Economics Working Paper 121

Arbitrage, Bubbles, and Valuation*

Jan Werner
University of Minnesota
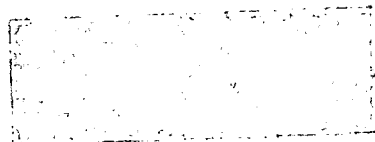
April 1995

Abstract

One of the important implications of the condition of the absence of arbitrage in asset markets is the "valuation principle". It asserts the existence of a strictly positive continuous linear operator assigning to the payoff of an asset, or the payoff of a portfolio of assets, its price. The operator extends valuation to claims which need not be payoffs of portfolios. It has various representations including martingale representation, and the present value pricing rule in terms of state prices.

We examine the validity of the valuation principle in infinite asset markets. We consider an example of an economy with infinitely many states of nature and infinitely many assets. Assets are Arrow securities for every state, and a riskless security. The price of the riskless security is not equal to the infinite sum of prices of Arrow securities. This apparent "mispricing" indicates the failure of the present value pricing rule. It does not, however, lead to an arbitrage opportunity. Moreover, portfolio demand is well-defined for a risk neutral investor, and for some risk averse investor. Although the valuation in terms of the present value pricing rule does not hold, we argue that there is a valuation relationship involving a pricing bubble for the riskless security.

Our example of infinite asset markets can be given a different interpretation of a model of term structure of interest rates. In this interpretation, there is a pricing bubble for a perpetuity.

1. Introduction

An arbitrage portfolio in asset markets is a portfolio that guarantees non-negative payoff in every possible future state of nature, a positive payoff in some state, and has zero price. The condition of the absence of arbitrage portfolio plays a fundamental role in the theory of financial markets. One of its implications in asset markets with finitely many assets is the "valuation principle". It asserts the existence of a valuation operator which is a strictly positive, continuous, linear operator assigning to the payoff of an asset, or the payoff of a portfolio of assets, its price. The operator extends valuation to contingent claims which need not be payoffs of portfolios. It has various equivalent representations including the martingale representation, and the present value pricing rule in terms of state prices.

Back and Pliska (1991) (following earlier work by Kreps (1981)) demonstrated that the valuation principle may fail to hold in certain infinite dimensional spaces of contingent claims with infinitely frequent asset trading. In this paper we provide another example of the failure of the valuation principle. It is a simple two–period, arbitrage–free asset trading model with infinitely many assets. The failure of the valuation principle in our model is accompanied by the failure of the present value pricing rule. Prices of some assets deviate from the present value of their payoffs. This indicates the presence of pricing bubbles in the sense of Gilles and LeRoy (1992). Our model is an example of pricing bubbles in a two–period asset trading model with infinitely many assets.

The space of contingent claims in which the valuation principle fails in our example is the L_2 space of contingent claims with finite variance. This version of the valuation principle is the one that has been most frequently used in the finance literature. With the space L_∞ of bounded contingent claims the valuation principle holds in our example. The L_∞ version of the valuation principle does not allow, however, a representation of asset prices by the present value pricing rule. Instead, asset prices can be decomposed into a fundamental value given by the present value pricing rule and a pricing bubble.

We analyze a simple example of asset markets with infinitely many assets. Asset

1

market models with infinitely many assets are an important class of models in financial economics. The well-known Arbitrage Pricing Theory of Ross (1976) takes the form of an infinite asset market model (see Chamberlain and Rothschild (1983), see also Brown and Werner (1993)). These models provide a framework to study diversification as a strategy of investing small fractions of wealth in a large number of assets, and implications of diversification for asset pricing.

Assets in our example are Arrow securities for every state of nature (there are countably many states), and a riskless security. The price of the riskless security exceeds the infinite sum of prices of Arrow securities. This apparent "mispricing" indicates the failure of the present value pricing rule.

An intriguing question is whether the "mispricing" of the riskless security leads to an arbitrage opportunity. In Section 3 we show that there is no arbitrage in form of an arbitrary finite portfolio. In Section 4 we demonstrate that our example can be considered as a viable model of equilibrium in securities markets. It passes the viability test introduced by Kreps (1981). We give two examples of investors whose portfolio choice problem has a well-defined solution among arbitrary finite portfolios. One investor is risk neutral, and thus her preferences satisfy the most stringent continuity requirements of the equilibrium theory with infinite dimensional commodity space (e.g., Mackey continuity introduced in Bewley (1972)). The wealth of the investor is restricted to be non-negative. The other investor is risk averse with utility of state contingent wealth equal to the minimum wealth.

In Section 5 we extend our analysis by allowing investors to hold portfolios of infinitely many securities. We consider the space of bounded portfolios. Prices and payoffs of individual securities specified in the example define in an unambiguous way prices and payoffs only for finite portfolios. Therefore, a specification of portfolio payoffs and prices for the space of infinite bounded portfolios involves an extension of the payoff operator (which assigns payoff to every portfolio), and an extension of the pricing operator (which assigns price to every portfolio) from the space of arbitrary finite portfolios to the whole space of infinite portfolios. Each pair of extensions is consistent with given prices and

2

payoffs of individual securities. We show that the extensions given by infinite summation of prices of individual securities, and infinite summation of payoffs of individual securities lead to an arbitrage opportunity in the space of bounded portfolios. There are, however, other extensions which preclude an arbitrage opportunity. One example is when the payoff operator is extended by infinite summation, while the extension of the pricing operator deviates from infinite summation (i.e., is not countable additive). Under these extensions, the (infinite) portfolio of one share of each Arrow security has the same payoff as the riskless security, and its price is equal to the price of the riskless security (and not the infinite sum of prices of Arrow securities). Therefore, the "mispricing" of the riskless security can be interpreted as lack of countable additivity of prices. We also argue that the arbitrage-free extensions of the pricing operator and the payoff operator are viable by showing that the risk neutral and the risk averse investors have well-defined optimal portfolio among bounded portfolios.

In Section 6 we study the validity of the valuation principle in our example. The question is whether the operator which assigns price to the payoff of every portfolio can be extended to a strictly positive, continuous linear operator on the whole space of contingent claims. In finite markets, the existence of such an extension follows from the absence of arbitrage opportunities. We consider two topological spaces of contingent claims: the L_2 space of square integrable random variables, and the L_∞ space of bounded random variables. A valuation operator for the L_2 space has a representation as the present value under a countably additive system of state prices (i.e., the value of a contingent claim is an infinite sum of claims in different states multiplied by state prices). We show that there is no valuation operator in our example for the L_2 space of contingent claims. Thus the absence of arbitrage opportunities does not imply (the L_2 version of) the valuation principle. Neither the viability of security prices implies the L_2 version of the valuation principle since the prices in our example are viable. The operator which assigns price to the payoff of a portfolio cannot be extended in a continuous way beyond the subspace of payoffs of finite portfolios.

3

It turns out, however, that there is a valuation operator for the space L_∞ of bounded contingent claims. This operator does not admit a representation of security prices as present value under a countably additive system of state prices. Instead, the representation involves a countably additive part and a purely finitely additive part. Following Gilles and LeRoy (1992), this valuation operator can be decomposed into a fundamental value and a pricing bubble. The fundamental value part obeys the present value pricing rule. Arrow securities are valued according to the fundamental present value rule, and have no pricing bubbles. The riskless security has a pricing bubble. The pricing bubble for the riskless security explains its "mispricing".

Section 7 provides some general results about pricing bubbles. We argue that the L_∞ version of the valuation principle holds under the condition of viability of asset prices. Consequently, the fundamental part of viable asset prices and their pricing bubble can always be identified. The question of the validity of the L_2 version of the valuation principle is essentially the question of the absence of pricing bubbles. We show that pricing bubble for a positive contingent claim cannot be negative, i.e., that the value exceeds or is equal to the fundamental value. Furthermore, we provide a characterization of contingent claims with zero pricing bubbles. Section 8 contains some concluding remarks.

Our example of securities markets can be given a different interpretation of a model of term structure of interest rates. Instead of states of nature one can think about future time periods (infinite time horizon), and about Arrow securities as bonds with different maturities. The riskless security is then interpreted as a perpetuity (e.g., console). There is a pricing bubble for the perpetuity.

2. An Example

We consider a market in which there is a countably infinite collection of securities available for trade at date zero. Securities are described by their payoffs at date one. Payoffs are random variables on the underlying state space $(S, \mathcal{S}, P)$. As in Back and Pliska (1991), the set of states is taken to be $S = \{1, 2, \ldots\}$ with $\mathcal{S}$ being the family of

4

all subsets of S (i.e., $\mathcal{S} = 2^S$), and the probability measure P given by $P(s) = 5(\frac{1}{6})^s$ for $s = 1, 2, \dots$.

The payoff of security n in state $s \in S$ is $r_n(s) \in R$, and we will use r_n to denote the sequence of payoffs $(r_n(1), r_n(2), \dots)$ of security n in all states. We take

$$r_n(s) = \begin{cases} 1, & \text{if } s = n \\ 0, & \text{otherwise} \end{cases}$$

for $n = 1, 2, \dots$, i.e., security n is the Arrow security for state $s = n$. In addition there is a riskless security with payoff $r_0(s) = 1$ for all $s \in S$.

The price of the riskless security at date 0 is $q_0 = 1$. The price of security n is $q_n = \frac{1}{2n(n+1)}$, $n = 1, 2, \dots$. Since $\sum_{n=1}^{\infty} q_n = \frac{1}{2}$, the infinite sum of prices of Arrow securities is not equal to the price of the riskless security.

Let $\theta = (\theta_0, \theta_1, \dots, \theta_N)$ be a (finite) portfolio formed from N securities for an arbitrary N. Every security can be held long or short, i.e., $\theta_n \in R$. The payoff of portfolio θ is $R(\theta)(s) = \sum_{n=0}^{N} \theta_n r_n(s)$ in state $s \in S$, its market value at date 0 is $\sum_{n=0}^{N} q_n \theta_n$.

The payoff r_n of security n and the payoff $R(\theta)$ of portfolio θ are examples of contingent claims. The space of all contingent claims X will be either the space $L_2(S, \mathcal{S}, P)$ of square integrable random variables or the space $L_\infty(S, \mathcal{S}, P)$ of bounded random variables, each equipped with its norm topology. The space $L_2(S, \mathcal{S}, P)$ is a representative of the class of $L_p(S, \mathcal{S}, P)$ spaces for $1 \leq p < \infty$.

3. The Absence of Arbitrage

We shall first consider the case when an investor can hold only a finite portfolio. Let $\Phi = \{\theta = (\theta_0, \theta_1, \dots) : \theta_n = 0 \text{ for } n \geq N_\theta \text{ for some } N_\theta\}$ be the space of finite portfolios. Portfolio $\theta \in \Phi$ is an *arbitrage portfolio*, if $R(\theta)(s) \geq 0$ for all $s \in S$, and $\sum_{n=0}^{N_\theta} q_n \theta_n \leq 0$, with at least one strict inequality.

If $\theta \in \Phi$, then $R(\theta)(s) = \theta_0 + \theta_s$ for $s \leq N_\theta$, and $R(\theta)(s) = \theta_0$ for $s > N_\theta$.

5

Consequently, $R(\theta) \geq 0$ implies $\theta_0 \geq 0$ and $\theta_0 + \theta_n \geq 0$ for $n \leq N_\theta$. Since $\sum_{n=1}^{N_\theta} q_n = \frac{1}{2}(1 - \frac{1}{N_\theta + 1}) < q_0$, we have $\sum_{n=0}^{N_\theta} q_n \theta_n \geq \left(\sum_{n=1}^{N_\theta} q_n\right)\theta_0 + \sum_{n=1}^{N_\theta} q_n \theta_n = \sum_{n=1}^{N_\theta} q_n(\theta_0 + \theta_n)$. Therefore, if $R(\theta) \geq 0$, then $\sum_{n=0}^{N_\theta} q_n \theta_n \geq 0$. If in addition $R(\theta) \neq 0$, then $\sum_{n=0}^{N_\theta} q_n \theta_n > 0$. Therefore, there is no finite arbitrage portfolio.

4. Viability

Having shown that the example of Section 2 does not admit a finite arbitrage portfolio, we proceed to demonstrate that it can be considered as a viable model of equilibrium in securities markets. A criterion of viability is the existence of an optimal portfolio for some investor who prefers more to less (see Kreps (1981)).

An investor is described by a strictly increasing, quasi-concave, continuous utility function $U : X_+ \to R$ which assigns to state contingent consumption (wealth) $x \in X_+$ at date 1, the utility level of $U(x)$. The investor's portfolio choice problem is

$$
\begin{aligned}
&\text{maximize} \quad U(R(\theta)) \\
&\text{subject to} \quad \theta \in \Phi, \quad \sum_{n=0}^{N_\theta} q_n \theta_n = w_0, \\
&\qquad R(\theta) \geq 0,
\end{aligned}
$$

where $w_0 > 0$ is the initial wealth. We point out that the investor is restricted to have non-negative wealth (a restriction not seen in Kreps (1981)).

Let us consider a risk neutral investor with $U(x) = E(x) = \sum_{s=1}^{\infty} P(s)x(s)$. It is easy to see that her optimal portfolio choice $\bar{\theta}$ is given by $\bar{\theta}_1 = 4w_0$, and $\bar{\theta}_n = 0$ for all $n = 0, 2, 3, \ldots$, i.e., investing entire wealth w_0 in security 1. Indeed, security 1 has the highest ratio of expected return to price (of $\frac{10}{3}$ as compared to 1 for the riskless security, or $\frac{5}{3}$ for security 2, etc.). Therefore the investor would like to invest as much as possible in security 1 even by selling short other securities. The non-negativity constraint $R(\theta) \geq 0$ (which implies $\theta_0 \geq 0$) prevents her from investing more than w_0

6

in security 1. The utility function of the risk neutral investor is strictly increasing and continuous in the norm topology of each contingent claim space L_2 and L_∞. Moreover, it is continuous in the Mackey topology $\tau(L_\infty, L_1)$ of the contingent claim space L_∞ (see Bewley (1972)), and uniformly norm proper on L_2 (see MasColell (1986)).

Next, let us consider a risk averse investor with $U(x) = 3\inf_s x(s) + E(x)$ for $x \in X_+$. The optimal portfolio choice $\bar{\theta}$ in this case is given by $\bar{\theta}_0 = w_0$, and $\bar{\theta}_n = 0$ for all $n = 1, 2, 3, \dots$, i.e., investing entire wealth w_0 in riskless security 0. In this case the riskless security yields 4 "utiles" for each dollar invested (more than $\frac{10}{3}$ for security 1) provided that no short sales are undertaken. One can easily show that the investor will not be willing to sell short. The utility function of the risk averse investor is strictly increasing and continuous in the (sup) norm topology of the contingent claim space L_∞. It is upper semi–continuous, but not lower semi–continuous in the Mackey topology and in the norm topology of L_2.

5. Bounded Portfolios

So far we have considered only finite portfolios. In this section we shall introduce infinite portfolios and reconsider the questions of the absence of an arbitrage opportunity and viability of security prices $\{q_n\}$.

Suppose that the portfolio space is the space ℓ_∞ of all bounded sequences. Clearly, ℓ_∞ includes all finite portfolios as well as some infinite portfolios. Security prices $\{q_n\}$ and payoffs $\{r_n\}$ define in an unambiguous way prices and payoffs only for finite portfolios. When extending the space of portfolios beyond the finite portfolios Φ we have to specify prices and payoffs of portfolios which are not in Φ. The price system $\{q_n\}$ defines a portfolio pricing operator $Q : \Phi \to R$ by $Q(\theta) = \sum_{n=0}^{N_\theta} q_n \theta_n$ for a finite portfolio $\theta \in \Phi$. Similarly, we have the payoff operator $R : \Phi \to X$ given by $R(\theta) = \sum_{n=0}^{N_\theta} r_n \theta_n$ for $\theta \in \Phi$. Our goal is to extend Q and R to (norm) continuous, linear operators on ℓ_∞ in such

a way as to avoid an arbitrage portfolio in ℓ_∞.

Let us first consider extensions $\hat{R}$ and $\hat{Q}$ having the countably additive form $\hat{R}(\theta) = \sum_{n=0}^{\infty} \theta_n r_n$, and $\hat{Q}(\theta) = \sum_{n=0}^{\infty} \theta_n q_n$ for $\theta \in \ell_\infty$. This pair of extensions leads to an arbitrage opportunity. Indeed, portfolio $(0,1,1,\ldots)$ of one share of each Arrow security has the same payoff as the riskless security but a different price of $\frac{1}{2}$ under such extensions. There is, however, a multitude of other extensions of R and Q which do not admit an arbitrage portfolio. One pair of such extensions consists of the countably additive extension $\hat{R}$, and an extension $\tilde{Q}$, which is constructed as follows:

It is well known that every norm continuous linear functional Ψ on ℓ_∞ has a representation as $\Psi(\theta) = \sum_{n=0}^{\infty} \theta_n \alpha_n + \int_N \theta_n \nu(dn)$ for $\theta \in \ell_\infty$, where $\{\alpha_n\}_{n=0}^{\infty}$ is a sequence of numbers such that $\sum_{n=0}^{\infty} |\alpha_n| < \infty$, and ν is a purely finitely additive bounded measure (set function) on 2^N (i.e., $\alpha \in \ell_1$, and $\nu \in ba(N)$, see Dunford and Schwartz (1957), and Rao and Rao (1983)). Let $\nu = b\mu$ for some $b \in R$, and the density measure μ. Density measure μ is a purely finitely additive measure such that $\int \theta_n \mu(dn) = \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N} \theta_n$ for $\theta \in \ell_\infty$, whenever the limit exists (see Rao and Rao (1983)). We are looking for a functional $\tilde{Q}$ which is an extension of Q. Therefore we require that $\tilde{Q}(e_n) = Q(e_n) = q_n$ for every $n = 0,1,2,\ldots$, where e_n denotes portfolio consisting of one share of security n, i.e., nth unit vector in ℓ_∞. Furthermore, in order to avoid an arbitrage opportunity it must be the case that portfolio $(0,1,1,\ldots)$ has the same price as the riskless security, since its payoff under the operator $\hat{R}$ is the same as the payoff of the riskless security. Thus, $\tilde{Q}((0,1,1,\ldots)) = q_0$. Using the representation of $\tilde{Q}$ and the specification $\nu = b\mu$, the pricing operator $\tilde{Q}$ satisfying these conditions is given by $\alpha_n = q_n$, for $n = 0,1,\ldots$, and $b = \frac{1}{2}$. Thus we have $\tilde{Q}(\theta) = \sum_{n=0}^{\infty} q_n \theta_n + \frac{1}{2} \int_N \theta_n \mu(dn)$ for $\theta \in \ell_\infty$.

We shall prove that the extensions $\hat{R}$ and $\tilde{Q}$ do not admit an arbitrage portfolio

8

in ℓ_∞. Let $\theta \in \ell_\infty$ be a portfolio such that $\hat{R}(\theta)(s) \geq 0$ for every $s \in S$. Since $\hat{R}(\theta)(s) = \theta_0 + \theta_s$, we have $\theta_0 + \theta_n \geq 0$ for every $n = 1, \ldots$. A little algebra shows that $\tilde{Q}(\theta) = \sum_{n=1}^{\infty} q_n(\theta_0 + \theta_n) + \frac{1}{2} \int_N (\theta_0 + \theta_n)\mu(dn)$. Therefore, $\tilde{Q}(\theta) \geq 0$. Clearly, if $\hat{R}(\theta) \neq 0$, then $\tilde{Q}(\theta) > 0$. There is no arbitrage portfolio in ℓ_∞.

Another pair of extensions which does not admit an arbitrage portfolio is the countably additive extension $\hat{Q}$ of the pricing operator, and an extension $\tilde{R}$ of the payoff operator given by $\tilde{R}(\theta)(s) = \sum_{n=0}^{\infty} r_n(s)\theta_n - \frac{1}{2} \int_N \theta_n \mu(dn)$ for $\theta \in \ell_\infty$. Under this pair of extensions, portfolio $(0, 1, 1, \ldots)$ has riskless payoff of $\frac{1}{2}$, and a price of $\frac{1}{2}$. Furthermore, there is a continuum of arbitrage–free extensions such that neither the pricing operator nor the payoff operator are countably additive.

Each of the arbitrage–free pairs of operators $(\hat{R}, \tilde{Q})$, and $(\tilde{R}, \hat{Q})$ is viable with respect to the portfolio space ℓ_∞. Indeed, one can show that the solutions to the portfolio choice problem of Section 4 for the risk neutral investor and for the risk averse investor remain unchanged in this larger portfolio space. (The proof for the risk neutral investor makes use of the Mackey continuity of the utility function; for the risk averse investor the proof is straightforward).

The system of prices $\{q_n\}$ and payoffs $\{r_n\}$ can be considered as being a part (restriction) of an arbitrage–free (and viable) system of pricing and payoff operators on the space of all bounded portfolios. The fact that each such system of operators is not countably additive indicates that the "mispricing" of the riskless security is due to the lack of countable additivity of prices or payoffs.

We remark that one could equip the portfolio space ℓ_∞ with a different topology, say the Mackey topology $\tau(\ell_\infty, \ell_1)$, and require pricing and payoff extensions to be Mackey continuous. Then, the payoff operator R and the pricing operator Q have unique extensions which are the countably additive operators $\hat{R}$ and $\hat{Q}$ (Φ is Mackey dense in ℓ_∞). This pair of extensions leads to an arbitrage opportunity. The suitability of the

9

Mackey topology for a portfolio space is, however, less apparent than for a consumption space as the standard arguments in favor of the Mackey topology (see Bewley (1972)) do not readily apply.

6. Valuation, and Pricing Bubbles

Let $M = R(\Phi) \subset X$ be the subspace of payoffs of all finite portfolios, i.e,

$$M = \{x \in X : x(s) = R(\theta)(s) \quad \text{for every} \quad s \in S, \quad \text{for some} \quad \theta \in \Phi\}.$$

We define the *induced pricing operator* $\Pi : M \to R$ by $\Pi(x) = Q(\theta)$ for $x \in M$, where θ is such that $R(\theta) = x$. A *valuation operator* is a continuous, strictly positive, linear functional $V : X \to R$ such that the restriction of V to M is identical to Π. Valuation operator assigns price to the payoff of every security, and the payoff of every portfolio. Moreover, it assigns value to contingent claims which need not be payoffs of portfolios. In asset market models with finitely many assets, the existence of a valuation operator follows from the absence of an arbitrage opportunity.

We shall first show that there is no valuation operator in our example, if $X = L_2$. For the existence of a valuation operator it is necessary that the induced pricing operator Π is continuous on the subspace M of payoffs of all finite portfolios. One can easily see that Π is not continuous on M in the norm topology of L_2. Indeed, let x^k be given by $x^k(s) = 1$ for $s \le k$, and $x^k(s) = 0$ for $s > k$, $k = 1, 2, \ldots$. Clearly, $x^k \in M$ for every k, since $x^k = R(\theta^k)$ for portfolio θ^k of holding one share of every Arrow security for states $s = 1, \ldots, k$. In the norm topology of L_2 the sequence $\{x^k\}_{k=1}^{\infty}$ converges to the payoff $r_0 \equiv 1$ of the riskless security. We have $\Pi(x^k) = Q(\theta^k) = \sum_{n=1}^{k} q_n \theta_n^k = \sum_{n=1}^{k} \frac{1}{2n(n+1)} = \frac{1}{2}(1 - \frac{1}{k+1})$. Thus $\lim_{k \to \infty} \Pi(x^k) = \frac{1}{2} \neq \Pi(r_0) = 1$, i.e., Π is discontinuous.

Next, we shall consider the contingent claim space $X = L_\infty$ and show that in this case the induced pricing operator Π can be extended to a valuation operator. We first

10

point out that in the norm topology of L_∞ the sequence of payoffs $\{x^k\}_{k=1}^\infty$ considered above does not converge to the payoff of the riskless security, and that Π is continuous on M (see Clark (1993, Theorem 2)).

We construct a L_∞ valuation operator in the following way (for a discussion of general conditions for the existence of a L_∞ valuation operator see Section 7, see also Clark (1993)): Let $\bar\nu \in ba(S)$ be defined by $\bar\nu(A) = \sum_{s \in A} q_s + \frac{1}{2}\mu(A)$, for the density measure $\mu \in ba(S)$. Cosider an operator $V : L_\infty \to R$ given by $V(x) = \int x(s)\bar\nu(ds)$. We have $V(x) = \sum_{s=1}^\infty q_s x(s) + \frac{1}{2}\int x(s)\mu(ds)$, and $\int x(s)\mu(ds) = \lim_{t \to \infty} \frac{1}{t}\sum_{s=1}^t x(s)$, for $x \in L_\infty$ whenever the limit exists. In particular, $V(R(\theta)) = \sum_{s=1}^\infty q_s(\theta_0 + \theta_s) + \frac{1}{2}\theta_0 = \sum_{n=0}^{N_\theta} q_n\theta_n$ for a finite portfolio $\theta \in \Phi$. Thus V coincides with the payoff pricing operator Π on M. Since V is strictly positive, it is a valuation operator.

Using the terminology of Gilles and LeRoy (1992), the value $V(x) = \sum_{s=1}^\infty q_s x(s) + \frac{1}{2}\int x(s)\mu(ds)$ of a contingent claim $x \in L_\infty$ can be decomposed into the *fundamental value* $\sum_{s=1}^\infty q_s x(s)$, and the *bubble* $\frac{1}{2}\int x(s)\mu(ds)$. The fundamental value results from countably additive valuation with state prices (which are equal to prices of Arrow securities). The pricing bubble for every Arrow security is zero (as it is for any portfolio that has non-zero payoff in only finitely many states). The pricing bubble for the riskless security is $\frac{1}{2}$, and explains its "mispricing".

The question of the existence of a valuation operator can also be analyzed in the case when the portfolio space is the space ℓ_∞ of bounded portfolios. Let $(\hat R, \hat Q)$, and $(\tilde R, \hat Q)$ be the two arbitrage free pairs of payoff and pricing operators on ℓ_∞ discussed in Section 5. Clearly, there cannot be a valuation operator if the payoff space is $X = L_2$ since both pairs are extensions of the system of payoffs $\{r_n\}$ and prices $\{q_n\}$. We shall therefore consider the case $X = L_\infty$. The subspace of payoffs of bounded portfolios is $M_1 = \hat R(\ell_\infty)$ for the payoff operator $\hat R$, and $M_2 = \tilde R(\ell_\infty)$ for the payoff operator $\tilde R$.

11

One can easily see that $M_1 = M_2 = L_\infty$, i.e., that markets are complete with bounded portfolios under each payoff operator. The induced pricing operator is therefore defined on the whole space L_∞, and is given by $\Pi_1(x) = \tilde{Q}(\hat{R}^{-1}(x))$, and $\Pi_2(x) = \hat{Q}(\tilde{R}^{-1}(x))$ for $x \in L_\infty$, respectively. We claim that $\Pi_1 = \Pi_2 = V$, i.e. that the valuation operator V derived in the case of finite portfolios coincides with each induced pricing operator. Indeed, $V(\hat{R}(\theta)) = \sum_{s=1}^\infty q_s \hat{R}(\theta)(s) + \frac{1}{2} \int \hat{R}(\theta)(s)\mu(ds) = \sum_{s=1}^\infty q_s(\theta_0 + \theta_s) + \frac{1}{2} \int (\theta_0 + \theta_s)\mu(ds) = \sum_{n=0}^\infty q_n \theta_n + \frac{1}{2} \int \theta_n \mu(dn) = \tilde{Q}(\theta)$. Similarly, $V(\tilde{R}(\theta)) = \hat{Q}(\theta)$ for every $\theta \in \ell_\infty$.

The fact that Π_1, Π_2, and V are all identical shows that the explanations of the "mispricing" of the riskless security by the lack of countable additivity of prices or payoffs and by the presence of a pricing bubble are equivalent.

7. Theory of Valuation with Bubbles

In this section we shall establish general conditions for the existence of a valuation operator with pricing bubbles (i.e., a L_∞ valuation operator), and investigate some of its properties.

The model underlying this section is a generalization of the setup of the example of Section 2. There is a countable set of securities indexed by $n = 0, 1, \ldots$ with payoffs r_n, and prices q_n. We shall retain the countable state space $(S, \mathcal{S}, \mathcal{P})$ for simplicity. We assume that $r_n \in L_\infty$ for every n, and that security $n = 0$ is riskless with payoff $r_0 \equiv 1$.

Let $M \subset L_\infty$ be the subspace of payoffs of all finite portfolios (i.e., $M = R(\Phi)$), and let Π be the induced pricing operator on M. Suppose that security prices $\{q_n\}$ do not admit an arbitrage opportunity in the portfolio space Φ. Then Π is a strictly positive linear functional on M, and by the Krein-Rutman Theorem (see Narici and Beckenstein (1985)) it can be extended to a positive linear functional on L_∞, continuous in the norm topology. The existence of a strictly positive extension (i.e., a valuation operator) is not

guaranteed in general. However, if the security prices $\{q_n\}$ are viable in the sense of Section 4, then the existence of a strictly positive valuation operator V on L_∞ follows. We prove this result by adapting the arguments of Kreps (1981, Theorem 1) to the case of an investor restricted to hold non-negative wealth.

Proposition 1: Let $X = L_\infty$. If security prices $\{q_n\}$ are viable, then there is a valuation operator.

Proof: Viability of prices $\{q_n\}$ means that there is an investor with norm continuous, quasi-concave, and strictly increasing utility function $U : L_\infty^+ \to I\!\!R$ and initial wealth $w_0 > 0$ such that some $\theta^* \in \Phi$ is her optimal portfolio. Let $x^* = R(\theta^*)$. We have $\Pi(x^*) = w_0 > 0$.

Consider the sets $P = \{x \in L_\infty^+ : U(x) > U(x^*)\}$ and $B = \{x \in M : \Pi(x) \leq \Pi(x^*)\}$. These sets are convex, disjoint, and P has nonempty interior. By the Separating Hyperplane Theorem there is a continuous functional $V : L_\infty \to I\!\!R$, $V \neq 0$, such that $V(x) \geq V(x^*) \geq V(x')$ for every $x \in P$ and $x' \in B$. Clearly, V is positive. Therefore, $V(r_0) > 0$ since $r_0 \equiv 1$ is in the interior of L_∞^+. Since $\Pi(r_0) > 0$ we can renormalize V so that $V(r_0) = \Pi(r_0)$. We claim that $V(x) = \Pi(x)$ for every $x \in M$. To this end, we first observe that if $x \in M$ and $\Pi(x) = 0$, then $\lambda x \in B$ for every $\lambda \in I\!\!R$ and therefore $V(x) = 0$. For an arbitrary $x \in M$ we have $\Pi(x) = \gamma\Pi(r_0)$ for some γ. Therefore, $\Pi(x - \gamma r_0) = 0$ which implies $V(x - \gamma r_0) = 0$ and $V(x) = \gamma V(r_0) = \gamma\Pi(r_0) = \Pi(x)$.

It remains to be shown that V is strictly positive. Let $z \in L_\infty$ satisfy $z > 0$. We have $U(x^* + z) > U(x^*)$. For ϵ small enough, we have $U((1 - \epsilon)x^* + z) > U(x^*)$. Consequently, $(1 - \epsilon)x^* + z \in P$ and therefore $V((1 - \epsilon)x^* + z) \geq V(x^*)$. This implies $V(z) \geq \epsilon V(x^*) = \epsilon\Pi(x^*) > 0$, and V is strictly positive. ∎

The fact that the positive orthant of the contingent claim space (which is the con-

sumption set of an investor in the definition of viability) has non-empty interior is crucial for Proposition 1. We have seen in Section 6 that with an empty interior of the consumption set (as in the L_2 space), the result may fail to hold.

Let $V : L_\infty \to R$ be a valuation operator. It has a representation as $V(x) = \int x(s)\nu(ds)$ for some finitely additive measure $\nu \in ba(S)$, $v \geq 0$. By the Yosida-Hewitt Theorem (see Rao and Rao (1983)), ν can be uniquely decomposed as $\nu = \nu_c + \nu_p$ for a purely finitely additive measure $\nu_p \geq 0$ and a countably additive measure $\nu_c \geq 0$. Since the state space S is countable, the measure ν_c is identified by the sequence of numbers $p_s = \nu_c(\{s\})$ for $s = 1, \ldots,$ so that $\int x(s)\nu_c(ds) = \sum_{s=1}^{\infty} p_s x(s)$. Let $F(x) = \sum_{s=1}^{\infty} p_s x(s)$ and $B(x) = \int x(s)\nu_p(ds)$. We have $V(x) = F(x) + B(x)$, and $F(x)$ is the (countably additive) *fundamental value* of x, while $B(x)$ is the pricing *bubble* (see Gilles and LeRoy (1992)).

Our next result is concerned with contingent claims that have zero pricing bubbles.

Proposition 2: Every contingent claim $x \in L_\infty$ which is non-zero in only finitely many states has zero pricing bubble. Moreover, the set of contingent claims which have zero pricing bubble is a norm closed linear subspace.

Proof: It follows from the definition of a purely finitely additive measure (see Rao and Rao (1983)) that $\nu_p(E) = 0$ for every finite set $E \subset S$. Therefore $B(x) = 0$ whenever x is non-zero in only finitely many states of nature. Since B is linear and norm continuous, the second part of the proposition easily follows. ∎

Consider a contingent claim which pays one dollar in state s and zero in all other states (Arrow security). By Proposition 2 the value of such contingent claim equals its fundamental value which is p_s. Therefore, p_s is the (implicit) state price of state s. Since the valuation operator V is strictly positive, state price p_s is strictly positive, too.

14

Furthermore, the fundamental value operator F is strictly positive.

If the pricing bubble $B(x)$ is non-negative for a contingent claim $x \in L_\infty$, then the value $V(x)$ exceeds or is equal to the fundamental value $F(x)$. The following Proposition 3 establishes that the pricing bubble is non-negative for every positive contingent claim. This result follows essentially from the fact that the purely finitely additive measure ν_p in the Yosida-Hewitt decomposition is positive. Nevertheless, we provide a proof which we believe is insightful.

Proposition 3 (*No negative bubbles*): The value $V(x)$ of a contingent claim x exceeds or is equal to its fundamental value $F(x)$ for every $x \in L_\infty$, $x \geq 0$.

Proof: We shall prove that if V is a positive operator, then $V(x) \geq F(x)$ for every $x \geq 0$. Let $x \in L_\infty$ be a non-negative contingent claim. Let x^s be a contingent claim which is the same as x in all states $\{1, \ldots, s\}$ and zero in all other states. By Proposition 1, $V(x^s) = F(x^s)$. Since $x - x^s \geq 0$ for every s, we have $V(x - x^s) \geq 0$, hence $V(x) \geq V(x^s) = F(x^s)$. We claim that $F(x^s)$ converges to $F(x)$. Indeed, the sequence $\{x^s\}$ converges to x in the weak* topology $\sigma(L_\infty, L_1)$, and F is a weak* continuous functional on L_∞ since it has a representation with countably additive measure ν_c. Consequently, $V(x) \geq F(x)$. $\blacksquare$

In both Propositions 2 and 3, if contingent claim x is attainable by a portfolio, i.e., if $x \in R(\theta)$ for some $\theta \in \Phi$, then the value $V(x)$ is equal to its price $\Pi(x) = q\theta$. We also point out that in Proposition 3 it is merely the positivity (and not strict positivity) of the valuation operator V that matters. This allows us to conclude that the presence of negative bubbles is incompatible with the absence of arbitrage opportunities.

We conclude this section with a discussion of Propositions 2 and 3 in the context of our example of Section 2. As we pointed out in Section 6, contingent claims which are non-

15

zero in only finitely many states (in particular Arrow securities) have zero bubbles. The pricing bubble for the riskless security is positive, i.e., the price exceeds the fundamental value. Let us consider a variation of the example with an arbitrary price q_0 of the riskless security. The reader can verify that whenever $q_0 \geq \frac{1}{2}$ security prices are arbitrage–free and viable. If $q_0 = \frac{1}{2}$, then there is no pricing bubble for the riskless security, otherwise there is a positive pricing bubble of $q_0 - \frac{1}{2}$. On the other hand, if $q_0 < \frac{1}{2}$, then there is a finite arbitrage portfolio.

8. Concluding Remarks

The example of this paper demonstrates the possibility of failure of the L_2 version of the valuation principle in large asset markets. Asset prices in our example do not allow a representation as present value under (countably additive) state prices. The asset prices are arbitrage–free, and they can be seen as arising in an asset market equilibrium. The failure of the L_2 valuation principle indicates the presence of pricing bubbles.

The question of the validity of the L_2 valuation principle is essentially the question of the absence of pricing bubbles. Back and Pliska (1991, Proposition 1) provided conditions under which there exists a valuation operator on the contingent claim space L_∞, continuous in the Mackey $\tau(L_\infty, L_1)$ topology. Such an operator has a representation as the present value under (countably additive) state prices, i.e., it provides valuation without bubbles. This form of valuation is slightly weaker than the L_2 valuation. Back and Pliska (1991) conditions require that asset prices be viable for an investor with Mackey continuous utility function, and that her optimal (end-of-period) wealth be bounded away from zero. Both these conditions appear restrictive. An important example of a utility function not continuous in the Mackey topology is the utility of the risk averse investor considered in Section 4.

16

References

Back, K. and S. Pliska (1991), "On the Fundamental Theorem of Asset Pricing with an Infinite State Space", *Journal of Mathematical Economics*, **20**, 1-18.

Bhaskara Rao, K.P.S., and M. Bhaskaro Rao (1983), "Theory of Charges". Academic Press, New York.

Bewley, T. (1972), "Existence of Equilibria in Economies with Infinitely Many Commodities", *Journal of Economic Theory*, **4**, 514-540.

Brown, D. and J. Werner (1992), "Arbitrage and Existence of Equilibrium in Infinite Asset Markets", Cal Tech Working Paper 825, *Review of Economic Studies*, forthcoming.

Chamberlain, G. and M. Rothschild, (1983), "Arbitrage, Factor Structure, and Mean-Variance Analysis of Large Asset Markets", *Econometrica*, **51**, 1281-1304.

Clark, S. (1992), "The Valuation Problem in Arbitrage Pricing Theory", *Journal of Mathematical Economics*, **22**, 463-478.

Dunford, N. and J.T. Schwartz (1957), "Linear Operators Part I: General Theory", Interscience, New York.

Gilles, Ch. and S. LeRoy (1991), "Bubbles and Charges", *International Economic Review*, **33**, 323-339.

Kreps, D. (1981), "Arbitrage and Equilibrium in Economies with Infinitely Many Commodities", *Journal of Mathematical Economics*, **8**, 15-35.

Mas-Colell, A. (1986), "The Price Equilibrium Existence Problem in Topological Vector Lattices", *Econometrica*, **54**, 1039-1054.

Narici, L., and E. Beckenstein (1985), "Topological Vector Spaces", Marcel Dekker, Inc., New York.

Ross, S. (1976), "The Arbitrage Theory of Asset Pricing", *Journal of Economic Theory*, **13**, 341-360.

WORKING PAPERS LIST

1. Albert Marcet and Ramon Marimon
    Communication, Commitment and Growth. (June 1991) [Published in *Journal of Economic Theory* Vol. 58, no. 2, (December 1992)]

2. Antoni Bosch
    Economies of Scale, Location, Age and Sex Discrimination in Household Demand. (June 1991) [Published in *European Economic Review* 35, (1991) 1589-1595]

3. Albert Satorra
    Asymptotic Robust Inferences in the Analysis of Mean and Covariance Structures. (June 1991) [Published in *Sociological Methodology* (1992), pp. 249-278, P.V. Marsden Edt. Basil Blackwell: Oxford & Cambridge, MA]

4. Javier Andrés and Jaume Garcia
    Wage Determination in the Spanish Industry. (June 1991) [Published as "Factores determinantes de los salarios: evidencia para la industria española" in J.J. Dolado et al. (eds.) *La industria y el comportamiento de las empresas españolas (Ensayos en homenaje a Gonzalo Mato)*, Chapter 6, pp. 171-196, Alianza Economia]

5. Albert Marcet
    Solving Non-Linear Stochastic Models by Parameterizing Expectations: An Application to Asset Pricing with Production. (July 1991)

6. Albert Marcet
    Simulation Analysis of Dynamic Stochastic Models: Applications to Theory and Estimation. (November 1991), 2d. version (March 1993) [Published in *Advances in Econometrics* invited symposia of the Sixth World Congress of the Econometric Society (Eds. JJ. Laffont i C.A. Sims). Cambridge University Press (1994)]

7. Xavier Calsamiglia and Alan Kirman
    A Unique Informationally Efficient and Decentralized Mechanism with Fair Outcomes. (November 1991) [Published in *Econometrica*, vol. 61, 5, pp. 1147-1172 (1993)]

8. Albert Satorra
    The Variance Matrix of Sample Second-order Moments in Multivariate Linear Relations. (January 1992) [Published in *Statistics & Probability Letters* Vol. 15, no. 1, (1992), pp. 63-69]

9. Teresa Garcia-Milà and Therese J. McGuire
    Industrial Mix as a Factor in the Growth and Variability of States'Economies. (January 1992) [Forthcoming in *Regional Science and Urban Economics*]

10. Walter Garcia-Fontes and Hugo Hopenhayn
    Entry Restrictions and the Determination of Quality. (February 1992)

11. Guillem López and Adam Robert Wagstaff
    Indicadores de Eficiencia en el Sector Hospitalario. (March 1992) [Published in *Moneda y Crédito* Vol. 196]

12. Daniel Serra and Charles ReVelle
    The PQ-Median Problem: Location and Districting of Hierarchical Facilities. Part I (April 1992) [Published in *Location Science*, Vol. 1, no. 4 (1993)]

13. Daniel Serra and Charles ReVelle
    The PQ-Median Problem: Location and Districting of Hierarchical Facilities. Part II: Heuristic Solution Methods. (April 1992) [Published in *Location Science*, Vol. 2, no. 2 (1994)]

14. Juan Pablo Nicolini
    Ruling out Speculative Hyperinflations: a Game Theoretic Approach. (April 1992)

15. Albert Marcet and Thomas J. Sargent
    Speed of Convergence of Recursive Least Squares Learning with ARMA Perceptions. (May 1992) [Forthcoming in *Learning and Rationality in Economics*]

16. Albert Satorra
    Multi-Sample Analysis of Moment-Structures: Asymptotic Validity of Inferences Based on Second-Order Moments. (June 1992) [Published in *Statistical Modelling and Latent Variables* Elsevier, North Holland. K.Haagen, D.J.Bartholomew and M. Deistler (eds.), pp. 283-298.]

Special issue   Vernon L. Smith
    Experimental Methods in Economics. (June 1992)

17. Albert Marcet and David A. Marshall
    Convergence of Approximate Model Solutions to Rational Expectation Equilibria Using the Method of Parameterized Expectations.

18. M. Antònia Monés, Rafael Salas and Eva Ventura
    Consumption, Real after Tax Interest Rates and Income Innovations. A Panel Data Analysis. (December 1992)

19. Hugo A. Hopenhayn and Ingrid M. Werner
    Information, Liquidity and Asset Trading in a Random Matching Game. (February 1993)

20. Daniel Serra
The Coherent Covering Location Problem. (February 1993) [Forthcoming in *Papers in Regional Science*]

21. Ramon Marimon, Stephen E. Spear and Shyam Sunder
Expectationally-driven Market Volatility: An Experimental Study. (March 1993) [Forthcoming in *Journal of Economic Theory*]

22. Giorgia Giovannetti, Albert Marcet and Ramon Marimon
Growth, Capital Flows and Enforcement Constaints: The Case of Africa. (March 1993) [Published in *European Economic Review* 37, pp. 418-425 (1993)]

23. Ramon Marimon
Adaptive Learning, Evolutionary Dynamics and Equilibrium Selection in Games. (March 1993) [Published in *European Economic Review* 37 (1993)]

24. Ramon Marimon and Ellen McGrattan
On Adaptive Learning in Strategic Games. (March 1993) [Forthcoming in *A. Kirman and M. Salmon eds.* "Learning and Rationality in Economics" Basil Blackwell]

25. Ramon Marimon and Shyam Sunder
Indeterminacy of Equilibria in a Hyperinflationary World: Experimental Evidence. (March 1993) [Forthcoming in *Econometrica*]

26. Jaume Garcia and José M. Labeaga
A Cross-Section Model with Zeros: an Application to the Demand for Tobacco. (March 1993)

27. Xavier Freixas
Short Term Credit Versus Account Receivable Financing. (March 1993)

28. Massimo Motta and George Norman
Does Economic Integration cause Foreign Direct Investment? (March 1993) [Published in *Working Paper University of Edinburgh* 1993:1]

29. Jeffrey Prisbrey
An Experimental Analysis of Two-Person Reciprocity Games. (February 1993) [Published in *Social Science Working Paper 787* (November 1992)]

30. Hugo A. Hopenhayn and Maria E. Muniagurria
Policy Variability and Economic Growth. (February 1993)

31. Eva Ventura Colera
A Note on Measurement Error and Euler Equations: an Alternative to Log-Linear Approximations. (March 1993) [Published in *Economics Letters*, 45, pp. 305-308 (1994)]

32. Rafael Crespí i Cladera
Protecciones Anti-Opa y Concentración de la Propiedad: el Poder de Voto. (March 1993)

33. Hugo A. Hopenhayn
The Shakeout. (April 1993)

34. Walter Garcia-Fontes
Price Competition in Segmented Industries. (April 1993)

35. Albert Satorra i Brucart
On the Asymptotic Optimality of Alternative Minimum-Distance Estimators in Linear Latent-Variable Models. (February 1993) [Published in *Econometric Theory*, 10, pp. 867-883]

36. Teresa Garcia-Milà, Therese J. McGuire and Robert H. Porter
The Effect of Public Capital in State-Level Production Functions Reconsidered. (February 1993)

37. Ramon Marimon and Shyam Sunder
Expectations and Learning Under Alternative Monetary Regimes: an Experimental Approach. (May 1993)

38. José M. Labeaga and Angel López
Tax Simulations for Spain with a Flexible Demand System. (May 1993)

39. Daniel Serra and Charles ReVelle
Market Capture by Two Competitors: The Pre-Emptive Location Problem. (May 1993) [Published in *Journal of Regional Science*, Vol. 34, no.4 (1994)]

40. Xavier Cuadras-Morató
Commodity Money in the Presence of Goods of Heterogenous Quality. (July 1993) [Published in *Economic Theory* 4 (1994)]

41. M. Antònia Monés and Eva Ventura
Saving Decisions and Fiscal Incentives: A Spanish Panel Based Analysis. (July 1993)

42. Wouter J. den Haan and Albert Marcet
Accuracy in Simulations. (September 1993) [Published in *Review of Economic Studies*, (1994)]

43. Jordi Galí
Local Externalities, Convex Adjustment Costs and Sunspot Equilibria. (September 1993) [Forthcoming in *Journal of Economic Theory*]

95.   Marc Sáez Zafra and Jorge V. Pérez-Rodríguez.
       Modelos Autorregresivos para la Varianza Condicionada Heteroscedástica (ARCH) (October 1994)

96.   Daniel Serra and Charles ReVelle.
       Competitive Location in Discrete Space (November 1994) [Forthcoming in Zvi Drezner (ed.): *Facility Location: a Survey of Applications and Methods*. Springer-Verlag New York.

97.   Alfonso Gambardella and Walter García-Fontes.
       Regional Linkages through European Research Funding (October 1994) [Forthcoming in *Economic of Innovation and New Technology*]

98.   Daron Acemoglu and Fabrizio Zilibotti.
       Was Prometheus Unbound by Chance? Risk, Diversification and Growth (November 1994)

99.   Thierry Foucault.
       Price Formation and Order Placement Strategies in a Dynamic Order Driven Market (June 1994)

100.   Ramon Marimon and Fabrizio Zilibotti.
       'Actual' versus 'Virtual' Employment in Europe: Why is there Less Employment in Spain? (December 1994)

101.   María Sáez Martí.
       Are Large Windows Efficient? Evolution of Learning Rules in a Bargaining Model (December 1994)

102.   María Sáez Martí.
       An Evolutionary Model of Development of a Credit Market (December 1994)

103.   Walter García-Fontes and Ruben Tansini and Marcel Vaillant.
       Cross-Industry Entry: the Case of a Small Developing Economy (December 1994)

104.   Xavier Sala-i-Martin.
       Regional Cohesion: Evidence and Theories of Regional Growth and Convergence (October 1994)

105.   Antoni Bosch-Domènech and Joaquim Silvestre.
       Credit Constraints in General Equilibrium: Experimental Results (December 1994)

106.   Casey B. Mulligan and Xavier Sala-i-Martin.
       A Labor-Income-Based Measure of the Value of Human Capital: an Application to the States of the United States. (December 1994)

107.   José M. Bailén and Luis A. Rivera-Bátiz.
       Human Capital, Heterogeneous Agents and Technological Change (March 1995)

108.   Xavier Sala-i-Martin.
       A Positive Theory of Social Security (February 1995)

109.   J. S. Marron and Frederic Udina.
       Interactive Local Bandwidth Choice (February 1995)

110.   Marc Sáez and Robert M. Kunst.
       ARCH Patterns in Cointegrated Systems (March 1995)

111.   Xavier Cuadras-Morató and Joan R. Rosés.
       Bills of Exchange as Money: Sources of Monetary Supply during the Industrialization in Catalonia (1844-74) (April 1995)

112.   Casey B. Mulligan and Xavier Sala-i-Martin.
       Measuring Aggregate Human Capital (January 1995)

113.   Fabio Canova.
       Does Detrending Matter for the Determination of the Reference Cycle and the Selection of Turning Points? (March 1995)

114.   Sergiu Hart and Andreu Mas-Colell.
       Bargaining and Value (February 1995)

115.   Teresa Garcia-Milà, Albert Marcet and Eva Ventura.
       Supply Side Interventions and Redistribution (June 1995)

116.   Robert J. Barro and Xavier Sala-i-Martin.
       Technological Diffusion, Convergence, and Growth (May 1995)

117.   Xavier Sala-i-Martin.
       The Classical Approach to Convergence Analysis (June 1995)

118.   Serguei Maliar and Vitali Perepelitsa.
       LCA Solvability of Chain Covering Problem (May 1995)

119.   Serguei Maliar, Igor' Kozin and Vitali Perepelitsa.
       Solving Capability of LCA (June 1995)

120.   Antonio Ciccone and Robert E. Hall.
       Productivity and the Density of Economic Activity (May 1995)

121. Jan Werner.
Arbitrage, Bubbles, and Valuation (April 1995)