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Asymptotic Robustness in Multi–Sample Analysis of Multivariate Linear Relations*

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Abstract

Standard methods for the analysis of linear latent variable models often rely on the assumption that the vector of observed variables is normally distributed. This normality assumption (NA) plays a crucial role in assessing optimality of estimates, in computing standard errors, and in designing an asymptotic chi-square goodness-of-fit test. The asymptotic validity of NA inferences when the data deviates from normality has been called asymptotic robustness. In the present paper we extend previous work on asymptotic robustness to a general context of multi-sample analysis of linear latent variable models, with a latent component of the model allowed to be fixed across (hypothetical) sample replications, and with the asymptotic covariance matrix of the sample moments not necessarily finite. We will show that, under certain conditions, the matrix Γ of asymptotic variances of the analyzed sample moments can be substituted by a matrix Ω that is a function only of the cross-product moments of the observed variables. The main advantage of this is that inferences based on Ω are readily available in standard software for covariance structure analysis, and do not require to compute sample fourth-order moments. An illustration with simulated data in the context of regression with errors in variables will be presented.

1 Introduction

Structural equation models have been widely used in economic, social and behavioral studies to analyze the relationship among variables, some of which may be unobservable (latent) or subject to measurement error (see e.g., Bollen, 1989, and references therein). The assumption that the vector of observed variables is normally distributed is typically used to justify the estimation method used, to compute standard errors and to construct a chi-square goodness-of-fit test statistic for the whole model. This normality assumption is often violated, since in many applications data is skewed or has non-normal kurtosis.

The typical approach to moment structure analysis is to use minimum discrepancy (MD) methods to fit a structured vector of population moments to the vector of sample moments (e.g., Browne 1984). Clearly, the asymptotic variance matrix Γ of the vector of analyzed sample moments s plays a fundamental role in designing an efficient MD analysis and in assessing the sampling variability of the statistics of interest. In the case of non-normal data, the matrix Γ involves the fourth-order moments of the observed variables. When the observed variables are normally distributed, Γ is a function of the first and second-order moments only. The form of Γ under the normality assumption is denoted in the present paper as $\Gamma^\star$. In the case of analyzing an augmented moment matrix, the matrix $\Gamma^\star$ is singular and is the difference of two terms: matrix Ω, a function of population cross-product moments only, and a matrix Υ that is a function of population first-order moments (see (15)). We investigate conditions under which the replacement of Γ by $\Gamma^\star$, or simply by Ω, gives asymptotically valid inferences when the data is non-normal and hence $\Gamma^\star$ (or Ω) is different from Γ. The results of the present paper are developed in the general case of simultaneous analysis of several samples of observations.

In multi-sample analysis of structural equation models, independent samples from several populations (groups) are analyzed simultaneously under a common model, with parameter restrictions across populations; interest focuses in studying the between-population variation of model parameters. Multi-sample analysis gives also the possibility of integrating into a single analysis sample information that arises from different sources, as in meta-analysis studies (Hedges and Olkin, 1985). It has been shown that the presence of missing data in structural equation models can be dealt also using a multi-sample analysis approach (Allison, 1987; Arminger and Sobel, 1989; Muthén, Kaplan and Hollis, 1987).

The validity of inferences based on the normality assumption when the data is not normally distributed has been called asymptotic robustness (Anderson, 1987). Asymptotic robustness in linear latent variable models was first investigated by Amemiya (1985), Anderson and Amemiya (1988), Amemiya and Anderson (1990) and Browne (1987), and further developed by Browne and Shapiro (1987), Anderson (1987, 1989), Kano (1993), Satorra and Bentler (1990, 1991) and Mooijaart and Bentler (1991). The above mentioned work, however, was confined to one-sample analysis of covariance structures. More recently, Browne (1990), Satorra (1992) and Satorra and Neudecker (1994) extended asymptotic robustness issues to the more general context of models with structured means as well as variances and covariances of observed variables. The extension of asymptotic robustness to the case of multi-sample analysis of mean and covariance structures was first investigated by Satorra (1993a,1993b).

The present paper extends the work on asymptotic robustness in several aspects. As in Satorra (1993a, 1993b), we consider asymptotic robustness in the general setting of multi-sample analysis. Now, however, by adopting Anderson-Amemiya's device of a Taylor series expansion around a parameter value that has some sample specific components, we are able to relax the assumption of finiteness of the matrix Γ and we can accommodate the case of a latent component fixed across sample replications.

The results to be developed in the present paper concern a wide class of structural equation models that include regression with errors in variables, path analysis models, econometric simultaneous equation models, multivariate regression, and so forth. In fact the results are of direct relevance to the general methods for structural equation models implemented in standard computer software such as LISREL of Jöreskog & Sörbom (1989), EQS of Bentler (1989), LISCOMP of Muthén (1989) and CALIS of SAS (1990), among others. The class of models and estimation methods considered in the present paper are more general than the ones considered in the previous work on asymptotic robustness.

The plan of the paper is as follows. Section 2 presents the assumptions and reviews the basic theory of moment structure analysis. Section 3 develops the general results of asymptotic robustness. Section 4 provides an illustration with Monte Carlo data of the performance in finite samples of the asymptotic results of the paper.

The following notation will be used. We denote by $\text{vec}\{a_i; i = 1, \ldots, I\}$ the vector formed by vertical stacking of the vectors a_i, and by $\text{diag}\{A_i; i = 1, \ldots, I\}$ the block-diagonal matrix formed with the square matrices A_i as diagonal blocks. For any matrix A, $\text{vec}(A)$ denotes the column vector formed by vertical stacking of the columns of A, one below the other. When A is a symmetric matrix, $\text{v}(A)$ is the vector obtained from $\text{vec}(A)$ by eliminating the duplicated elements associated with the symmetry of A. We use the "duplication" and "elimination" matrices D and D^+ (Magnus and Neudecker, 1993), such that $\text{vec}(A) = D\text{v}(A)$ and $\text{v}(A) = D^+\text{vec}(A)$, for a symmetric matrix A. The matrices D and D^+ will be of varying order determined by the context. The standard notation $O_p(1)$ and $o_p(1)$ for orders of magnitude will be used. By $\{a_i; i = 1, \ldots, I\}$ are iid, we mean that the a_i are mutually independent and identically distributed random quantities.

2 Multi-sample analysis of moment structures

In this section we present the type of data analyzed, the models, and the basic assumptions used in the paper. We will also review the standard theory for multi-sample analysis of moment structures, under both asymptotic distribution-free (DF) and normal theory framework. Four fundamental assumptions will be discussed: Multivariate linear relation (MLR), Model assumption (MA), Identification (I) and the Normality assumption (NA).

We deal with multi-sample data

$$\{z_{gi},\ i = 1, \ldots, n_g,\ g = 1, \ldots, G\},$$

where z_{gi} is a p_g-dimensional vector of observed variables associated with individual i in group g, G is the number of groups, n_g is the sample size of group g, $n \equiv \sum_{i=1}^{G} n_g$ is the *total* sample size, and $p \equiv \sum_g p_g$. Note that the number p_g of observed variables may vary across groups. Without lost of generality, we suppose that the first component of z_{gi} is a constant equal to 1. The presence in z_{gi} of this component equal to 1 allow us to structure means as well as variances and covariances of the observed variables. We assume that $\frac{n_g}{n} \to \pi_g$ when $n \to +\infty$, with the π_g being positive numbers.

For each group g ($g = 1, \ldots, G$), consider the matrix of (uncentered) sample cross-product moments

$$S_g \equiv n_g^{-1} \sum_{i=1}^{n_g} z_{gi} z_{gi}'. \tag{1}$$

Let $s \equiv \text{vec}\{s_g; g = 1, \ldots, G\}$, with $s_g \equiv \text{v}(S_g)$, be the overall vector of sample moments. Throughout the paper we restrict to the case where the samples from the different groups are independent, so that the s_g are statistically independent vectors. The asymptotic variance matrix Γ of $\sqrt{n}s$ is thus the block diagonal matrix

$$\Gamma = \text{diag}\left\{\pi_g^{-1}\Gamma_g; g = 1, \ldots, G\right\}, \tag{2}$$

where Γ_g is the asymptotic variance matrix of $\sqrt{n_g}s_g$.

When the Γ_g are finite, and under suitable regularity conditions (see, e.g., Chamberlain, 1982), an (unbiased) consistent estimate of Γ is

$$\begin{aligned} \hat{\Gamma} &\equiv \text{diag}\{\frac{n}{n_g}\hat{\Gamma}_g; g = 1, \ldots, G\}, && (3) \\ \hat{\Gamma}_g &\equiv \frac{1}{n_g - 1}\sum_{i=1}^{n_g}(d_{gi} - s_g)(d_{gi} - s_g)', && (4) \\ d_{gi} &\equiv \text{v}(z_{gi} z_{gi}'). \end{aligned}$$

The above matrix (3) will be called the asymptotic robust (AR) estimate of Γ, to distinguish it from the estimate of Γ based on the NA assumption to be introduced below.

A key assumption for the theory to be developed in the present paper is the following one.

ASSUMPTION 1 (MULTIVARIATE LINEAR RELATION, MLR) *For $g = 1, \ldots, G$,*

$$z_{gi} = \sum_{\ell=0}^{L_g} \Pi_{g\ell}\xi_{gi\ell}, \tag{5}$$

where $\xi_{gi\ell}$ is a $m_{g\ell}$-dimensional vector of (observed or latent) variables and $\Pi_{g\ell}$ is a $p_g \times m_{g\ell}$ matrix of coefficients. Further,

1. *the* $\{\xi_{gi0}; i = 1, \ldots, n_g\}$ *are fixed (across hypothetical sample replications) vectors such that*

$$\lim_{n_g \to +\infty} \frac{1}{n_g} \sum_{i=1}^{n_g} u_i^2 = \phi_u, \tag{6}$$

a finite value, for each component u_i *of* ξ_{gi0}. [1]

2. *except (possibly) for* $\ell = 0$, *the* $\{\xi_{gi\ell}; i = 1, \ldots, n_g\}$ *are iid, and mutually independent across* $\ell = 1, \ldots, L_g$.

3. *the* $\{\xi_{giL_g}; i = 1, \ldots, n_g\}$ *are iid normally distributed.*

When $\xi_{gi\ell}$ *is stochastic, it has zero mean and finite variance matrix* $\Phi_{g\ell}$.

Clearly, MLR implies

$$S_g \xrightarrow{P} \Sigma_g, \tag{7}$$

when $n_g \to +\infty$, where $\Sigma_g \geq 0$ is a $p_g \times p_g$ (finite) positive semidefinte matrix. The matrix Σ_g will be called the matrix of *population cross-product moments* . This matrix is assumed to be positive definite.

The above assumption MLR imply a structure on the matrix of population cross-product moments. In the case of multi-sample analysis, the structured population matrices are fitted simultaneously to the corresponding sample matrices. In the present paper we restrict attention to the following moment structures.

ASSUMPTION 2 (MOMENT STRUCTURE, MS) *For* $g = 1, \ldots, G$,

$$\Sigma_g = \sum_{\ell=0}^{L_g} \Pi_{g\ell} \Phi_{g\ell} \Pi_{g\ell}', \tag{8}$$

where the matrices $\Pi_{g\ell}$ *and* $\Phi_{g\ell}$ *are of dimension* $p_g \times m_{g\ell}$ *and* $m_{g\ell} \times m_{g\ell}$ *respectively, and* $\Pi_{g\ell} = \Pi_{g\ell}(\tau)$ *and* $\Phi_{gL_g} = \Phi_{gL_g}(\tau)$ *are twice continuously differentiable matrix-valued functions of a t-dimensional parameter vector* τ.

Note that MS does not restrict the variance matrices $\Phi_{g\ell}$, except for Φ_{gL_g}. It should be noted that equality (8) is implied already by MLR. Assumption

[1] By the Cauchy-Schwarz inequality, (6) implies that $\lim_{n_g \to +\infty} n_g^{-1} \sum_{i=1}^{n_g} \xi_{gi0} \xi_{gi0}' = \Phi_{g0}$, a finite matrix.

MS, however, structures the matrices $\Pi_{g\ell}$ and Φ_{gL_g} to be functions of τ. We have stated MS independently of MLR since, in practice, MS can be analyzed without the necessity of MLR to hold.

Assumption 2 (MS) allows us to write $\Sigma_g = \Sigma_g(\vartheta_g)$, where $\Sigma_g(.)$ is a twice continuously differentiable matrix-valued function of $\vartheta_g \equiv (\tau', \phi_g{}')'$, with $\phi_g \equiv \text{vec}\{\phi_{g\ell}; \ell = 0, \ldots, L_g - 1\}$ and $\phi_{g\ell} \equiv \text{v}(\Phi_{g\ell})$. Note that ϕ_{gL_g} is not included in ϕ_g.

Let $\sigma_g \equiv v(\Sigma_g)$; then we have

$$\sigma_g = \sum_{\ell=0}^{L_g} \left[D^+(\Pi_{gl}(\tau) \otimes \Pi_{g\ell}(\tau))D \right] \phi_{gl}. \tag{9}$$

Define $\phi \equiv \text{vec}\{\phi_g; g = 1, \ldots, G\}$ and $\vartheta \equiv (\tau', \phi')'$, then the vector $\sigma \equiv \text{vec}\{\sigma_g; g = 1, \ldots, G\}$ can be expressed as

$$\sigma = \sigma(\vartheta),$$

where $\sigma(.)$ is a twice continuously differentiable vector-valued function of ϑ. We denote by t^* the length of ϑ.

Given the sample moment matrices S_g, estimation of the parameter vector ϑ is possible. A general method of estimation is weighted least-squares (WLS) (see, e.g., Browne, 1984, Chamberlain, 1982, Satorra, 1989), where the estimator of ϑ is defined as

$$\hat{\vartheta} \equiv \arg\min_{\vartheta \in \Theta} [s - \sigma(\vartheta)]' \hat{V} [s - \sigma(\vartheta)], \tag{10}$$

with a $p \times p$ matrix $\hat{V}$ that converges in probability to V, a positive definite matrix. The parameter space Θ is assumed to be a compact subset of the Euclidean t^*-dimensional space (R^{t^*}). For further use, we introduce the vector of fitted moments $\hat{\sigma} \equiv \sigma(\hat{\vartheta})$, and the subvector $\hat{\tau}$ of $\hat{\vartheta}$. Consider the Jacobian matrix $\Delta \equiv \partial\sigma(\theta)/\partial\sigma'$, and let $\hat{\Delta}$ be the matrix Δ evaluated at $\vartheta = \hat{\vartheta}$.

We now state a standard assumption of model identification that is assumed to hold throughout the paper.

Assumption 3 (Identification, I) *When $s = \sigma$ and $\hat{V} = V$ the optimization problem of (10) has (locally) a unic solution ϑ that lies in the interior of Θ. Furthermore, the matrix $\Delta' V \Delta$ is nonsingular.*

As has been shown elsewhere (e.g., Satorra, 1989), under standard regularity conditions (clearly implied by MS and I), $\hat{\vartheta}$ is a consistent and asymptotically normal estimator of θ, with asymptotic variance (avar) matrix given by

$$\text{avar}(\hat{\vartheta} \mid V, \Gamma) = n^{-1}(\Delta' V \Delta)^{-1} \Delta' V \Gamma V \Delta (\Delta' V \Delta)^{-1}. \tag{11}$$

When in (11) we replace $\hat{\Gamma}$, $\hat{V}$ and $\hat{\Delta}$ by Γ, V and Δ respectively, we obtain the so-called asymptotic robust (AR) (as no normality assumption is involved) estimator of the variance matrix of $\hat{\vartheta}$. In practice, however, one wants to avoid the use of $\hat{\Gamma}$, since as (4) shows it involves computing the fourth-order sample moments of the data. Further, the use of $\hat{\Gamma}$ raises concern about robustness in small samples. The use of $\hat{\Gamma}$ requires also further regularity conditions, such as the finiteness of Γ.

When $V \Gamma V = V$, or $\Gamma V \Gamma = \Gamma$ and Δ is in the column space of Γ (see Satorra and Neudecker, 1994), then clearly the variance matrix (11) simplifies to

$$\text{avar}(\hat{\vartheta} \mid V) = n^{-1}(\Delta' V \Delta)^{-1}. \tag{12}$$

In this case, we say that the corresponding WLS estimator is asymptotically optimal. See Satorra and Neudecker (1994) for detail discussion on the asymptotic optimality of alternative WLS estimators in the context of linear latent variable models.

We now introduce the normality assumption (NA), used very often to draw statistical inferences in structural equation modeling.

ASSUMPTION 4 (NORMALITY ASSUMPTION, NA) *For* $g = 1, \ldots, G$, *and* $\ell = 0, \ldots, L_g$, *equation (5) holds with* $\{\xi_{gi\ell}; i = 1, \ldots, n_g\}$ *being iid normally distributed.*

REMARK *Clearly, NA implies that* $\{z_{gi}; i = 1, \ldots, n_g\}$ *are iid normally distributed.*

Inferences derived under this assumption will be called NA inferences. In fact, the major importance of the present paper is to develop results showing that some of the NA inferences are valid even though NA does not hold. This is of practical relevance since under NA the computation of the sample fourth-order moments of the data can be avoided.

Now we review the formulae for asymptotic inferences under NA. The asymptotic variance matrix Γ_g of $\sqrt{n_g} s_g$ based on NA will be denoted as $\Gamma_g^\star$;

and that of Γ as Γ^*. Clearly, we have

$$\Gamma^* = \text{diag}\{\pi_g^{-1}\Gamma_g^*; g = 1,\ldots,G\}.$$

When NA holds the asymptotic variance matrix of $\hat{\vartheta}$ is given by (11) with Γ^* substituted for Γ.

Define also,

$$\begin{aligned} \Omega &\equiv \text{diag}\{\pi_g^{-1}\Omega_g; g = 1,\ldots,G\}, & (13)\\ \Omega_g &\equiv 2D^+(\Sigma_g \otimes \Sigma_g)D^{+\prime}, & (14) \end{aligned}$$

and denote by $\hat{\Omega}$ the estimator of Ω obtained by substituting $\frac{n_g}{n}$ for π_g in (13) and S_g for Σ_g in (14). .

Under assumption NA, the matrix Γ can be expressed as (see, e.g., Satorra 1992b)

$$\Gamma^* = \Omega - \Upsilon, \qquad (15)$$

where

$$\begin{aligned} \Upsilon &\equiv \text{diag}\{\pi_g^{-1}\Upsilon_g; g = 1,\ldots,G\}, & (16)\\ \Upsilon_g &\equiv 2D^+(\mu_g\mu_g{}' \otimes \mu_g\mu_g{}')D^{+\prime}, &\\ \mu_g &\equiv \mathcal{E}(z_g), & \end{aligned}$$

with " $\mathcal{E}$" denoting mathematical expectation.

Under MLR and NA,

$$\mu_g = \Pi_{g0}\mu_{g0},$$

where $\mu_{g0} \equiv \mathcal{E}(\xi_{g0})$; thus,

$$\begin{aligned} \Upsilon_g &= 2D^+(\Pi_{g0} \otimes \Pi_{g0})(\mu_{g0}\mu_{g0}{}' \otimes \mu_{g0}\mu_{g0}{}')(\Pi_{g0} \otimes \Pi_{g0})'D^{+\prime} & (17)\\ &= \Lambda_g M_g \Lambda_g', & \end{aligned}$$

where $\Lambda_g \equiv D^+(\Pi_{g0} \otimes \Pi_{g0})D$ and $M_g \equiv 2D^+(\mu_{g0}\mu_{g0}{}' \otimes \mu_{g0}\mu_{g0}{}')D^{+\prime}$.[2] Consequently, when MLR and NA holds,

$$\Gamma^* = \Omega - \Lambda_0 M \Lambda_0', \qquad (18)$$

[2]This is due to the fact that

$$(\mu_{g0}\mu_{g0}{}' \otimes \mu_{g0}\mu_{g0}{}') = DD^+(\mu_{g0}\mu_{g0}{}' \otimes \mu_{g0} \otimes \mu_{g0}{}')(DD^+)'$$

as $(\mu \otimes \mu) = N(\mu \otimes \mu)$ and $N = DD^+$.

where $M \equiv \text{diag}\{M_g; g = 1, \ldots, G\}$ and $\Lambda_0 \equiv \text{diag}\{\Lambda_{g0}; g = 1, \ldots, G\}$.

Define also

$$V^{\star} \equiv \text{diag}\{\pi_g V_g^{\star}; g = 1, \ldots, G\}, \tag{19}$$

$$V_g^{\star} \equiv \frac{1}{2} D'(\Sigma_g{}^{-1} \otimes \Sigma_g{}^{-1}) D, \tag{20}$$

and denote by $\hat{V}^{\star}$ the estimator of $V^{\star}$ obtained by substituting $\frac{n_g}{n}$ for π_g in (19) and S_g for Σ_g in (20). The use of $\hat{V} = \hat{V}^{\star}$ in (10) produces the so-called NA-WLS estimators, which have been shown to be asymptotically equivalent to the maximum likelihood estimators under NA (e.g., Satorra, 1992b). Note that $V^{\star} = \Omega^{-1}$ (and $\hat{V}^{\star} = \hat{\Omega}^{-1}$).

When NA holds and Δ is in the column space of $\Gamma^{\star}$, the asymptotic variance matrix (11) of the NA-WLS estimator reduces to

$$\text{avar}(\hat{\vartheta} \mid V = V^{\star}) = n^{-1}(\Delta' V^{\star} \Delta)^{-1}, \tag{21}$$

since $\Gamma^{\star} V^{\star} \Gamma^{\star} = \Gamma^{\star}$ (see Lemma 2 of Satorra, 1992b), and hence (12) is attained. The standard errors extracted from (21) will be called the NA standard errors. Clearly, the NA standard errors may not be correct when NA does not hold.

When the z_g are normally distributed, it can be shown (e.g., by direct application of results of Meredith and Tisak, 1990) that the log-likelihood function is an affine transformation of

$$F_{ML} \equiv \sum_{g=1}^{G} \frac{n_g}{n} [\log \mid \Sigma_g(\vartheta) \mid + \text{trace}\{S \Sigma_g(\vartheta)^{-1}\} - \log \mid S_g \mid - p_g]; \tag{22}$$

thus, the minimization of $F_{ML} = F_{ML}(\vartheta)$ yields the maximum likelihood (ML) estimator of ϑ. Since $\frac{\partial^2 F_{ML}}{\partial \sigma \partial \sigma} = V^{\star}$ (see, e.g., Neudecker and Satorra, 1991), ML estimation is asymptotically equivalent to NT-WLS (Shapiro, 1985; Newey, 1988).

In addition to parameter estimation, we are interested in testing the goodness of fit of the model. Consider the following goodness-of-fit test statistic

$$T_V^{\star} = n(s - \hat{\sigma})'(\hat{P} \hat{V}^{\star} \hat{P}')^{+}(s - \hat{\sigma}), \tag{23}$$

where

$$\hat{P} \equiv I - \hat{\Delta}(\hat{\Delta}' \hat{V} \hat{\Delta})^{-1} \hat{\Delta}' V, \tag{24}$$

with the superscript "+" denoting Moore-Penrose inverse. Denote by P the asymptotic limit of $\hat{P}$. It will be shown below that under MLR and MS, T_V^* is asymptotically chi-square distributed with

$$r = \text{rank}\{PV^*P'\} \tag{25}$$

degrees of freedom (df). Clearly, when $V = V^*$, an alternative expression of $T^{**} \equiv T_{V^*}^*$ is

$$T^{**} = n(s - \hat{\sigma})'(\hat{V}^* - \hat{V}^*\hat{\Delta}(\hat{\Delta}'\hat{V}^*\hat{\Delta})^{-1}\hat{\Delta}'\hat{V}^*)(s - \hat{\sigma}).$$

When Γ is finite, and the AR estimate $\hat{\Gamma}$ (see (3)) is available, an alternative goodness-of-fit test statistic is

$$T_V = n(s - \hat{\sigma})'(\hat{P}\hat{\Gamma}\hat{P}')^{+}(s - \hat{\sigma}). \tag{26}$$

When MS and I holds (not necessarely MLR), T_V can be shown to be asymptotically chi-square distributed with the degrees of freedom of (25) regardless of whether NA holds or not (e.g., Satorra, 1993b). As T_V is an asymptotic chi-square statistic without the requirement of NA to hold (it requires only MS and the regularity conditions for $\hat{\Gamma}$ to be a consistent estimate of Γ), we call T_V the AR goodness-of-fit test statistic.

Under NA single-group covariance structure analysis, T_V is Browne's (1984) residual-based chi-square goodness-of-fit statistic. The asymptotic robustness of T_V^* (for $V = I, V^*$) has been investigated in single-sample analysis by Satorra and Bentler (1989), and for multi-sample analysis of mean and covariance structures by Satorra (1993a, 1993b). It can be shown that T^{**} is asymptotically equivalent to the usual goodness-of-fit test statistics developed under NA using the likelihood ratio principle (Satorra, 1992b, 1993b).

In the next section, we will show that some of the above formulae for NA inferences retain their validity even when the assumption NA is violated, and thus Γ may be different from Γ^*. Furthermore, we will allow the possibility even that Γ is not finite.

3 Asymptotic robustness of NA inferences

To develop the results of this section, we first need to obtain compact expressions for some of the statistics and parameter matrices introduced in the section above. In relation with (5), let

$$\Pi_g \equiv [\Pi_{g0}, \Pi_{g1}, \ldots, \Pi_{gL_g}]$$

and

$$\xi_{gi} \equiv \text{vec}\{\xi_{gi\ell};\, \ell = 0, \ldots, L_G\},$$

and write

$$z_{gi} = \Pi_g \xi_{gi}. \tag{27}$$

Note that ξ_{gi} is a vector of length $m_g = \sum_{l=1}^{L_g} m_{gl}$. Clearly,

$$S_g = \Pi_g Q_g {\Pi_g}', \tag{28}$$

where

$$Q_g \equiv n_g^{-1} \sum_{i=1}^{n_g} \xi_{gi} {\xi_{gi}}'. \tag{29}$$

The matrix Q_g is composed of blocks

$$Q_{grt} \equiv {n_g}^{-1} \sum_{i=1}^{n_g} \xi_{gri} {\xi_{gti}}'; \tag{30}$$

$$Q_g = \begin{pmatrix} Q_{g00} & Q_{g01} & \cdots & Q_{g0L_g} \\ Q_{g10} & Q_{g11} & \cdots & Q_{g1L_g} \\ \vdots & \vdots & \ddots & \vdots \\ Q_{gL_g0} & Q_{gL_g1} & \cdots & Q_{gL_gL_g} \end{pmatrix}.$$

Further, we decompose Q_g as

$$Q_g = \bar{Q}_g + \tilde{Q}_g, \tag{31}$$

where

$$\tilde{Q}_g \equiv \begin{pmatrix} \text{diag}\{Q_{g\ell\ell};\, \ell = 0, \ldots, L_g - 1\} & 0 \\ 0 & 0_{m_{gL_g} \times m_{gL_g}} \end{pmatrix};$$

so that

$$\bar{Q}_g = \begin{pmatrix} 0 & Q_{g01} & \cdots & Q_{g0(L_g-1)} & Q_{g0L_g} \\ Q_{g10} & 0 & \cdots & Q_{g1(L_g-1)} & Q_{g1L_g} \\ \vdots & \vdots & \ddots & \vdots & \\ Q_{g(L_g-1)0} & Q_{g(L_g-1)1} & \cdots & 0 & Q_{g(L_g-1)L_g} \\ Q_{gL_g0} & Q_{gL_g1} & \cdots & Q_{gL_g(L_g-1)} & Q_{gL_gL_g} \end{pmatrix}. \tag{32}$$

This decomposition of Q_g implies

$$S_g = \Pi_g \bar{Q}_g \Pi_g' + \Pi_g \tilde{Q}_g \Pi_g';$$

hence, for $s_g \equiv \mathrm{v}(S_g)$, we have

$$s_g = \mathrm{v}(\Pi_g \bar{Q}_g \Pi_g') + \mathrm{v}(\Pi_g \tilde{Q}_g \Pi_g') = \Xi_g(\mathrm{v}\,\bar{Q}_g + \mathrm{v}\,\tilde{Q}_g), \tag{33}$$

where $\Xi_g \equiv D^+(\Pi_g \otimes \Pi_g)D$.

Let $\bar{q}_g$ and $\tilde{q}_g$ be the vectors of unique non-zero elements of the matrices $\bar{Q}_g$ and $\tilde{Q}_g$ respectively. That is, $\mathrm{v}(\bar{Q}_g) = \bar{J}_g \bar{q}_g$ and $\mathrm{v}(\tilde{Q}_g) = \tilde{J}_g \tilde{q}_g$, where $\bar{J}_g$ and $\tilde{J}_g$ are appropriate selection matrices with elements 0 and 1. Note that

$$\bar{q}_g = \bar{J}_g^+ \mathrm{v}(\bar{Q}_g) \tag{34}$$

and

$$\tilde{q}_g = \tilde{J}_g^+ \mathrm{v}(\tilde{Q}_g). \tag{35}$$

Consequently, (33) can be expressed as

$$s_g = \bar{\Xi}_g \bar{q}_g + \tilde{\Xi}_g \tilde{q}_g, \tag{36}$$

where $\bar{\Xi}_g \equiv \Xi_g \bar{J}_g$ and $\tilde{\Xi}_g \equiv \Xi_g \tilde{J}_g$. For brevity, let $\bar{q} \equiv \mathrm{vec}\{\bar{q}_g;\ g = 1, \ldots, G\}$, $\tilde{q} \equiv \mathrm{vec}\{\tilde{q}_g;\ g = 1, \ldots, G\}$, $\bar{\Xi} \equiv \mathrm{diag}\{\bar{\Xi}_g; g = 1, \ldots, G\}$ and $\tilde{\Xi} \equiv \mathrm{diag}\{\tilde{\Xi}_g; g = 1, \ldots, G\}$, so that

$$s = \bar{\Xi}\bar{q} + \tilde{\Xi}\tilde{q}. \tag{37}$$

Now let

$$\Phi_g = \mathrm{diag}\{\Phi_{g\ell}; \ell = 1, \ldots, L_G\},$$

where the $\Phi_{g\ell}$ are the parameter matrices introduced in MS. In parallel to the decomposition of Q in (31), consider

$$\Phi_g = \bar{\Phi}_g + \tilde{\Phi}_g, \tag{38}$$

where

$$\bar{\Phi}_g = \begin{pmatrix} 0_{(m_g - m_{gL_g}) \times (m_g - m_{gL_g})} & 0 \\ 0 & \Phi_{gL_g}(\tau) \end{pmatrix}. \tag{39}$$

In analogy with (37), we have

$$\sigma = \bar{\Xi}\bar{\phi} + \tilde{\Xi}\tilde{\phi}, \tag{40}$$

where $\bar{\phi} \equiv \text{vec}\,\{\bar{\phi}_g;\, g = 1, \ldots, G\}$ and $\tilde{\phi} \equiv \text{vec}\,\{\tilde{\phi}_g;\, g = 1, \ldots, G\}$, with $\bar{\phi}_g \equiv \bar{J}_g^+ \bar{\Phi}_g$ and $\tilde{\phi}_g \equiv \tilde{J}_g^+ \tilde{\Phi}_g$. Note that owing to MS, Π_g and Φ_{gL_g} are twice continuously differentiable matrix-valued functions of τ; therefore $\bar{\Xi}$, $\tilde{\Xi}$ and $\bar{\phi}$ are also twice continuously differentiable matrix- (vector-) valued functions of τ.

Following Anderson-Amemiya approach (e.g., Anderson & Amemiya, 1988; Amemiya & Anderson, 1990), we consider the "mixture" parameter vector

$$\tilde{\vartheta} \equiv \begin{pmatrix} \tau \\ \tilde{q} \end{pmatrix}, \tag{41}$$

in which the components of τ are population values and the components of $\tilde{q}$ are sample-specific. Denote by $\tilde{\sigma} \equiv \sigma(\tilde{\vartheta})$ the vector of fitted moments evaluated at the "mixture" parameter value $\vartheta = \tilde{\vartheta}$. Due to Assumptions MLR and MS, (37) and (40) imply

$$(s - \tilde{\sigma}) = \bar{\Xi}(\bar{q} - \bar{\phi}). \tag{42}$$

Now we consider the asymptotic variance matrix of various sample moments. Let u and v be asymptotically normal (vector-valued) statistics. Denote by Γ_u the asymptotic variance matrix of $\sqrt{n}u$, and by Γ_{uv} the matrix of asymptotic covariances between $\sqrt{n}u$ and $\sqrt{n}v$. Denote by $\Gamma_u^\star$ and $\Gamma_{uv}^\star$ the analogous matrices computed under the assumption NA. Under this convention, Γ_s and $\Gamma_s^\star$ are the asymptotic variance matrices Γ and $\Gamma^\star$ of $\sqrt{n}s$ under distribution free (DF) and NA assumption, respectively. We note that under MLR and MS, Lemma 1 introduced below guarantees that $\Gamma_{\bar{q}}$ is a finite matrix.

When in addition to MLR and MS, NA holds, then the asymptotic variance matrix $\Gamma_s^\star$ of $\sqrt{n}s$ is finite and

$$\Gamma_s^\star = \bar{\Xi}\Gamma_{\bar{q}}^\star\bar{\Xi}' + \tilde{\Xi}\Gamma_{\tilde{q}}^\star\tilde{\Xi}' + \bar{\Xi}\Gamma_{\bar{q},\tilde{q}}^\star\tilde{\Xi}' + \tilde{\Xi}\Gamma_{\tilde{q},\bar{q}}^\star\bar{\Xi}', \tag{43}$$

due to (37).[3]

Now before formulating the theorem with the results on asymptotic robustness, we present two lemmas that describe the asymptotic distribution of the vectors $\bar{q}$ and $\tilde{q}$ when MLR and MS hold.

[3] In fact, in this case, $\Gamma_s^\star$ is the limit of the finite sample size variance of $\sqrt{n}s$.

LEMMA 1 *When MLR and MS hold,*

$$\sqrt{n}(\bar{q} - \bar{\phi}) = O_p(1) \tag{44}$$

and

$$(\tilde{q} - \tilde{\phi}) = o_p(1). \tag{45}$$

Proof: See Appendix □

Note that Lemma 1 relates the matrices $\bar{Q}_g$ and $\bar{\Phi}_g$ of (32) and (39) respectively. The results of Lemma 1 allow also for some degree of model misspecification, since it requires only that $\sqrt{n}(\bar{q} - \bar{\phi})$ be bounded in probability, with an asymptotic mean that may differ from zero.[4] Combining (42) and (44) we see that Lemma 1 ensures that $\sqrt{n}(s - \tilde{\sigma})$ is bounded in probability; it does not ensure, however, the convergence on distribution of of $\sqrt{n}(s - \sigma)$; the asymptotic variance matrix Γ of $\sqrt{n}s$ need not be finite.

LEMMA 2 *When MLR and MS hold,*

$$\sqrt{n}\bar{q} \xrightarrow{L} N(0, \Gamma^*_{\bar{q}}), \tag{46}$$

where $\Gamma^*_{\bar{q}}$ is the asymptotic variance matrix of $\sqrt{n}\bar{q}$ under the NA assumption.

Proof: See Appendix □

Lemma 2 stablishes that under MLR and MS, NA inferences apply to $\bar{q}$ (i.e. to the matrices $\bar{Q}_g$). The following Theorem states the major results of the paper.

THEOREM 1 *When MLR, MS and I hold,*

1. *$\hat{\vartheta}$ is a consistent estimator of ϑ.*

2. *$\sqrt{n}(s - \hat{\sigma})$ is asymptotically normally distributed, with zero mean and variance matrix determined only by τ, V and Γ^*.*

3. *$\sqrt{n}(\hat{\tau}-\tau)$ is asymptotically normally distributed with $avar(\hat{\tau}) = avar(\hat{\tau} \mid V, \Gamma^*)$ (i.e. the corresponding submatrix of (11) with $\Gamma = \Gamma^*$).*

[4] A sequence of local alternatives, however, is implied when such asymptotic mean is finite but different from zero.

4. *when $V = V^\star$, then $avar(\hat{\tau})$ is the $t \times t$ leading principal submatrix of $\Delta' V^\star \Delta$, and $\hat{\tau}$ is an efficient estimator within the class of WLS estimators of (10).*

5. *the asymptotic distribution of the goodness-of-fit test statistic $T_V^\star$ of (23) is chi-square with degrees of freedom given by (25).*

Proof: See Appendix ▫

Note that NA is not listed in the conditions of the theorem; i.e., MLR and MS are enough to guaranty that NA inferences concerning the residual vector $(s - \hat{\sigma})$, the subvector $\hat{\tau}$ of $\hat{\vartheta}$, and the goodness-of-fit test statistic $T_V^\star$, remain (asymptotically) valid despite of NA holds or not. Furthermore, under MLR and MS we attain valid inferences when Γ is replaced by Ω (an incorrect asymptotic variance matrix for $\sqrt{n}s$ even when NA is verified). As it is illustrated in the next section, the validity of using Ω for statistical inferences allows the use of standard software of covariance structure analysis (such as LISREL of Jöreskog & Sörbom (1989), or EQS of Bentler (1989)) to analyze mean- and covariance structures .

The theorem guarantees also that when MLR and MS hold, the NA standard errors of $\hat{\tau}$ coincide (asymptotically) with the AR standard errors based on the matrix $\hat{\Gamma}$ of fourth-order sample moments; the NA-WLS estimator $\hat{\tau}$ is asymptotically efficient; and $T_V^\star$ has the same asymptotic distribution as the AR test statistic T_V. [5]

Clearly, this theorem encompasses the results on asymptotic robustness that were mentioned in Section 1. In the next section we present a small simulation study that shows the performance of the asymptotic results in the case of finite samples and a specific model context. The specific model set-up of the next section will serve also to illustrate details to be taken into account when applying the results of Theorem 1.

[5] The results of the Theorem apply also to the case of a Fisher-consistent estimate $\hat{\vartheta}$ of ϑ that is asymptotically equal to $g(s)$, where $g(.)$ is a smooth function. Obviously, the results of the Theorem applies also in the case where the matrices S_g and Σ_g are replaced by the sample and population variance matrices.

4 Regression with error in variables

This section presents a small Monte Carlo study where it is investigated the performance in finite samples of NA-WLS inferences in the context of a (multi-sample) regression with error in variables, and data deviating from the normality assumption. We distinguish between de structural model, where the true regressor varies randomly across sample replications, and the functional model, where the true regressor is a set of values that are fixed across sample replications. Regression with errors in variables is widely discussed by Fuller (1987).

We consider the following regression model with errors in variables:

$$\begin{cases} Y = \alpha + \beta x + \zeta \\ X_1 = x + \epsilon_1 \\ X_2 = x + \epsilon_2 \end{cases} \qquad (47)$$

Two-sample data is considered with the variable X_2 missing in the second subsample. This is a regression model in which Y is the dependent variable and X_1 and X_2 are two indicators of the "true" regressor x. We distinguish between the cases of x assumed "fixed" across replications (the functional model) or considered to be random (the structural model). Note that this model could serve in an applicaton of a regression with error in variables where two replicates X_1 and X_2 are available for a subsample of cases (group $g = 1$), while only one replicate is available in the rest of the sample (group $g = 2$).

We use the estimation method NA-WLS described in Section 3. In the illustration, we are concerned with correct asymptotic inferences when no distributional assumptions are imposed on the true regressor x, or on the disturbance regression term ζ. However, normality will be assumed for the distribution of the measurement error variables ϵ_1 and ϵ_2. [6] The measurement error variances and the parameters α and β are set equal across groups. The second-order moments of the random constituents of the model are assumed to be finite. In the functional model case, we need to assume also that the uncentered second order moments of x converge as $n \to +\infty$ (Condition 1 in MLR).

[6] Note that this assumption could be tested by exploring the distribution of $X_1 - X_2$ in the subsample where both X_1 and X_2 are measured .

4.1 The Functional Model

The equations describing the two-sample model are a measurement model for the first group

$$z_{1,i} \equiv \begin{pmatrix} 1 \\ X_{1,1i} \\ X_{1,2i} \\ Y_{1,i} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ x_{1,i} \\ Y_{1,i} \end{pmatrix} + \begin{pmatrix} 0 \\ \epsilon_{1,1i} \\ \epsilon_{1,2i} \\ 0 \end{pmatrix}, \quad i = 1, \ldots, n_1, \quad (48)$$

a measurement model for the second group

$$z_{2,i} \equiv \begin{pmatrix} 1 \\ X_{2,1i} \\ Y_{2,i} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ x_{2,i} \\ Y_{2,i} \end{pmatrix} + \begin{pmatrix} 0 \\ \epsilon_{2,1i} \\ 0 \end{pmatrix}, \quad i = 1, \ldots, n_2, \quad (49)$$

and a structural equation that is common to both groups

$$\begin{pmatrix} 1 \\ x_{g,i} \\ Y_{g,i} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \alpha & \beta & 0 \end{pmatrix} \begin{pmatrix} 1 \\ x_{g,i} \\ Y_{g,i} \end{pmatrix} + \begin{pmatrix} 1 \\ x_{g,i} \\ \zeta_{g,i} \end{pmatrix}, \quad i = 1, \ldots, n_g, \quad g = 1, 2. \quad (50)$$

Clearly, assumptions MLR of (27) hold with

$$\Pi_1 = \left[\Lambda_1 (I - B)^{-1}, \begin{bmatrix} 0_{2\times 2} \\ I_{2\times 2} \end{bmatrix} \right], \quad (51)$$

$$\Pi_2 = \left[\Lambda_2 (I - B)^{-1}, \begin{bmatrix} 0_{2\times 2} \\ I_{1\times 2} \end{bmatrix} \right], \quad (52)$$

$$\Lambda_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \Lambda_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \alpha & \beta & 0 \end{pmatrix},$$

$$\xi_{1,i} = (1, x_{1,i}, \zeta_{1,i}, \epsilon_{1,1i}, \epsilon_{1,2i})'$$

and

$$\xi_{2,i} = (1, x_{2,i}, \zeta_{2,i}, \epsilon_{2,1i})'.$$

Thus we have the correspondence $\xi_{g,0i} = (1, x_{g,i})$, $\xi_{g,1i} = (\zeta_{g,i})'$, $\xi_{1,2i} = (\epsilon_{1,1i}, \epsilon_{1,2i})'$ and $\xi_{2,2i} = (\epsilon_{2,1i})'$, for $g = 1, 2$, Note also that assumption MS holds with the matrices Π_g of (51) and (52) being functions of β and α, and

$$\Phi_g = \text{diag}\{\Phi_{g,00}, \Phi_{g,11}, \Phi_{g,22}\},$$

where

$$\Phi_{g,00} = \begin{pmatrix} \nu_g & \phi_{g,01} \\ \phi_{g,01} & \phi_{g,00} \end{pmatrix}, \quad \Phi_{g,11} = \begin{pmatrix} \phi_{g,11} \end{pmatrix},$$

$$\Phi_{1,22} = \begin{pmatrix} \psi_{11} & 0 \\ 0 & \psi_{22} \end{pmatrix}, \quad \Phi_{2,22} = \begin{pmatrix} \psi_{11} \end{pmatrix}.$$

for $g = 1, 2$, where ψ_{11} is the common variance of $\epsilon_{1,1}$ and $\epsilon_{2,1}$, ψ_{22} is the variance of $\epsilon_{2,1}$, and $\phi_{g,01}$ and $\phi_{g,00}$ are respectively the first and second-order moments of the unobservable variable x_g, $g = 1, 2$. We thus have the 12-dimensional parameter vector $\vartheta = (\tau', \phi')'$ with

$$\tau = (\alpha, \beta, \psi_{11}, \psi_{22})'$$

and

$$\phi = (\nu_1, \phi_{1,01}, \phi_{1,00}, \phi_{1,11}, \nu_2, \phi_{2,01}, \phi_{2,00}, \phi_{2,11})'.$$

4.2 The Structural Model

Here, x_g is a random constituent of the model with mean μ_x, set equal across groups in the present example. We have the measurement equations of (48) and (49), and the structural equation common to both groups,

$$\begin{pmatrix} 1 \\ x_{g,i} \\ Y_{g,i} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ \mu_x & 0 & 0 \\ \alpha & 0 & \beta \end{pmatrix} \begin{pmatrix} 1 \\ x_{g,i} \\ Y_{g,i} \end{pmatrix} + \begin{pmatrix} 1 \\ x_{g,i} - \mu_g \\ \zeta_{g,i} \end{pmatrix}, \quad g = 1, 2. \quad (53)$$

Note that now a new parameter, the mean μ_x of x, has been introduced. Here also MLR is satisfied with the same Π_g as in (51) and (52) and with the appropriate substitution of the matrix B. Further,

$$\xi_{1,i} = (1, \zeta_{1,i}, x_{1,i} - \mu_x, \epsilon_{1,1i}, \epsilon_{1,2i})$$

and

$$\xi_{2,i} = (1, \zeta_{2,i}, x_{2,i} - \mu_x, \epsilon_{2,1i})$$

We have the correspondence $\xi_{g,0i} = 1$, $\xi_{g,1i} = \zeta_{g,i}$, $\xi_{g,2i} = (x_{1,i} - \mu)$, $\xi_{1,3i} = (\epsilon_{1,1i}, \epsilon_{1,2i})'$, for $g = 1, 2$, and $\xi_{2,3i} = \epsilon_{2,1i}$. The matrix Φ in MS is

$$\Phi_g = \operatorname{diag}\{\nu_g, \Phi_{g,1}, \Phi_{g,2}, \Phi_{g,3}(\tau)\}$$

where $\Phi_{g,1} = (\phi_{g,11})$ corresponds to the variance of ζ_g, $\Phi_{g,2} = (\phi_{g,2})$ corresponds to the variance of x_g, $\Phi_{1,3} =$ diag $\{\psi_{11}, \psi_{22}\}$ and $\Phi_{2,3} = \left(\psi_{11} \right)$; where ψ_{11} is the common variance of $\epsilon_{1,1}$ and $\epsilon_{2,1}$ and ψ_{22} is the variance of $\epsilon_{2,1}$. Here $\vartheta = (\tau', \phi')'$ has 11 components,

$$\tau = (\alpha, \beta, \mu_x, \psi_{11}, \psi_{22})'$$

and

$$\phi = (\nu_1, \phi_{1,11}, \phi_{1,22}, \nu_2, \phi_{2,11}, \phi_{2,22})'.$$

For both functional and structural models, ν_g is a "pseudo" parameter that is unrestricted across groups and that has a population value of 1. Note that in the case of the functional model it made no sense to restrict the means of x_g to be equal across groups.

The total number of distinct moments is 16 $(= \frac{4\times5}{2} + \frac{3\times4}{2})$. Hence, the number of degrees of freedom of the goodness-of-fit test statistic T^{**} is 4 $(= 16 - 12)$ in the case of the functional model, and 5 $(= 16 - 11)$ in the case of the structural model.

The Monte Carlo study generated 1000 sets of two-sample data according to the data generating process described below and the NA-WLS analysis of each two-sample data. A summary of the Monte Carlo results is presented in Tables 1 (the functional model) and 2 (the structural model).

Non-normal data was simulated for the x's and the ζ's as independent draws from the chi-square distribution with 1 df, conveniently scaled. The $\{\epsilon_{g,1i}; i = 1, \ldots, n_g\}$ and $\{\epsilon_{g,2i}; i = 1, \ldots, n_g\}$, $g = 1, 2$, were generated as iid draws from independent normal variables. Sample size was $n_1 = 800$ and $n_2 = 400$ in the structural model example, and $n_1 = 2800$ and $n_2 = 2200$ in the functional model example . The number of replications was set to 1000. In the functional model, a fixed set of values of x were used across the 1000 replications.

Given the described data generating process, conditions MLR and MS for Theorem 1 are clearly verified, for both the functional and structural models. We thus expect asymptotic correctness of the NA standard errors for the τ parameters (i.e., intercept and slope) and variances of $\epsilon_{g,1}$ and $\epsilon_{g,2}$. By Theorem 1, the goodness-of-fit test T^{**} is also expected to be asymptotically chi-square distributed. Note that this asymptotic correctness holds despite non-normality of ζ_g (disturbance term) and x_g ("true" values of the regressor), and the consequent non-normality of the observed variables. Correctness of the NA standard errors for estimates of the ϕ type of parameters (variances and covariances of non-normal constituents of the model) is not guaranteed. Such expectations of robustness are corroborated with the results shown by Tables 1 and 2.

Comparison of the entries in column $E(\hat{\vartheta})$ with the true values demonstrates the consistency of parameter estimates. Comparison of NA standard errors (column $E(se)$) with the empirical standard errors in column $sd(\hat{\vartheta})$ shows the consistency of standard errors for all parameters except for the variances of ζ_g in the functional model, and variances of ζ_g and x_g in the structural model. This consistency of standard errors for certain parameter estimates is corroborated by inspecting the deviations from 1 in column $\frac{E(se)}{sd(\hat{\vartheta})}$. The inspection of the columns giving empirical tails of a z-statistic shows also asymptotic normality and correctness of standard errors for the estimates of τ parameters. The empirical distribution of the goodness-of-fit test shows also a reasonably accurate fit to a chi-square distribution with the corresponding df. [7]

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[7] The described WLS analysis can be carried out using standard software like LISREL or EQS. The "input" for a LISREL run on both type of models (functional and structural) is available from the author upon request.

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5 Appendix

In this Appendix we prove Lemmas 1 and 2, two additional Lemmas, and the Theórem 1.

5.1 Proof of Lemma 1

Each component of $\bar{q}$ is of the form $\bar{q}_\alpha = \frac{1}{n_g}\sum_{i=1}^{n_g} u_i v_i$. We will prove (44) by showing that such a $\bar{q}_\alpha$ is asymptotically normally distributed. Hence the whole vector $\sqrt{n}\bar{q}$ is bounded in probability. Note that MLR and MS imply at least one of the following conditions:

1. $\{u_i\}$ and $\{v_i\}$ are mutually independent iid sequences.
2. $\{u_i\}$ are fixed values such that $\lim_{n_g \to +\infty} n_g^{-1} \sum_{i=1}^{n_g} u_i^2 = \phi_u$, a finite value and $\{v_i\}$ are iid.
3. $\{u_i\}$ and $\{v_i\}$ are iid normally distributed, and mutually independent (not necessarely identically distributed).

In all the conditions above, whenever u_i (v_i) is random, it is of zero mean and finite variance ϕ_u (ϕ_v).

In case 1, we have that $\mathcal{E}(u_i v_i)^2 = \mathcal{E}(u_i^2)\mathcal{E}(v_i^2)$, hence applying the standard version of the central limit theorem (iid terms with finite variance), we obtain the asymptotic normality of $\sqrt{n_g}\bar{q}_\alpha$. The same standard version of the central limit theorem will be used in case 3.

In case 2, the asymptotic normality of $\sqrt{n_g}\bar{q}_\alpha$ is attained using the Lindeberg-Feller version of the central limit theorem to the sum $\frac{1}{\sqrt{n_g}}\sum_{i=1}^{n_g} u_i v_i$ (e.g., Serfling, 1980). This ends the proof of the result (44) of Lemma 1.

The result (45) claimed by Lemma 1 follows as a direct consequence of applying the standard law of large numbers (e.g., Serfling, 1980) to the sums $\frac{1}{n_g}\sum_{i=1}^{n_g} u_i v_i$. □

5.2 Proof of Lemma 2

Note first that $\bar{q}$ collects the unique non-zero element of $\bar{Q}_g$, $g = 1, \ldots, G$. Given the structure of $\bar{Q}_g$ displayed in (32) (the diagonal blocks $Q_{g,\ell\ell}$, except Q_{g,L_gL_g}, are excluded from $\bar{Q}_g$), (30), and the conditions 1-3 of MLR, each component of the vector $\bar{q}$ has the form ${n_g}^{-1}\sum_{i=1}^{n_g} u_i v_i$, where the u_i and v_i satisfy at least one of the conditions 1 to 3 listed in the proof of Lemma 1.

Let $\bar{q}_\alpha = n_g^{-1}\sum_{i=1}^{n_g} u_i v_i$ and $\bar{q}_\beta = n_g^{-1}\sum_{i=1}^{n_g} w_i h_i$, with identities possibly among the u_i, v_i, w_i, h_i. Note that the elements of the matrix $\Gamma_{\bar{q}}$ of Lemma 2 are the covariances $(\Gamma_{\bar{q}_\alpha,\bar{q}_\beta})$ among the $\bar{q}_\alpha$ and $\bar{q}_\beta$. We have

$$\begin{aligned}\Gamma_{\bar{q}_\alpha,\bar{q}_\beta} &= \lim_{n\to+\infty} \pi_g^{-1} \operatorname{cov}\left(\frac{1}{\sqrt{n_g}}\sum_{i=1}^{n_g} u_i v_i, \frac{1}{\sqrt{n_g}}\sum_{j=1}^{n_g} w_j h_j\right) \\ &= \lim_{n\to+\infty} \pi_g^{-1} \operatorname{cov}\left(\frac{1}{\sqrt{n_g}}\sum_{i=1}^{n_g} u_i v_i, \frac{1}{\sqrt{n_g}}\sum_{i=1}^{n_g} w_j h_j\right) \\ &= \lim_{n\to+\infty} \pi_g^{-1}\frac{1}{n_g}\sum_{i=1}^{n_g} \operatorname{cov}(u_i v_i, w_i r_i)\,.\end{aligned}$$

When $\bar{q}_\alpha$ and $\bar{q}_\alpha$ correspond to different groups then, clearly, $\Gamma_{\bar{q}_\alpha,\bar{q}_\beta} = 0$, the same value as under NA. Further, when all the variables u_i, v_i, w_i and h_i are elements of ξ_{g,L_g} (and therefore are normally distributed), then obviously $\Gamma_{\bar{q}_\alpha,\bar{q}_\beta}$ is the same as under NA.

We are left with the case where at least one of the variables u_i, v_i, w_i, h_i is not a component of ξ_{g,L_g}. Clearly, we have

$$\operatorname{cov}(u_i v_i, w_i h_i) = \mathcal{E}(u_i v_i w_i h_i)$$

since either $\mathcal{E}(u_i v_i)$ or $\mathcal{E}(w_i h_i)$ are zero. Further

$$\mathcal{E}(u_i v_i w_i h_i) = \mathcal{E}(u_i w_i)\mathcal{E}(v_i h_i),$$

since when u_i, v_i, w_i, h_i are all random and not all of them elements of ξ_{g,L_g}, then either (u_i, w_i) is statistically independent of (v_i, h_i), or $\bar{q}_\alpha$ or $\bar{q}_\beta$ is an element of $Q_{g,L_g L_g}$.

Consequently,

$$\Gamma_{\bar{q}_\alpha,\bar{q}_\beta} = \lim \pi_g^{-1}\frac{1}{n_g}\sum_{i=1}^{n_g}\mathcal{E}(u_i w_i)\mathcal{E}(v_i h_i) = \pi_g^{-1}\phi_{uw}\phi_{vh}$$

where $\phi_{uw} = \mathcal{E}(u_i w_i)$ when u_i and w_i are random, and $\phi_{uw} = \lim_{n_g\to+\infty} n_g^{-1}\sum_{i=1}^{n_g} u_i w_i$ when the u_i and w_i are sequences of fixed values. Thus, only the elements of the cross-product matrices $\Phi_{g,\ell}$ are involved in the asymptotic covariances $\Gamma_{\bar{q}_\alpha,\bar{q}_\beta}$, i.e. in this case also $\Gamma_{\bar{q}_\alpha,\bar{q}_\beta}$ is the same as under NA. □

5.3 Proof of Theorem 1

The vector s converges in probability to σ, the vector of population moments, as $n \longrightarrow +\infty$. Since $F(s, \sigma(\vartheta)) \equiv (s - \sigma(\vartheta))'V(s - \sigma(\vartheta))$ is a continuous function of ϑ, the estimator $\hat{\vartheta}$ converges in probability to the population value ϑ. Since the mixture vector $\tilde{\vartheta}$ also converges in probability to the true value ϑ, we get that $\tilde{\vartheta} - \hat{\vartheta} = o_p(1)$.

Under assumption MS and I, the familiar implicit function theorem (e.g., Dijkstra, 1983) states that

$$\sqrt{n}(\hat{\vartheta} - \tilde{\vartheta}) = (\tilde{\Delta}'V\tilde{\Delta})^{-1}\tilde{\Delta}'V\sqrt{n}(s - \tilde{\sigma}) + o_p(1), \tag{54}$$

where $\tilde{\Delta} = \frac{\partial \sigma(\vartheta)'}{\partial \vartheta} \mid_{\vartheta = \tilde{\vartheta}}$. Thus, using (42), we have

$$\sqrt{n}(\hat{\vartheta} - \tilde{\vartheta}) = (\tilde{\Delta}'V\tilde{\Delta})^{-1}\tilde{\Delta}'V\bar{\Xi}\sqrt{n}(\bar{q} - \bar{\phi}) + o_p(1). \tag{55}$$

Consequently, since $\sqrt{n}(\hat{\tau} - \tau) = J\sqrt{n}(\hat{\vartheta} - \bar{\vartheta})$, where J is a 0-1 selection matrix, we obtain

$$\sqrt{n}(\hat{\tau} - \tau) = \bar{W}\sqrt{n}(\bar{q} - \bar{\phi}) + o_p(1), \tag{56}$$

where $\bar{W} \equiv J(\Delta'V\Delta)^{-1}\Delta'V\bar{\Xi}$. The fact that $\tilde{\Delta} - \Delta = o_p(1)$ and (44) were used to justify the replacement of $\tilde{\Delta}$ by Δ. [8]

The result (56), combined with (44) and the fact that $\bar{\Xi}$ and Δ are functions of ϑ only, guarantees that $\sqrt{n}(\hat{\tau} - \tau)$ converges to a distribution that is determined by ϑ, V and the asymptotic distribution of $\bar{q}$.

To proceed with the proof of the theorem, we need two additional Lemmas: Lemma 3 which proves that under MLR and MS the residual vector $\sqrt{n}(s - \hat{\sigma})$ is bounded in probability as $n \longrightarrow +\infty$; and Lemma 4 which show that the asymptotic distribution of the residual vector is determined by the asymptotic distribution of $\bar{q}$ (recall the asymptotic distribution of $\bar{q}$ stated in Lemma 2).

LEMMA 3 *When MLR and MS hold,*

$$\sqrt{n}(s - \hat{\sigma}) = O_p(1). \tag{57}$$

[8] Note that we can replace $\tilde{\Delta}$ and $\hat{\Delta}$ by Δ when the corresponding multiplying factor is bounded in probability. This boundedness condition is crucial in the present paper, since we do not assume, for instance, that $\sqrt{n}(\bar{q} - \tilde{\phi})$ is bounded in probability.

PROOF Given the definition of $\hat{\theta}$, we have

$$\hat{\Delta}'\hat{V}(s - \hat{\sigma}) = 0;$$

hence

$$\sqrt{n}(s - \hat{\sigma}) = \hat{P}\sqrt{n}(s - \hat{\sigma}), \tag{58}$$

where

$$\hat{P} = I - \hat{\Delta}(\hat{\Delta}'\hat{V}\hat{\Delta})^{-1}\hat{\Delta}'V$$

is an idempotent matrix orthogonal to $\hat{\Delta}$ ($\hat{P}\hat{\Delta} = 0$) .

Given MLR and MS, identities (37) and (40) are satisfied, and since $\hat{P}\hat{\bar{\Xi}} = 0$, we have

$$\begin{aligned} \sqrt{n}\hat{P}(s - \hat{\sigma}) &= \sqrt{n}\hat{P}(\bar{\Xi}\bar{q} + \tilde{\Xi}\tilde{q} - \hat{\bar{\Xi}}\hat{\bar{\phi}} - \hat{\tilde{\Xi}}\hat{\tilde{\phi}}) &(59)\\ &= \sqrt{n}(\hat{P}\bar{\Xi}\bar{q} + \hat{P}\tilde{\Xi}\tilde{q} - \hat{P}\hat{\bar{\Xi}}\hat{\bar{\phi}}). &(60)\end{aligned}$$

Now, given that $\sqrt{n}(\hat{\tau} - \tau) = O_p(1)$, and that $\tilde{\Xi}$ is a continuously differentiable matrix-valued function of τ, we have

$$\sqrt{n}\hat{P}\tilde{\Xi} = \sqrt{n}\hat{P}\hat{\tilde{\Xi}} + O_p(1) = O_p(1).$$

Consequently, replacing $\hat{\bar{\Xi}}$ by $\bar{\Xi}$, [9] we get

$$\begin{aligned} \hat{P}\sqrt{n}(s - \hat{\sigma}) &= \sqrt{n}\hat{P}\bar{\Xi}\bar{q} - \sqrt{n}\hat{P}\hat{\bar{\Xi}}\hat{\bar{\phi}} + O_p(1)\\ &= \hat{P}\bar{\Xi}\sqrt{n}(\hat{\bar{\phi}} - \bar{q}) + O_p(1) = O_p(1), \end{aligned}$$

which combined with (58) concludes the proof of the Lemma. ∎

LEMMA 4 *When MLR and MS hold,*

$$\hat{P}\sqrt{n}(s - \hat{\sigma}) = P\bar{\Xi}\sqrt{n}(\bar{q} - \bar{\phi}) + o_p(1). \tag{61}$$

PROOF We have

$$(s - \hat{\sigma}) = (s - \tilde{\sigma}) - (\hat{\sigma} - \hat{\tilde{\sigma}}) - (\hat{\tilde{\sigma}} - \tilde{\sigma}),$$

[9] To justify the replacement of $\hat{\bar{\Xi}}$ by $\bar{\Xi}$ and to guarantee that $\sqrt{n}(\hat{\bar{\phi}} - \bar{q})$ is bounded in probability, the result $\sqrt{n}(\hat{\tau} - \tau) = O_p(1)$ was used

where $\hat{\tilde{\sigma}} = \sigma(\hat{\tilde{\vartheta}})$ and $\hat{\tilde{\vartheta}} = (\hat{\tau}', \tilde{q}')'$. Denote by $\hat{\tilde{\phi}}$ the estimator of $\tilde{\phi}$ determined by $\hat{\tau}$. Then, using (40),

$$(\hat{\sigma} - \hat{\tilde{\sigma}}) = \hat{\tilde{\Xi}}(\hat{\tilde{\phi}} - \tilde{q}).$$

Since, $\hat{P}'\hat{\tilde{\Xi}} = 0$, we have

$$\hat{P}\sqrt{n}(s - \hat{\sigma}) = \hat{P}\sqrt{n}(s - \tilde{\sigma}) - \hat{P}\sqrt{n}(\hat{\tilde{\sigma}} - \tilde{\sigma}). \quad (62)$$

Note that

$$\hat{\tilde{\sigma}} = \tilde{\sigma} + \tilde{\Delta}(\hat{\tilde{\vartheta}} - \tilde{\vartheta}) + o_p(\hat{\tilde{\vartheta}} - \tilde{\vartheta})$$

implies

$$\sqrt{n}(\hat{\tilde{\sigma}} - \tilde{\sigma}) = \tilde{\Delta}\sqrt{n}(\hat{\tilde{\vartheta}} - \tilde{\vartheta}) + o_p(1),$$

where the result $\sqrt{n}(\hat{\tilde{\vartheta}} - \tilde{\vartheta}) = O_p(1)$, implied by $\sqrt{n}(\hat{\tau} - \tau) = O_p(1)$, was used. We can thus write (62) as

$$\begin{aligned} \hat{P}\sqrt{n}(s - \hat{\sigma}) &= \hat{P}\sqrt{n}(s - \tilde{\sigma}) - \hat{P}\tilde{\Delta}\sqrt{n}(\hat{\tilde{\vartheta}} - \tilde{\vartheta}) + o_p(1) \\ &= \hat{P}\sqrt{n}(s - \tilde{\sigma}) - \hat{P}\hat{\Delta}\sqrt{n}(\hat{\tilde{\vartheta}} - \tilde{\vartheta}) + o_p(1). \end{aligned}$$

Now since

$$\hat{P}\tilde{\Delta}\sqrt{n}(\hat{\tilde{\vartheta}} - \tilde{\vartheta}) = \hat{P}\hat{\Delta}\sqrt{n}(\hat{\tilde{\vartheta}} - \tilde{\vartheta}),$$

(we used again that $\sqrt{n}(\hat{\tilde{\vartheta}} - \tilde{\vartheta})$ is bounded in probability) and (57), we obtain

$$\hat{P}\sqrt{n}(s - \hat{\sigma}) = \hat{P}\sqrt{n}(s - \tilde{\sigma}) + o_p(1). \quad (63)$$

Combining (63) with (42), we obtain

$$\begin{aligned} \hat{P}\sqrt{n}(s - \hat{\sigma}) &= \hat{P}\bar{\Xi}\sqrt{n}(\bar{q} - \bar{\phi}) + o_p(1) \\ &= P\bar{\Xi}\sqrt{n}(\bar{q} - \bar{\phi}) + o_p(1), \end{aligned}$$

since $\sqrt{n}(\bar{q} - \bar{\phi})$ is bounded in probability, which completes the proof of the Lemma. □

Lemmas 3 and 4 allow us to proceed with the proof of Theorem 1. Combining the result (61), equality (58), and Lemma 2, we deduce the claim of the theorem with regard to the asymptotic distribution of the residual vector $\sqrt{n}(s - \hat{\sigma})$. Now we will concentrate on the statments of Theorem 1 concerning the asymptotic distribution of $\hat{\tau}$.

From (40) we obtain

$$\Delta = \left[\frac{\partial(\bar{\Xi}\bar{\phi})}{\partial\tau'} + (\bar{\psi}' \otimes I)\frac{\partial \mathrm{vec}\bar{\Xi}}{\partial\tau'}, \bar{\Xi}\right];$$

thus, due to (45),

$$\tilde{\Delta} - \Delta = \left[((\tilde{q} - \bar{\phi}') \otimes I)\frac{\partial \mathrm{vec}\bar{\Xi}}{\partial\tau'}, 0\right] = o_p(1).$$

Note also that, from (9), there is a reordering of the parameters such that the matrix Δ partitions as $\Delta = [\Delta_a, \Delta_0]$, where Δ_0 is the same matrix as in $\Gamma^\star$ of (18).

The delta-method applied to (56) and Lemma 2, gives the following expression for the asymptotic variance matrix of $\hat{\tau}$:

$$\Gamma_{\hat{\tau}} = \bar{W}\Gamma_{\bar{q}}\bar{W}' = \bar{W}\Gamma^\star_{\bar{q}}W' =$$

$$J(\Delta'V\Delta)^{-1}\Delta'V\Gamma^\star_s V\Delta(\Delta'V\Delta)^{-1}J', \tag{64}$$

where (43) and $\tilde{W} = J(\Delta'V\Delta)^{-1}\Delta'V\bar{\Xi} = 0$ where used. Finally, given $\Delta = [\Delta_a, \Delta_0]$ and (18), it holds that

$$\Gamma_{\hat{\tau}} = J(\Delta'V\Delta)^{-1}\Delta'V\Omega V\Delta(\Delta'V\Delta)^{-1}J',$$

which is the asymptotic variance matrix of $\hat{\tau}$ claimed in 3 of Theorem 1.

Using the well-known result that, for any non-negative matrix V, [10]

$$(\Delta'V\Delta)^{-1}\Delta'V\Omega V\Delta(\Delta'V\Delta)^{-1} \geq (\Delta'\Omega^{-1}\Delta)^{-1},$$

we obtain

$$\Gamma_{\hat{\tau}} \geq J(\Delta'\Omega^{-1}\Delta)^{-1}J'. \tag{65}$$

The statement of asymptotic efficiency claimed in 4 of Theorem 1 is obtained directly from (65) by recalling that equality is attained when $V = \Omega^{-1}$ ($V^\star$).

[10]Here the notation $A \geq B$, with A and B matrices, is used to denote that $A - B$ is a positive semidefinite matrix.

This concludes the proof of the results concerning parameter estimates. [11]

Now we consider the results concerning the asymptotic distribution of T_V^*. First note that the asymptotic distribution of $\hat{P}\sqrt{n}(s - \hat{\sigma})$ is the same as the asymptotic distribution of $P\sqrt{n}(s - \hat{\sigma})$, since $\sqrt{n}(s - \hat{\sigma})$ is bounded in probability, and $\hat{\vartheta}$ is consistent for ϑ. Lemma 4 implies that $P\sqrt{n}(s - \hat{\sigma})$ is asymptotically normally distributed, with asymptotic variance matrix $P\bar{\Xi}\Gamma_{\bar{q}}\bar{\Xi}'P'$. Under MS and MLR,

$$P\bar{\Xi}\Gamma_{\bar{q}}\bar{\Xi}'P' = P\bar{\Xi}\Gamma_{\bar{q}}^*\bar{\Xi}'P' = P\Gamma_s^*P' = P\Omega P',$$

where (43), (18) and $P\Delta_0 = 0$ where used. Note that $\hat{P}\hat{\Omega}\hat{P}'$ is a consistent estimator of $P\Omega P'$; thus, the Wald test statistic (see, e.g., Moore, 1974, for details on constructing Wald type test statistics)

$$T_V^* = n(s - \hat{\sigma})'\hat{P}'(\hat{P}\hat{\Omega}\hat{P}')^+\hat{P}(s - \hat{\sigma}) \stackrel{a}{=} n(s - \hat{\sigma})'(P\Omega P')^+(s - \hat{\sigma})$$

has the asymptotic distribution claimed in 5 of Theorem 1. [12] □

[11]These developments apply also to the case of a Fisher-consistent estimate $\hat{\vartheta} = g(s)$ of ϑ, where $g(.)$ is a smooth function. Effectively, assumption MS allows us to write

$$\sqrt{n}(\hat{\vartheta} - \tilde{\vartheta}) = \Delta\sqrt{n}(s - \bar{\sigma}) + o_p(1),$$

and, consequently,

$$\sqrt{n}(\hat{\tau} - \tau) = J^*\sqrt{n}(\hat{\vartheta} - \tilde{\vartheta}) = J^*\Delta\sqrt{n}(s - \bar{\sigma}) + o_p(1),$$

which implies the Lemma 4 directly.

[12]Note that to prove Theorem 1 it was not needed that Γ_s is a finite matrix.

Table 1: Results of the simulation study. Structural model: $Y = \alpha + \beta x + \zeta$, $X_1 = x + \epsilon_1$, $X_2 = x + \epsilon_2$. Two subsamples of sizes $n_1 = 800$ and $n_2 = 400$ with the equation corresponding to X_2 missing in the second subsample. Distribution of x and ζ are independent chi-square of 1 df. Distribution of ϵ_1 and ϵ_2 is normal. Number of replications is 1000.

	Estimates and standard errors					
Parameters	$E(\hat{\vartheta})^a$	$sd(\hat{\vartheta})^b$	$E(se)^c$	$\frac{E(se)}{sd(\hat{\vartheta})}$	5% [d]	10%
$Var(e_1)$= 1	1.00	0.15	0.09	0.63	22.40	30.20
$Var(x_1)$= 1	0.99	0.14	0.06	0.42	43.90	52.00
$Var(u1)$ =0.3	0.30	0.02	0.02	1.04	3.90	10.00
$Var(u2)$ =0.4	0.40	0.03	0.03	0.96	6.90	11.90
β =2	2.00	0.05	0.05	0.98	5.10	9.30
α =1	0.99	0.16	0.16	0.99	4.90	10.40
μ =3	3.00	0.03	0.03	0.97	6.40	10.80
$Var(e_2)$=1	0.98	0.23	0.16	0.68	17.80	26.20
$Var(x_2)$= 1	0.99	0.19	0.08	0.45	38.10	45.90
		Goodness-of-fit test, T^{**} (df=5)				
		Mean	Var	1%	5%	10%
		5.13	10.49	1.10	4.90	9.80

[a] empirical mean across the 1000 replications

[b] standard deviation across 1000 replications

[c] empirical mean of the NA se

[d] 5% and 10% (nominal) two-sided tails for z-statistic $\frac{\hat{\vartheta}-\vartheta}{se(\hat{\vartheta})}$

Table 2: Results of the simulation study. Functional model: $Y = \alpha + \beta x + \zeta$, $X_1 = x + \epsilon_1$, $X_2 = x + \epsilon_2$. Two subsamples of sizes $n_1 = 2800$ and $n_2 = 2200$, with the equation corresponding to X_2 missing in the second subsample. Distribution of the disturbance term ϵ is chi-square of 1 df. The values of the x's are fixed across replications (and were generated as independent values from a uniform distribution). Distribution of ϵ_1 and ϵ_2 is normal. Number of replications is 500.

	Estimates and standard errors					
Parameters	$\mathrm{E}(\hat{\vartheta})^a$	$\mathrm{sd}(\hat{\vartheta})^b$	$\mathrm{E(se)}^c$	$\frac{E(se)}{sd(\hat{\vartheta})}$	5% [d]	10%
$\mathrm{Var}(e_1) = 1$	1.00	0.08	0.05	2.66	24.60	31.60
$\mathrm{Var}(u_1) = 0.3$	0.30	0.01	0.01	1.01	6.00	11.20
$\mathrm{Var}(u_2) = 0.4$	0.40	0.02	0.01	1.08	5.40	10.80
$\beta = 2$	2.00	0.01	0.01	1.03	6.40	10.60
$\alpha = 1$	1.00	0.02	0.02	1.01	5.80	11.60
$\mathrm{Var}(e_2) = 1$	1.00	0.10	0.08	1.56	10.60	17.60
	Goodness-of-fit test, T^{**} (df=4)					
	Mean	Var	5%	10%	20%	
	4.02	8.14	4.8	10.6	21.4	

[a] empirical mean across the 1000 replications

[b] standard deviation across 1000 replications

[c] empirical mean of the NA se

[d] 5% and 10% (nominal) two-sided tails for z-statistic $\frac{\hat{\vartheta}-\vartheta}{se(\hat{\vartheta})}$

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