Economics Working Paper 127

Compact Matrix Expressions for Generalized Wald Tests of Equality of Moment Vectors[*]

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August 1995

Abstract

Asymptotic chi-squared test statistics for testing the equality of moment vectors are developed. The test statistics proposed are generalized Wald test statistics that specialize for different settings by inserting and appropriate asymptotic variance matrix of sample moments. Scaled test statistics are also considered for dealing with situations of non-iid sampling. The specialization will be carried out for testing the equality of multinomial populations, and the equality of variance and correlation matrices for both normal and non-normal data. When testing the equality of correlation matrices, a scaled version of the normal theory chi-squared statistic is proven to be an asymptotically exact chi-squared statistic in the case of elliptical data.

1 Introduction

Testing the equality of population moments is of wide generality in multivariate analysis. Specific examples are the test of equality of variance or correlation matrices and the test of the hypothesis of equality of means and variances across populations. In the present paper a family of generalized Wald test statistics will be considered. The tests to be developed share a common general formulation that is easily adapted to different settings by just inserting an appropriate asymptotic variance matrix of sample moments. The test statistics will be specialized to the test of equality of variance or correlation matrices, for the cases of normality, ellipticity and also in distribution-free settings. Testing the equality of moments arises also in meta-analysis studies, where the results of independent studies have to be compared (Hedges and Olkin, 1985).

The test of equality of variance matrices is usualy carried out under the assumption that the variables are normally distributed (Anderson, 1987, Chapter 10). Under the normality assumption also, a very common test for the equality of variance matrices is Bartlett's modified likelihood ratio test (Muirhead, 1982, pp. 298-309). Tests for equality of correlation matrices have also been considered under the normality assumption by Jennrich (1970). In practice, however, data deviate often from the normality assumption and it is of interest to develop methods that are free of the normality assumption. The test statistics developed in the present paper apply also to the case of non-normal data. Recently, bootstrap techniques have been introduced for testing equality of variance matrices when data can be non-normal (Zhang and Boos, 1992). Such techniques, however, require intensive computations. In contrast, our test statistics are fairly simple to com-

pute. A robust test for comparing correlation matrices was also investigated recently by Modarres and Jernigan (1993), though they did not provide a general formulation for the statistic.

The plan of the paper is as follows. Section 2 develops the general expressions for the test statistics. Section 3 specializes the test statistics to different testing settings.

With regard to notation, D and D^+ will denote respectively the "duplication" and "elimination" matrices for symmetry, so that vec $A = D\mathrm{v}(A)$ for symmetric matrix A, where "vec" is the usual columnwise vectorization operator and $\mathrm{v}(A)$ is obtained from vec A after eliminating the duplicated elements due to the symmetry of A. It holds that $\mathrm{v}(A) = D^+\mathrm{vec}\,A$ where $D^+ \equiv (D'D)^{-1}D'$ is the Moore-Penrose inverse of D (see Magnus and Neudecker, 1988). In the present paper the matrices D will be of varying orders to be determined by the context. We denote by E_{gg} the gth unit matrix of order G, by $1_G \equiv (1, \ldots, 1)'$ the G-dimensional column vector of ones, by $E \equiv 1_G 1_G'$ the $G \times G$ matrix of ones, and by e_g the gth unit column vector of order G (note that $E_{gg} = e_g e_g'$). Further $A \geq 0$ indicates that A is a positive semidefinite matrix and $\mathcal{M}(A)$ denotes the column space of A. The matrix A^- will be any generalized inverse of A (i.e. satisfying $AA^-A = A$), whereas A^+ will be the Moore-Penrose inverse. We use the notation of $A_d \equiv I \times A$ with I denoting an identity matrix and $\times$ denoting the Hadamard product of matrices. We also introduce the duplication and elimination matrices for zero-axial symmetry $\tilde{D}$ and $\tilde{D}^+$, respectively, where $\mathrm{vec}A = \tilde{D}w(A)$ and $w(A)$ is obtained from vec$A$ after eliminating the zero diagonal and upper triangular elements. Clearly $w(A) = \tilde{D}^+\mathrm{vec}A$, where $\tilde{D}^+ = \frac{1}{2}\tilde{D}'$. Given a set of vectors a_i, $i = 1, \ldots, 1$, we denote by vec$[a_i \mid i = 1, \ldots, I]$ the column vector formed by stacking the vectors a_i one below the

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other. Finally, the notation $" \xrightarrow{P} "$ will be used for convergence in probability and $" \xrightarrow{L} "$ for convergence in distribution.

2 General test for equality of population moments

Let r_g, $g = 1, \ldots, G$, be $p \times 1$ vectors of sample moments based on independent samples from G populations. Assume $r_g \xrightarrow{P} \rho_g$, where ρ_g is a $p \times 1$ vector of population moments, and

$$\sqrt{n_g}(r_g - \rho_g) \xrightarrow{L} N(0, \Gamma_g), \tag{1}$$

where the $p \times p$ matrix $\Gamma_g \geq 0$ is the asymptotic variance matrix of $\sqrt{n_g} r_g$ and n_g is the sample size for group g.

Consider the multi-sample vectors of sample and population moments $r \equiv \text{vec}[r_g \mid g = 1, \ldots, G]$ and $\rho \equiv \text{vec}[\rho_g \mid g = 1, \ldots, G]$ respectively. Clearly, from (1) and the independence of the G samples, follows

$$\sqrt{n}(r - \rho) \xrightarrow{L} N(0, \Gamma), \tag{2}$$

where $n \equiv \sum_{g=1}^{G} n_g$ is the overall sample size and Γ is the block-diagonal matrix

$$\Gamma = \sum_{g=1}^{G} \frac{n}{n_g} (E_{gg} \otimes \Gamma_g). \tag{3}$$

We will assume that the fractions $\frac{n_g}{n}$ do not vary when $n \to +\infty$.

Often the following assumption of the equality of the Γ_g can be made.

ASSUMPTION A. *It holds that* $\Gamma_g = \bar{\Gamma}$, $g = 1, \ldots, G$, *with* $\bar{\Gamma}$ *a* $p \times p$ *positive semidefinite matrix.*

Under assumption A, we have

$$\Gamma = \Lambda^{-1} \otimes \bar{\Gamma}, \tag{4}$$

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where

$$\Lambda \equiv \sum_{g=1}^{G} \frac{n_g}{n} E_{gg}.$$ (5)

In the present paper we are concerned with the test of the following hypothesis of equality of population moments

$$H_0 : \rho_g = \vartheta; \quad g = 1, 2, \ldots, G,$$ (6)

where ϑ is an unknown p-dimensional parameter vector. Generalized Wald test statistics (Moore, 1977) will be developed to test this hypothesis.

It will be convenient to write H_0 as the following multiple-group moment structure:

$$H_0 : \rho = \Delta \vartheta,$$ (7)

where $\Delta \equiv 1_G \otimes I_p$ is a $Gp \times p$ matrix of full column rank, and ϑ is a $p-$vector of unknown parameters.

Consider now the weighted least-squares (WLS) estimation of the moment structure (7), with weight matrix W of the following general form:

$$W \equiv \sum_{g=1}^{G} \frac{n_g}{n} (E_{gg} \otimes \bar{W}) = \Lambda \otimes \bar{W},$$ (8)

where $\bar{W}$ is a $p \times p$ positive definite matrix. Note that $1_G' \Lambda 1_G = 1$ and hence $E\Lambda E = E$. This yields the WLS estimator

$$\hat{\vartheta} \equiv (\Delta' W \Delta)^{-1} \Delta' W r = (1_G' \Lambda \otimes I_p) r = \sum_{g=1}^{G} \frac{n_g}{n} r_g,$$ (9)

of the parameter vector ϑ, and the estimator

$$\hat{\rho} \equiv \Delta \hat{\vartheta} = \Delta (\Delta' W \Delta')^{-1} \Delta W r = (1_G \otimes I_p)(1_G' \Lambda \otimes I_p) r$$ (10)

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$$= (E\Lambda \otimes I_p)r = \sum_{g=1}^{G} \frac{n_g}{n}(1_G \otimes r_g)$$

of ρ. The vector of residuals $e \equiv r - \Delta\hat{\vartheta}$ clearly satisfies

$$e = [(I_G - E\Lambda) \otimes I_p]r = Qr, \tag{11}$$

where $Q \equiv (I_G - E\Lambda) \otimes I_p$ is an idempotent matrix of rank $p(G-1)$. Note that $Q\Delta = 0$ and that the estimators $\hat{\vartheta}$ and $\hat{\rho}$ do not depend on the choice of $\bar{W}$.

In some instances we will need to consider the case where the null hypothesis H_0 holds only approximately. The following alternative hypothesis will then be considered

$$H_1 : \rho = \Delta\vartheta + n^{-1/2}\eta, \tag{12}$$

where η is a pG -dimensional vector. This is an alternative hypothesis of a sequence of local alternatives that is typically used to investigate the asymptotic non-null distribution of test statistics (e.g., Foutz and Srivastava, 1977). Clearly, $\eta = 0$ under H_0.

Using (2), we obtain

$$\sqrt{n}Q(r - \rho) \xrightarrow{L} N(0, Q\Gamma Q'). \tag{13}$$

Thus, under H_1, we have the distributional result

$$\sqrt{n}e \xrightarrow{L} N(Q\eta, Q\Gamma Q'), \tag{14}$$

as $Q\Delta = 0$.

Let Y be a positive semidefinite matrix of the same rank as Γ and such that $Y \xrightarrow{P} \Gamma$. We will consider the following generalized Wald test statistic (Moore, 1977)

$$T \equiv ne'(QYQ')^+e = nr'Q'(QYQ')^+Qr. \tag{15}$$

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We used (11) to derive the right-hand side of (15). [1] We note that when $r \in \mathcal{M}(Y)$, then T of (15) is equal to

$$T = nr'Q'(QYQ')^- Qr, \tag{16}$$

as

$$nr'Q'(QYQ')^- Qr = n\ell'YQ'(QYQ')^- QY\ell$$

$$= n\ell'Y^{1/2}Y^{1/2}Q'(QYQ')^- QY^{1/2}Y^{1/2}\ell,$$

where $r = Y\ell$ and $Z'(ZZ')^+ Z = Z'(ZZ')^- Z$ for any matrix Z.[2]

Given the distributional result (14), the following theorem is obtained

THEOREM 1. *When H_1 holds, T of (15) satisfies*

$$T \xrightarrow{L} \chi^2(k, \lambda), \tag{17}$$

where $k = \text{rank}(Q\Gamma Q')$ and

$$\lambda = \eta'Q'(Q\Gamma Q')^+ Q\eta. \tag{18}$$

When Γ is non-singular, then $k = p(G - 1)$. When $r \in \mathcal{M}(Y)$ and $\eta \in \mathcal{M}(\Gamma)$ then the Moore-Penrose inverse in (18) can be replaced by a generalized inverse, T and λ being invariant with respect to the choice of generalized inverse. Under H_0, $\lambda = 0$, since $\eta = 0$.

PROOF: Consider the spectral decomposition $Q\Gamma Q' = CUC'$, where $C'C = I_k$, and $U = U_d > 0$. By premultiplying both sides of (14) by C', we obtain

$$\sqrt{n}C'e \xrightarrow{L} N(C'Q\eta, U), \tag{19}$$

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[1] Note that $Q'(QYQ')^- Q$ is a g-inverse of QYQ'.
[2] Let $A = Z'(ZZ')^+ Z - Z'(ZZ')^- Z = Z'\{(ZZ')^+ - (ZZ')^-\}Z$. Since $\sum_{ij} a_{ij}^2 = \text{trace} AA' = 0$, $A = 0$.

since $C'Q\Gamma Q'C = U$. Hence,

$$n\, e'(Q\Gamma Q')^+ e = n\, e'CU^{-1}C'e \xrightarrow{L} \chi^2(k,\lambda).$$

The proof concludes by noting that $(QYQ')^+ \xrightarrow{P} (Q\Gamma Q')^+$ since $Y \xrightarrow{P} \Gamma$ and generally $\text{plim}A^+ = (\text{plim}A)^+$ from the four defining equations for the Moore-Penrose inverse. $\blacksquare$

Note that the stated theorem would hold also in the case of Q being a stochastic matrix that converges to a finite probability limit $\tilde{Q}$. [3] See also Andrews (1987) for some key remarks concerning conditions for the construction of generalized Wald test statistics in the case of singular variance matrices.

Some alternative expressions for the test statistic T will now be developed.

Since Γ is a block-diagonal matrix, we take

$$Y \equiv \sum_{g=1}^{G} \frac{n_g}{n}(E_{gg} \otimes Y_g), \tag{20}$$

where the $Y_g \geq 0$ are $p \times p$ positive semidefinite matrices. When Assumption A holds, then

$$Q\Gamma Q' = (\Lambda^{-1} - E) \otimes \bar{\Gamma}.$$

Inspired by Assumption A the matrix Y will be taken to be of the form

$$Y = \Lambda^{-1} \otimes \bar{Y}, \tag{21}$$

where $\bar{Y} \geq 0$ and $\bar{Y} \xrightarrow{P} \bar{\Gamma}$.

If we partition Q as

$$Q = [Q_1, Q_2, \ldots, Q_G], \tag{22}$$

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[3]Suppose $Q \to \tilde{Q}$, then $k = \text{rank}(\tilde{Q}\Gamma\tilde{Q}')$ and $\lambda = \eta'\tilde{Q}'(\tilde{Q}\Gamma\tilde{Q}')^+\tilde{Q}\eta$

comformably with the partition of Y, the test statistic of (15) will have the alternative expression

$$T = n(\sum_{g=1}^{G} Q_g r_g)'(\sum_{g=1}^{G} \frac{n_g}{n} Q_g Y_g Q_g')^+(\sum_{g=1}^{G} Q_g r_g). \tag{23}$$

The following two lemmas will be of use.

LEMMA 1. *Under the definitions given above, when Y is positive definite, we have*

$$Q'(QYQ')^- Q = Y^{-1} - Y^{-1}\Delta(\Delta'Y^{-1}\Delta)^-\Delta'Y^{-1} = \Delta_\perp(\Delta_\perp'Y\Delta_\perp)^-\Delta_\perp', \tag{24}$$

where $\Delta_\perp$ denotes an orthogonal complement of the matrix Δ (i.e. a matrix of full column rank such that $\Delta_\perp'\Delta = 0$).

PROOF: It is easy to see that the matrix

$$Y^{1/2}Q'(QYQ')^- QY^{1/2} + Y^{-1/2}\Delta(\Delta'Y^{-1}\Delta)^-\Delta'Y^{-1/2}$$

$$= Y^{1/2}Q'(QYQ')^+ QY^{1/2} + Y^{-1/2}\Delta(\Delta'Y^{-1}\Delta)^+\Delta'Y^{-1/2}$$

is symmetric idempotent of full rank, hence it equals I_{pG}. This yields the first part of the Lemma. Further, we have

$$Q'(QYQ')^- Q = Q'(QYQ')^+ Q = ZC'(CZ'YZC')^+ CZ' =$$

$$ZC'C(Z'YZ)^+ C'CZ' = Z(Z'YZ)^+ Z' = Z(Z'YZ)^- Z',$$

due to the singular-value decomposition $Q = CZ'$, with $C'C = I_{p(G-1)}$. Clearly, Z is an orthogonal complement of Δ, as $Q\Delta = 0$. ∎

The Lemma can be adapted to the case of positive semidefinite Y, when further $\mathcal{M}(Q') \subset \mathcal{M}(Y)$. This yields

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LEMMA 2. *Under the definitions given above, when Y is singular and $\mathcal{M}(Q') \subset$*

$\mathcal{M}(Y)$, *we have*

$$Q'(QYQ')^-Q = Y^+ - Y^+\Delta(\Delta'Y^+\Delta)^-\Delta'Y^+ = \Delta_\perp(\Delta_\perp'Y\Delta_\perp)^-\Delta_\perp' \qquad (25)$$

PROOF: We use the spectral decomposition $Y = \bar{Z}M\bar{Z}'$, where $\bar{Z}'\bar{Z} = I_q$, q is the

rank of Y and $M = M_d > 0$. By Lemma 1 we have

$$\bar{Z}'Q'(Q\bar{Z}M\bar{Z}'Q')^-Q\bar{Z} = M^{-1} - M^{-1}\bar{Z}'\Delta(\Delta'\bar{Z}M^{-1}\bar{Z}'\Delta)^-\Delta'\bar{Z}M^{-1}$$

as $Q\bar{Z}\bar{Z}' = L'Y^+\bar{Z}\bar{Z}' = L'Y^+ = Q$. (We used $Q' = Y^+L$ and $Y^+ = \bar{Z}M^{-1}\bar{Z}'$). Hence

$$\bar{Z}\bar{Z}'Q'(QYQ')^-Q\bar{Z}\bar{Z}' = Y^+ - Y^+\Delta(\Delta'Y^+\Delta)^-\Delta'Y^+$$

or

$$Q'(QYQ')^-Q = Y^+ - Y^+\Delta(\Delta'Y^+\Delta)^-\Delta'Y^+.$$

Furthermore, since $Q' = Y^+L$ we can write

$$Q'(QYQ')^-Q = Q'(QYQ')^+Q = ZC'(CZ'YZC')^+CZ'$$

$$= ZC'C(Z'YZ)^+C'CZ' = Z(Z'YZ)^+Z' = Z(Z'YZ)^-Z',$$

as $Z = Y^+LC$. ▣

An explicit form for $\Delta_\perp'$ can easily be seen to be given by $\Delta_\perp' \equiv J' \otimes I_p$, where J'

denotes the Helmert matrix of order G with the first row omitted.[4] Partitioning J' as

$$J' = ((J').1, \ldots, (J').G) = \begin{pmatrix} (J')_{1.} \\ \vdots \\ (J')_{(G-1).} \end{pmatrix},$$

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[4]Helmert matrices have been described in Searle (1982, p. 71). We have available the Matlab function Helmert(G) that produces the Helmert matrix of order G.

where $(J')_{.g}$ and $(J')_{i.}$ denote respectively the gth column and ith row of J', we have

$$(J')_{i.} = (\frac{1}{i\sqrt{(i+1)}}1'_i, \frac{-1}{\sqrt{(i+1)}}, 0'_{G-i-1}),\tag{26}$$

Note that $\Delta_\perp' = (\Delta_{\perp 1}, \dots, \Delta_{\perp g}, \dots, \Delta_{\perp G})$ where $\Delta_{\perp g} = (J')_{.g} \otimes I_p$.

Consequently, when Y is nonsingular, we can use Lemma 1 to write the test statistic T of (15) as

$$T = n(\sum_{g=1}^{G} \Delta_{\perp g} r_g)'(\sum_{g=1}^{G} \frac{n_g}{n} \Delta_{\perp g} Y_g \Delta_{\perp g}')^+(\sum_{g=1}^{G} \Delta_{\perp g} r_g),\tag{27}$$

and, by virtue of Lemma 2, the same expression holds when Y is singular but $\mathcal{M}(Q') \subset \mathcal{M}(Y)$. Note that in the above expression for T the matrix to be inverted is of dimension $p(G-1) \times p(G-1)$, which is slightly less than the dimension $pG \times pG$ as encountered in (15) .

We are now able to state the following theorem which provides a simple expression for the test statistic T of (15).

THEOREM 2.   *When* $Y = \Lambda^{-1} \otimes \bar{Y}$ *and* $r \in \mathcal{M}(Y)$, *then* T *of (15) equals*

$$T = nr'(H \otimes \bar{Y}^-)r.\tag{28}$$

where $H \equiv \Lambda - \Lambda E \Lambda$, $E = 1_G 1'_G$ *and* Λ *was defined in (5) .*

PROOF:

$$T = nr'\left[(I - \Lambda E) \otimes I\right]\left[(\Lambda^{-1} - E) \otimes \bar{Y}\right]^+\left[(I - E\Lambda) \otimes I\right]r$$

$$= nr'\left[(I - \Lambda E) \otimes I\right]\left[(\Lambda^{-1} - E)^+ \otimes \bar{Y}^+\right]\left[(I - E\Lambda) \otimes I\right]r$$

$$= nr'\left[(I - \Lambda E)(\Lambda^{-1} - E)^+(I - E\Lambda) \otimes \bar{Y}^+\right]r$$

$$= nr'(H \otimes \bar{Y}^+)r = n\ell'Y(H \otimes \bar{Y}^+)Y\ell$$

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$$= n\ell'(\Lambda^{-1} \otimes \bar{Y})(H \otimes \bar{Y}^+)(\Lambda^{-1} \otimes \bar{Y})\ell = n\ell'(\Lambda^{-1}H\Lambda^{-1} \otimes \bar{Y}\bar{Y}^+\bar{Y})\ell$$

$$= n\ell'(\Lambda^{-1}H\Lambda^{-1} \otimes \bar{Y}\bar{Y}^-\bar{Y})\ell = n\ell'(\Lambda^{-1} \otimes \bar{Y})(H \otimes \bar{Y}^-)(\Lambda^{-1} \otimes \bar{Y})\ell$$

$$= n\ell'Y(H \otimes \bar{Y}^-)Y\ell = nr'(H \otimes \bar{Y}^-)r,$$

where $r = Y\ell$ and the equality [5]

$$H = \Lambda - \Lambda E\Lambda = \Lambda(\Lambda^{-1} - E)\Lambda$$

$$= \Lambda(\Lambda^{-1} - E)(\Lambda^{-1} - E)^+(\Lambda^{-1} - E)\Lambda = (I - \Lambda E)(\Lambda^{-1} - E)^+(I - E\Lambda)$$

was used. [6] ∎

In some applications one may be using a misspecified expression for $\bar{Y}$, i.e. a test statistic of the general form

$$T_{\bar{V}} \equiv nr'(H \otimes \bar{V}^+)r, \tag{29}$$

where $\bar{V} \xrightarrow{P} \bar{\Omega}$ with $\bar{\Omega} \geq 0$ a $p \times p$ matrix with possibly $\bar{\Omega} \neq \bar{\Gamma}$. We define $V = \Lambda^{-1} \otimes \bar{V}$ and $\Omega = \Lambda^{-1} \otimes \bar{\Omega}$. Note that $V \xrightarrow{P} \Omega$. Following the line of proof of Theorem 2, it can easily be seen that when $r \in \mathcal{M}(V)$ then

$$T_{\bar{V}} \equiv nr'(H \otimes \bar{V}^-)r. \tag{30}$$

The following theorem establishes the limit distribution of this general class of test statistics.

THEOREM 3. *Let $T_{\bar{V}}$ denote the quadratic form statistic given in (29). Then, under* H_1

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[5]We are indebted to Anna Cuxart of Universitat Pompeu Fabra for providing the proof of this equality involving a g-inverse instead of the Moore-Penrose inverse

[6]We note the properties of $H \geq 0$, $\text{rank}(H) = G - 1$ and $H1_G = 0$.

a) $T_{\bar{V}} \xrightarrow{L} \sum_{i=1}^{k} \alpha_i(u_i + \omega_i)^2$, *where the u_i's are independent standard normal variables,*

the ω_i's are the components of $\omega \equiv A^{-1}R'\Gamma^{1/2}(H \otimes \bar{\Omega}^+)\eta$, $\Gamma^{1/2}(H \otimes \bar{\Omega}^+)\Gamma^{1/2} =$

RAR', $R'R = I_k$, $A = A_d$, the α_i's are the diagonal (positive) elements of A and

$k \equiv rank\{\Gamma^{1/2}(H \otimes \bar{\Omega}^+)\Gamma^{1/2}\}$;

b) *when $\Gamma = \Lambda^{-1} \otimes \bar{\Gamma}$ and $\bar{\Omega} = \bar{\Gamma}$ then $T_{\bar{V}}$ is the statistic T of (15) and thus the result*

(17) applies.

Note that under H_0 we have $\eta = 0$, and hence $\omega = 0$. When $r \in \mathcal{M}(V)$ then the

Moore-Penrose inverse can be replaced by a generalized inverse.

PROOF. Note that

$$T_{\bar{V}} \equiv nr'(H \otimes \bar{V}^+)r = n(r - \Delta\vartheta)'(H \otimes \bar{V}^+)(r - \Delta\vartheta), \tag{31}$$

since $H1_G = 0$. Consequently, since under H_1

$$\sqrt{n}(r - \Delta\vartheta) \xrightarrow{L} \mathcal{N}(\eta, \Gamma),$$

we obtain that

$$T_{\bar{V}} \xrightarrow{L} z'(H \otimes \bar{\Omega}^+)z, \tag{32}$$

where $z = \mathcal{N}(\eta, \Gamma)$. The theorem now follows by straightforward application of standard

results on quadratic forms in normal variables (see Dik and Gunst, 1985; also Neudecker,

1994), applied to the right-hand side of (32). ∎

2.1   Scaled chi-squared test statistics

As in Rao and Scott (1984), when the asymptotic chi-squaredness of the test statistic

is not guaranteed, it may be of interest to consider a first order adjustment of the non

necessarily asymptotic chi-sqyared test statistic. Consider the scaled statistic

$$\bar{T}_V = T_V/a, \tag{33}$$

where T_V is defined in (29) and a is a consistent estimator of

$$\alpha \equiv \frac{1}{k}\mathrm{tr}\, Q'(Q\Omega Q')^+ Q\Gamma,$$

where k is given in Theorem 3. Note that Theorem 3 implies that under H_0, the asymptotic mean of T_V is equal to $\mathrm{tr}\,\{Q'(Q\Omega Q')^+ Q\Gamma\}$, thus under H_0 the asymptotic mean of the scaled statistic $\bar{T}_V$ is the same as the mean of the the $\chi_k{}^2$; this suggest the use of $\bar{T}_V$ as an approximately chi-squared statistic when T_V is not exactly chi-squared (Rao and Scott, 1984).

Generally, we have

$$\alpha = \mathrm{tr}\{(Q\Omega Q')^+(Q\Gamma Q')\}/k$$

$$= \mathrm{tr}(\Theta' Q\Gamma Q'\Theta)/k = \sum_{g=1}^{G} \frac{n}{n_g}\mathrm{tr}(\Theta' Q_g\Gamma_g Q'_g\Theta)/k,$$

where $\Theta\Theta' = (Q\Omega Q')^+$. A consistent estimator of α can thus be constructed as

$$a = \sum_{g=1}^{G} \frac{n}{n_g}\mathrm{tr}\{B' Q_g Y_g Q'_g B\}/k, \tag{34}$$

where $BB' = (QVQ')^+$. Often, sample moments are of the form

$$r_g \equiv \frac{1}{n_g}\sum_{i=1}^{n_g} d_{gi}, \quad g = 1,\ldots,G, \tag{35}$$

where the $\{d_{gi}\}_{i=1}^{n_g}$, $g = 1,\ldots,G$, are mutually independent iid (independent and identically distributed) sequences of p-dimensional random vectors. In this set-up a (distribution-free) consistent estimator of Γ_g will be

$$Y_g = \frac{1}{n_g}\sum_{i=1}^{n_g}(d_{gi} - \bar{d}_g)(d_{gi} - \bar{d}_g)', \tag{36}$$

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where $\bar{d}_g$ denotes the sample mean of $\{d_{gi}\}_{i=1}^{n_g}$. Further, under assumption A, the corresponding "pooled" estimator of the common matrix $\bar{\Gamma}$ will be $Y = \Lambda^{-1} \otimes \bar{Y}$ with

$$\bar{Y} = \sum_{g=1}^{G} \frac{n_g}{n} Y_g. \tag{37}$$

Thus in the case of iid sampling, we can write (34) as

$$a = \frac{1}{k} \sum_{g=1}^{G} \frac{n}{n_g} \left(\frac{1}{n_g} \sum_{i=1}^{n_g} b_{gi}' b_{gi} \right),$$

where $b_{gi} \equiv B' Q_g (d_{gi} - \bar{d}_g)$. Further, when $\Gamma = \Lambda^{-1} \otimes \bar{\Gamma}$ holds, then

$$Q'(Q\Omega Q')^+ Q\Gamma = (H \otimes \bar{\Omega}^+)(\Lambda^{-1} \otimes \bar{\Gamma}) = H\Lambda^{-1} \otimes \bar{\Omega}^+ \bar{\Gamma},$$

hence

$$\alpha = \frac{G-1}{k} \operatorname{tr} \bar{\Omega}^+ \bar{\Gamma},$$

since $\operatorname{tr} H\Lambda^{-1} = G - 1$. Thus a consistent estimator of α in that case will be

$$a = \frac{G-1}{k} \operatorname{tr} \bar{V}^+ \bar{Y}.$$

Note that the scaling correction will be automaticaly inactive when $\operatorname{tr} \bar{\Omega}^+ \bar{\Gamma} = p$ and $k = (G-1)p$. Note also that $\operatorname{tr} \bar{\Omega}^+ \bar{\Gamma} = p$ when $\bar{\Omega}^+ = \bar{\Gamma}^{-1}$. When also $r \in \mathcal{M}(V)$ the Moore-Penrose inverse can be replaced by a g-inverse.

In the case of non-iid sampling, as for example in multi-stage clustered sampling, the estimator a of α would have the same expression as above, but with the expression Y_g of (36) modified so that the new Y_g is a consistent estimator of Γ_g.

3 Applications to specific testing settings

The above described test statistics will now be applied to different cases of equality of moment matrices.

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3.1 Equality of multinomial populations

Consider the problem of

$$H_0 : \rho_g = \bar{\rho}, \quad g = 1, \ldots, G,$$

where ρ_g is a p-dimensional vector of positive numbers (proportions) satisfying $1_p'\rho = 1$.

Let r_g be the p-vector of sample proportions for which $1_p'r_g = 1$, $g = 1, \ldots, G$, and

r_g converges to ρ_g in probability. In the case of a multinomial distribution the variance

matrix of r_g is known to be

$$\Gamma_g = P_g - \rho_g \rho_g',$$

where $P_g = \text{dg}(\rho_g)$. Here $\text{dg}(a)$ for a vector a denotes the diagonal matrix with the

elements of a on the diagonal. Note that Γ_g is of rank $p - 1$. We define the pooled

estimator of the common matrix Γ_g as

$$\bar{Y} \equiv \bar{R} - \bar{r}\bar{r}',$$

where $\bar{r} = \sum_{g=1}^{G} \frac{n_g}{n} r_g$ and $\bar{R} = \text{dg}(\bar{r})$, and we let $Y \equiv \Lambda^{-1} \otimes \bar{Y}$. We have that under

H_0, $Y \xrightarrow{P} \Gamma$. Note that in the present testing setting, the null hypothesis H_0 implies

Assumption A of equality of the variance matrices Γ_g.

Note that $(r - \frac{1}{p}1) \in \mathcal{M}(Y)$, since $\bar{Y}$ is of rank $p - 1$, $1_p'\bar{Y} = 0$ and $1_p'r_g = 1$. Thus,

the test statistic of (28) will be

$$T = nr'(H \otimes \bar{R}^{-1})r, \tag{38}$$

as $nr'(H \otimes \bar{R}^{-1})r = n((r - \frac{1}{p}1))'(H \otimes \bar{R}^{-1})(r - \frac{1}{p}1)$ (we use again $1'H = 0$) and $\bar{R}^{-1}$ is

a generalized inverse of the variance matrix $\bar{Y} = \bar{R} - \bar{r}\bar{r}'$. Further, it is easy to see that

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$k = \mathrm{rank}\,(Q\Gamma Q') = (G-1)(p-1)$, since $\mathrm{rank}(\bar{\Gamma}) = p - 1$. [7]

In the case of non-iid sampling, the scaled statistic $\bar{T} = a^{-1}T$ provides an approximate chi-squared test statistic. Under Assumption A, we have $a = \frac{G-1}{k}\mathrm{tr}\bar{R}\bar{Y} = \mathrm{tr}\bar{R}\bar{Y}/p$, where $\bar{Y}$ is a consistent estimator of the true common variance matrix of the r_g.

3.2 Equality of variance and augmented moment matrices

Consider $H_0 : \Sigma_g = \bar{\Sigma}, \quad g = 1, \ldots, G$, where Σ_g is the $h \times h$ variance matrix of the gth group (population). Consider $\{z_{gi}\}_{i=1}^{n_g}$, $g = 1, \ldots, G$, to be mutually independent iid sequences of $h \times 1$ vectors, and

$$S_g \equiv \frac{1}{n_g}\sum_{i=1}^{n_g}(z_{gi} - \bar{z}_g)(z_{gi} - \bar{z}_g)', \tag{39}$$

be the usual sample variance matrix for the gth group. Here $\bar{z}_g$ denotes the sample mean of $\{z_{gi}\}_{i=1}^{n_g}$. We define the pooled sample variance matrix $\bar{S} \equiv \sum_{g=1}^{G} \frac{n_g}{n}S_g$. Define $s = \mathrm{vec}[s_g \mid g = 1, \ldots, G]$ and $\sigma = \mathrm{vec}[\sigma_g \mid g = 1, \ldots, G]$, where $s_g \equiv D^+\mathrm{vec}S_g$ and $\sigma_g \equiv D^+\mathrm{vec}\Sigma_g$. Note that s and σ are pG-dimensional vectors where $p = 2^{-1}h(h + 1)$.

Denote by Φ_g the asymptotic variance matrix of $\sqrt{n_g}s_g$, and by $\bar{\Phi}$ the same variance matrix when it is common to all groups. Clearly, regardless the distribution of the z_i, consistent estimators of Φ_g and $\bar{\Phi}$ are respectively

$$W_g \equiv \frac{1}{n_g}\sum_{i=1}^{n_g}v_{gi} \tag{40}$$

and

$$\bar{W} \equiv \sum_{g=1}^{G}\frac{n_g}{n}W_g, \tag{41}$$

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[7]
$$Q\Gamma Q' = [(I - E\Lambda) \otimes I_p](\Lambda^{-1} \otimes \bar{\Gamma})[(I - E\Lambda) \otimes I_p] =$$
$$(I - E\Lambda)\Lambda^{-1}(I - E\Lambda) \otimes \bar{\Gamma} = (\Lambda^{-1} - E) \otimes \bar{\Gamma} = (I - E\Lambda)\Lambda^{-1} \otimes \bar{\Gamma}.$$
Consequently, $\mathrm{rank}(Q\Gamma Q') = \mathrm{rank}\{(I - E\Lambda)\Lambda^{-1}\}\mathrm{rank}(\bar{\Gamma}) = \mathrm{rank}\{(I - E\Lambda)\}\mathrm{rank}(\bar{\Gamma}) = (G - 1)(p - 1).$

where $v_{gi} = D^+ \text{vec} (z_{gi} - \bar{z}_g)(z_{gi} - \bar{z}_g)'$.

When $\{z_{gi}\}_{i=1}^{n_g}$ are iid normally distributed, then Φ_g takes the following normal-theory (NT) expression

$$\Phi_g^\star \equiv 2\, D^+(\Sigma_g \otimes \Sigma_g)D^{+\prime}, \tag{42}$$

which is consistently estimated by

$$W_g{}^\star \equiv 2\, D^+(S_g \otimes S_g)D^{+\prime}. \tag{43}$$

The corresponding matrices $\bar{\Phi}^\star$ and $\bar{W}^\star$ are obtained by replacing the matrix Σ_g in the expression of $\Phi_g^\star$ by $\bar{\Sigma}$ and $\bar{S}$ respectively. Note that under NT, the null hypothesis H_0 implies also Assumption A of equality of the matrices Φ_g.

The asymptotic distribution-free (DF) and the normal-theory (NT) of (28) will thus be

$$T = n\, s'(H \otimes \bar{W}^-)s \tag{44}$$

and

$$T^\star = n\, s'(H \otimes \bar{W}^{\star-})s, \tag{45}$$

respectively. The number of degrees of freedom of the test is equal to $k = (G-1)p$. Note that $T^\star$ is simpler to compute than T, since it requires only the inversion of matrices of dimension $h \times h$, while T requires the inversion of a $p \times p$ matrix.

The scaled version (33) of $T^\star$ will be

$$\bar{T} = T^\star/a, \tag{46}$$

where $a = \text{tr}(\bar{W}^{\star-1}\bar{W})/p$.

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Consider now the case where $z_{gi} = (y_{gi}', 1)'$ is an augmented moment vector, and we define $\Sigma_g \equiv \mathcal{E}(z_{gi} z_{gi}')$ and $S_g \equiv \frac{1}{n_g} \sum_{i=1}^{n_g} z_{gi} z_{gi}'$ as the population and sample augmented moment matrices respectively. In this case, H_0 is the hypothesis of equality of mean vector and variance matrix across groups. It holds that W_g of (40) and $\bar{W}$ of (41) still give consistent estimators of the DF expressions of Φ_g and $\bar{\Phi}$ respectively. Now, however, the NT form of Φ_g is (e.g. , Satorra 1992)

$$\Phi_g^* \equiv 2 D^+ (\Sigma_g \otimes \Sigma_g - \mu_g \mu_g' \otimes \mu_g \mu_g') D^{+'}, \tag{47}$$

where $\mu_g = \mathcal{E} z_{gi}$. It can easily be verified that $2^{-1} D'(\Sigma_g^{-1} \otimes \Sigma_g^{-1}) D = \Phi_g^{*-}$, thus the same expressions of T, T^* and $\bar{T}$ as reported in (44), (45) and (46) respectively hold in the case of augmented moment matrices.

3.3   Testing the equality of correlation matrices

Consider $H_0 : P_g = \bar{P}$, $\quad g = 1, \ldots, G$, where P_g is the $(h \times h)$ correlation matrix of an $h-$dimensional vector in the gth group. Let $R_g \equiv (S_g)_d^{-1/2} S_g (S_g)_d^{-1/2}$ be the sample correlation matrix, where S_g is the gth sample variance matrix. Let $r \equiv \text{vec}[r_g \mid g = 1, \ldots, G]$, where $r_g \equiv \tilde{D} \text{vec} R_g$. The asymptotic variance matrix of $\sqrt{n_g} r_g$ is (Neudecker and Wesselman, 1990)

$$\Psi_g = \tilde{D} \Pi_g \Phi_g \Pi_g' \tilde{D}', \tag{48}$$

where

$$\Pi_g = [I - (I \otimes P_g) K_d] \left[(\Sigma_g)_d^{-1/2} \otimes (\Sigma_g)_d^{-1/2} \right] \tag{49}$$

and Φ_g is the asymptotic variance matrix of s_g described in section above. Here K is the commutation matrix (see Magnus and Neudecker, 1988), and note that $\tilde{D}$ is the

duplication matrix for zero-axial symmetry. The expression for $\bar{\Psi}$, the variance matrix common to all groups, is obtained by replacing in the expression Ψ_g the matrices Π_g, Φ_g, P_g and Σ_g by $\bar{\Pi}$, $\bar{\Phi}$, $\bar{P}$ and $\bar{\Sigma}$ respectively.

Consistent estimators of the DF and NT expressions of Ψ_g are

$$A_g \equiv \tilde{D}\hat{\Pi}_g W_g \hat{\Pi}'_g \tilde{D}'$$

and

$$A_g^\star \equiv \tilde{D}\hat{\Pi}_g W_g^\star \hat{\Pi}'_g \tilde{D}'$$

respectively, where W_g and $W_g^\star$ are given in (40) and (43) respectively, and $\hat{\Pi}_g$ is the matrix of (49) with P_g and Σ_g replaced by R_g and S_g respectively. The corresponding expressions for $\bar{A}$ and $\bar{A}^\star$ are obtained by replacing in the expressions above W_g, S_g and R_g by $\bar{W}$, $\bar{S}$ and $\bar{R}$ respectively. Here $\bar{R}$ is the correlation matrix associated with $\bar{S}$. In contrast with the test of equality of augmented moment matrices discussed in the last section, now under normality (NT) the null hypothesis H_0 does not imply Assumption A.

The DF and NT chi-squared tests statistics of (28) will then be

$$T = n \; r'(H \otimes \bar{A}^{-1})r$$

and

$$T^\star = n \; r'(H \otimes \bar{A}^{\star -1})r.$$

respectively. We note that to compute $T^\star$ the following equality can be useful (Jennrich, 1970)

$$A_g^{\star -1} = \frac{1}{2}\tilde{D}' \left[R_g^{-1} \otimes R_g^{-1} - 2(I \otimes R_g^{-1})JU_g^{-1}J'(R_g^{-1} \otimes I) \right] \tilde{D}, \qquad (50)$$

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where $U_g \equiv I + R_g \times R_g^{-1}$ and J' is the matrix that converts $\text{vec}X$ to $x = X_d 1$, (i.e. $J'\text{vec}X = X_d 1$) for any square matrix X.

The statistic T^* can be scaled to $\bar{T} = T^*/a$, where $a = \text{tr}(\bar{A}^{*-1}\bar{A})/p$. When the distribution of the observed variables is elliptical, then Ψ_g is of the form $\Psi_g = (1+\kappa_g)\Phi_g^*$, where κ_g is a kurtosis parameter (Neudecker, 1994) and Φ_g^* was given in (42). [8] Further, when $\kappa_g = \bar{\kappa}$, then $\alpha = (1+\bar{\kappa})/p$ and $\bar{T}$ will then be asymptotically an exact chi-squared statistic when H_0 holds.

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[8]We recall that in the case of an elliptical distribution

$$\Phi_g = 2(1 + \kappa_g)D^+(\Sigma_g \otimes \Sigma_g)D^{+\prime} + \kappa_g D^+(\text{vec}\Sigma_g)(\text{vec}\Sigma_g)'D^{+\prime} \tag{51}$$

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