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Solving Capability of LCA

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Abstract

The aim of this research is to study the properties of Linear Convolution Algorithms (LCA) used for solving multicriteria problems of mathematical programming given on n–vertices N–weighted graphs. It is known that for many problems of vector optimization on graphs LCA do not guarantee finding the complete set of pareto optimal solutions. Consequently we are interested in the following question: if LCA is not able to find the whole set of pareto optimal elements, then what is the "solving capability" of LCA, i.e. how many pareto optimal elements this class of algorithms is able to find. In the paper we study in details solving capability of LCA for the spanning tree problem. We also obtain some estimates characterizing solving capability of LCA for the other well–known problems of discrete multicriteria optimization.

1 Introduction

The aim of this research is to study the properties of Linear Convolution Algorithms (LCA) used for solving multicriteria problems of mathematical programming given on $n-$vertices $N-$weighted graphs. It is known that for many problems of vector optimization on graphs LCA do not guarantee finding the complete set of pareto optimal solutions. Consequently we are interested in the following question: if LCA is not able to find the whole set of pareto optimal elements, then what is the "solving capability" of LCA, i.e. how many pareto optimal elements this class of algorithms is able to find. In the paper we study in details solving capability of LCA for the spanning tree problem. We also obtain some estimates characterizing solving capability of LCA for the other well-known problems of discrete multicriteria optimization.

2 Notations and statement of the problem

In the paper we consider the problems of discrete multicriteria optimization with Vector Objective Function *(VOF)*

$F(x) = (F_1(x), ..., F_v(x), ..., F_N(x))$

defined on a finite set of feasible solutions $X = \{x\}$.

Criteria of VOF are to be optimized (minimized or maximized)

$F_v(x) \to \min, \quad v = 1, ..., N.$

A feasible solution $\tilde{x} \in X$ is called Pareto Optimum (PO) if there is no element $x^* \in X$ such that $F(x^*) \geq F(\tilde{x})$ and $F(x^*) \neq F(\tilde{x})$. The set $\tilde{X}$ of all PO is called Pareto Set (PS).

Subset $\hat{X} \subseteq \tilde{X}$ is called a Complete Set of Alternatives (CSA) if it's cardinality $|\hat{X}|$ is minimum and equality $F(X^*) = F(\hat{X})$ is satisfied, $F(X^*) = \{F(x) : x \in X^*\}, \forall X^* \in X$.

For the problem given by a pair $(X, F(x))$ we need to find a sufficiently effective algorithm that is capable of finding and describing CSA.

In the paper we consider the following multicriteria problems on graphs:

Z_1 is a perfect matching problem given on a graph $G = (V, E)$, where a feasible solution $x = (V, E_x)$ is a perfect matching;

Z_2 is a spanning tree problem given on a graph $G = (V, E)$, where a feasible solution $x = (V, E_x)$ is a spanning tree;

Z_3 is a problem of path between a pair of vertices given on a graph $G = (V, E)$, where a feasible solution $x = (V_x, E_x)$ is a simple path between two vertices u_1 and u_2;

Z_4 is a perfect matching problem given on a graph $G = (V_1, V_2, E)$, $|V_1| = |V_2| = n$, where a feasible solution $x = (V, E_x)$ is a perfect matching;

Z_5 is a travelling salesman problem given on a graph $G = (V, E)$, where a feasible solution $x = (V_x, E_x)$ is a simple Hamiltonian cycle;

Z_6 is a chain covering problem given on a graph $G = (V, E)$, where a feasible solution $x = (V, E_x)$ is a spanning tree such that every connected component of x is a given $h-$chain;

Z_7 is a problem of covering of a graph $G = (V, E)$ by chains of a given set $H \subseteq \{2, ..., n\}$, where a feasible solution $x = (V, E_x)$ is a spanning tree such that every connected component of x is a given $h-$chain and $h \in H$;

Z_8 is a star covering problem of a graph $G = (V, E)$, where a feasible solution $x = (V, E_x)$ is a spanning tree such that every connected component of x is a star of a given type;

Z_9 is a schedule problem given on a graph $G = (V, E)$, where a feasible solution $x = x_{ij}$ is Boole $n \times n$ matrix such that $\sum\limits_{j=1}^{n} x_{ij} = 1$, $i = 1, ..., n$ and $\sum\limits_{i=1}^{n} x_{ij} = 1$, $j = 1, ..., n$;

Z_{10} is a whole-number transport problem.

Most widely recognized class of algorithms used for finding CSA of multi-criteria problems Z_q, $q \in \{1, ..., 10\}$ is the one called by Linear Convolution Algorithms. Each LC algorithm is based on the following property: for any vector

$$\lambda \in \Lambda_N \left\{ \lambda = (\lambda_1, ..., \lambda_N) : \sum_{v=1}^{N} \lambda_v = 1, \ \lambda_v > 0 \right\}$$

an element x^* minimizing on the FSS X the linear convolution

$$F^{\lambda}(x) = \sum_{v=1}^{N} \lambda_v \ F_v(x)$$

of VOF is PO.

$N-$criteria mass MOP is called unsolvable by means of LCA if there is an instance problem of this MOP such that

$$\exists\, \tilde{x} \in \tilde{X} : F^{\lambda}(\tilde{x}) > \min\left\{F^{\lambda}(x) : x \in X\right\},\ \forall \lambda \in \Lambda_N,$$

where $F^{\lambda}(x)$ is the linear convolution of the criteria.

It has been already shown by [1], [2], etc. that for many of instance problems belonging to the classes Z_q, $q = \{1, ..., 10\}$ the CSA can not be found by means of LCA. Consequently, in the paper we study the following question: if for a given problem CSA $\widehat{X}$ can not be found by means of LCA, then what is a maximum cardinality of the set $X_\alpha \subseteq \widehat{X}$ that can be found by means of LCA i.e. we want to estimate the solving capability of LCA.

According to [3] any problem of whole number programming Z_q, $q \in \{1, ..., 10\}$ can be restated as a problem of Boole programming. Consequently, in the paper we study only the problem of Boole programming described below.

Let each element e from the set $E = \{e\}$ be weighed by the numbers

$w_k(e) \in M_k$, $M_k \subseteq \{1, 2, ..., r\}$, $k = 1, ..., N$.

The cardinality of the sets $|M_k| = r_k$ and $|E| = m$.

Feasible Set of Solutions $X = \{x\}$ is defined by the set of subsets $E_x \subseteq E$.

Define a feasible vector $x = \{x(e)\} \in X$, $|x| = m$ corresponding to every subset E_x such that

$x(e) = 1$, if $e \in E_x$;

$x(e) = 0$, if $e \notin E_x$.

Define on the set $X = \{x\}$ the *VOF* $F(x) = (F(x_1), ..., F(x_N))$ with the following criteria:

$$(1)\quad F_k(x) = \sum_{e \in E_x} u_k(e) \cdot x(e) \to \min, \quad k = 1, ..., N.$$

For describing of instance problems of the given mass problem let us number the elements $e \in E$ such that $E = \{e_i\}$, $i = 1, ..., m$ and define on them the *VOF* $F(x)$.

Thus, we define a vector

$w = (w^{(1)}, ..., w^{(N)})$,

where each element

$w^{(k)} = (w_k(e_1), ..., w_k(e_m))$

is a vector of criteria parameters $F_k(x)$, $k = 1, ..., N$ corresponding to the given *VOF*.

Denote the *VOF* defined by the vector w by $F^w(x)$. Note, that the *FSS* X and the vector w generate an instance problem.

Mass problem is defined by the set $W = \{w\}$, where $W \subseteq (Z_+^m)^N$ and the set $\aleph = \{X\}$, $X \subseteq \{0, 1\}^m$.

Let $F^{\lambda, w}(x) = \sum_{k=1}^{N} \lambda_k F_k^w(x)$ be a linear convolution of criteria $F_k^w(x)$ of *VOF* $F^w(x)$.

The solution $x^* \in \widetilde{X}$ is called an Effective Point *(EP)* for the problem with *VOF* $F^w(x)$ if there is exists a vector

$\lambda^* = (\lambda_1^*, ..., \lambda_N^*) \in \Lambda_N$ such that $F^{\lambda^*, w}(x) = \min_{x \in X} \sum_{k=1}^{N} \lambda_k^* F_k^w(x)$.

Denote by

$$Y_X^w = \left\{ x^* \in X \mid \exists \lambda^* \in \Lambda_N,\ F^{\lambda^*, w}(x) = \min_{x \in X} \sum_{k=1}^{N} \lambda_k^* F_k^w(x) \right\}$$

the set *EP* of the given problem.

We call by the solving capability of *LCA* the value

$$(2)\quad \rho_q = \max_{X \in \aleph} \max_{w \in W} |F^w(Y_X^w)|.$$

The value ρ_q represents a high boundary of cardinality of *EP* set in a criteria space. Consequently, the value ρ_q corresponding to the given problem can be interpreted as a maximum subset of *CSA* $\widehat{X}$ that can be found by means of *LCA*.

3 SPANNING TREE PROBLEM

According to [6] for every instance problem Z_q, $q \in \{1, ..., 10\}$ with VOF $F^w(x)$ given on the FSS X the low boundary of the cardinality of CSA $\widehat{X}$ satisfies to following inequality:

$$(3)\quad \left|\widehat{X}\right| \leq |F(Y_X^w)|.$$

From (3) and results obtained by [6] follows that there is such $N-$criteria problem Z_q, $q \notin \{1, ..., 10\}$, for which the low boundary of the ρ_q value can be estimated as

$$(4)\quad \rho_q \geq N!.$$

Most likely that the problem Z_q is "artificial", i.e. the value of $F^w(x)$, $x \in X$ is given by a table. For the "natural" problems Z_q, $q \in \{1, ..., 10\}$ it is not known yet whether or not for every N and $n = n(N)$ there exists a problem for which ρ_q satisfies the inequality (4).

Subsequently we concentrate on estimating the high boundary for the solving capability of LCA. We consider the value ρ_q as a function of the parameters N and n i.e. $\rho_q = \rho_q(N, n)$.

First let us study the case when $N = 2$ and $n = 2$ for a spanning tree problem Z_2 with VOF

$$(5)\quad F^w(x) = (F_1(x),\ F_2(x)),$$

where

$$(6)\quad F_k^w(x) = w_k(x) \to \min,\ k = 1, 2.$$

For our purpose it is convenient to consider the convolution of criteria (6) in the form

$$(7)\quad F^t(x) = t \cdot F_1^w(x) + F_2^w(x),$$

where $t \in (0, +\infty)$.

Note, that there is unique relationship $t = \frac{\lambda_1}{\lambda_2}$ between the elements of the set

$\Lambda_2 = \{(\lambda_1, \lambda_2) \mid \lambda_1 + \lambda_2 = 1,\ \lambda_1, \lambda_2 > 0\}$

and the set

$\Gamma = \{t \mid t \in (0, +\infty)\}$.

Consequently if for an effective point $x^* \in Y_X^w$ the equality

$F^{\lambda^*,w}(x^*) = \min_{x\in X} (\lambda_1 \cdot F_1^w(x) + \lambda_2 \cdot F_2^w(x)),$
is satisfied then, the equality
$F^{(t^*)}(x^*) = \min_{x\in X} F^{(t^*)}(x)$
is also satisfied for $t^* = \frac{\lambda_1^*}{\lambda_2^*}$ and visa versa.

Let the *FSS* X of the problem Z_2 be determined by $n-$vertices $2-$weighted graph $G = (V, E)$ and the sets *PS* $\widetilde{X}$, *CSA* $\widehat{X}$, *EP* $Y_X^w \subseteq \widetilde{X}$, in turn, be determined by *VOF* (5) $-$ (6). We denote the graph $G = (V, E)$ by $G^{(t)}$, $t \in \Gamma$ if each edge $e \in E$ of the graph G is weighted by the convolutions of weights

(8) $f_e(t) = t \cdot w_1(e) + w_2(e).$

LCA finds an effective point $\widetilde{x} \in \widetilde{X}$ in the following way: for a given graph $G^{(t)}$ an algorithm finds such a spanning tree x that has a minimum weight. We can apply, for example, well-known *Kraskal's* algorithm [5] that arranges the edges $e \in E$ of the graph G in sequence $e_1, ..., e_2$, $m = |E|$ according to diminishing of their weights:

(9) $f_{e_1}(t) \leq f_{e_2}(t) \leq ... \leq f_{e_m}(t).$

Then, it scans the sequence (9) in direction of increase of the index i and marks each edge e_i if e_i does not close a cycle in a $l-$net, $l \geq 0$, where the $l-$net consists of edges e_k, $k \leq i-1$ marked by the algorithm before. The construction of the spanning tree finishes when $n-1$ edges are marked.

For a fixed value $t \in \Gamma$ let $X^{(t)}$ be the set of all spanning trees of the graph $G^{(t)}$. For any $x = (V, E_x) \in X^{(t)}$ let us number the edges $e \in E_x$ by indices $j = 1, ..., n-1$ according to the sequence of weights $E_x = \{e_1, ..., e_{n-1}\}$ given by (9). The obtained sequence of weights we call by the structure of weights.

(10) $f(x, t) = \{f_{e_1}(t), f_{e_2}(t), ..., f_{e_m}(t)\}.$

Lemma 3.1

Any optimum trees $x' = (V, E_{x'}) \in X^{(t)}$ *and* $x'' = (V, E_{x''}) \in X^{(t)}$ *have identical structures of weights* $f(E_{x'}, t) = f(E_{x''}, t)$.

Proof:

The proof of this well-known fact for the algorithms of *Prima* and *Kraskal* can be found, for example, in [4] .□

For each edge e of the graph $G = (V, E)$ consider the functions $f_e(t)$, $t \in (0, +\infty)$. If for a pair of edges $e', e'' \in E$ the graphs of the functions $f_{e'}(t)$ and $f_{e''}(t)$ cross, then denote the point of their crossing by $t_{e'e''}$. In the case $w_1(e') \neq w_2(e'')$ we obtain the following equality:

(11) $t_{e'e''} = \frac{w_2(e'')-w_2(e')}{w_1(e')-w_1(e'')}$, $w_1(e') \neq w_2(e'').$

We call the value $t_{e'e''}$ by a critical point.

Lemma 3.2

For 2−weighted graph $G = (V, E)$ fix a number $t_0 > 0$, such that t_0 does not coincide with any of the critical points. Then, the VOF (5) − (6) has the same value for all optimum points $X^{(t_0)}$.

Proof:

The case $|X^{(t_0)}| = 1$ is trivial. Consider the case $|X^{(t_0)}| \geq 2$. Choose in $X^{(t_0)}$ any pair of minimum spanning trees $x' = (V, E_{x'})$ *and* $x'' = (V, E_{x''})$. For chosen trees consider the structures of weights $f(E_{x'}, t_0)$ and $f(E_{x''}, t_0)$. It follows from **Lemma 3.1** that $f(E_{x'}, t_0) = f(E_{x''}, t_0)$, i.e. the following equalities are satisfied:

(12) $f_{e'_j}(t) = f_{e''_j}(t)$ $j = 1, ..., n-1$.

Suppose that $F^w(x') \neq F^w(x'')$. Then, it follows that $w_1(x') \neq w_2(x'')$. Consequently, for the structures of weights $f(E_{x'}, t_0)$ and $f(E_{x''}, t_0)$ there is a number $j \in \{1, ..., n-1\}$ such that

(13) $w_1(e'_j) \neq w_2(e''_j)$, $e'_j \in E_{x'}$, $e''_j \in E_{x''}$.

Taking into account (10) − (13) we obtain:

$t_0 = t_{e'e''} = \frac{w_2(e'')-w_2(e')}{w_1(e')-w_1(e'')}$,

that means that t_0 is a critical point, a contradiction.□

Lemma 3.3

Assume that for $t_1 \neq t_2$, $t_1, t_2 > 0$ the optimal point $x' \in X^{(t_1)}$ does not belong to the set of the optimum solutions $X^{(t_2)}$. Then, either the point t_1 is critical or there is a critical point between t_1 and t_2.

Proof:

In addition to the given point $x' \in X^{(t_1)}$ consider any optimum point $x'' \in X^{(t_2)}$. The point $x' \notin X^{(t_2)}$ and consequently the trees $x' = (V, E_{x'})$ and $x'' = (V, E_{x''})$ have not only different sets of edges $(E_{x'} \neq E_{x''})$, but different structures of weights given by (10) for $t = t_1$ and $t = t_2$. Then, according to (8), it is possible to find the number j of the edge $e'_j \in E_{x'}$ such that the following inequalities are satisfied:

(14) $t_1 \cdot w_1(e'_j) + w_2(e'_j) \leq t_1 \cdot w_1(e''_j) + w_2(e''_j)$;

$t_2 \cdot w_1(e'_j) + w_2(e'_j) > t_2 \cdot w_1(e''_j) + w_2(e''_j)$.

Note that from (14) follows that these inequalities can be satisfied together only if $w_1(e'_j) \neq w_1(e''_j)$.

Rewrite (14) as follows:

(15) $t_1 \cdot (w_1(e'_j) - w_1(e''_j)) \leq w_2(e'_j) - w_2(e''_j)$;

$t_2 \cdot (w_1(e_j') + w_1(e_j'')) > w_2(e_j') - w_2(e_j'')$.

Calculate the value of the critical point for the pair of edges e_j', e_j''

$t_{e'e''} = \frac{w_2(e'')-w_2(e')}{w_1(e')-w_1(e'')}$.

Taking into account (15) we conclude that only cases are feasible:

a). $t_1 \leq t_{e'e''}$ and $t_2 > t_{e'e''}$, if $w_1(e_j') > w_1(e_j'')$;

b). $t_1 \geq t_{e'e''}$ and $t_2 < t_{e'e''}$, if $w_1(e_j') < w_1(e_j'')$.

From the last two expressions follows the statement of **Lemma 3.3**.□

Lemma 3.4

For given graph G let $[t', t'']$, $t' > 0$ be a closed interval that does not contain critical points. Then, for any pair of points $t_1, t_2 \in [t', t'']$ the optimal sets $X^{(t_1)}$ and $X^{(t_2)}$ coincide.

Proof:

Suppose not, i.e. there is a pair of points $t_1, t_2 \in [t', t'']$ such that $X^{(t_1)} \neq X^{(t_2)}$. Then according to **Lemma 3.3** there is a critical point $t_{e'e''} \in [t', t'')$, a contradiction.□

Consider an arbitrary instance problem Z_2 with VOF (5) – (6). The set of edges E of a given $n-$vertices graph $G = (V, E)$ determines FSS X. The vector $w = (w^{(1)}, w^{(2)})$ of the edges weights $e_i \in E$, $w^{(k)} = (w_k(e_1), ..., w_k(e_m))$, $m = |E|$, $k = 1, 2$ determines the value of VOF on the FSS X.

Let $\Gamma(w, E) = \{t_k\}$ be the set of all critical points calculated according to (11) and ordered by the numbers $k = 1, ..., L$ by their values. Note that $L = |\Gamma(w, E)|$. Taking into account that for any graph G the cardinality $|E| = m \leq \binom{n}{2}$ and the number of all possible pair of edges $e', e'' \in E$ is equal to $\binom{m}{2}$ we conclude that for any $n-$vertices graph G and for any vector of weights the number of critical points L satisfies the following inequality:

(16) $L \leq \binom{m}{2}$, $m \leq \binom{n}{2}$.

Each pair of critical points t_k, t_{k+1}, $1 \leq k < L$ defines the interval $I_k = (t_k, t_{k+1})$. Let us also define the intervals $I_L = (t_L, +\infty)$ and I_0 such that $I_0 = (0, t_1)$ if $t_1 > 0$ and $I_0 = \emptyset$ if $t_1 = 0$. Denote by $J = \{I_k\}$ the set of all defined intervals I_k. Note, that the cardinality of the set J is

(17) $|J| \leq L + 1$.

Theorem 3.1

For 2−criteria spanning trees problem Z_2 the solving capability of LCA satisfies the following inequality:

(18) $\rho(2, n) \leq \frac{1}{4} \cdot (n^4 - 2n^3 - n^2 + 2n) + 1$.

Proof:

Note, that the LCA applied to the given problem Z_2 for each fixed number $t \in \Gamma$ finds a pareto optimal solution $x^{(t)}$. Divide the set Γ into two subsets Γ_1 and Γ_2 such that $\Gamma_1 = \bigcup_{k=0}^{N} I_k = \Gamma \setminus \Gamma_2$ and where the set $\Gamma_2 = \Gamma(w, E) = \{t_k\}$ is the set of all critical points.

Consequently the set $X^* \subseteq Y_X^w \subseteq \widetilde{X}$ found by LCA can be presented as a union of the sets $X^{(1)}$ and $X^{(2)}$, where $X^{(1)}$ is the set of all trees $x^{(t)}$ found by LCA for all $t \in \Gamma_1$ and $X^{(2)}$ is the set of all trees $x^{(t)}$ found by LCA for all $t \in \Gamma_2$.

Consider the interval I_k, $k = 0, 1, ..., L$. According to **Lemmas 3.2, 3.4** the value of VOF $F(x^{(t)}) = (f_1^{(k)}, f_2^{(k)})$ found by LCA for the solution $x^{(t)}$ is the same for $\forall t \in I_k$. Thus, taking into account (16) and (17) we can write

$$(19)\ \ |F(X^{(1)})| \leq \binom{m}{2},\ m \leq \binom{n}{2}.$$

According to definition of the set $X^{(2)}$ and (16) we obtain

$$(20)\ \ |F(X^{(2)})| \leq \binom{m}{2} + 1,\ m \leq \binom{n}{2}.$$

From the fact that the inequalities (19) – (20) are satisfied for any 2–criteria problem Z_2 and taking into account that $|F(X^*)| \leq |F(X^{(1)})| + |F(X^{(2)})|$ we conclude that the solving capability of LCA satisfies the following inequality:

$$\rho(2, N) \leq 2 \cdot \binom{m}{2} + 1,\ m \leq \binom{n}{2}.$$

After straightforward calculations from the last expression follows the result (18). The calculated estimate for solving capability are true for any LCA applied to a spanning tree problem.□

4 GENERAL ESTIMATES

Consider 2–criteria problem Z_q, $q \in \{1, ..., 10\}$.

Theorem 4.1

If for $\forall x \in X$ *the cardinality* $|E_x| \leq n$ *and* $\max_{e \in E_x} w_k(e) \leq n^{c_k}$, *where* $c_k =$ *const,* $k = 1, 2$, *then the solving capability of LCA satisfies the following inequality:*

$$(21)\ \ \rho(2, n) \leq \min\left\{n^{c_1+1}, n^{c_2+2}, \upsilon_1, \upsilon_2\right\},$$

where $\upsilon_k = O(n^{2k-1})$, $k = 1, 2$.

Proof:

By definition we have

$F^{(\lambda)}(x) = \sum\limits_{e \in E_x} (\lambda_1 \cdot w_1(e) + \lambda_2 \cdot w_2(e)),\ \lambda_1, \lambda_2 > 0,\ \lambda_1 + \lambda_2 = 1.$

Fix an element x and define $\lambda = \lambda_2$. Note, that $F^{(\lambda)}(x)$ is a linear function of λ.:

$F^{(\lambda)}(x) = \Phi_x(\lambda) = w_1(x) + \lambda \cdot w_2(x),$

where $w_1(x) = \sum\limits_{e \in E_x} w_1(e)$, $w_2(x) = \sum\limits_{e \in E_x} (w_2(e) - w_1(e))$ and $\lambda \in (0,1)$.

If the linear convolution $F^{(\lambda^*)}(x)$ for a fixed value of $\lambda^* \in (0,1)$ reaches its minimum at the point x^*, then for any $x \in X$

$\Phi_x(\lambda^*) \geq \Phi_{x^*}(\lambda^*)$.

Let x_1 and x_2 be the minimums of linear convolution $F^{(\lambda^*)}(x)$ for $\lambda^* = \lambda_1^*$ and $\lambda^* = \lambda_2^*$ respectively. Then, we can write

(22) $\Phi_{x_1}(\lambda_1^*) \geq \Phi_{x_2}(\lambda_1^*)$ and $\Phi_{x_2}(\lambda_2^*) \geq \Phi_{x_2}(\lambda_2^*)$.

Define $\Phi(\lambda) = \Phi_{x_2}(\lambda) - \Phi_{x_1}(\lambda)$. Taking into account (22) and the fact that $\Phi(\lambda)$ is a linear function of the variable λ we can conclude that there exists $\lambda^* \in [\lambda_1^*, \lambda_2^*]$ such that $\Phi(\lambda^*) = 0$. Geometrically it means that the graphs of linear functions $\Phi_1(\lambda^*)$ and $\Phi_2(\lambda^*)$ cross at the point $\lambda = \lambda^*$.

Consequently, there exist coefficients $w_k(x_1)$ and $w_k(x_2)$ that are not identical, i.e.

$\sum\limits_{e \in E_{x_1}} w_k(e) \neq \sum\limits_{e \in E_{x_2}} w_k(e),\ k = 1, 2.$

From it follows that the solving capability of LCA does not exceed the minimum number of different sums $\sum\limits_{e \in E_x} w_k(e)$, that can be obtained from the numbers $w_k(e)$, $k = 1, 2$. Taking into account that all $w_k(e) > 0$ and $\max\limits_{e \in E_x} w_k(e) \leq n^{c_k}$ we can write:

$\sum\limits_{e \in E_x} w_k(e) \leq n^{c_k + 1}.$

Also note, that the number of different elements in the sum $\sum\limits_{e \in E_x} w_k(e)$ does not exceed r_k. Taking into account that the cardinality of $|E_x| \leq n$ we conclude that the number of different sums that can be obtained from the elements $w_k(e)$ does not exceed $O(n^{r_k - 1})$.

The result (21) directly follows from obtained estimates.□

We can extend the obtained results for the N−criteria problems Z_q, $q \in \{1, ..., 10\}$, where $N > 2$.

Theorem 4.2

If for $\forall x \in X$ the cardinality $|E_x| \leq n$ and $\max\limits_{e \in E_x} w_k(e) \leq n^{c_k}$, *where $c_k = const$, $k = 1, ...N$, then the solving capability of LCA satisfies the following inequality:*

(23) $\rho(N, n) \leq \min\left\{n^{c+N-1}, \upsilon\right\},$

where $c = \sum_{k=1}^{N} c_k - \max_{1 \le k \le N} c_k$,

and $v = \frac{1}{\max_{1\le k\le N} v_k} \prod_{k=1}^{N} v_k$, $v_k = O(n^{r_k - 1})$, $k = 1, ..., N$.

Proof:

By definition $F^{(\lambda)}(x) = \sum_{k=1}^{N} \lambda_k F_k(x)$. Taking into account (1) we also can write

(24) $F^{(\lambda)}(x) = \sum_{k=1}^{N} \lambda_k \sum_{e \in E_x} w_k(x),$

where $\lambda = (\lambda_1, ..., \lambda_N)$, $\sum_{k=1}^{N} \lambda_k = 1$, $\lambda_k > 0$, $k = 1, ..., N$.

Without loss of generality assume that $c_N = \max_{1 \le k \le N} c_k$.

From (24) we obtain:

(25) $F^{(\lambda)}(x) = \sum_{k=1}^{N-1} (\sum_{e \in E_x} w_k(x) - w_N(e)) \cdot \lambda_k + \sum_{e \in E_x} w_N(e).$

Let x_1 and x_2 be pareto optimum elements found by LCA for $\lambda = \lambda_1^*$ and $\lambda = \lambda_2^*$ respectively.

Then,

(26) $F^{(\lambda_1^*)}(x_1) < F^{(\lambda_1^*)}(x_2)$, $F^{(\lambda_2^*)}(x_1) < F^{(\lambda_2^*)}(x_2)$.

Define $\Phi(\lambda) = F^{(\lambda)}(x_1) - F^{(\lambda)}(x_2)$. It follows from the inequalities (26) that $\Phi(\lambda_1^*) < 0$ and $\Phi(\lambda_2^*) < 0$, i.e. $\Phi(\lambda) \ne const$. Consequently $\exists k \in \{1, 2, ..., N\}$ such that the inequality

$\sum_{e \in E_{x_1}} w_k(e) \ne \sum_{e \in E_{x_2}} w_k(e)$

is satisfied.

Thus, the cardinality of the set of PO solutions that can be found by LCA does not exceed the number of different sums

$\sum_{e \in E_x} w_1(e), ..., \sum_{e \in E_x} w_N(e)$

that can be obtained from the given elements $w_k(e)$.

Taking into account that all $w_k(e) > 0$, $\max_{e \in E_x} w_k(e) \le n^{c_k}$ and $|E_x| \le n$ we can write

$\sum_{e \in E_x} w_k(e) \le n^{c_k + 1}.$

Consequently, the number of different elements in the sum $\sum_{e \in E_x} w_k(e)$ does not exceed $r_k = |M_k|$ and thus the number of different sums that can be obtained from $w_k(e)$ elements does not exceed $O(n^{r_k - 1})$.

The result (23) directly follows from obtained estimates.□

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