat{w}_{\phi,t} > \hat{\lambda}_{\phi,S,t}$ for all S. It follows that $\hat{w}_{\phi,t} = 0$. Therefore, $\Pi_t(v) = \sum_i \Delta_{i,t}(0)$. Note that $\boldsymbol{r} = \mathbf{0}$ and $S = \emptyset$ is feasible for $\Phi_t(v)$ and the objective function value associated with this solution is $\sum_i \Delta_{i,t}(0)$. Therefore $\Phi_t(v) \geq \sum_i \Delta_{i,t}(0) = \Pi_t(v)$.

Case 2: Suppose that $\hat{\mathcal{C}}_i$ is nonempty for some i. We consider two subcases.
Case 2.a. $\hat{\mathcal{I}}^+$ is empty. We choose a resource l such that $\hat{\mathcal{C}}_l$ is nonempty and choose $\hat{S} \in \hat{\mathcal{C}}_l$ such that $B_{l,\hat{S}}(\hat{\lambda}, \hat{w})$ is nonempty. By part (i) of Lemma 5, $B_{i,\hat{S}}(\hat{\lambda}, \hat{w})$ is nonempty for all $i \in \mathcal{I}_{\hat{S}}$ and $\hat{w}_{\phi,t} = \hat{\lambda}_{\phi,\hat{S},t}$. So, for all $i \in \mathcal{I}_{\hat{S}}$, we have $\hat{w}_{i,t} = \hat{\lambda}_{i,\hat{S},t} - \sum_{j \in \mathcal{J}_i} P_j(\hat{S})\psi_{i,t+1}(\hat{r}_i) + \Delta_{i,t}(\hat{r}_i)$, where $\hat{r}_i \in B_{i,\hat{S}}(\hat{\lambda}, \hat{w})$. Note that $\hat{r}_i \geq 1$ for all $i \in \mathcal{I}_{\hat{S}}$. On the other hand, since $\hat{\mathcal{I}}^+$ is empty. we have $\hat{w}_{i,t} = \Delta_{i,t}(0)$ for all i. In particular, $\hat{w}_{i,t} = \Delta_{i,t}(0)$ for all $i \notin \mathcal{I}_S$. Putting everything together,

$$
\begin{aligned}
\Pi_t(v) &= \hat{w}_{\phi,t} + \sum_{i \in \mathcal{I}_{\hat{S}}} \hat{w}_{i,t} + \sum_{i \notin \mathcal{I}_{\hat{S}}} \hat{w}_{i,t} \\
&= \hat{\lambda}_{\phi,\hat{S},t} + \sum_{i \in \mathcal{I}_{\hat{S}}} \left[\hat{\lambda}_{i,\hat{S},t} - \sum_{j \in \mathcal{J}_i} P_j(\hat{S})\psi_{i,t+1}(\hat{r}_i) + \Delta_{i,t}(\hat{r}_i) \right] + \sum_{i \notin \mathcal{I}_{\hat{S}}} \Delta_{i,t}(0) \\
&= R(\hat{S}) - \sum_{j \in S} \sum_{i \in \mathcal{I}_j} \mathbb{1}_{[\hat{S} \in \mathcal{C}_i]} P_j(\hat{S})\psi_{i,t+1}(\hat{r}_i) + \sum_{i \in \mathcal{I}_{\hat{S}}} \Delta_{i,t}(\hat{r}_i) + \sum_{i \notin \mathcal{I}_{\hat{S}}} \Delta_{i,t}(0) \\
&\leq \Phi_t(v)
\end{aligned}
$$

where the last equality follows since $(\hat{\lambda}, \hat{w})$ satisfies constraint (23) and

$$
\begin{aligned}
\sum_{i \in \mathcal{I}_S} \sum_{j \in \mathcal{J}_i} P_j(S)\psi_{i,t+1}(r_i) = \sum_i \sum_{j \in \mathcal{J}_i} \mathbb{1}_{[i \in \mathcal{I}_S]} P_j(S)\psi_{i,t+1}(r_i) &= \sum_j \sum_{i \in \mathcal{I}_j} \mathbb{1}_{[i \in \mathcal{I}_S]} P_j(S)\psi_{i,t+1}(r_i) \\
&= \sum_{j \in S} \sum_{i \in \mathcal{I}_j} \mathbb{1}_{[S \in \mathcal{C}_i]} P_j(S)\psi_{i,t+1}(r_i).
\end{aligned}
$$

The inequality follows from (18) by noting that $\hat{S}$ and $\boldsymbol{r}$, where $r_i = \hat{r}_i$ for $i \in \mathcal{I}_{\hat{S}}$ and $r_i = 0$ otherwise, is feasible to $\Phi_t(v)$. To see this, observe that for all $j \in \hat{S}$, if $\mathbb{1}_{[i \in \mathcal{I}_j]} = 1$, then $i \in \mathcal{I}_{\hat{S}}$ and so $r_i = \hat{r}_i \geq 1$. Therefore, for all $j \in \hat{S}$, $\mathbb{1}_{[i \in \mathcal{I}_j]} \leq r_i$ for all i and so $\hat{S} \subset \mathcal{Q}(\boldsymbol{r})$.

Case 2.b. $\hat{\mathcal{I}}^+$ is nonempty. Let $\hat{S} = \cap_{i \in \hat{\mathcal{I}}^+} \hat{\mathcal{C}}_i$, which by Lemma 7 is nonempty. Note that $\hat{S} \in \hat{\mathcal{C}}_i \subset \mathcal{C}_i$ for all $i \in \hat{\mathcal{I}}^+$. Now every $i \in \hat{\mathcal{I}}^+$ satisfies $\hat{w}_{i,t} > \Delta_{i,t}(0)$. Therefore if $i \notin \mathcal{I}_{\hat{S}}$ then $\hat{w}_{i,t} = \Delta_{i,t}(0)$. Since $\hat{S} \in \hat{\mathcal{C}}_l$ for some l, $B_{l\hat{S}}(\hat{\lambda}, \hat{w})$ is nonempty. Part (i) of Lemma 5 implies that $B_{i,\hat{S}}(\hat{\lambda}, \hat{w})$ is nonempty for all $i \in \mathcal{I}_{\hat{S}}$ and that $\hat{w}_{\phi,t} = \hat{\lambda}_{\phi,\hat{S},t}$. Since $B_{i,\hat{S}}(\hat{\lambda}, \hat{w})$ is nonempty for $i \in \mathcal{I}_{\hat{S}}$, there exists $\hat{r}_i \in \{1, \ldots, r_i^1\}$ such that $\hat{w}_{i,t} = \hat{\lambda}_{i,\hat{S},t} - \sum_{j \in \mathcal{J}_i} P_j(\hat{S})\psi_{i,t+1}(\hat{r}_i) + \Delta_{i,t}(\hat{r}_i)$ ($\hat{r}_i$ need not be unique, we can pick any $r \in B_{i,\hat{S}}(\hat{\lambda}, \hat{w})$). We have

$$
\begin{aligned}
\Pi_t(v) &= \hat{\lambda}_{\phi,\hat{S},t} + \sum_{i \in \mathcal{I}_{\hat{S}}} \left[\hat{\lambda}_{i,\hat{S},t} - \sum_{j \in \mathcal{J}_i} P_j(\hat{S})\psi_{i,t+1}(\hat{r}_i) + \Delta_{i,t}(\hat{r}_i) \right] + \sum_{i \notin \mathcal{I}_{\hat{S}}} \Delta_{i,t}(0) \\
&= R(\hat{S}) - \sum_{j \in \hat{S}} P_j(\hat{S}) \sum_{i \in \mathcal{I}_j} \mathbb{1}_{[\hat{S} \in \mathcal{C}_i]} \psi_{i,t+1}(\hat{r}_i) + \sum_{i \in \mathcal{I}_{\hat{S}}} \Delta_{i,t}(\hat{r}_i) + \sum_{i \notin \mathcal{I}_{\hat{S}}} \Delta_{i,t}(0) \\
&\leq \Phi_t(v)
\end{aligned}
$$

where the inequality follows from (18) by noting that $\hat{S}$ and $\boldsymbol{r}$ is feasible to $\Phi_t(v)$ where $r_i = \hat{r}_i \geq 1$ for $i \in \mathcal{I}_{\hat{S}}$ and $r_i = 0$ otherwise. □

Lemma 3 and Proposition 4 together imply that $\Phi_t(v) = \Pi_t(v)$. We are now ready to show that $V^{PL} \geq V^{LRo}$. Note that we can write PL as $V^{PL} = \min_v \sum_i v_{i,1}(r_i^1)$ subject to $0 \geq \Phi_t(v)$ for all t with the boundary condition that $v_{i,\tau+1}(\cdot) = 0$. Letting $\underline{V} = \min_v \sum_i v_{i,1}(r_i^1) + \sum_t \Phi_t(v)$ subject to $0 \geq \Phi_t(v)$ for all t with the same boundary condition, it follows that $V^{PL} \geq \underline{V}$. Using the fact that $\Phi_t(v) = \Pi_t(v)$ together with the linear programming formulation $LP_{\Pi_t(v)}$, of $\Pi_t(v)$, we can write $\underline{V}$ as

$$\underline{V} = \min_v \quad \sum_i v_{i,1}(r_i^1) + \sum_t \left(w_{\phi,t} + \sum_i w_{i,t} \right)$$

s.t

$$w_{i,t} + v_{i,t}(r_i) \geq \max_{S \subset \mathcal{Q}_i(r_i)} \left\{ \mathbb{1}_{[S \in \mathcal{C}_i]} \lambda_{i,S,t} + \sum_{j \in \mathcal{J}_i} P_j(S) \left[v_{i,t+1}(r_i - 1) - v_{i,t+1}(r_i) \right] + v_{i,t+1}(r_i) \right\} \quad \forall t, r_i \in \mathcal{R}_i$$

$$w_{\phi,t} \geq \max_{S \subset \mathcal{J}} \{\lambda_{\phi,S,t}\} \quad \forall t$$

$$w_{\phi,t} \geq 0$$

$$\lambda_{\phi,S,t} + \sum_{i \in \mathcal{I}_S} \lambda_{i,S,t} = R(S) \quad \forall t, S \subset \mathcal{J}$$

$$v_{i,\tau+1}(\cdot) = 0.$$

With the change of variables $\vartheta_{i,t}(r_i) = v_{i,t}(r_i) + \sum_{s=t}^{\tau} w_{i,s}$ and $\vartheta_{\phi,t} = \sum_{s=t}^{\tau} w_{\phi,s}$, we can write the above linear program as

$$\underline{V} = \min_\vartheta \quad \sum_i \vartheta_{i,1}(r_i^1) + \vartheta_{\phi 1}$$

s.t

$$\vartheta_{i,t}(r_i) \geq \max_{S \subset \mathcal{Q}_i(r_i)} \left\{ \mathbb{1}_{[S \in \mathcal{C}_i]} \lambda_{i,S,t} + \sum_{j \in \mathcal{J}_i} P_j(S)[\vartheta_{i,t+1}(r_i - 1) - \vartheta_{i,t+1}(r_i)] + \vartheta_{i,t+1}(r_i) \right\} \quad \forall t, r_i \in \mathcal{R}_i$$

$$\vartheta_{\phi,t} \geq \max_{S \subset \mathcal{J}} \{\lambda_{\phi,S,t}\} + \vartheta_{\phi,t+1} \quad \forall t$$

$$\lambda_{\phi,S,t} + \sum_{i \in \mathcal{I}_S} \lambda_{i,S,t} = R(S) \quad \forall t, S \subset \mathcal{J}$$

$$\vartheta_{i,\tau+1}(\cdot) = 0.$$

which is exactly LRo. Therefore, $V^{PL} \geq V^{LRo}$ □

We conclude this section with a comment on the number and form of the Lagrange multipliers. Requiring $\sum_{i \in \mathcal{I}_S} \lambda_{i,S,t} = R(S)$ can be overly restrictive and we do not necessarily have $V^{PL} = V^{LRo}$ if we impose this constraint on the Lagrange multipliers. We give an example below which illustrates this. Therefore, the Lagrange multipliers have to be chosen carefully in order to make the Lagrangian relaxation equivalent to the piecewise linear approximation.

Example 2: Consider a network revenue management problem with two products, two resources and a single time period in the booking horizon. The first product uses only the first resource, while the second product uses only the second resource, and we have a single unit of capacity on each resource. Note that in the airline context, this example corresponds to a parallel flights network. The revenues associated with the products are $f_1 = 99$ and $f_2 = 101$. The choice probabilities are given in Table

S	$P_1(S)$	$P_2(S)$
$\{1\}$	50/99	0
$\{2\}$	0	51/101
$\{1,2\}$	1/2	1/2

Table 2: Choice probabilities

2. Letting $S_1 = \{1\}, S_2 = \{2\}$ and $S_3 = \{1, 2\}$, we have $R(S_1) = 50$, $R(S_2) = 51$ and $R(S_3) = 100$, $\mathcal{I}_{S_1} = \{1\}, \mathcal{I}_{S_2} = \{2\}$ and $\mathcal{I}_{S_3} = \{1, 2\}$. If we impose the constraint $\sum_{i\in\mathcal{I}_S} \lambda_{i,S,t} = R(S)$ on the Lagrange multipliers, then we have $\lambda_{1,S_1,1} = 50, \lambda_{2,S_2,1} = 51$ and $\lambda_{1,S_3,1} + \lambda_{2,S_3,1} = 100$. We have $\vartheta^{\lambda}_{1,1}(1) = \max\{50, \lambda_{1,S_3,1}\}$ and $\vartheta^{\lambda}_{2,1}(1) = \max\{51, \lambda_{2,S_3,1}\}$. It can be verified that

$$\min_{\lambda \,|\, \lambda_{1,S_1,1}=50, \lambda_{2,S_2,1}=51, \lambda_{1,S_3,1}+\lambda_{2,S_3,1}=100} \vartheta^{\lambda}_{1,1}(1) + \vartheta^{\lambda}_{2,1}(1) = 101 > 100 = V^{PL}.$$

5 Computational experiments

In this section, we compare the upper bounds obtained by the Lagrangian relaxation using offer-set specific multipliers with the upper bounds obtained by other benchmark solution methods. Our test problems are drawn from Meissner and Strauss [14]. In all of the test problems, we have multiple customer segments and within each segment, choice is governed by the MNL model.

We begin by describing the MNL choice model with multiple customer segments and explain how we modify our Lagrangian-based approach for the case of multiple customer segments. We then describe the different benchmark solution methods and the experimental setup.

5.1 MNL choice model with multiple customer segments

We consider the case where the total demand is comprised of demand from multiple customer segments with each customer segment being interested only in a small subset of the products. There are $g = 1, \ldots, G$ customer segments, each with distinct purchase behavior. In each period, we have exactly one customer arrival and an arriving customer belongs to segment g with probability α^g. Since the total arrival rate is 1, we have $\sum_g \alpha^g = 1$. Customer segment g has a *consideration set* $\mathcal{J}^g \subset \mathcal{J}$ of products that it considers for purchase. A segment g customer is indifferent to a product outside its consideration set, in the sense that the customer's choice probabilities are not affected by products offered outside its consideration set. We assume that the consideration sets of the different customer segments are known to the firm by a previous process of estimation and analysis. As we mentioned in the Introduction, we also assume that the consideration sets of the different segments are small enough for its power set to be enumerable.

Within each segment choice is according to the MNL model. The MNL model associates a preference weight ω^g_j with product j that is in the consideration set of segment g. Similarly it associates a preference weight ω^g_0 with a segment g arrival not purchasing anything. The probability that a segment g arrival purchases product j when S is the offer set is (Ben-Akiva and Lerman [1])

$$P^g_j(S) = \frac{\omega^g_j}{\omega^g_0 + \sum_{k\in S\cap\mathcal{J}^g} \omega^g_k},$$

while the probability it purchases none of the offered products is

$$P_0^g(S) = \frac{\omega_0^g}{\omega_0^g + \sum_{k \in S \cap \mathcal{J}^g} \omega_k^g}.$$

Note that since the customer is indifferent to products outside its consideration set, we have $P_j^g(S) = P_j^g(S \cap \mathcal{J}^g) = P_j^g(S^g)$, where $S^g = S \cap \mathcal{J}^g$. Letting $R^g(S) = R^g(S^g) = \sum_{j \in S^g} f_j P_j^g(S^g)$ denote the expected revenue from offering set S given a segment g arrival, we have $R(S) = \sum_g \alpha^g R^g(S^g)$.

5.2 Modification of piecewise-linear approximation for multiple segments

We modify the Lagrangian relaxation approach for the case with multiple customer segments by associating multipliers with every subset S^g of each segment's consideration set $\mathcal{J}^g$. Letting $\lambda = \{\lambda_{\phi,S^g,t}^g, \lambda_{\phi,S^g,t}^g \,|\, \forall t, g, S^g \subset \mathcal{J}^g, i \in \mathcal{I}_{S^g}\}$ denote the set of Lagrange multipliers, we solve the optimality equation

$$\vartheta_{i,t}^\lambda(r_i) = \sum_g \alpha^g \max_{\{S^g \subset \mathcal{Q}_i(r_i) \,|\, S^g \subset \mathcal{J}^g\}} \left\{ \mathbb{1}_{[S^g \in \mathcal{C}_i]} \lambda_{i,S^g,t}^g + \sum_{j \in \mathcal{J}_i} P_j^g(S^g) \left[\vartheta_{i,t+1}^\lambda(r_i - 1) - \vartheta_{i,t+1}^\lambda(r_i) \right] \right\} + \vartheta_{i,t+1}^\lambda(r_i), \tag{24}$$

for resource i, with the boundary condition that $\vartheta_{i,\tau+1}^\lambda(\cdot) = 0$. We solve the optimality equation

$$\vartheta_{\phi,t}^\lambda = \sum_g \alpha^g \max_{S^g \subset \mathcal{J}^g} \{\lambda_{\phi,S^g,t}^g\} + \vartheta_{\phi,t+1}^\lambda \tag{25}$$

for "resource" ϕ with the boundary condition that $\vartheta_{\phi,\tau+1}^\lambda = 0$. We refer to this modification as the segment-based Lagrangian relaxation (*sLRo*). Many of the results from §3 carry over to the segment-based Lagrangian relaxation. In particular, $V_t(\boldsymbol{r}) = \sum_i \vartheta_{i,t}^\lambda(r_i) + \vartheta_{\phi,t}^\lambda$ gives us an upper bound on the value function for every set of Lagrange multipliers that satisfy $\lambda_{\phi,S^g,t}^g + \sum_{i \in \mathcal{I}_{S^g}} \lambda_{i,S^g,t}^g = R^g(S^g)$ for all t, g and $S^g \subset \mathcal{J}^g$. We find the tightest upper bound by solving

$$V^{sLRo} = \min_{\left\{\lambda \,|\, \lambda_{\phi,S^g,t}^g + \sum_{i \in \mathcal{I}_{S^g}} \lambda_{i,S^g,t} = R^g(S^g),\, \forall S^g, g, t\right\}} V_1^\lambda(\boldsymbol{r}_1). \tag{26}$$

$V_t^\lambda(\boldsymbol{r})$ is a convex function of λ and an expression for its subgradient can be derived in a manner analogous to Proposition 2; we omit the details. Therefore, we can solve the above minimization problem using subgradient search. However, unlike in Proposition 3, it is not the case that *sLRo* is equivalent to *PL* and we can have $V^{PL} < V^{sLRo}$. Although *sLRo* can be weaker than *PL*, an advantage of the segment-based approach is computational tractability. Note that solving *PL* through *LRo* requires $\mathcal{O}(2^{|\mathcal{J}|})$ Lagrange multipliers which quickly gets intractable. On the other hand, *sLRo* requires $\mathcal{O}(\sum_g 2^{|\mathcal{J}^g|})$ Lagrange multipliers, a much more manageable number provided the consideration sets for each segment are small. Moreover, in our computational experiments, we find that *sLRo* tends to generate upper bounds that are quite close to the *PL* upper bounds.

5.3 Benchmark methods

Choice Deterministic Linear Program (CDLP) This is the solution method described in Section 2.4. Since customer choice is according to the MNL model, we use the sales based formulation of *CDLP* described in Gallego, Ratliff, and Shebalov [6] and solve it as a compact linear program.

Affine Approximation (AF) This is the solution method described in Section 2.5. We use the reduced formulation described in Vossen and Zhang [23] to solve AF. For the MNL choice model, the separation problem of AF can be solved as a linear mixed integer program (Zhang and Adelman [25]).

Piecewise-linear Approximation (PL) This is the solution method described in Section 2.6. Meissner and Strauss [14] refer to PL as the time-and-inventory-sensitive approach with complete disaggregation ($TISA - C$).

Time and Inventory Sensitive Approach with Aggregation (TISA-K) This solution method is proposed by Meissner and Strauss [14] as a more tractable alternative to PL. $TISA - K$ essentially divides the capacity of each resource into K ranges and assumes that the marginal value of capacity remains the same in each range. The number of ranges K is an input to the solution method. If $K = 1$, then the marginal value of capacity remains the same across all capacity levels and the method reduces to the affine approximation. On the other hand, if the number of ranges is the same as the number of units of capacity, we have a different marginal value for each unit of capacity and the method becomes PL.

Segment-based Lagrangian Relaxation (sLRo) This is the solution method described in Section 5.2, where we associate Lagrange multipliers with each subset of a customer segment's consideration set. In our computational experiments, we use subgradient search to solve problem (26). We use a step size of $5000/\sqrt{k}$ at iteration k of the subgradient algorithm. We run the subgradient search algorithm for a maximum of 3000 iterations. If the norm of the subgradient at any iteration is less than 0.1, we terminate the algorithm. Although, our step size selection does not guarantee convergence, it provided good solutions and stable performance in our test problems.

5.4 Hub-and-Spoke Network A

We have a network with a single hub serving two spokes. There is one flight from each spoke to the hub and one flight from the hub to each spoke, so that there are four flights in total. Note that the flight legs correspond to the resources in our network RM formulation. There are six origin-destination pairs in total (two spoke-to-hub, two hub-to-spoke and two spoke-to-spoke pairs). There are two fare products associated with each origin-destination pair, of which one is a low-fare product and the other is a high-fare product. We assoicate a customer segment with each origin-destination pair and each customer segment is only interested in the fare products connecting the origin-destination pair that it is associated with. As mentioned, within each segment, customer choice is governed by the MNL model. Meissner and Strauss [14] contains more details on the network.

We measure the tightness of the leg capacities using the nominal load factor. Letting

$$S^{g*} \in \ \text{argmax}_{\ S^g \subset \mathcal{J}^g} R^g(S^g) \tag{27}$$

be the offer set that maximizes expected revenues from segment g when there is ample capacity on all the flight legs, the nominal load factor is

$$\zeta = \frac{\sum_t \sum_i \left[\sum_g \alpha^g [\sum_{j \in \mathcal{J}_i} P^g_j(S^{g*})]\right]}{\sum_i r^1_i}.$$

The number of time periods in our test problems is $\kappa\hat{\tau}$, while the initial resource capacities are given by $\kappa\hat{\boldsymbol{r}}^1$, where $\hat{\tau} = 20$ and $\hat{\boldsymbol{r}}^1 = [2, 4, 4, 2]$ and κ represents the factor by which we scale $\hat{\tau}$ and $\hat{\boldsymbol{r}}^1$ to obtain different test problems. We label the test problems by the pair (ζ, κ) where $\zeta \in \{1.0, 1.2, 1.6\}$ and $\kappa \in \{1, 2, 4\}$, which gives us a total of nine test problems.

Table 3 compares the upper bounds obtained by the different solution methods. The first column in the table describes the problem characteristics by using (ζ, κ). The second to fifth columns, respectively, give the upper bounds obtained by $CDLP, AF, PL, TISA-K$ and *sLRo*. The last four columns, respectively, give the percentage gap between the upper bounds obtained by $CDLP, AF, PL$ and $TISA-K$ with respect to *sLRo*. We use the compact formulation of $CDLP$ and solve it to optimality. On the other hand, the reduced formulation of AF has an exponential number of constraints. We solve AF using constraint generation and stop when we are within 1% of optimality. We use the upper bounds reported in Meissner and Strauss [14] for PL and $TISA-K$. As mentioned, PL corresponds to $TISA-C$ in Meissner and Strauss [14]. Meissner and Strauss [14] use a value of $K = 2$ for $TISA-K$. That is, the capacity of each resource is split into two equal ranges and the marginal value of capacity is assumed to be the same across each range. We note that Meissner and Strauss [14] use column generation of solve PL and $TISA-K$ and stop when the solution is within 1% of optimality.

sLRo obtains significantly tighter upper bounds that $CDLP$ and AF. The average gap between the upper bounds obtained by $CDLP$ and *sLRo* is around 12%, while that between AF and *sLRo* is around 7%. The *sLRo* upper bounds are on average around 3% tighter than the $TISA-K$ upper bounds. We observe one instance where the *sLRo* bound is weaker than the $TISA-K$ bound. However since the gap is less than the 1% optimality stopping criterion for $TISA-K$, it is not immediately possible to conclude whether this gap is meaningful. Compared to PL, the *sLRo* bounds are slightly weaker. The average gap between the PL and *sLRo* bounds is around -0.7%.

Problem	Upper Bound					% Gap with *sLRo*			
(ζ, κ)	$CDLP$	AF	PL	$TISA-K$	*sLRo*	$CDLP$	AF	PL	$TISA-K$
(1.0, 1)	925	851	766	775	772	19.73	10.12	-0.83	0.34
(1.0, 2)	1,850	1,803	1,661	1,788	1,669	10.80	8.01	-0.50	7.10
(1.0, 4)	3,699	3,654	3,488	3,653	3,502	5.64	4.35	-0.39	4.32
(1.2, 1)	1,077	961	864	877	872	23.53	10.20	-0.89	0.60
(1.2, 2)	2,154	2,051	1,878	2,026	1,889	14.03	8.58	-0.58	7.26
(1.2, 4)	4,308	4,220	3,953	4,214	3,976	8.35	6.15	-0.57	5.99
(1.6, 1)	1,200	1,086	997	1,008	1,010	18.89	7.60	-1.24	-0.15
(1.6, 2)	2,400	2,322	2,153	2,299	2,173	10.49	6.88	-0.90	5.82
(1.6, 4)	4,801	4,745	4,529	4,738	4,558	5.32	4.09	-0.64	3.95

Table 3: Comparison of the upper bounds on the optimal expected total revenue for the first hub-and-spoke network.

5.5 Hub-and-Spoke Network B

We have a network with a single hub serving two spokes (from Meissner and Strauss [14] again). Now, there are two flights from the first hub to the spoke and two flights from the hub to the second spoke, so that there are four flights in total. There are four fare products connecting the first spoke to the hub, four connecting the hub to the second spoke and eight fare products connecting the first spoke to the second one. Half of the fare products are high-fare products while the remaining are low-fare products. We have a customer segment associated with each origin-destination pair and each customer segment is only interested in the fare products connecting the origin-destination pair that it is associated with. We label our test problems using $(\zeta, \kappa) \in \{1.0, 1.2, 1.6\} \times \{1, 2, 4\}$ so that we have a total of nine test problems.

Table 4 compares the upper bounds obtained by the different solution methods. The columns have the same interpretation as before. $sLRo$ continues to obtain significantly tighter upper bounds than $CDLP$. The average gap between the $CDLP$ and $sLRo$ upper bounds is around 2%. The gaps with AF, $TISA-K$ and PL are smaller compared to the previous set of hub-and-spoke test problems. The average gap between the upper bounds obtained by AF and $sLRo$ is around 0.6%, while that between $TISA-K$ and $sLRo$ is around 0.2%. The gaps tend to increase with the load factor ζ. The $sLRo$ bound is again slightly weaker than the PL bound and the average gap between the two is around -0.6%.

Problem	Upper Bound					% Gap with $sLRo$			
(ζ, κ)	$CDLP$	AF	PL	$TISA-K$	$sLRo$	$CDLP$	AF	PL	$TISA-K$
(1.0, 1)	1,293	1,254	1,235	1,243	1,244	3.95	0.77	-0.75	-0.10
(1.0, 2)	2,587	2,554	2,523	2,548	2,541	1.80	0.50	-0.71	0.27
(1.0, 4)	5,174	5,146	5,109	5,140	5,137	0.71	0.17	-0.55	0.05
(1.2, 1)	1,495	1,450	1,430	1,440	1,440	3.81	0.69	-0.69	0.00
(1.2, 2)	2,990	2,945	2,917	2,939	2,934	1.90	0.37	-0.58	0.17
(1.2, 4)	5,980	5,935	5,897	5,930	5,926	0.90	0.15	-0.49	0.07
(1.6, 1)	1,816	1,751	1,715	1,736	1,728	5.12	1.31	-0.75	0.46
(1.6, 2)	3,633	3,587	3,537	3,573	3,558	2.10	0.82	-0.59	0.42
(1.6, 4)	7,266	7,226	7,170	7,220	7,208	0.80	0.25	-0.52	0.17

Table 4: Comparison of the upper bounds on the optimal expected total revenue for the second hub-and-spoke network.

Problem	Upper Bound				% Gap with $sLRo$		
$(\beta, (\omega_0^H, \omega_0^L))$	$CDLP$	AF	$TISA-K$	$sLRo$	$CDLP$	AF	$TISA-K$
(0.6, (1,5))	36,187	35,811	35,775	35,677	1.43	0.38	0.27
(0.6, (5, 10))	33,158	32,745	32,728	32,483	2.08	0.81	0.75
(0.6, (10, 20))	29,960	29,556	29,531	29,316	2.20	0.82	0.73
(0.8, (1,5))	43,202	42,780	42,862	42,688	1.21	0.22	0.41
(0.8, (5, 10))	38,900	38,530	38,551	38,469	1.12	0.16	0.21
(0.8, (10, 20))	34,678	34,431	34,403	33,848	2.45	1.72	1.64
(1.0, (1,5))	48,822	48,440	48,496	48,531	0.60	-0.19	-0.07
(1.0, (5, 10))	43,767	43,446	43,417	43,262	1.17	0.42	0.36
(1.0, (10, 20))	35,103	35,103	35,101	34,939	0.47	0.47	0.46
(1.2, (1,5))	53,564	53,241	53,238	53,388	0.33	-0.27	-0.28
(1.2, (5, 10))	44,690	44,690	44,637	44,454	0.53	0.53	0.41
(1.2, (10, 20))	35,103	35,103	35,102	35,101	0.01	0.01	0.00
(1.4, (1,5))	55,257	55,161	55,084	54,770	0.89	0.71	0.57
(1.4, (5, 10))	44,690	44,690	44,640	44,690	0.00	0.00	-0.11
(1.4, (10, 20))	35,103	35,103	35,102	35,103	0.00	0.00	0.00

Table 5: Comparison of the upper bounds on the optimal expected total revenue for the small airline network.

5.6 Small Airline Network

We consider an airline network consisting of seven flights that connect three spokes with a hub. There are 22 fare products, of which half are high-fare products whereas the remaining are low-fare products. Each origin-destination pair is associated with two customer segments. The first segment is interested only in the high-fare products connecting the origin-destination pair, while the second segment is interested only in the low-fare products connecting the same origin-destination pair. All problem parameters are the same as in Meissner and Strauss [14]. Let ω_0^H and ω_0^L denote the no-purchase preference weights associated with the customer segments interested in the high-fare and low-fare products, respectively. We vary the no-purchase preference weights and scale the flight leg capacities by a parameter β to obtain different test problems. We label our test problems by the tuple $(\beta, (\omega_0^H, \omega_0^L)) \in \{0.6, 0.8.1.0, 1.2, 1.4\} \times \{(1,4), (5,10), (10,20)\}$. This gives us a total of 15 test problems.

Table 5 compares the upper bounds obtained by the different solution methods. We do not include the PL upper bounds as they are not reported in Meissner and Strauss [14]. As for $TISA-K$, the level of aggregation K depends on the flight leg and varies from 1 to 4; we refer the reader to Meissner and Strauss [14] for details. $sLRo$ provides a small but consistent improvement over $CDLP$, AF and $TISA-K$. On average it obtains upper bounds that are 0.9% tighter than $CDLP$ and around 0.4% tighter than AF and $TISA-K$. We observe test problems where the improvements over the $CDLP$, AF and $TISA-K$ upper bounds can be as high as 2.45%, 1.72% and 1.64%, respectively. The improvements in the upper bounds tend to be larger when the flight leg capacities are somewhat scarce. If there is ample capacity, as in the case where $\beta = 1.4$, all solution methods tend to perform roughly the same.

Since we solve $sLRo$ using subgradient search, it becomes important to have a good starting solution. To this end, we consider the following four ways of initializing the Lagrange multipliers. First, we consider the initialization where $\lambda^g_{\phi,S^g,t} = R^g(S^g)$ and $\lambda^g_{i,S^g,t} = 0$ for all $i \in \mathcal{I}_{S^g}$ and S^g. That is, we allocate all the revenue associated with an offer set to the resource ϕ and allocations on the remaining resources are set to zero. We refer to this initialization as IN-1. Next, we consider the initialization where $\lambda^g_{\phi,S^g,t} = 0$ and $\lambda^g_{i,S^g,t} = R^g(S^g)/|\mathcal{I}_{S^g}|$. That is, we allocate the revenue associated with an offer set equally among all the resources $i \in \mathcal{I}_{S^g}$ and allocate nothing to resource ϕ. We refer to this initialization as IN-2. Third, we consider the allocation$\lambda^g_{\phi,S^g,t} = \lambda^g_{i,S^g,t} = R^g(S^g)/(1 + |\mathcal{I}_{S^g}|)$, where we allocate the revenue equally across all the resources including resource ϕ. We refer to this initialization as IN-3. Finally, we consider an initial solution that builds on the heuristic arguments in §4.1. We initialize the Lagrange multipliers according to equation (17), where we approximate ψ^* by the optimal dual values corresponding to constraints (3) in $CDLP$ and σ_t^* by the optimal dual value corresponding to constraint (4) for time period t in $CDLP$. We refer to this initialization as IN-4.

Figure 2 shows how the upper bounds obtained by the four initializations compare with upper bound obtained by the optimal Lagrange multipliers on the different test problems. We observe that although no initialization uniformly dominates the others, IN-1 and IN-4 typically tend to work well. If capacity is scarce, the initialization based on the $CDLP$ dual solution, IN-4, tends to work well. As the overall capacity increases, IN-1, which allocates all of the revenue to the resource ϕ, tends to be better. In some cases, IN-1 essentially gives us the optimal solution. We provide some intuition for this. Suppose there is so much capacity that the capacity constraint is never binding. In this case, the solution to the network revenue management dynamic program DP is $V^{DP} = \tau \sum_g \alpha^g R^g(S^{g*})$, where S^{g*} is given by (27). Now consider initialization the Lagrange multipliers according to IN-1. Since $\lambda^g_{i,S^g,t} = 0$, we have $\vartheta^\lambda_{i,t}(\cdot) = 0$ for all i and t. On the other hand, we have $\vartheta^\lambda_{\phi,t} = \sum_g \alpha^g \max_{S^g \subset \mathcal{J}^g} \{\lambda^g_{\phi,S^g,t}\} + \vartheta^\lambda_{\phi,t+1} = \sum_g \alpha^g \max_{S^g \subset \mathcal{J}^g} \{R^g(S^g)\} + \vartheta^\lambda_{\phi,t+1} = \sum_g \alpha^g R^g(S^{g*}) + \vartheta^\lambda_{\phi,t+1}$. Consequently, we have $V_1^\lambda(\boldsymbol{r}_1) = \tau \sum_g \alpha^g R^g(S^{g*}) = V^{DP}$ and it follows that IN-1 gives the optimal set of Lagrange multipliers. In our computational experiments, we compare the upper bounds obtained by the four initializations and choose the initialization which gives the tightest upper bound as the starting solution for the subgradient search algorithm.

Table 6 gives the CPU seconds required by $CDLP$, AF and $sLRo$ to solve the different test

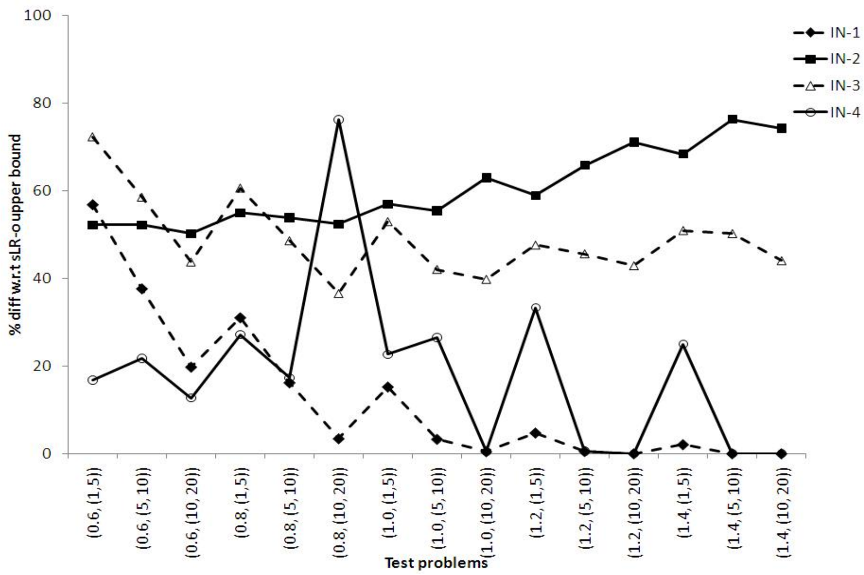

Figure 2: Comparison of the upper bounds obtained by the different initializations of the Lagrange multipliers.

Problem	CPU secs.		
$(\beta, (\omega_0^H, \omega_0^L))$	$CDLP$	AF	$sLRo$
(0.6, (1,5))	0.1	15	122
(0.6, (5, 10))	0.1	17	129
(0.6, (10, 20))	0.1	8	137
(0.8, (1,5))	0.1	10	164
(0.8, (5, 10))	0.1	7	170
(0.8, (10, 20))	0.1	3	178
(1.0, (1,5))	0.1	7	204
(1.0, (5, 10))	0.1	4	210
(1.0, (10, 20))	0.1	1	224
(1.2, (1,5))	0.1	4	241
(1.2, (5, 10))	0.1	1	251
(1.2, (10, 20))	0.0	1	1
(1.4, (1,5))	0.1	2	276
(1.4, (5, 10))	0.1	1	1
(1.4, (10, 20))	0.1	1	1

Table 6: CPU seconds for $CDLP, AF$ and $sLRo$ for the small airline network test problems.

problems. We do not include the running times for $TISA-K$ reported in Meissner and Strauss [14] in Table 6 since the running times depend on the implementation as well as the computer hardware. All the computational experiments are carried out on a Pentium Core 2 Duo desktop with 3-Ghz CPU and 4-GB RAM. We use CPLEX 12.1 to solve all the linear programs. $CDLP$ runs in a fraction of a second, while the running time of AF is in seconds and that of $sLRo$ in minutes. We observe some instances where $sLRo$ terminates in a second. These correspond to test problems with relatively large flight leg capacities, where IN-1 essentially gives the optimal set of Lagrange multipliers. Consequently, the norm of the subgradient turns out to be small and the algorithm terminates almost immediately. We note that the implementations of $CDLP$ and AF exploit the functional form of the MNL choice probabilities. On the other hand, $sLRo$ is agnostic to the choice model and we do not expect the running times to vary significantly for different choice models.

6 Conclusions

In this paper we showed that the piecewise-linear relaxation problem for network revenue management under a general discrete-choice model can be solved as a linear program with $\mathcal{O}(2^{|\mathcal{J}|}\tau \sum_i r_i^1)$ constraints. This is opposed to an original formulation with $\mathcal{O}(2^{|\mathcal{J}|}\tau \prod_i r_i^1)$ constraints (and whose separation problem is NP-complete). Moreover, by showing that it can be solved as a Lagrangian relaxation problem, we are able to use convexity and subgradient search as an alternative to linear programming. This makes the problem tractable for small consideration sets, with practical implications, as there are many operational situations that can be modeled as customers considering only a small set of products. We show by numerical experiments that our ability to solve the piecewise-linear relaxation provides significant benefits. Finally, our results apply to general discrete-choice models. It would be interesting to see if improvements in complexity can be obtained by specializing to specific models such as MNL or nested logit.

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Appendix

Proof of Lemma 1

We introduce some notation to simplify the expressions. Fixing a resource l, we let $\mathcal{R}_l(r_l) = \{\boldsymbol{x} \in \mathcal{R} \,|\, x_l = r_l\}$ be the set of capacity vectors where the capacity on resource l is fixed at r_l. Given a separable piecewise-linear approximation $v = \{v_{i,t}(r_i) \,|\, \forall t, i, r_i \in \mathcal{R}_i\}$, we let

$$\epsilon_{l,t}(r_l, v) = \min_{\boldsymbol{r} \in \mathcal{R}_l(r_l), S \subset \mathcal{Q}(\boldsymbol{r})} \left\{ \sum_i v_{i,t}(r_i) - \sum_j P_j(S) \left[f_j + \sum_{i \in \mathcal{I}_j} [v_{i,t+1}(r_i - 1) - v_{i,t+1}(r_i)] \right] - \sum_i v_{i,t+1}(r_i) \right\}$$

where the argument v emphasizes the dependence on the given approximation. Note that if $v = \{v_{i,t}(r_i) \,|\, \forall t, i, r_i \in \mathcal{R}_i\}$ is feasible to (PL), then $\epsilon_{i,t}(r_i, v) \geq 0$ for all t, i and $r_i \in \mathcal{R}_i$. We begin with a preliminary result.

Lemma 8. *There exists an optimal solution $\hat{v} = \{\hat{v}_{i,t}(r_i) \,|\, \forall t, i, r_i \in \mathcal{R}_i\}$ to (PL) such that for all t, i and $r_i \in \mathcal{R}_i$, we have $\epsilon_{i,t}(r_i, \hat{v}) = 0$.*

Proof. Let $v = \{v_{i,t}(r_i) \,|\, \forall t, i, r_i \in \mathcal{R}_i\}$ be an optimal solution to problem (PL). Let s be the largest time index such that there exists a resource l and $r_l \in \mathcal{R}_l$ with $\epsilon_{l,s}(r_l, v) > 0$. Since v is feasible, this means that $\epsilon_{i,t}(r_i, v) = 0$ for all $t > s$, i and $r_i \in \mathcal{R}_i$. We consider decreasing $v_{l,s}(r_l)$ alone by $\epsilon_{l,s}(r_l, v)$ leaving all the other components of v unchanged. That is, let $\hat{v} = \{\hat{v}_{i,t}(r_i) \,|\, \forall t, i, r_i \in \mathcal{R}_i\}$ where

$$\hat{v}_{i,t}(x) = \begin{cases} v_{i,t}(x) - \epsilon_{l,s}(r_l, v) & \text{if } i = l, t = s, x = r_l \\ v_{i,t}(x) & \text{otherwise.} \end{cases} \tag{28}$$

Note that since $\hat{v}_{i,t}(r_i) \leq v_{i,t}(r_i)$ for all t, i and $r_i \in \mathcal{R}_i$, we have $\sum_i \hat{v}_{i,1}(r_{i,1}) \leq \sum_i v_{i,1}(r_{i,1})$. Next, we show that $\hat{v}$ is feasible. Since $\hat{v}$ differs from v only in one component, we only have to check those constraints where $\hat{v}_{l,s}(r_l)$ appears. Observe that $\hat{v}_{l,s}(r_l)$ appears only in the constraints for time periods $s-1$ and s. For time period $s-1$, we have

$$\begin{aligned}
&\sum_j P_j(S) \left[f_j + \sum_{i \in \mathcal{I}_j} \hat{v}_{i,s}(r_i - 1) \right] + \sum_i \left[1 - \sum_{j \in \mathcal{J}_i} P_j(S) \right] \hat{v}_{i,s}(r_i) \\
&\quad \leq \sum_j P_j(S) \left[f_j + \sum_{i \in \mathcal{I}_j} v_{i,s}(r_i - 1) \right] + \sum_i \left[1 - \sum_{j \in \mathcal{J}_i} P_j(S) \right] v_{i,s}(r_i) \\
&\quad \leq \sum_i v_{i,s-1}(r_i) \\
&\quad = \sum_i \hat{v}_{i,s-1}(r_i)
\end{aligned}$$

for all $\boldsymbol{r} \in \mathcal{R}$ and $S \subset \mathcal{Q}(\boldsymbol{r})$, where the first inequality follows since $\hat{v}_{i,s}(r_i) \leq v_{i,s}(r_i)$ and $\sum_{j \in \mathcal{J}_i} P_j(S) \leq 1$, the second inequality follows from the feasibility of v and the equality follows from (28). For time period s, $\hat{v}_{l,s}(r_l)$ appears only in constraints corresponding to $\boldsymbol{r} \in \mathcal{R}_l(r_l)$. For $\boldsymbol{r} \in \mathcal{R}_l(r_l)$, we have

$$\begin{aligned}
&\sum_i \hat{v}_{i,s}(r_i) \\
&\quad = \sum_i v_{i,s}(r_i) - \epsilon_{l,s}(r_l, v) \\
&\quad \geq \sum_j P_j(S) \left[f_j + \sum_{i \in \mathcal{I}_j} v_{i,s+1}(r_i - 1) - v_{i,s+1}(r_i) \right] + \sum_i v_{i,s+1}(r_i) \\
&\quad = \sum_j P_j(S) \left[f_j + \sum_{i \in \mathcal{I}_j} \hat{v}_{i,s+1}(r_i - 1) - \hat{v}_{i,s+1}(r_i) \right] + \sum_i \hat{v}_{i,s+1}(r_i)
\end{aligned}$$

for all $S \subset \mathcal{Q}(\boldsymbol{r})$, where the inequality follows from the definition of $\epsilon_{l,s}(r_l, v)$ and the last equality follows from (28). Therefore $\hat{v}$ is feasible, which implies that $\epsilon_{i,t}(r_i, \hat{v}) \geq 0$ for all t, i and $r_i \in \mathcal{R}_i$. Next, we note from (28) that $\epsilon_{i,t}(r_i, \hat{v}) = 0$ for all $t > s$, i and $r_i \in \mathcal{R}_i$. For time period s, since $\hat{v}_{i,s}(r_i) \leq v_{i,s}(r_i)$ and $\hat{v}_{i,s+1}(r_i) = v_{i,s+1}(r_i)$, it follows that $\epsilon_{i,s}(r_i, \hat{v}) \leq \epsilon_{i,s}(r_i, v)$. Therefore, if $\epsilon_{i,s}(r_i, v)$ was zero, then $\epsilon_{i,s}(r_i, \hat{v})$ is also zero. Moreover, $\epsilon_{l,s}(r_l, \hat{v}) = 0 < \epsilon_{l,s}(r_l, v)$.

To summarize, $\hat{v}$ is an optimal solution with $\epsilon_{i,t}(r_i, \hat{v}) = 0$ for all $t > s$, i and $r_i \in \mathcal{R}_i$ and $|\{\epsilon_{i,s}(r_i, \hat{v}) \,|\, \epsilon_{i,s}(r_i, \hat{v}) > 0\}| < |\{\epsilon_{i,s}(r_i, v) \,|\, \epsilon_{i,s}(r_i, v) > 0\}|$. We repeat the above procedure finitely many times to obtain an optimal solution $\hat{v}$ with $\epsilon_{i,t}(r_i, \hat{v}) = 0$ for all $t \geq s$, i and $r_i \in \mathcal{R}_i$. Repeating the entire procedure for time periods $s-1, \ldots, 1$ completes the proof. □

We are ready to prove Lemma 1. By Lemma 8, we can assume without loss of generality that the optimal solution $\hat{v} = \{\hat{v}_{i,t}(r_i) \,|\, \forall t, i, r_i \in \mathcal{R}_i\}$ satisfies $\epsilon_{i,t}(r_i, \hat{v}) = 0$ for all t, i and $r_i \in \mathcal{R}_i$. The proof proceeds by induction on the time periods. It is easy to see that statements hold for time period τ. Assuming that the statements hold for all time periods $s > t$, we show below that the statements hold for time period t as well.

Lemma 9. *Assume that statements (i)-(iv) of Lemma 1 hold for time periods $t > s$, then statement (i) holds for time period t.*

Proof. Fix a resource l. For $r_l > 1$, Lemma 8 implies that there exists $\boldsymbol{x} \in \mathcal{R}_l(r_l - 1)$ and $S \subset \mathcal{Q}(\boldsymbol{x})$ such that

$$\begin{aligned}
&\hat{v}_{l,t}(r_l - 1) + \sum_{i \neq l} \hat{v}_{i,t}(x_i) \\
&\quad = \sum_j P_j(S) \left[f_j + \sum_{i \neq l} \mathbb{1}_{[i \in \mathcal{I}_j]} \left[\hat{v}_{i,t+1}(x_i - 1) - \hat{v}_{i,t+1}(x_i) \right] + \mathbb{1}_{[l \in \mathcal{I}_j]} \left[\hat{v}_{l,t+1}(r_l - 2) - \hat{v}_{l,t+1}(r_l - 1) \right] \right] \\
&\qquad\qquad + \hat{v}_{l,t+1}(r_l - 1) + \sum_{i \neq l} \hat{v}_{i,t+1}(x_i). \qquad (29)
\end{aligned}$$

Now consider the capacity vector $\boldsymbol{y}$ with $y_i = x_i$ for $i \neq l$ and $y_l = r_l$. Since $\boldsymbol{x} \leq \boldsymbol{y}$, $\mathcal{Q}(\boldsymbol{x}) \subset \mathcal{Q}(\boldsymbol{y})$

and it follows that $S \subset \mathcal{Q}(\boldsymbol{y})$. Since $\hat{v}$ is feasible, we have

$$\begin{aligned}
&\hat{v}_{l,t}(r_l) + \sum_{i \neq l} \hat{v}_{i,t}(x_i) \\
&\quad \geq \sum_j P_j(S) \left[f_j + \sum_{i \neq l} \mathbb{1}_{[i \in \mathcal{I}_j]} \left[\hat{v}_{i,t+1}(x_i - 1) - \hat{v}_{i,t+1}(x_i) \right] + \mathbb{1}_{[l \in \mathcal{I}_j]} \left[\hat{v}_{l,t+1}(r_l - 1) - \hat{v}_{l,t+1}(r_l) \right] \right] \\
&\qquad\qquad\qquad + \hat{v}_{l,t+1}(r_l) + \sum_{i \neq l} \hat{v}_{i,t+1}(x_i). \qquad (30)
\end{aligned}$$

Subtracting (29) from (30), we get

$$\begin{aligned}
&\hat{v}_{l,t}(r_l) - \hat{v}_{l,t}(r_l - 1) \\
&\quad \geq \sum_j P_j(S) \mathbb{1}_{[l \in \mathcal{I}_j]} \left[2\hat{v}_{l,t+1}(r_l - 1) - \hat{v}_{l,t+1}(r_l) - \hat{v}_{l,t+1}(r_l - 2) \right] + \hat{v}_{l,t+1}(r_l) - \hat{v}_{l,t+1}(r_l - 1) \\
&\quad \geq \hat{v}_{l,t+1}(r_l) - \hat{v}_{l,t+1}(r_l - 1)
\end{aligned}$$

where the last inequality follows from the induction assumption that $2\hat{v}_{l,t+1}(r_l - 1) - \hat{v}_{l,t+1}(r_l) - \hat{v}_{l,t+1}(r_l - 2) \geq 0$. The case $r_l = 1$ can be shown to hold in a similar manner. Therefore, part (ii) of Lemma 1 holds for time period t. □

Lemma 10. *Assume that statements (i)-(iv) of Lemma 1 hold for time periods $t > s$, then statement (ii) holds for time period t.*

Proof. For $r_l \in \mathcal{R}_l \backslash \{0, r_l^1\}$, Lemma 8 implies that there exists $\boldsymbol{x} \in \mathcal{R}_l(r_l + 1)$ and $S \subset \mathcal{Q}(\boldsymbol{x})$ such that

$$\begin{aligned}
&\hat{v}_{l,t}(r_l + 1) + \sum_{i \neq l} \hat{v}_{i,t}(x_i) \\
&\quad = \sum_j P_j(S) \left[f_j + \sum_{i \neq l} \mathbb{1}_{[i \in \mathcal{I}_j]} \left[\hat{v}_{i,t+1}(x_i - 1) - \hat{v}_{i,t+1}(x_i) \right] + \mathbb{1}_{[l \in \mathcal{I}_j]} \left[\hat{v}_{l,t+1}(r_l) - \hat{v}_{l,t+1}(r_l + 1) \right] \right] \\
&\qquad\qquad\qquad + \hat{v}_{l,t+1}(r_l + 1) + \sum_{i \neq l} \hat{v}_{i,t+1}(x_i). \qquad (31)
\end{aligned}$$

Now consider the capacity vector $\boldsymbol{y}$ with $y_i = x_i$ for $i \neq l$ and $y_l = r_l$. Since, $r_l \geq 1$, $\mathcal{Q}(\boldsymbol{y}) = \{j \,|\, \mathbb{1}_{[j \in \mathcal{J}_i]} \leq y_i, i \neq l; \mathbb{1}_{[j \in \mathcal{J}_l]} \leq r_l\} = \{j \,|\, \mathbb{1}_{[j \in \mathcal{J}_i]} \leq x_i, i \neq l; \mathbb{1}_{[j \in \mathcal{J}_l]} \leq r_l + 1\} = \mathcal{Q}(\boldsymbol{x})$. Therefore, $S \subset \mathcal{Q}(\boldsymbol{y})$ and since $\hat{v}$ is feasible

$$\begin{aligned}
&\hat{v}_{l,t}(r_l) + \sum_{i \neq l} \hat{v}_{i,t}(x_i) \\
&\quad \geq \sum_j P_j(S) \left[f_j + \sum_{i \neq l} \mathbb{1}_{[i \in \mathcal{I}_j]} \left[\hat{v}_{i,t+1}(x_i - 1) - \hat{v}_{i,t+1}(x_i) \right] + \mathbb{1}_{[l \in \mathcal{I}_j]} \left[\hat{v}_{l,t+1}(r_l - 1) - \hat{v}_{l,t+1}(r_l) \right] \right] \\
&\qquad\qquad\qquad + \hat{v}_{l,t+1}(r_l) + \sum_{i \neq l} \hat{v}_{i,t+1}(x_i). \qquad (32)
\end{aligned}$$

Subtracting (32) from (31), we get

$$\begin{aligned}
&\hat{v}_{l,t}(r_l+1) - \hat{v}_{l,t}(r_l) \\
&\quad \le \sum_j P_j(S)\mathbb{1}_{[l\in\mathcal{I}_j]}\left[2\hat{v}_{l,t+1}(r_l) - \hat{v}_{l,t+1}(r_l-1) - \hat{v}_{l,t+1}(r_l+1)\right] + \hat{v}_{l,t+1}(r_l+1) - \hat{v}_{l,t+1}(r_l) \\
&\quad \le 2\hat{v}_{l,t+1}(r_l) - \hat{v}_{l,t+1}(r_l-1) - \hat{v}_{l,t+1}(r_l+1) + \hat{v}_{l,t+1}(r_l+1) - \hat{v}_{l,t+1}(r_l) \\
&\quad = \hat{v}_{l,t+1}(r_l) - \hat{v}_{l,t+1}(r_l-1) \\
&\quad \le \hat{v}_{l,t}(r_l) - \hat{v}_{l,t}(r_l-1),
\end{aligned}$$

where the second inequality follows from the induction assumption that $\hat{v}_{l,t+1}(r_l) - \hat{v}_{l,t+1}(r_l-1) \ge \hat{v}_{l,t+1}(r_l+1) - \hat{v}_{l,t+1}(r_l)$ and the fact that $\sum_j P_j(S)\mathbb{1}_{[l\in\mathcal{I}_j]} \le 1$. The last inequality follows from Lemma 9. Therefore, part (iii) of Lemma 1 holds for time period t. ☐

Lemma 11. *Assume that statements (i)-(iv) of Lemma 1 hold for time periods $t > s$, then statement (iii) holds for time period t.*

Proof. By the induction assumption, $\hat{v}_{i,t+1}(r_i) \ge \hat{v}_{i,t+1}(r_i-1)$ for $r \in \mathcal{R}_i\backslash\{0\}$. The result now follows from Lemma 9. ☐

Lemma 12. *Assume that statements (i)-(v) of Lemma 1 hold for time periods $t > s$, then statement (iv) holds for time period t.*

Proof. We first show that $\hat{v}_{i,t}(0) \ge \hat{v}_{i,t+1}(0)$. Suppose there exists l with $\hat{v}_{l,t}(0) < \hat{v}_{l,t+1}(0)$. Since $\hat{v}$ is feasible, it satisfies constraint (6) for the state vector $\boldsymbol{r} = \mathbf{0}$ and $S = \emptyset$. That is

$$\hat{v}_{l,t}(0) + \sum_{i\ne l}\hat{v}_{i,t}(0) \ge \hat{v}_{l,t+1}(0) + \sum_{i\ne l}\hat{v}_{i,t+1}(0)$$

where we use $P_j(\emptyset) = 0$ for all j. This implies there exists k with $\hat{v}_{k,t}(0) > \hat{v}_{k,t+1}(0)$. Letting $\delta = \min\{\hat{v}_{l,t+1}(0) - \hat{v}_{l,t}(0), \hat{v}_{k,t}(0) - \hat{v}_{k,t+1}(0)\} > 0$ and

$$\bar{v}_{i,s}(x) = \begin{cases} \bar{v}_{i,s}(x) + \delta & \text{if } i = l, s = t, x \in \mathcal{R}_l \\ \bar{v}_{i,s}(x) - \delta & \text{if } i = k, s = t, x \in \mathcal{R}_k \\ \hat{v}_{i,s}(x) & \text{otherwise,} \end{cases}$$

it can be verified that $\bar{v}$ is also an optimal solution to (PL). Moreover, since $\hat{v}$ satisfies properties (i)-(iii) for time periods $s \ge t$, so does $\bar{v}$. If $\bar{v}_{l,t}(0) < \bar{v}_{l,t+1}(0)$, then by repeating the above arguments, there exists a resource k' with $\bar{v}_{k',t}(0) > \bar{v}_{k',t+1}(0)$. Repeating the above procedure finitely many times, we obtain an optimal solution $\bar{v}$ with $\bar{v}_{i,t}(0) \ge \bar{v}_{i,t+1}(0)$ for all i.

Now assume that $\hat{v}_{i,t}(r_i-1) \ge \hat{v}_{i,t+1}(r_i-1)$. Lemma 9 implies that $\hat{v}_{i,t}(r_i) \ge \hat{v}_{i,t+1}(r_i)$. ☐

Proof of Lemma 2

The first inequality follows from [14]. So, we only show the second inequality. For $\lambda \in \Lambda$, let $\bar{v}_{i,t}(r_i) = \vartheta^{\lambda}_{i,t}(r_i) + \vartheta^{\lambda}_{\phi,t}/m$ for all t, i and $r_i \in \mathcal{R}_i$, where m is the total number of resources and

$\vartheta^\lambda_{i,t}(\cdot)$ and $\vartheta^\lambda_{\phi,t}$ are given by (9) and (10), respectively. For $\boldsymbol{r}$ and $S \subset \mathcal{Q}(\boldsymbol{r})$

$$\begin{aligned}
&\sum_i \bar{v}_{i,t}(r_i) = \sum_i \left[\vartheta^\lambda_{i,t}(r_i) + \vartheta^\lambda_{\phi,t}/m\right] \\
&\geq \sum_i \left[\mathbb{1}_{[S\in\mathcal{C}_i]}\lambda_{i,S,t} + \sum_{j\in\mathcal{J}_i} P_j(S)\left[\vartheta^\lambda_{i,t+1}(r_i-1) - \vartheta^\lambda_{i,t+1}(r_i)\right] + \vartheta^\lambda_{i,t+1}(r_i) + \lambda_{\phi,S,t}/m + \vartheta^\lambda_{\phi,t+1}/m\right] \\
&= R(S) + \sum_j \sum_{i\in\mathcal{I}_j} P_j(S)\left[\vartheta^\lambda_{i,t+1}(r_i-1) - \vartheta^\lambda_{i,t+1}(r_i)\right] + \sum_i \left[\vartheta^\lambda_{i,t+1}(r_i) + \vartheta^\lambda_{\phi,t+1}/m\right] \\
&= \sum_j P_j(S)\left[f_j + \sum_{i\in\mathcal{I}_j} \bar{v}_{i,t+1}(r_i-1) - \bar{v}_{i,t+1}(r_i)\right] + \sum_i \bar{v}_{i,t+1}(r_i)
\end{aligned}$$

where the inequality holds since $S \subset \mathcal{Q}(\boldsymbol{r}) \subset \mathcal{Q}_i(r_i)$ for all i, from (9) and (10), and the fact that $\lambda \in \Lambda$. The second equality holds since $\lambda \in \Lambda$, and the last equality follows from (5). Therefore, $\bar{v}$ is feasible for PL and its objective function value is $\sum_i \bar{v}_{i,1}(r^1_i) = \sum_i \vartheta^\lambda_{i,1}(r^1_i) + \vartheta^\lambda_{\phi,1} = V^\lambda_1(\boldsymbol{r}^1)$. Therefore, $V^{PL} \leq V^\lambda_1(\boldsymbol{r}^1)$. □

Proof of Proposition 2

We show the results by induction over the time periods. We begin with the first part of the proposition. It is easy to show the result for the last time period. So we assume that the result holds for time period $t+1$ and show that it holds for time period t. We have

$$\begin{aligned}
&\vartheta^{\hat\lambda}_{i,t}(r_i) \geq \mathbb{1}_{\left[S^\lambda_{i,t}(r_i)\in\mathcal{C}_i\right]}\hat\lambda_{i,S^\lambda_{i,t}(r_i),t} + \sum_{j\in\mathcal{J}_i} P_j(S^\lambda_{i,t}(r_i))\left[\vartheta^{\hat\lambda}_{i,t+1}(r_i-1) - \vartheta^{\hat\lambda}_{i,t+1}(r_i)\right] + \vartheta^{\hat\lambda}_{i,t+1}(r_i) \\
&\geq \mathbb{1}_{\left[S^\lambda_{i,t}(r_i)\in\mathcal{C}_i\right]}\lambda_{i,S^\lambda_{i,t}(r_i),t} + \mathbb{1}_{\left[S^\lambda_{i,t}(r_i)\in\mathcal{C}_i\right]}\left[\hat\lambda_{i,S^\lambda_{i,t}(r_i),t} - \lambda_{i,S^\lambda_{i,t}(r_i),t}\right] \\
&\quad + \sum_{j\in\mathcal{J}_i} P_j(S^\lambda_{i,t}(r_i))\left[\vartheta^\lambda_{i,t}(r_i-1) + \sum_{k=t+1}^{\tau}\sum_S \mathbb{1}_{[S\in\mathcal{C}_i]}\Pr\{S^\lambda_{i,k}(X_{i,k}) = S \,|\, X_{i,t+1} = r_i - 1\}\left[\hat\lambda_{i,S,k} - \lambda_{i,S,k}\right]\right] \\
&\quad + \left[1 - \sum_{j\in\mathcal{J}_i} P_j(S^\lambda_{i,t}(r_i))\right]\left[\vartheta^\lambda_{i,t+1}(r_i) + \sum_{k=t+1}^{\tau}\sum_S \mathbb{1}_{[S\in\mathcal{C}_i]}\Pr\{S^\lambda_{i,k}(X_{i,k}) = S \,|\, X_{i,t+1} = r_i\}\left[\hat\lambda_{i,S,k} - \lambda_{i,S,k}\right]\right] \\
&= \vartheta^\lambda_{i,t}(r_i) + \sum_{k=t}^{\tau}\sum_S \mathbb{1}_{[S\in\mathcal{C}_i]}\Pr\{S^\lambda_{i,k}(X_{i,k}) = S \,|\, X_{i,t} = r_i\}\left[\hat\lambda_{i,S,k} - \lambda_{i,S,k}\right],
\end{aligned}$$

where the first inequality holds since $S^\lambda_{i,t}(r_i)$ is a feasible solution to problem (9) when the Lagrange multipliers are $\hat\lambda$. The second inequality holds by the induction assumption, while the last equality follows by noting that

$$
\begin{aligned}
&\Pr\left\{S^{\lambda}_{i,k}(X_{i,k}) = S \,|\, X_{i,t} = r_i\right\} = \\
&\quad \Pr\left\{S^{\lambda}_{i,k}(X_{i,k}) = S \,|\, X_{i,t} = r_i, X_{i,t+1} = r_i\right\} \Pr\left\{X_{i,t+1} = r_i \,|\, X_{i,t} = r_i\right\} \\
&\qquad + \Pr\left\{S^{\lambda}_{i,k}(X_{i,k}) = S \,|\, X_{i,t} = r_i, X_{i,t+1} = r_i - 1\right\} \Pr\left\{X_{i,t+1} = r_i - 1 \,|\, X_{i,t} = r_i\right\} \\
&\quad = \Pr\left\{S^{\lambda}_{i,k}(X_{i,k}) = S \,|\, X_{i,t+1} = r_i\right\} \Pr\left\{X_{i,t+1} = r_i \,|\, X_{i,t} = r_i\right\} \\
&\qquad + \Pr\left\{S^{\lambda}_{i,k}(X_{i,k}) = S \,|\, X_{i,t+1} = r_i - 1\right\} \Pr\left\{X_{i,t+1} = r_i - 1 \,|\, X_{i,t} = r_i\right\} \\
&\quad = \Pr\left\{S^{\lambda}_{i,k}(X_{i,k}) = S \,|\, X_{i,t+1} = r_i\right\} \left[1 - \sum_{j \in \mathcal{J}_i} P_j(S^{\lambda}_{i,t}(r_i))\right] \\
&\qquad + \Pr\left\{S^{\lambda}_{i,k}(X_{i,k}) = S \,|\, X_{i,t+1} = r_i - 1\right\} \sum_{j \in \mathcal{J}_i} P_j(S^{\lambda}_{i,t}(r_i)).
\end{aligned}
$$

This concludes the proof of the first part of the proposition.

The proof of the second part proceeds in a similar fashion. We assume that the result holds for time period $t+1$ and show that it holds for time period t. We have

$$
\begin{aligned}
\vartheta^{\hat{\lambda}}_{\phi,t} \quad &\geq \quad \hat{\lambda}_{\phi, S^{\lambda}_{\phi t}, t} + \vartheta^{\hat{\lambda}}_{\phi,t+1} \\
&\geq \quad \lambda_{\phi, S^{\lambda}_{\phi,t}, t} + \left[\hat{\lambda}_{\phi, S^{\lambda}_{\phi,t}, t} - \lambda_{\phi, S^{\lambda}_{\phi,t}, t}\right] + \vartheta^{\lambda}_{\phi,t+1} + \sum_{k=t+1}^{\tau} \sum_{S} \mathbb{1}_{\left[S^{\lambda}_{\phi,t} = S\right]} \left[\hat{\lambda}_{\phi,S,k} - \lambda_{\phi,S,k}\right] \\
&= \quad \vartheta^{\lambda}_{\phi,t} + \sum_{k=t}^{\tau} \sum_{S} \mathbb{1}_{\left[S^{\lambda}_{\phi,k} = S\right]} \left[\hat{\lambda}_{\phi,S,k} - \lambda_{\phi,S,k}\right],
\end{aligned}
$$

where the first inequality holds because $S^{\lambda}_{\phi,t}$ is feasible but not necessarily optimal for problem (10) when the Lagrange multipliers are $\hat{\lambda}$ and the second inequality follows from the induction assumption. □