On the closed-form approximation of short-time random strike options.

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Abstract

In this paper we propose a general technique to develop first and second order closed-form approximation formulas for short-time options with random strikes. Our method is based on Malliavin calculus techniques and allows us to obtain simple closed-form approximation formulas depending on the derivative operator. The numerical analysis shows that these formulas are extremely accurate and improve some previous approaches on two-assets and three-assets spread options as Kirk's formula or the decomposition mehod presented in Alòs, Eydeland and Laurence (2011).
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1 Introduction

This paper is devoted to the study of options with random strikes (as two-asset and three-asset spread options), whose payoff is of the form

$$(S_T - K_T)_+,$$

where S denotes the asset price and $\{K_t, t \in [0,T]\}$ is a random process. In the particular case that $\{S_t\}$ and $\{K_t\}$ are two geometric Brownian motions (that may be correlated), the corresponding option price is given by the Margrabe fomula (see Margrabe (1978)), which can be deduced from the fact that $\{S_t/K_t, t \in [0,T]\}$ is a log-normal process. Thus, in this case, the spread option value can be expressed as the classical Black-Scholes call price with initial

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asset price S_0, where we take the strike equal to the expected value of K_T and volatility equal to $\sqrt{\sigma^2 - 2\rho\sigma\sigma' + (\sigma')^2}$. Here σ and σ' are the volatility parameters of S_t and K_t, respectively, and ρ denotes the correlation. If $K_t = S_t^{'} + K$ (the two-asset spread case), the strike process is not log-normal and then the arguments used in the deduction of Margrabe formula cannot be applied anymore. In this context a successful method was suggested by Kirk (1995), who proposed to approach $S/(S' + K)$ by a log-normal random process and then to apply the Margrabe formula. Nowadays Kirk's formula is the most popular option pricing approximation expression for spread options due to its accuracy and its simplicity. A similar idea was proposed by Alòs, Eydeland and Laurence (2011), where the authors extended this approach to the tree-assets case (i.e. $K_t = S_t^1 + S_t^2 + K$) and proposed a closed-form approximation as the price of a vanilla option with strike equal to $K_0 = S_0^1 + S_0^2 + K$ and a suitably adjusted volatility.

Both classical Kirk's formula and its extension given in Alòs, Eydeland and Laurence (2011) are very simple and accurate. Nevertheless, there is not, up to our knowledge, an analytical study of their goodness of fit or a systematic method to improve them. Notice that in both approximations, the adjusted volatility (deduced from the log-normal approximation of the strike process) does not depend on the asset price S_0, so we can consider these adjusted volatilities as first-order approximations of the corresponding implied volatilities as functions of S_0.

In this paper we propose a systematic method to develop closed-form first and second order short-time approximation formulas for options with random strikes. The proposed first-order approximation formula will consist in the price of an European call option with asset price S_0 and strike price K_0 and a adjusted volatility that does not depend on the asset price S_0. Then, by means of Malliavin calculus we decompose the option price as the sum of this closed-form approximation and two error terms. This decomposition gives us an extension of the Margrabe formula that allows us to find an expression for the short-time skew slope for spread options. Finally, the obtained expression for this skew will give us a tool to construct an improvement of the approximation of the implied volatility

The organization of this paper is as follows. In Section 2 we introduce the framework of this paper. Section 3 is devoted to present an extended Margrabe formula that gives us a systematic method to construct first-order approximation prices. Moreover, this formula is used in Section 4 to figure up an expression for the derivative of the implied volatility with respect to the log-stock price. The short time behaviour of this derivative is analyzed in Section 5, and from this study we deduce a method to obtain second-order approximation formulas. Finally in Section 6 we apply our results to the study of two-assets and three-assets spread options.

2 Statement of the model and notation

In this paper we consider the following model for the log-price of a stock under a risk-neutral probability measure Q:

$$dX_t = \left(r - \frac{\sigma_t^2}{2}\right)dt + \sigma_t(\sum_{i=1}^{d}\rho_{i,d+1}dW_t^i + \sqrt{1-\sum_{i=1}^{d}\rho_{i,d+1}^2}dB_t),\ t\in[0,T]. \quad (1)$$

Here, r is the instantaneous interest rate, $W=(W^1,\ldots,W^d)$ is a d-dimensional Brownian motion, B is a standard Brownian motions and $\rho_{i,d+1}\in(-1,1)$, $i\in\{1,\ldots,d\}$ and satisfy that $\sum_{i=1}^d \rho_{i,d+1}^2<1$. In the remaining of this paper we assume that W and B are independent, and that, for the sake of simplicity, the volatility process σ is a square-integrable deterministic function which is right-continuous. We denote by $\mathcal{F}^W$ and $\mathcal{F}^B$ the augmentation under the underlying probability measure of the filtrations generated by W and B, respectively. We define $\mathcal{F}:=\mathcal{F}^W\vee\mathcal{F}^B$.

In this paper we consider European call options with payoff $h(X_T):=\left(e^{X_T}-K_T\right)_+$, where we allow the strike K_T to be random. More precisely, we assume that $\{K_t, t\in[0,T]\}$ is a square-integrable, positive, continuous, bounded and $\mathcal{F}_t^W$-measurable process. Notice that this choice includes some popular classes of options as spread and basket options.

It is well-known that the price of an European call with random strike K_T is given by the formula

$$V_t = e^{-r(T-t)}E\left((e^{X_T}-K_T)_+|\mathcal{F}_t\right). \quad (2)$$

In the sequel, we will make use of the following notation:

- $M_t^T := E\left[K_T|\,\mathcal{F}_t^W\right]$. Observe that, by the martingale representation theorem (see, for instance, Karatzas and Shreve (1991)), there exist d $\mathcal{F}_t^W$-measurable processes $m^1(T,\cdot),...,m^d(T,\cdot)$ such that

$$M_t^T = E\left(K_T\right) + \int_0^t \sum_{i=1}^{d} m^i(T,s)dW_s^i. \quad (3)$$

- $v_t := \left(\frac{Y_t}{T-t}\right)^{\frac{1}{2}}$, with $Y_t := \int_t^T a_s^2 ds$, where $a_s^2 ds := \sigma_s^2 ds - 2\frac{d\langle M^T, X\rangle_s}{M_s^T} + \frac{d\langle M^T,M^T\rangle_s}{(M_s^T)^2}$. Note that

$$\begin{aligned} a_s^2 &= \sigma_s^2 - 2\sigma_s\sum_{i=1}^{d}\rho_{i,d+1}\frac{m^i(T,s)}{M_s^T} + \sum_{i=1}^{d}\frac{\left(m^i(T,s)\right)^2}{(M_s^T)^2} \\ &= \sigma_s^2\left(1-\sum_{i=1}^{d}\rho_{i,d+1}^2\right) + \sum_{i=1}^{d}\left(\sigma_s\rho_{i,d+1} - \frac{m^i(T,s)}{M_s^T}\right)^2 \end{aligned}$$

is a positive quantity. Although the right-hand-side of the last equality depends on T, we denote it by a_s^2 in order to simplify the notation. Now it is easy to see that there is a constant C such that $a_s^2 \geq C\sigma_s^2$.

- $BS(t, x, K, \sigma)$ denotes the price of an European call option under the classical Black-Scholes model with constant volatility σ, current log stock price x, time to maturity $T-t$, strike price K and interest rate r. Remember that in this case:

$$BS(t, x, K, \sigma) = e^x N(d_+) - Ke^{-r(T-t)} N(d_-),$$

where N denotes the cumulative probability function of the standard normal law and

$$d_\pm := \frac{x - x_t^*}{\sigma\sqrt{T-t}} \pm \frac{\sigma}{2}\sqrt{T-t},$$

with $x_t^* := \ln K - r(T-t)$.

- $\mathcal{L}_{BS}\left(\sigma^2\right)$ stands for the Black-Scholes differential operator, in the log variable, with volatility σ :

$$\mathcal{L}_{BS}\left(\sigma^2\right) = \partial_t + \frac{1}{2}\sigma^2\partial_{xx}^2 + (r - \frac{1}{2}\sigma^2)\partial_x - r\cdot$$

It is well known that $\mathcal{L}_{BS}\left(\sigma^2\right) BS(\cdot, \cdot, \sigma) = 0$.

Now we describe some basic notation that is used in this article. For this, we assume that the reader is familiar with the elementary results of the Malliavin calculus, as given for instance in Nualart (2006).

Let us consider a standard Brownian motion $Z = \{Z_t,\, t \in [0, T]\}$ defined on a complete probability space $(\Omega, \mathcal{F}, P)$. The set $\mathbb{D}_Z^{1,2}$ is the domain of the derivative operator D^Z in the Malliavin calculus sense. $\mathbb{D}_Z^{1,2}$ is a dense subset of $L^2(\Omega)$ and D^Z is a closed and unbounded operator from $L^2(\Omega)$ into $L^2([0, T] \times \Omega)$. We also consider the iterated derivatives $D^{Z,n}$, for $n > 1$, whose domains is denoted by $\mathbb{D}_Z^{n,2}$.

The adjoint of the derivative operator D^Z, denoted by δ^Z, is an extension of the Itô integral in the sense that the set $L_a^2([0, T] \times \Omega)$ of square integrable and adapted processes (with respect to the filtration generated by Z) is included in Domδ^Z and the operator δ^Z restricted to $L_a^2([0, T] \times \Omega)$ coincides with the Itô integral. We make use of the notation $\delta^Z(u) = \int_0^T u_t dZ_t$ and $\delta^Z(u\mathbf{1}_{[t,T]}) = \int_t^T u_t dZ_t$. We recall that $\mathbb{L}_Z^{n,2} := L^2([0, T]; \mathbb{D}_Z^{n,2})$ is included in the domain of δ^Z for all $n \geq 1$.

3 A decomposition result and a first order approximation formula

Before proving an extension of the Hull and White formula, we state the following result, which is nedeed in the remaining of the paper.

Lemma 1 *Let K be bounded, $0 \leq t \leq s < T$ and $\mathcal{G}_t := \mathcal{F}_t \vee \mathcal{F}_T^W$. Then, for any $n \geq 0$, there exists $C = C(n, \sum_{i=1}^d \rho_{i,d+1}^2)$ such that*

$$\begin{aligned} &\left| E\left((\partial_x^{n+2} - \partial_x^{n+1}) BS(s, X_s, M_s^T, v_s) | \mathcal{G}_t \right) \right| \\ \leq \; & C \left(\int_t^T \sigma_\theta^2 d\theta \right)^{-\frac{1}{2}(n+1)}. \end{aligned}$$

Proof: In order to show this result, we proceed as in the proof of Lemma 4.1 in Alòs, León and Vives (2007) and we use the fact that K is a bounded and $\mathcal{F}^W-$measurable and adapted process to obtain that

$$\begin{aligned} &\left| E\left((\partial_x^{n+2} - \partial_x^{n+1}) BS(s, X_s, M_s^T, v_s) | \mathcal{G}_t \right) \right| \\ \leq \; & C \left((1 - \sum_{i=1}^d \rho_{i,d+1}^2) \int_t^s \sigma_\theta^2 d\theta + \int_s^T a_\theta^2 d\theta \right)^{-\frac{1}{2}(n+1)}. \end{aligned}$$

Notice that, as $\sum_{i=1}^d \rho_{i,d+1}^2 < 1$, there exists a positive constant C such that $\int_s^T a_\theta^2 d\theta \geq C \int_s^T \sigma_\theta^2 d\theta$, from where the result follows. ■

Now we are able to prove the main result of this section, the extended Hull and White formula. We will need the following hypothesis:

(H1) The process $a^2 \in \mathbb{L}_{W^i}^{1,2}$, for all $i = 1, ..., d$.

Theorem 2 *Consider the model (1) and assume that hyptothesis (H1) holds. Then it follows that*

$$\begin{aligned} V_t = & E\left(BS(t, X_t, M_t^T, v_t) \middle| \mathcal{F}_t \right) \\ & + \frac{1}{2} E\left(\sum_{i=1}^d \left\{ \rho_{i,d+1} \int_t^T e^{-r(s-t)} \left(\partial_{xxx}^3 - \partial_{xx}^2 \right) BS(s, X_s, M_s^T, v_s) \sigma_s \Lambda_s^{W^i} ds \right.\right. \\ & \left.\left. + \int_t^T e^{-r(s-t)} \partial_K \left(\partial_{xx}^2 - \partial_x \right) BS(s, X_s, M_s^T, v_s) \Lambda_s^{W^i} m^i(T, s) ds \right\} \middle| \mathcal{F}_t \right), \quad (4) \end{aligned}$$

where $\Lambda_s^{W^i} := \left[D_s^{W^i} \int_s^T a^2(r) dr \right]$, $i = 1, ..., d$.

Proof: This proof is similar to the one of the main theorem in Alòs, León and Vives (2007), so we only sketch it. Notice that $BS(T, X_T, M_T^T, v_T) = V_T$. Then, from (2), we have

$$e^{-rt} V_t = E(e^{-rT} BS(T, X_T, K_T, v_T) | \mathcal{F}_t).$$

Now, using the Itô's formula to the process

$$t \to e^{-rt} BS(t, X_t, M_t^T, v_t)$$

and proceeding as in Alòs, León and Vives (2007) (see also Alòs and Nualart (1998), Alòs (2006) or Nualart (2006)), we can write

$$\begin{aligned}
& e^{-rT} BS(T, X_T, M_T^T, v_T) \\
= \quad & e^{-rt} BS(t, X_t, M_t^T, v_t) \\
& + \int_t^T e^{-rs} \mathcal{L}_{BS}(v_s^2) BS(s, X_s, M_s^T, v_s) ds \\
& + \int_t^T e^{-rs} \partial_x BS(s, X_s, M_s^T, v_s) \sigma_s \left(\sum_{i=1}^d (\rho_{i,d+1} dW_s^i) + \sqrt{1 - \sum_{i=1}^d \rho_{i,d+1}^2} dB_s \right) \\
& + \int_t^T e^{-rs} \partial_K BS(s, X_s, M_s^T, v_s) dM_s^T \\
& + \int_t^T e^{-rs} \partial_{xK}^2 BS(s, X_s, M_s^T, v_s) d \left\langle M^T, X \right\rangle_s \\
& + \frac{1}{2} \int_t^T e^{-rs} \partial_\sigma BS(s, X_s, M_s^T, v_s) \frac{v_s^2 - a_s^2}{v_s(T-s)} ds \\
& + \int_t^T e^{-rs} \partial_{x\sigma}^2 BS(s, X_s, M_s^T, v_s) \frac{\sigma_s \sum_{i=1}^d \rho_{i,d+1} \Lambda_s^{W^i}}{2v_s(T-s)} ds \\
& + \int_t^T e^{-rs} \partial_{K\sigma}^2 BS(s, X_s, M_s^T, v_s) \frac{\sum_{i=1}^d \Lambda_s^{W^i} m^i(T,s)}{2v_s(T-s)} ds \\
& + \frac{1}{2} \int_t^T e^{-rs} \left(\partial_{xx}^2 - \partial_x \right) BS(s, X_s, M_s^T, v_s) \left(\sigma_s^2 - v_s^2 \right) ds \\
& + \frac{1}{2} \int_t^T e^{-rs} \partial_{KK}^2 BS(s, X_s, M_s^T, v_s) d \left\langle M^T, M^T \right\rangle_s .
\end{aligned}$$

Hence, the fact that $\mathcal{L}_{BS}(v_s^2)BS(s,X_s,M_s^T,v_s)=0$, multiplying by $e^{rt}$and taking conditional expectations we can establish

$$
\begin{aligned}
& E\left(e^{-r(T-t)}BS(T,X_T,M_T^T,v_T)\middle| \mathcal{F}_t\right) \\
= & \ E\left\{ BS(t,X_t,M_t^T,v_t)+\int_t^T e^{-r(s-t)}\partial_{xK}^2 BS(s,X_s,M_s^T,v_s)d\left\langle M^T,X\right\rangle_s \right. \\
& +\frac{1}{2}\int_t^T e^{-r(s-t)}\partial_\sigma BS(s,X_s,M_s^T,v_s)\frac{v_s^2-a_s^2}{v_s(T-s)}ds \\
& +\int_t^T e^{-r(s-t)}\partial_{x\sigma}^2 BS(s,X_s,M_s^T,v_s)\frac{\sigma_s\sum_{i=1}^d \rho_{i,d+1}\Lambda_s^{W^i}}{2v_s(T-s)}ds \\
& +\int_t^T e^{-r(s-t)}\partial_{K\sigma}^2 BS(s,X_s,M_s^T,v_s)\frac{\sum_{i=1}^d \Lambda_s^{W^i}m^i(T,s)}{2v_s(T-s)}ds \\
& +\frac{1}{2}\int_t^T e^{-r(s-t)}\left(\partial_{xx}^2-\partial_x\right)BS(s,X_s,M_s^T,v_s)\left(\sigma_s^2-v_s^2\right)ds \\
& \left. +\frac{1}{2}\int_t^T e^{-r(s-t)}\partial_{KK}^2 BS(s,X_s,M_s^T,v_s)d\left\langle M^T,M^T\right\rangle_s \middle| \mathcal{F}_t\right\}.
\end{aligned}
$$

Consequently, the classical relationships between the greeks

$$\partial_{xx}^2 BS-\partial_x BS=\partial_\sigma BS\frac{1}{\sigma(T-t)}$$

$$\partial_{xK}^2 BS=-\partial_\sigma BS\frac{1}{K\sigma(T-t)}$$

$$\partial_{KK}^2 BS=\partial_\sigma BS\frac{1}{K^2\sigma(T-t)}$$

give

$$
\begin{aligned}
&E\left(e^{-r(T-t)}BS(T,X_T,M_T^T,v_T)\Big|\mathcal{F}_t\right)\\
&= E\Bigg\{BS(t,X_t,M_t^T,v_t)-\int_t^T e^{-r(s-t)}\partial_\sigma BS(s,X_s,M_s^T,v_s)\frac{1}{M_s^T v_s(T-s)}d\left\langle M^T,X\right\rangle_s\\
&\quad+\frac{1}{2}\int_t^T e^{-r(s-t)}\partial_\sigma BS(s,X_s,M_s^T,v_s)\frac{v_s^2-a_s^2}{v_s(T-s)}ds\\
&\quad+\int_t^T e^{-r(s-t)}\partial_{x\sigma}^2 BS(s,X_s,M_s^T,v_s)\frac{\sigma_s\sum_{i=1}^d\rho_{i,d+1}\Lambda_s^{W^i}}{2v_s(T-s)}ds\\
&\quad+\int_t^T e^{-r(s-t)}\partial_{K\sigma}^2 BS(s,X_s,M_s^T,v_s)\frac{\sum_{i=1}^d\Lambda_s^{W^i}m^i(T,s)}{2v_s(T-s)}ds\\
&\quad+\frac{1}{2}\int_t^T e^{-r(s-t)}\partial_\sigma BS(s,X_s,M_s^T,v_s)\left(\sigma_s^2-v_s^2\right)\frac{1}{v_s(T-s)}ds\\
&\quad+\frac{1}{2}\int_t^T e^{-r(s-t)}\partial_\sigma BS(s,X_s,M_s^T,v_s)\frac{1}{M_s^2 v_s(T-s)}d\left\langle M^T,M^T\right\rangle_s\Bigg|\mathcal{F}_t\Bigg\}.
\end{aligned}
$$

That is,

$$
\begin{aligned}
&E\left(e^{-r(T-t)}BS(T,X_T,M_T^T,v_T)\Big|\mathcal{F}_t\right)\\
&= E\Bigg\{BS(t,X_t,M_t^T,v_t)+\int_t^T e^{-r(s-t)}\frac{\partial_\sigma BS(s,X_s,M_s^T,v_s)}{v_s(T-t)}\\
&\quad\times\left[-\frac{d\left\langle M^T,X\right\rangle_s}{M_s^T}+\frac{1}{2}\left(v_s^2-a_s^2\right)ds+\frac{1}{2}\left(\sigma_s^2-v_s^2\right)ds+\frac{1}{2}\frac{d\left\langle M^T,M^T\right\rangle_s}{\left(M_s^T\right)^2}\right]\\
&\quad+\int_t^T e^{-r(s-t)}\partial_{x\sigma}^2 BS(s,X_s,M_s^T,v_s)\frac{\sigma_s\sum_{i=1}^d\rho_{i,d+1}\Lambda_s^{W^i}}{2v_s(T-s)}ds\\
&\quad+\int_t^T e^{-r(s-t)}\partial_{K\sigma}^2 BS(s,X_s,M_s^T,v_s)\frac{\sum_{i=1}^d\Lambda_s^{W^i}m^i(T,s)}{2v_s(T-s)}ds\Bigg|\mathcal{F}_t\Bigg\}.
\end{aligned}
$$

Since, $a_s^2ds:=\sigma_s^2ds-2\frac{d\left\langle M^T,X\right\rangle_s}{M_s^T}+\frac{d\left\langle M^T,M^T\right\rangle_s}{(M_s^T)^2}$ we obtain

$$
\begin{aligned}
&E\left(e^{-r(T-t)}BS(T,X_T,M_T^T,v_T)\Big|\mathcal{F}_t\right)\\
&= E\Bigg\{BS(t,X_t,M_t^T,v_t)\\
&\quad+\int_t^T e^{-r(s-t)}\partial_{x\sigma}^2 BS(s,X_s,M_s^T,v_s)\frac{\sigma_s\sum_{i=1}^d\rho_{i,d+1}\Lambda_s^{W^i}}{2v_s(T-s)}ds\\
&\quad+\int_t^T e^{-r(s-t)}\partial_{K\sigma}^2 BS(s,X_s,M_s^T,v_s)\frac{\sum_{i=1}^d\Lambda_s^{W^i}m^i(T,s)}{2v_s(T-s)}ds\Bigg|\mathcal{F}_t\Bigg\},
\end{aligned}
$$

as we wanted to prove. ■

Remark 3 *Notice that, from the above decomposition result,*

$$E\left(BS(t, X_t, M_t^T, v_t) \middle| \mathcal{F}_t \right)$$

can be seen as a first order approximation formula for short-time random strike options. Notice that the adjusted volatility v_t *is constant as a function of the log-asset price* X_t*. Moreover, for short-time options, a Taylor expansion gives us that* v_t *can be approximated by* $\sqrt{a_t^2}$ *and then we can consider* $BS(t, X_t, M_t^T, \sqrt{a_t^2})$ *as a first-order approximation formula for random strike options. In fact,* $BS(t, X_t, M_t^T, \sqrt{a_t^2})$ *recovers Kirk's formula and its tree-assets extension presented in Alòs, Eydeland and Laurence (2011), as we will see in the following examples.*

Example 4 *Assume the model (1) with constant volatility* σ, $d = 1$ *and interest rate* r*. We consider a call spread option with strike equal to* $K_T = S_T^1 + K$*, where* K *is a non-negative deterministic constant and* S^1 *is another stock price of the form* $S_t^1 = \exp\left(X_t^1\right)$*, where*

$$dX_t^1 = \left(r - \frac{(\sigma_1)^2}{2} \right) dt + \sigma_1 dW_t^1, \, t \in [0, T],$$

for some positive constant σ_1*. Then we can easily check that* $m^1(T, \theta) = \exp(r(T-\theta))S_\theta^1 \sigma_1$ *and* $M_\theta^T = \exp(r(T-\theta))S_\theta^1 + K$*. So*

$$a_\theta^2 := \sigma^2 - \frac{2\rho_{1,2}\sigma\sigma_1 \exp(r(T-\theta))S_\theta^1}{\exp(r(T-\theta))S_\theta^1 + K} + \frac{(\sigma_1)^2 \left(\exp(r(T-\theta))S_\theta^1\right)^2}{\left(\exp(r(T-\theta))S_\theta^1 + K\right)^2}.$$

which coincides with Kirk's square implied volatility approximation. Notice that, if $K = 0$,

$$a_\theta^2 = \sigma^2 - 2\rho_{1,2}\sigma\sigma_1 + \sigma_1^2$$

and $D_s^{W^1} a^2(\theta) = 0$*. Then, equality (4) reduces to*

$$V_t = BS\left(t, X_t, \exp\left(r\left(T-t\right)\right) S_t^1, \sqrt{\sigma^2 - 2\rho_{1,2}\sigma\sigma_1 + \sigma_1^2}\right),$$

and we recover the well-known Margrabe formula (see Margrabe (1978)).

Remark 5 *Notice that, in the context of the previous example, when* K *is negative, the call option on the spread* $S_T - S_T^1$ *is equivalent to the corresponding put option on the spread* $S_T^1 - S_T$ *with positive strike* $-K$*. Then, without loss of generality, we can assume that the spread option is written with a positive* K*.*

Example 6 *Assume the model (1) with constant volatility* $\sigma, d = 2$ *and interest rate* $r = 0$*, for the sake of simplicity. We consider a call spread option with strike* *equal to* $K_T = S_T^1 + S_T^2 + K$*, where* K *is a non-negative deterministic constant and* S^1*,*S^2 *are two asset prices of the form* $S^i = \exp(X^i)$*. Here*

$$\begin{aligned} dX_t^1 &= \left(-\frac{\sigma_1^2}{2}\right) dt + \sigma_1 dW_t^1, \ t \in [0,T] \\ dX_t^2 &= \left(-\frac{\sigma_2^2}{2}\right) dt + \sigma_2 \left(\rho_{1,2} dW_t^1 + \sqrt{1-\rho_{1,2}^2} dW_t^2\right), \ t \in [0,T] \end{aligned}$$

In this case we have that

$$m^1(T,\theta) = S_\theta^1 \sigma_1 + S_\theta^2 \sigma_2 \rho_{1,2}, m^2(T,\theta) = S_\theta^2 \sigma_2 \sqrt{1-\rho_{1,2}^2}, M_\theta^T = S_\theta^1 + S_\theta^2 + K.$$

Hence, similar arguments as in the previous example give us that

$$\begin{aligned} a_\theta^2 &= \sigma^2 - 2\rho_{1,3}\sigma\sigma_1 \frac{S_\theta^1}{S_\theta^1 + S_\theta^2 + K} - 2\tilde{\rho}_{2,3}\sigma\sigma_2 \frac{S_\theta^2}{S_\theta^1 + S_\theta^2 + K} \\ &\quad + \frac{\left(S_\theta^1 \sigma_1\right)^2}{\left(S_\theta^1 + S_\theta^2 + K\right)^2} + \frac{2\rho_{1,2}\left(S_\theta^1 \sigma_1\right)\left(S_\theta^2 \sigma_2\right)}{\left(S_\theta^1 + S_\theta^2 + K\right)^2} + \frac{\left(S_\theta^2 \sigma_2\right)^2}{\left(S_\theta^1 + S_\theta^2 + K\right)^2} \\ &= \sigma^2 - 2\rho_{1,3}\sigma\sigma_1 a - 2\tilde{\rho}_{2,3}\sigma\sigma_2 b + \sigma_1^2 a^2 + 2\sigma_1\sigma_2\rho_{1,2} ab + \sigma_2^2 b^2, \end{aligned}$$

where

$$\tilde{\rho}_{2,3} = d\left\langle \rho_{1,2} W^1 + \sqrt{1-\rho_{1,2}^2} W^2, \rho_{1,3} W^1 + \rho_{2,3} W^2 \right\rangle_t = \frac{d\left\langle X^2, X \right\rangle_t}{\sigma\sigma_2},$$

$a := \frac{S_\theta^1}{S_\theta^1 + S_\theta^2 + K}$ *and* $b := \frac{S_\theta^2}{S_\theta^1 + S_\theta^2 + K}$*. This expression coincides with the square implied volatility approximation proposed in Alòs, Eydeland and Laurence (2011).*

4 Derivative of the implied volatility

Let $I_t(X_t)$ denote the implied volatility process, which satisfies by definition $V_t = BS(t, X_t, M_t^T, I_t(X_t))$. In this section we prove a formula for its at-the-money derivative that we use in Section 5 to study the short-time behavior of the implied volatility and its dependence on the asset price.

Proposition 7 *Assume that the model (1) holds with* $a \in \mathbb{L}_{W^i}^{1,2}$*, for all* $i \in \{1, ..., d\}$ *and that, for every fixed* $t \in [0,T)$*,* $\left(\int_t^T \sigma_\theta^2 d\theta\right)^{-1} < \infty$*. Then it follows that*

$$\frac{\partial I_t}{\partial X_t}(x_t^*) = \left. \frac{E(\int_t^T e^{-r(s-t)}(\partial_x F(s, X_s, M_s^T, v_s) - \frac{1}{2} F(s, X_s, M_s^T, v_s)) ds | \mathcal{F}_t)}{\partial_\sigma BS(t, x_t^*, M_t^T, I_t(x_t^*))} \right|_{X_t = x_t^*}, \ a.s.$$

where

$$F(s,X_s,M_s^T,v_s) = \frac{1}{2}\Bigg[\left(\partial_{xxx}^3-\partial_{xx}^2\right)BS(s,X_s,M_s^T,v_s)\sigma_s\sum_{i=1}^d \rho_{i,d+1}\Lambda_s^{W^i} + \partial_K\left(\partial_{xx}^2-\partial_x\right)BS(s,X_s,M_s^T,v_s)\sum_{i=1}^d \Lambda_s^{W^i} m^i(T,s)\Bigg]$$

and $x_t^* = \ln(M_t^T) - r(T-t)$.

Proof: Using Theorem 2 and the expression $V_t = BS(t,X_t,M_t^T,I_t(X_t))$ we obtain

$$\frac{\partial V_t}{\partial X_t} = \partial_x BS(t,X_t,M_t^T,I_t(X_t)) + \partial_\sigma BS(t,X_t,M_t^T,I_t(X_t))\frac{\partial I_t}{\partial X_t}(X_t) \tag{5}$$

and

$$\frac{\partial V_t}{\partial X_t} = E(\partial_x BS(t,X_t,M_t^T,v_t)|\mathcal{F}_t) + E(\int_t^T e^{-r(s-t)}\partial_x F(s,X_s,M_s^T,v_s)ds|\mathcal{F}_t). \tag{6}$$

We can check that the conditional expectation $E(\int_t^T e^{-r(s-t)}\partial_x F(s,X_s,M_s^T,v_s)ds|\mathcal{F}_t)$is well defined and finite a.s. due to the fact that$\left(\int_t^T \sigma_\theta^2 d\theta\right)^{-1} < \infty$. Thus, (5) and (6) imply

$$\begin{aligned} &\frac{\partial I_t}{\partial X_t}(x_t^*) \qquad (7) \\ = \ &\frac{1}{\partial_\sigma BS(t,x_t^*,M_t^T,I_t(x_t^*))}\big[E(\partial_x BS(t,x_t^*,M_t^T,v_t)|\mathcal{F}_t) - \partial_x BS(t,x_t^*,M_t^T,I_t(x_t^*)) \\ &+E(\int_t^T e^{-r(s-t)}\partial_x F(s,X_s,M_s^T,v_s)ds|\mathcal{F}_t)\Big]\Big|_{X_t=x_t^*}. \end{aligned}$$

Notice that

$$\begin{aligned} &E(\partial_x BS(t,x_t^*,M_t^T,v_t)|\mathcal{F}_t) \\ &= \partial_x E(BS(t,x,M_t^T,v_t)|\mathcal{F}_t)\big|_{x=x_t^*} = \partial_x BS(t,x,M_t^T,I_t^0(x))|_{x=x_t^*}, \end{aligned} \tag{8}$$

where, by the Hull and White formula, $I_t^0(X_t)$ is the implied volatility of call option with constant strike M_t^T, for a certain stochastic volatility model where $\rho_{i,d+1}=0$ for all $i=1,...,d$ and the volatility process is given by a_t. Thus,

$$\begin{aligned} &\partial_x(BS(t,x,M_t^T,I_t^0(x))\big|_{x=x_t^*} \\ &= \partial_x BS(t,x_t^*,M_t^T,I_t^0(x_t^*)) + \partial_\sigma BS(t,x_t^*,M_t^T,I_t^0(x_t^*))\frac{\partial I_t^0}{\partial x}(x_t^*). \end{aligned} \tag{9}$$

From Renault and Touzi (1996) we know that $\frac{\partial I_t^0}{\partial x}(x_t^*) = 0$. Then, (7), (8) and (9) imply that

$$
\begin{aligned}
& \frac{\partial I_t}{\partial X_t}(x_t^*) \qquad (10)\\
= \quad & \frac{1}{\partial_\sigma BS(t, x_t^*, M_t^T, I_t(x_t^*))} \big[\partial_x BS(t, x_t^*, M_t^T, I_t^0(x_t^*)) - \partial_x BS(t, x_t^*, M_t^T, I_t(x_t^*)) \\
& + E(\int_t^T e^{-r(s-t)} \partial_x F(s, X_s, M_s^T, v_s) ds | \mathcal{F}_t) \big] \Big|_{X_t = x_t^*} .
\end{aligned}
$$

On the other hand, straightforward calculations lead us to

$$
\partial_x BS(t, x_t^*, M_t^T, \sigma) = e^{x_t^*} N(\frac{1}{2}\sigma\sqrt{T-t})
$$

and

$$
BS(t, x_t^*, M_t^T, \sigma) = e^{x_t^*}(N(\frac{1}{2}\sigma\sqrt{T-t}) - N(-\frac{1}{2}\sigma\sqrt{T-t})).
$$

Then

$$
\partial_x BS(t, x_t^*, M_t^T, \sigma) = \frac{1}{2}(e^{x_t^*} + BS(t, x_t^*, M_t^T, \sigma)),
$$

which yields

$$
\begin{aligned}
& \partial_x BS(t, x_t^*, M_t^T, I_t^0(x_t^*)) - \partial_x BS(t, x_t^*, M_t^T, I_t(x_t^*)) \\
& = \frac{1}{2}(BS(t, x_t^*, M_t^T, I_t^0(x_t^*)) - BS(t, x_t^*, M_t^T, I_t(x_t^*))) \\
& = \frac{1}{2} E(BS(t, x_t^*, M_t^T, v_t) - V_t | \mathcal{F}_t) \\
& = -\frac{1}{2} E(\int_t^T e^{-r(s-t)} F(s, X_s, M_s^T, v_s) ds | \mathcal{F}_t) \Big|_{X_t = x_t^*} .
\end{aligned}
$$

This, together with (10), implies that the result holds. ■

5 Short-time behaviour and second order approximation formulas

In this Section we study the short-time behaviour of the implied volatility in order to describe its dependence on the asset price. More precisely, this section is devoted to study the limit of $\frac{\partial I_t}{\partial X_t}(x_t^*)$ as $T \downarrow t$. This analysis will gives us a tool to improve the first-order approximation formula presented in Section.3. The following result is part of the tool needed for our results.

Lemma 8 *Assume the model (1) is satisfied. Then* $I_t(x_t^*)\sqrt{T-t} \to 0$ *a.s. as* $T \to t$.

Proof: Notice that the fact that K is a square-integrable and continuous random process and the dominated convergence theorem lead to get

$$\begin{aligned} V_t|_{X_t=x_t^*} &= E(e^{-r(T-t)}(e^{X_T}-K_T)_+|\mathcal{F}_t)\Big|_{X_t=x_t^*} \\ &= E\left(e^{-r(T-t)}(e^{X_T-X_t}e^{-r(T-t)}M_t^T-K_T)_+|\mathcal{F}_t\right)\Big|_{X_t=x_t^*} \\ &\leq E\left((e^{X_T-X_t}M_t^T-K_Te^{r(T-t)})_+|\mathcal{F}_t\right)\Big|_{X_t=x_t^*} \\ &= E\left(((e^{X_T-X_t}-e^{r(T-t)})M_t^T+e^{r(T-t)}(M_t^T-K_T))_+|\mathcal{F}_t\right)\Big|_{X_t=x_t^*} \\ &\leq E\left(|e^{X_T-X_t}-e^{r(T-t)}|M_t^T|\mathcal{F}_t\right)\Big|_{X_t=x_t^*} \\ &\quad +E\left(|M_t^T-K_T|e^{r(T-t)}|\mathcal{F}_t\right)\Big|_{X_t=x_t^*} \\ &\leq M_t^T E\left(|e^{X_T-X_t}-e^{r(T-t)}||\mathcal{F}_t\right)\Big|_{X_t=x_t^*} \\ &\quad +E\left(|M_t^T-K_T|e^{r(T-t)}|\mathcal{F}_t\right)\Big|_{X_t=x_t^*} \to 0 \ a.s., \end{aligned}$$

as $T \to t$. Hence, taking into account that, in the at-the-money case, $V_t|_{X_t=x_t^*} = BS(t, x_t^*, M_t^T, I_t(x_t^*))$, we deduce that

$$BS(t, x_t^*, M_t^T, I_t(x_t^*)) = 2M_t^T e^{-r(T-t)}\left[N\left(\frac{I(x_t^*)\sqrt{T-t}}{2}\right)-\frac{1}{2}\right] \longrightarrow 0 \ a.s.,$$

and this allows us to complete the proof. ■

Henceforth we consider the following hypotheses:

(H1') $a^2 \in \mathbb{L}_{W^i}^{2,2}$, $i \in \{1, ..., d\}$ and, moreover, there exists a positive constant C such that, for all $0 < s < \theta < r < T$, and $i, j \in \{1, ..., d\}$,

$$\left|D_s^{W^i} a_r^2\right| + \left|D_\theta^{W^i} D_s^{W^j} a_r^2\right| \leq C.$$

Notice that this hypothesis implies that (H1) holds.

(H2) There exist two positive constants c_1, c_2 such that for all $r \in [0, T]$ $c_1 \leq \sigma_r \leq c_2$. Notice that $a_s^2 \geq C\sigma_s^2$ for some positive constant $C > 0$.Thus this hypothesis implies that a_s^2 is lower bounded by a positive constant.

(H3) The processes $m^i(T, \cdot) \in \mathbb{L}_{W^j}^{1,2}$, $i, j \in \{1, ..., d\}$ and moreover, there exists a positive $\mathcal{F}_t-$adapted process C_t such that for all $T > s > r > t$ and $i, j \in \{1, ..., d\}$,

$$E\left(\left|m^i(T, r)\right|^2 \Big| \mathcal{F}_t\right) + E\left(\left|D_s^{W^i} m^j(T, r)\right|^2 \Big| \mathcal{F}_t\right) \leq C_t.$$

Proposition 9 *Assume that the model (1) and Hypotheses (H1')-(H3) hold. Also assume that there is a constant $c > 0$ such that $c < K_t$, for all $t \in [0, T]$. Then*

$$\begin{aligned}
&\partial_\sigma BS(t, x_t^*, M_t^T, I_t(x_t^*)) \frac{\partial I_t}{\partial X_t}(x_t^*) \\
&= \frac{1}{2} E\left(\left(\partial_x - \frac{1}{2}\right) \left(\partial_{xxx}^3 - \partial_{xx}^2\right) BS(t, x_t^*, M_t^T, v_t) \int_t^T \sigma_s \sum_{i=1}^d \rho_{i,d+1} \Lambda_s^{W^i} ds \right. \\
&\quad \left. + \partial_K \left(\partial_x - \frac{1}{2}\right) \left(\partial_{xx}^2 - \partial_x\right) BS(t, x_t^*, M_t^T, v_t) \int_t^T \sum_{i=1}^d \Lambda_s^{W^i} m^i(T, s) ds \right| \mathcal{F}_t \right) \\
&\quad + O(T-t).
\end{aligned}$$

as $T \to t$.

Proof: Proposition 7 gives us that

$$\begin{aligned}
&\partial_\sigma BS(t, x_t^*, M_t^T, I_t(x_t^*)) \frac{\partial I_t}{\partial X_t}(x_t^*) \\
&= \frac{1}{2} E\left(\int_t^T e^{-r(s-t)} (\partial_x - \frac{1}{2}) \left(\partial_{xxx}^3 - \partial_{xx}^2\right) BS(s, X_s, M_s^T, v_s) \sigma_s \sum_{i=1}^d \rho_{i,d+1} \Lambda_s^{W^i} ds \right. \\
&\quad + \int_t^T e^{-r(s-t)} (\partial_x - \frac{1}{2}) \partial_K \left(\partial_{xx}^2 - \partial_x\right) BS(s, X_s, M_s^T, v_s) \\
&\quad \left. \left. \times \sum_{i=1}^d \Lambda_s^{W^i} m^i(T, s) ds \right| \mathcal{F}_t \right) \Bigg|_{X_t = x_t^*} =: T_1 + T_2.
\end{aligned}$$

Now the proof is decomposed into two steps.

Step 1. Here we see that

$$T_1 = \frac{1}{2} E\left(L(t, x_t^*, M_t^T, v_t) \int_t^T \sigma_s \sum_{i=1}^d \rho_{i,d+1} \Lambda_s^{W^i} ds | \mathcal{F}_t \right) + O\left(T - t\right), \qquad (11)$$

where $L(s, X_s, M_s^T, v_s) = \left(\partial_x - \frac{1}{2}\right) \left(\partial_{xxx}^3 - \partial_{xx}^2\right) BS(s, X_s, M_s^T, v_s)$. In fact, applying Itô formula to

$$e^{-rs} L(s, X_s, M_s^T, v_s) (\int_s^T \sigma_r \sum_{i=1}^d \rho_{i,d+1} \Lambda_r^{W^i} dr)$$

as in the proof of Theorem 2 and taking conditional expectations with respect to $\mathcal{F}_t$, we obtain that

$$
\begin{aligned}
&\frac{1}{2}E(\int_t^T e^{-r(s-t)}L(s,X_s,M_s^T,v_s)\sigma_s\sum_{i=1}^d \rho_{i,d+1}\Lambda_s^{W^i}ds|\mathcal{F}_t)\\
&= \frac{1}{2}E\left(L(t,X_t,M_t^T,v_t)(\int_t^T \sigma_s\sum_{i=1}^d \rho_{i,d+1}\Lambda_s^{W^i}ds)|\mathcal{F}_t\right)\\
&\quad +\frac{1}{4}E(\int_t^T e^{-r(s-t)}(\partial_{xxx}^3-\partial_{xx}^2)L(s,X_s,M_s^T,v_s)\sigma_s(\sum_{i=1}^d \rho_{i,d+1}\Lambda_s^{W^i})\\
&\qquad \times\left(\int_s^T \sigma_r\sum_{j=1}^d \rho_{j,d+1}\Lambda_r^{W^j}dr\right)ds|\mathcal{F}_t)\\
&\quad +\frac{1}{4}E(\int_t^T e^{-r(s-t)}\partial_K(\partial_{xx}^2-\partial_x)L(s,X_s,M_s^T,v_s)\ (\sum_{i=1}^d \Lambda_s^{W^i}m^i(T,s))\\
&\qquad \times\left(\int_s^T \sigma_r\sum_{j=1}^d \rho_{j,d+1}\Lambda_r^{W^j}dr\right)ds|\mathcal{F}_t)\\
&\quad +\frac{1}{2}E(\int_t^T e^{-r(s-t)}\partial_x L(s,X_s,M_s^T,v_s)\ \sigma_s\sum_{i,j=1}^d \rho_{j,d+1}\\
&\qquad \times\left(\int_s^T (D_s^{W^j}\Lambda_r^{W^i})\rho_{i,d+1}\sigma_r dr\right)ds|\mathcal{F}_t)|\\
&\quad +\frac{1}{2}E(\int_t^T e^{-r(s-t)}\partial_K L(s,X_s,M_s^T,v_s)\sum_{i,j=1}^d m^j(T,s)\\
&\quad \times\left(\int_s^T (D_s^{W^j}\Lambda_r^{W^i})\rho_{i,d+1}\sigma_r dr\right)ds|\mathcal{F}_t)\\
&= \frac{1}{2}E\left(L(t,X_t,M_t^T,v_t)(\int_t^T \sigma_s\sum_{i=1}^d \rho_{i,d+1}\Lambda_s^{W^i}ds)|\mathcal{F}_t\right)\\
&\quad +S_1+S_2+S_3+S_4.
\end{aligned}
$$

Using Lemma 1 and Hypotheses (H1') and (H2), we can write

$$
\begin{aligned}
|S_1| &= \Bigg| \frac{1}{4} E\Big(\int_t^T e^{-r(s-t)} E\left[(\partial^3_{xxx} - \partial^2_{xx}) L(s, X_s, M_s^T, v_s) \middle| \mathcal{G}_t \right] \\
&\quad \times \sum_{i,j=1}^{d} \Big(\int_s^T \rho_{j,d+1} \Lambda_r^{W^j} \sigma_r dr \Big) \rho_{i,d+1} \Lambda_s^{W^i} \sigma_s ds | \mathcal{F}_t \Big) \Bigg| \\
&\leq C \sum_{k=4}^{6} E\left[\int_t^T \left(\int_s^T a_\theta^2 d\theta \right)^{-\frac{k}{2}} \sum_{i,j=1}^{d} \int_s^T |\rho_{j,d+1} \Lambda_r^{W^j} \sigma_r| dr |\rho_{i,d+1} \Lambda_s^{W^i} \sigma_s| ds \middle| \mathcal{F}_t \right] \\
&\leq C \sum_{k=4}^{6} E \int_t^T (T-s)^{-\frac{k}{2}} \sum_{i,j=1}^{d} \int_s^T |\Lambda_r^{W^j} \sigma_r| dr |\Lambda_s^{W^i} \sigma_s| ds \middle| \mathcal{F}_t \right]
\end{aligned}
$$

Hence, using Hypotheses (H1'), (H2), and (H3), we can write

$$
|S_1| \leq C \sum_{k=4}^{6} (T-t)^{-\frac{k}{2}+4} = O(T-t).
$$

Similarly, we have

$$
\begin{aligned}
|S_2| &= \Bigg| \frac{1}{4} E\Bigg(\int_t^T e^{-r(s-t)} E\left[\partial_K (\partial^2_{xx} - \partial_x) L(s, X_s, M_s^T, v_s) \middle| \mathcal{G}_t \right] \\
&\quad \times \sum_{i,j=1}^{d} \Big(\int_s^T \rho_{j,d+1} \Lambda_r^{W^j} \sigma_r dr \Big) \Lambda_s^{W^i} m^i(T,s) ds | \mathcal{F}_t \Bigg) \Bigg| .
\end{aligned}
$$

Therefore, the relation

$$
\frac{\partial^2 BS(t,x,K,\sigma)}{\partial_x \partial_K} = \frac{1}{k} \left(\frac{\partial BS(t,x,K,\sigma)}{\partial_x} - \frac{\partial^2 BS(t,x,K,\sigma)}{\partial_{x^2}} \right),
$$

togheter with the hypotheses of the Proposition, implies

$$
|S_2| \leq C_t \sum_{i=1}^{d} \sum_{k=3}^{6} E\left(\int_t^T (T-s)^{-\frac{\kappa}{2}+3} \left| m^i(T,s) \right| \middle| \mathcal{F}_t \right) = O(T-t).
$$

In a similar way,

$$
\begin{aligned}
|S_3| &= \frac{1}{2}\left| E\left(\int_t^T e^{-r(s-t)} \partial_x L(s, X_s, M_s, v_s)\ \sigma_s \sum_{i,j=1}^d \rho_{j,d+1} \right.\right. \\
&\quad \left.\left. \times \left(\int_s^T (D_s^{W^j} \Lambda_r^{W^i}) \rho_{i,d+1} \sigma_r dr \right) ds \right| \mathcal{F}_t \right) \Bigg| \\
\leq \quad & C \sum_{k=3}^4 E\left(\int_t^T (T-s)^{-\frac{\kappa}{2}} \left| \sigma_s \sum_{i,j=1}^d \rho_{j,d+1} \left(\int_s^T (D_s^{W^j} \Lambda_r^{W^i}) \rho_{i,d+1} \sigma_r dr \right) \right| ds \right| \mathcal{F}_t \right) \\
\leq \quad & C \sum_{k=3}^4 \left(\int_t^T (T-s)^{-\frac{\kappa}{2}+2} \right) = O(T-t).
\end{aligned}
$$

Finally, the same arguments give us that

$$
|S_4| = O(T-t).
$$

Step 2. In order to finish the proof we only need to proceed as in Step 1. Here we see that

$$
T_2 = \frac{1}{2} E\left(P(t, x_t^*, M_t^T, v_t) \int_t^T \sum_{i=1}^d \Lambda_s^{W^i} m^i(T, s) ds | \mathcal{F}_t \right) + O\left(T-t\right), \qquad (12)
$$

where $P(s, X_s, M_s^T, v_s) = (\partial_x - \frac{1}{2}) \partial_K \left(\partial_{xx}^2 - \partial_x \right) BS(s, X_s, M_s^T, v_s)$. In fact, applying Itô formula to

$$
e^{-rs} P(s, X_s, M_s^T, v_s) \left(\int_s^T \sum_{i=1}^d m^i(T, r) \Lambda_r^{W^i} dr \right)
$$

as in the proof of Theorem 2 and taking conditional expectations with respect to $\mathcal{F}_t$, we obtain that

$$
\begin{aligned}
&\frac{1}{2}E\left(\int_t^T e^{-r(s-t)}P(s,X_s,M_s^T,v_s)\sum_{i=1}^d \Lambda_s^{W^i} m^i(T,s)ds \middle| \mathcal{F}_t\right)\\
=\ & \frac{1}{2}E\left(P(t,X_t,M_t^T,v_t)(\int_t^T \sum_{i=1}^d m^i(T,s)\Lambda_s^{W^i} ds) \middle| \mathcal{F}_t\right)\\
&+\frac{1}{4}E\left(\int_t^T e^{-r(s-t)}(\partial_{xxx}^3-\partial_{xx}^2)P(s,X_s,M_s^T,v_s)\sigma_s \sum_{i=1}^d \rho_{i,d+1}\Lambda_s^{W^i}\right.\\
&\quad \times \left.\left(\int_s^T \sum_{j=1}^d m^j(T,r)\Lambda_r^{W^j} dr\right) ds \middle| \mathcal{F}_t\right)\\
&+\frac{1}{4}E\left(\int_t^T e^{-r(s-t)}\partial_k(\partial_{xx}^2-\partial_x)P(s,X_s,M_s^T,v_s)\sum_{i=1}^d \Lambda_s^{W^i} m^i(T,s)\right.\\
&\quad \times \left.\left(\int_s^T \sum_{j=1}^d \Lambda_r^{W^j} m^j(T,r)dr\right) ds \middle| \mathcal{F}_t\right)\\
&+\frac{1}{2}E\left(\int_t^T e^{-r(s-t)}\partial_x P(s,X_s,M_s^T,v_s)\ \sigma_s \sum_{i,j=1}^d \rho_{i,d+1}\right.\\
&\quad \times \left.\left(\int_s^T D_s^{W^i}\left(\Lambda_r^{W^j} m^j(T,r)\right)dr\right) ds|\mathcal{F}_t\right)\\
&+\frac{1}{2}E\left(\int_t^T e^{-r(s-t)}\partial_K P(s,X_s,M_s^T,v_s)\sum_{i,j=1}^d m^i(T,s)\right.\\
&\quad \times \left.\left(\int_s^T D_s^{W^i}\left(\Lambda_r^{W^j} m^j(T,r)\right)dr\right) ds \middle| \mathcal{F}_t\right).
\end{aligned}
$$

Now, following the same arguments as in Step 1 the proof is complete. ■

Remark 10 *This proof only needs some integrability and regularity conditions. So, depending on the coefficients of the model (1) and the process K, Hypotheses (H1')-(H3) can be substituted by appropiate integrability conditions.*

Now we can state the main result of this paper. Towards this end, we need to state the following assumptions:

(H4) Let $i \in \{1,...,d\}$ and $t \in [0,T]$. Assume that $m^i(\cdot,\cdot)$ has continous paths and that

$$\sup_{t<\theta\wedge s\wedge r<T} E\left(\sigma_s a_r - \frac{\sigma_t}{\tilde{a}_t}a_\theta^2 \middle| \mathcal{F}_t\right) \to 0 \quad \text{as} \quad T \to t, \text{ a.s.}$$

and

$$\sup_{t<\theta\wedge s\wedge r<T} E\left(m^i(T,s)a_r - \frac{m^i(t,t)}{\tilde{a}_t}a_\theta^2 \Big| \mathcal{F}_t\right) \to 0 \quad \text{as } \ T\to t, \text{ a.s.}$$

where, by convention,

$$\tilde{a}_t := \sigma_t^2 - 2\frac{\sigma_t}{K_t}\sum_{j=1}^d \rho_{j,d+1} m^j(t,t) + \sum_{j=1}^d \left(\frac{m^j(t,t)}{K_t}\right)^2.$$

(H5) Let $i\in\{1,...,d\}$. There exists an $\mathcal{F}_t$ -measurable random variable $D_t^{W^i+}a_t$ such that

$$\sup_{t<s<r<T}\left|E\left(\left(D_s^{W^i}a_r - D_t^{W^i+}a_t\right)\Big|\mathcal{F}_t\right)\right| \to 0, \quad \text{a.s.}$$

as $T\to t$.

Theorem 11 *Consider the model (1). Suppose that Hypotheses (H1')-(H5) hold and there exists a positive constant c such that $c<K$. Then*

$$\lim_{T\to t}\frac{\partial I_t}{\partial X_t}(x_t^*) = \frac{1}{2}\left(\frac{\sum_{i=1}^d m^i(t,t)D_t^{W^i+}a_t}{K_t} - \sigma_t\sum_{i=1}^d \rho_{i,d+1}D_t^{W^i+}a_t\right)\frac{1}{\tilde{a}_t^2}. \quad (13)$$

Proof: We can write

$$\partial_\sigma BS(t,x_t^*,M_t^T,I_t(x_t^*)) = \frac{M_t^T e^{-r(T-t)}e^{\frac{-I_t(x_t^*)^2(T-t)}{8}}\sqrt{T-t}}{\sqrt{2\pi}},$$

$$\left(\partial_x - \frac{1}{2}\right)\left(\partial_{xxx}^3 - \partial_{xx}^2\right)BS(t,x_t^*,M_t^T,v_t) = -M_t^T e^{-r(T-t)}\frac{1}{\sqrt{2\pi}}e^{-\frac{v_t^2(T-t)}{8}}v_t^{-3}(T-t)^{-\frac{3}{2}}$$

and

$$\begin{aligned}
&\partial_K\left(\partial_x - \frac{1}{2}\right)\left(\partial_{xx}^2 - \partial_x\right)BS(t,x_t^*,M_t^T,v_t)\\
&= M_t^T e^{-r(T-t)}\frac{1}{\sqrt{2\pi}}e^{-\frac{v_t^2(T-t)}{8}}v_t^{-1}(T-t)^{-\frac{1}{2}}\left(\frac{1}{M_t^T v_t^2(T-t)}\right).
\end{aligned}$$

Then we can write, due to Proposition 9,

$$\begin{aligned}
&\frac{\partial I_t}{\partial X_t}(x_t^*)\\
=\ & -\frac{1}{2}e^{\frac{I_t(x_t^*)^2(T-t)}{8}}(T-t)^{-2}E(e^{-\frac{v_t^2(T-t)}{8}}v_t^{-3}\int_t^T \sigma_s\sum_{i=1}^d \rho_{i,d+1}\Lambda_s^{W^i}ds|\mathcal{F}_t)\\
&+\frac{1}{2}e^{\frac{I_t(x_t^*)^2(T-t)}{8}}(T-t)^{-1}E\left(e^{-\frac{v_t^2(T-t)}{8}}v_t^{-1}\left(\frac{1}{M_t^T v_t^2(T-t)}\right)\int_t^T\sum_{i=1}^d \Lambda_s^{W^i}m^i(T,s)ds|\mathcal{F}_t\right)\\
&+O(T-t)^{\frac{1}{2}}\\
=\ & : S_1 + S_2 + O(T-t)^{\frac{1}{2}}.
\end{aligned}$$

By Lemma 8, we know that $I_t(x_t^*)^2(T-t) \to 0$ a.s. as $T \to t$. Then,

$$\lim_{T\to t} S_1 = -\frac{1}{2}\lim_{T\to t}\left[(T-t)^{-2} E(e^{-\frac{v_t^2(T-t)}{8}} v_t^{-3}\int_t^T \sigma_s \sum_{i=1}^d \rho_{i,d+1}\Lambda_s^{W^i} ds|\mathcal{F}_t)\right]$$

and

$$\begin{aligned}\lim_{T\to t} S_2 &= \frac{1}{2}\lim_{T\to t}\Bigg[(T-t)^{-1}E(e^{-\frac{v_t^2(T-t)}{8}} v_t^{-1}\left(\frac{1}{M_t^T v_t^2\,(T-t)}\right) \\ &\quad \times \int_t^T \sum_{i=1}^d \Lambda_s^{W^i} m^i(T,s)ds|\mathcal{F}_t)\Bigg]. \end{aligned} \tag{14}$$

Now, let us see that

$$\lim_{T\to t}\left(S_1 + \frac{\sigma_t}{2\tilde{a}_t^2}\sum_{i=1}^d \rho_{i,d+1} D_t^{W^i+} a_t\right) = 0 \text{ a.s.}. \tag{15}$$

In fact, we can establish

$$\lim_{T\to t}\left(S_1 + \frac{\sigma_t}{2\tilde{a}_t^2}\sum_{i=1}^d \rho_{i,d+1} D_t^{W^i+} a_t\right) = \lim_{T\to t} E\left(A_T B_T + \frac{\sigma_t}{2\tilde{a}_t^2}\sum_{i=1}^d \rho_{i,d+1} D_t^{W^i+} a_t\middle| \mathcal{F}_t\right)$$

where

$$A_T := \exp\left(-\frac{v_t^2(T-t)}{8}\right)\frac{1}{v_t}$$

and

$$B_T := -\frac{1}{v_t^2(T-t)^2}\int_t^T\int_s^T a_r \sigma_s \sum_{i=1}^d \rho_{i,d+1} D_s^{W^i} a_r dr ds.$$

Consequently

$$\begin{aligned} &\lim_{T\to t} E\left(A_T B_T + \frac{\sigma_t}{2\tilde{a}_t^2}\sum_{i=1}^d \rho_{i,d+1} D_t^{W^i+} a_t\middle| \mathcal{F}_t\right) \\ &= \lim_{T\to t} E\left(\left(A_T - \frac{1}{\tilde{a}_t}\right) B_T\middle| \mathcal{F}_t\right) + \frac{1}{\tilde{a}_t}\lim_{T\to t} E\left(B_T + \frac{\sigma_t}{2\tilde{a}_t}\sum_{i=1}^d \rho_{i,d+1} D_t^{W^i+} a_t\middle| \mathcal{F}_t\right) \\ &= \lim_{T\to t} U_1 + \frac{1}{\tilde{a}_t}\lim_{T\to t} U_2. \end{aligned}$$

Applying Schwartz inequality for conditional expectation, it follows that

$$U_1 \le \left[E\left(\left(A_T - \frac{1}{\tilde{a}_t}\right)^2\middle| \mathcal{F}_t\right)\right]^{\frac{1}{2}} \left[E\left(B_T^2\middle| \mathcal{F}_t\right)\right]^{\frac{1}{2}}.$$

From the dominated convergence theorem and (H2), it is easy to see that $E\left(\left(A_T-\frac{1}{\tilde{a}_t}\right)^2\middle|\mathcal{F}_t\right)$ tends to zero a.s. as $T\to t$, and a simple calculation gives us that (H1') and (H2) imply that $E\left(B_T^2\middle|\mathcal{F}_t\right)$ is bounded, from where we deduce that $\lim_{T\to t}U_1=0$.

Observe that we also have,

$$\begin{aligned}
|U_2| &= \left|\frac{1}{(T-t)^2}E\left(\int_t^T\int_s^T\left(\frac{\sigma_s a_r}{v_t^2}\sum_{i=1}^d\rho_{i,d+1}D_s^{W^i}a_r-\frac{\sigma_t}{\tilde{a}_t}\sum_{i=1}^d\rho_{i,d+1}D_t^{W^i+}a_t\right)drds\middle|\mathcal{F}_t\right)\right|\\
&\leq \frac{C}{(T-t)^2}\left|E\left(\int_t^T\int_s^T\left(\frac{\sigma_s a_r}{v_t^2}-\frac{\sigma_t}{\tilde{a}_t}\right)\sum_{i=1}^d\rho_{i,d+1}D_s^{W^i}a_r drds\middle|\mathcal{F}_t\right)\right|\\
&\quad+\frac{C}{(T-t)^2}\left|E\left(\sum_{i=1}^d\rho_{i,d+1}\int_t^T\int_s^T\left(D_s^{W^i}a_r-D_t^{W^i+}a_t\right)\middle|\mathcal{F}_t\right)drds\right|\\
&=: |U_{2,1}|+|U_{2,2}|.
\end{aligned}$$

Using Hypotheses (H1') and (H2) we obtain that

$$\begin{aligned}
|U_{2,1}| &\leq \frac{C}{(T-t)^2}E\left(\int_t^T\int_s^T\left|\frac{\sigma_s a_r}{v_t^2}-\frac{\sigma_t}{\tilde{a}_t}\right|drds\middle|\mathcal{F}_t\right)\\
&\leq \frac{C}{(T-t)^2}E\left(\int_t^T\int_s^T\left|\sigma_s a_r-\frac{\sigma_t}{\tilde{a}_t}v_t^2\right|drds\middle|\mathcal{F}_t\right)\\
&= \frac{C}{(T-t)^2}E\left(\int_t^T\int_s^T\left|\sigma_s a_r-\frac{\sigma_t}{\tilde{a}_t(T-t)}\int_t^T a_\theta^2 d\theta\right|drds\middle|\mathcal{F}_t\right)\\
&\leq \frac{C}{(T-t)^3}\int_t^T\int_s^T\int_t^T E\left(\left|\sigma_s a_r-\frac{\sigma_t}{\tilde{a}_t}a_\theta^2\right|\middle|\mathcal{F}_t\right)d\theta drds,
\end{aligned}$$

which tends to zero, a.s. as $T\to t$, because of Hypothesis (H4). Similarly,

$$|U_{2,2}|\leq\frac{C}{(T-t)^2}\left|\sum_{i=1}^d\rho_{i,d+1}\int_t^T\int_s^T E\left(\left(D_s^{W^i}a_r-D_t^{W^i+}a_t\right)\middle|\mathcal{F}_t\right)drds\right|,$$

which tends to zero by Hypothesis (H5). Thus we have proved (15) is true.

On the other hand, by (14) we can write

$$\lim_{T\to t}\left(S_2-\frac{1}{2K_t\tilde{a}_t^2}\sum_{i=1}^d m^i(t,t)D^{W^i+}a_t\right)=\lim_{T\to t}E\left(A_TB_T-\frac{1}{2K_t\tilde{a}_t^2}\sum_{i=1}^d m^i(t,t)D^{W^i+}a_t\middle|\mathcal{F}_t\right)$$

but now

$$A_T:=\exp\left(-\frac{v_t^2(T-t)}{8}\right)\frac{1}{M_t^Tv_t}$$

and

$$B_T := \frac{1}{v_t^2 (T-t)^2} \int_t^T \int_s^T a_r \sum_{i=1}^d m^i(T,s) D_s^{W^i} a_r dr ds.$$

Finally, proceeding similarly as before, we have (14) yields that S_2 converges to $\frac{\sum_{i=1}^d m^i(t,t) D_t^{W^i+} a_t}{2K_t \tilde{a}_t^2}$, which, together with (15), implies that (13) is satisfied. ■

Remark 12 *From (13) and by using Taylor expansions we see that for small times to maturities and for near-the-money options, the following approximation for the implied volatility holds*

$$\hat{I}_t(X_t) := \sqrt{\tilde{a}_t^2} + \frac{1}{2} \left(\frac{\sum_{i=1}^d m^i(t,t) D_t^{W^i+} a_t}{K_t} - \sigma_t \sum_{i=1}^d \rho_{i,d+1} D_t^{W^i+} a_t \right) \frac{1}{\tilde{a}_t^2} \left(X_t - x_t^* \right). \tag{16}$$

and

$$BS(t, X_t, M_t^T, \sqrt{\hat{I}_t(X_t)})$$

becomes a closed-form second-order approximation for the option price. In the following Section we will check the goodness of this approximation for two-assets and tree-assets spread options.

6 Application to the study of spread options

This section is devoted to apply the previous results to study the implied volatility behaviour for spread options. This study will allow us to easily improve, for short-time spread options, Kirk's formula and its three-assets extension proposed in Alòs, Eydeland and Laurence (2011).

6.1 Two-assets spread options

Consider an spread option with $K_T = S_T^{'} + K$ as in Example 4. For the sake of simplicity we will assume the interest rate $r = 0$ and we use the notation $\rho_{1,2} = \rho$. Then it is easy to see that

$$a_t^2 := \sigma^2 - 2\rho\sigma\sigma' \frac{S_t'}{S_t' + K} + (\sigma')^2 \frac{(S_t')^2}{(S_t' + K)^2}.$$

Therefore, for $\theta < t$,

$$\begin{aligned} D_\theta^W a_t^2 &= \left(-2\rho\sigma\sigma' \frac{K}{(S_t'+K)^2} + 2(\sigma')^2 \left(\frac{S_t'}{S_t'+K} \right) \frac{K}{(S_t'+K)^2} \right) \sigma' S_t' \\ &= 2(\sigma')^2 \left(-\rho\sigma + \sigma' \left(\frac{S_t'}{S_t'+K} \right) \right) \frac{S_t' K}{(S_t'+K)^2}. \end{aligned}$$

Hence, we deduce that

$$\begin{aligned} D_\theta^W a_t &= D_\theta^W \sqrt{a_t^2} = \frac{D_\theta^W a_t^2}{2\sqrt{a_t^2}} \\ &= \frac{1}{\sqrt{a_t^2}} \left(-\rho\sigma + \sigma' \left(\frac{S_t'}{S_t' + K} \right) \right) (\sigma')^2 \frac{S_t' K}{(S_t' + K)^2}. \end{aligned}$$

Then, from Theorem 11, we get

$$\begin{aligned} & \lim_{T \to t} \frac{\partial I_t}{\partial X_t}(x_t^*) \\ &= \frac{1}{2} \left(\frac{m^1(t,t)}{K_t} - \rho\sigma \right) \frac{D_t^+ a_t}{\tilde{a}_t^2} \\ &= \frac{1}{2} \left(\sigma' \left(\frac{S_t'}{S_t' + K} \right) - \rho\sigma \right)^2 \frac{1}{\left(\sqrt{a_t^2}\right)^3} (\sigma')^2 \frac{S_t' K}{(S_t' + K)^2}. \end{aligned} \tag{17}$$

Remark 13 *Notice that the above quantity is always positive. In the following examples we will study its behaviour as a function of* K *and* ρ.

Example 14 *In Figure 1 we plot* $\lim_{T \to t} \frac{\partial I_t}{\partial X_t}(x_t^*)$ *as a function of* K *for* $\rho = 0.9$ *(solid) and* $\rho = 1$ *(dash), and for* $S_t = 100, \sigma = 0.5$ *and* $\sigma' = 0.4$. *We can observe the limit skew* $\lim_{T \to t} \frac{\partial I_t}{\partial X_t}(x_t^*)$ *is zero in the case* $K = 0$. *This was expected from Example 4, where we found that in this case the implied volatility is constant, and then* $\frac{\partial I_t}{\partial X_t}(x_t^*) = 0$. *Notice also that, even this skew increases with* K, *this increment seems to be clearly bigger in the completely correlated case* $\rho = 1$.

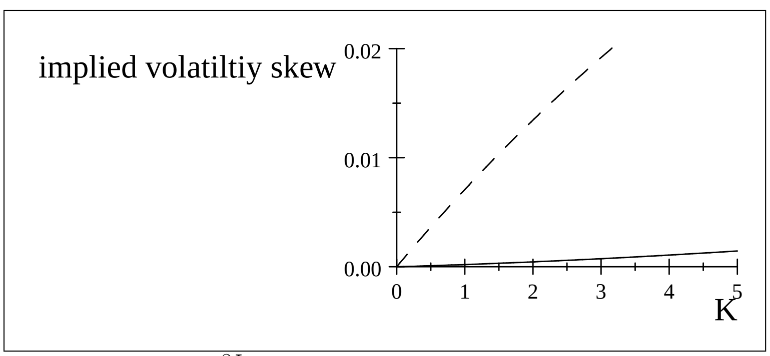

Figure 1: $\lim_{T \to t} \frac{\partial I_t}{\partial X_t}(x_t^*)$ as a function of K for $\rho = 0.9$ (solid) and $\rho = 1$ (dash). Here $\sigma = 0.5, \sigma' = 0.4$.

Example 15 *In Figure 2 we plot* $\lim_{T \to t} \frac{\partial I_t}{\partial X_t}(x_t^*)$ *as a function of* ρ *for* $K = 5$ *(solid) and* $K = 10$ *(dash), and for the same parameter values of Fig.1. We can observe the limit skew* $\lim_{T \to t} \frac{\partial I_t}{\partial X_t}(x_t^*)$ *has its maximum at the completely* *correlated case $\rho = 1$. Notice that this means that the constant volatility approximation given by Kirk's formula is expected to be less accurate in this case. This fact is consistent with numerical empirical evidence (see for example Baeva (2011) and Borovkova (2007)).*

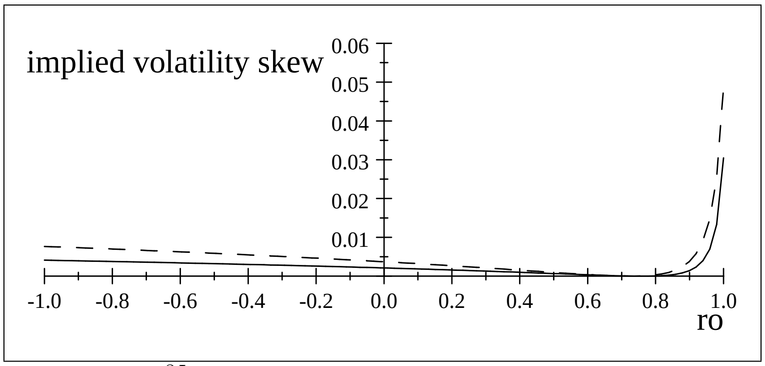

Figure 2: $\lim_{T\to t} \frac{\partial I_t}{\partial X_t}(x_t^*)$ as a function of ρ for $K = 5$ (solid) and $K = 10$ (dash). Here $\sigma = 0.5, \sigma' = 0.4$.

Example 16 *In Figure 3 we plot* $\lim_{T\to t} \frac{\partial I_t}{\partial X_t}(x_t^*)$ *as a function of ρ and K for the same parameter values of Fig.1 and Fig. 2. Notice that this limit skew is substantially bigger near the case $\rho = 1$.*

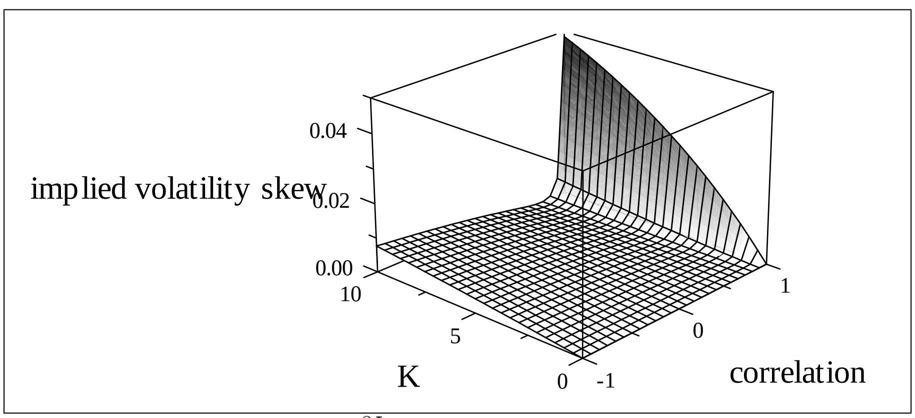

Figure 3: $\lim_{T\to t} \frac{\partial I_t}{\partial X_t}(x_t^*)$ as a function of ρ and K.

6.1.1 An improvement of Kirk's formula

Kirk's approximation for spread option prices is given by

$$BS(t, X_t, M_t^T, \sqrt{a_t^2}).$$

It is well-known that Kirk's formula is a very accurate approximation given its simplicity (see for example Baeva (2011), Bjerksund and Stensland (2011) or

Carmona and Durrleman (2011)). Nevertheless, it is well-known it may fail for highly correlated assets (see for example Baeva (2011)). The above results give an analytical reason for this phenomenon. In fact, notice that $\sqrt{a_t^2}$ (the volatility parameter in the Kirk's formula) is a process that does not depend on X_t. Then, Kirk's formula may not reproduce the short-time volatility skews that we have seen appear in the highly correlated case (ρ close to 1) and we can expect it can fail when ρ is near to one.

In this case we have

$$\hat{I}_t(X_t) := \sqrt{a_t^2} + \frac{1}{2}\left(\sigma'\left(\frac{S_t'}{S_t'+K}\right) - \rho\sigma\right)^2 \frac{1}{\left(\sqrt{a_t^2}\right)^3} (\sigma')^2 \frac{S_t' K}{(S_t'+K)^2} (X_t - x_t^*) .$$

And now we can consider the modified Kirk approximation given by

$$BS(t, X_t, M_t^T, \hat{I}_t(X_t)).$$

In the following example we will check numerically the goodness-of-fit of this approximation.

Example 17 *In the following table we can compare the prices given by Kirk's formula, by the modified Kirk's formula and by the Monte Carlo simulations, for different values for K and ρ and for the same parameters of Example 14. Here $T-t=0.5$. Notice that the modified Kirk's formula is extremely accurate and it reduces significatively the error of approximation, specially in the case of highly correlated assets.*

K/ρ		0.60	0.98	0.99	0.999
	Monte-Carlo	$9,4564$	$2,1890$	$1,8386$	$1,5011$
	Kirk	$9,4176$	$2,2159$	$1,8775$	$1,5420$
5	error (Kirk)	*−0,410%*	*1,230%*	*2,117%*	*2,725%*
	Modified Kirk	$9,4255$	$2,2067$	$1,8309$	$1,4829$
	error (Modified Kirk)	*−0,327%*	*0,809%*	*0,804%*	*0,414%*
	Monte-Carlo	$7,6404$	$1,2714$	$1,0207$	$0,7934$
	Kirk	$7,5988$	$1,3326$	$1,1015$	$0,8848$
10	error (Kirk)	*−0,545%*	*4,814%*	*7,913%*	*11,516%*
	Modified Kirk	$7,6060$	$1,2888$	$1,0367$	$0,8210$
	error (Modified Kirk)	*−0,451%*	*1,368%*	*1,660%*	*1,400%*

6.2 The three-assets case

Consider a random strike of the form $K_T = S_T^1 + S_T^2 + K$ as in Example 6. Using the same arguments as in the two-asset case, we obtain the following approximation for the implied volatility

$$\begin{aligned}\hat{I}_t(X_t) \ : \ &= \sqrt{a_t^2} \\ &+\frac{1}{2}\left(\frac{m^1(t,t)D_t^{W^1+}a_t + m^2(t,t)D_t^{\tilde{W}^2+}a_t}{K_t} - \sigma\left(\rho_{13}D_t^{W^1+}a_t + \rho_{23}D_t^{\tilde{W}^2+}a_t\right)\right)\frac{1}{a_t^2}(X_t - x_t^*) .\end{aligned}$$

Here a_t^2 is as in Example 6,

$$m^1(t,t) = S_t^1\sigma_1 + S_t^2\sigma_2\rho_{1,2},\ m^2(t,t) = S_t^2\sigma_2\sqrt{1-\rho_{1,2}^2}.$$

Also

$$\begin{aligned} D_t^{W^1+}a_t &= \frac{D_t^{W^1+}a_t^2}{2\sqrt{a_t^2}} \\ &= \frac{1}{\sqrt{a_t^2}}\Big(-\rho_{1,3}\sigma\sigma_1 D_t^{W^1+}a - \rho_{2,3}\sigma\sigma_2 D_t^{W^1+}b \\ &\quad + \sigma_1^2 a D_t^{W^1+}a + \sigma_1\sigma_2\rho_{1,2}\left(aD_t^{W^1+}b + bD_t^{W^1+}a\right) + \sigma_2^2 bD_t^{W^1+}b\Big) \end{aligned}$$

and

$$\begin{aligned} D_t^{W^2+}a_t &= \frac{D_t^{W^2+}a_t^2}{2\sqrt{a_t^2}} \\ &= \frac{1}{\sqrt{a_t^2}}\Big(-\rho_{1,3}\sigma\sigma_1 D_t^{W^2+}a - \tilde{\rho}_{2,3}\sigma\sigma_2 D_t^{W^2+}b \\ &\quad + \sigma_1^2 a D_t^{W^2+}a + \sigma_1\sigma_2\rho_{1,2}\left(aD_t^{W^2+}b + bD_t^{W^2+}a\right) + \sigma_2^2 bD_t^{W^2+}b\Big), \end{aligned}$$

where

$$D_t^{W^1+}a = \frac{\left(S_t^2+K\right)\sigma_1 a}{\left(S_t^1+S_t^2+K\right)} - \sigma_2\rho_{1,2}ab,\ D_t^{W^1+}b = \frac{\left(S_t^1+K\right)\sigma_2\rho_{1,2}b}{\left(S_t^1+S_t^2+K\right)} - \sigma_1 ab$$

and

$$D_t^{W^2+}a = -ab\sigma_2\sqrt{1-\rho_{1,2}^2},\ D_t^{W^2+}b = \frac{\left(S_t^1+K\right)b\sigma_2\sqrt{1-\rho_{1,2}^2}}{\left(S_s^1+S_s^2+K\right)}.$$

In the following example we compare the results given by the approximation formula proposed in Alòs, Eydeland and Laurence with the results obtained by this modified approximation.

Example 18 *Take* $T = 0.5, (\rho_{1,2}, \rho_{1,3}, \tilde{\rho}_{2,3}) = (0.99, 0.96, 0.94),\ (\sigma_1, \sigma_2, \sigma) = (0.5, 0.45, 0.2), \left(S_0^1, S_0^2, K\right) = (50, 2, 1)$. *In the following table we compare the errors given by the extended Kirk's approximation prices obtained in Alòs, Eydeland and Laurence (2011) (AEL) with the modified Alòs, Eydeland and Laurence approximation (MAEL) given by*

$$BS(t, X_t, M_t^T, \hat{I}_t(X_t)).$$

Then benchmarks have been obtained from a Monte Carlo simulation procedure with 1000,000 trials.

S_0	Monte Carlo	AEL	Error(AEL)	MAEL	Error(MAEL)
48	0.09256	0.00988	6.749 1%	0.009 14	−1.234 2%
50	0.34575	0.355 34	2.773 7%	0.345 97	0.0636%
52	0.93606	0.944 11	0.8600%	0.939 68	0.386 7%

Notice that the error is again significatively reduced.

7 Conclusions

By means of Malliavin calculus we have developed a general technique to find closed-form approximation formulas for short-time random strike options. The obtained approximations are simple and easy to apply and the numerical analysis show they are extremely accurate even in the case when some other approaches (as the case of Kirk's formula and the decomposition method presented in Alòs, Eydeland and Laurence (2011)) fail.

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