

Intergenerational justice when future worlds are uncertain[☆]

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Abstract

Let there be a positive (exogenous) probability that, at each date, the human species will disappear. We postulate an Ethical Observer (EO) who maximizes intertemporal welfare under this uncertainty, with expected-utility preferences. Various social welfare criteria entail alternative von Neumann- Morgenstern utility functions for the EO: utilitarian, Rawlsian, and an extension of the latter that corrects for the size of population. Our analysis covers, first, a cake-eating economy (without production), where the utilitarian and Rawlsian recommend the same allocation. Second, a productive economy with education and capital, where it turns out that the recommendations of the two EOs are in general different. But when the utilitarian program diverges, then we prove it is optimal for the extended Rawlsian to ignore the uncertainty concerning the possible disappearance of the human species in the future. We conclude by discussing the implications for intergenerational welfare maximization in the presence of global warming.

Keywords: Discounted utilitarianism, Rawlsian, sustainability, maximin, uncertainty, expected utility, von Neumann-Morgenstern, dynamic welfare maximization.

JEL Classification Numbers: D63, D81, O40, Q54, Q56.

1. Introduction

2 We study the problem of intergenerational welfare maximization when the existence of future
3 worlds is uncertain. One of the major examples of this problem today concerns global warming,
4 and how to structure resource use intertemporally in its presence. The theoretical issues raised by
5 uncertainty are quite complex, and in the interest of clarity, we will study only two simple models
6 in this article – and neither of them explicitly models the effect of production on the biosphere
7 and global temperature. In a companion paper (Llavador, Roemer, and Silvestre, 2009), we
8 study a more complex version of the second model of this article, which does take into account
9 the biosphere as a renewable resource: but that paper studies only the case with no uncertainty

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10 concerning the existence of future generations. The conclusions of the present paper suggest
11 some inferences for the more complex problem.

12 We study several (intergenerational) social welfare functions: utilitarian, Rawlsian, ‘extended
13 Rawlsian,’ and ‘Rawlsian with growth.’ The Rawlsian function is identified with the view of
14 *sustainability*, in a model with production.² Sustainability, in our parlance, means sustaining
15 human welfare over time at the highest possible level. This is often called ‘weak sustainability,’
16 to be contrasted with ‘strong sustainability’, which advocates sustaining the physical stock of
17 bio-resources – species variety, forests, and so on. (See, for instance, Neumayer, 2003, and
18 the articles in Asheim, 2007.) In another dimension, it is to be contrasted with the discounted-
19 utilitarian approach, which does not advocate sustaining human welfare over time, but rather the
20 maximization of a weighted sum of generational welfare levels.

21 There is a literature on Rawlsian social choice in the dynamic context, beginning with Arrow
22 (1973), Dasgupta (1974), Solow (1974) and (Phelps and Riley, 1978). As far as we know, how-
23 ever, there is no literature on the Rawlsian problem when the existence of future generations is
24 uncertain.

25 In the next section, we introduce an Ethical Observer (EO) who has von Neumann – Mor-
26 genstern preferences over the future history of the world. These preferences can be utilitarian,
27 Rawlsian or extended Rawlsian. We show that the EO’s expected utility, evaluated at the lottery
28 which specifies stochastically when the human species will come to an end, gives rise either to
29 ‘discounted utilitarianism’ or ‘discounted sustainabilitarianism,’ depending on the EO’s prefer-
30 ences. We apply these criteria to two alternative economies.

31 First (Section 3), we consider a ‘cake-eating’ model: there is a single non-produced con-
32 sumption good that must be allocated over all future generations. The perhaps surprising result
33 is that the sustainabilitarian and the utilitarian recommend exactly the same solution to the cake-
34 eating problem (Theorem 1). Thus, these two apparently very different social preference orders
35 do not differ in their optimal choice in this simple economy.

36 We introduce in Section 4 a generalization of the classical Solow economic growth model.
37 There are two links between generations: investment, which determines the change in capital
38 stock, and education, which determines the transmission of skill to the next generation. It is
39 obvious that the utilitarian and sustainabilitarian cannot in general choose the same path in this
40 model, for with some parameter values, the discounted utilitarian program diverges, while the
41 discounted sustainabilitarian program always has a (finite) solution. Nevertheless, we show that
42 if the discounted utilitarian program converges, and if the initial capital-labor ratio of the econ-
43 omy is sufficiently large, then the two programs do have the same solution (Corollary, Section
44 4.4). A fundamental result for this model is a Turnpike Theorem (Theorem 4), which we prove.

45 More important, perhaps, is the case when the discounted utilitarian program diverges – in-
46 deed, given the characterization of when this occurs (Theorem 5), this may be the empirically
47 salient case. The remarkable result is that in this case, the solutions of the *discounted* sustainabil-
48 itarian program (in the sense of the extended Rawlsian EO) and *undiscounted* sustainabilitarian
49 program are identical (Theorem 6). This case occurs when the economy is sufficiently produc-
50 tive, and the result says that great productivity renders it optimal for the sustainabilitarian EO to
51 ignore the uncertainty concerning the possible disappearance of the human species in the future.
52 We consider this the most important result of our analysis.

²Calling the intergenerational welfare function ‘Rawlsian’ may lead to some confusion. We mean ‘maximin’ applied to the society consisting of an infinite number of generations. It is well known that Rawls himself, however, did not advocate ‘maximin’ for the intergenerational problem.

53 Some readers may find ‘sustainability,’ as we model it, too stark, as it precludes the increase
 54 in the welfare of the representative generational agent over time. In Section 4.5, we introduce
 55 growth, and study optimal paths when it is specified that welfare should grow at some exoge-
 56 nously specified rate g over time.

57 As noted above, when the initial capital-labor ratio is above a certain lower bound, the dis-
 58 counted utilitarian and sustainabilitarian programs have the same solution. In the Appendix we
 59 compute an example showing how the optimal paths of these two programs differ when the initial
 60 capital-labor ratio is below this bound and the utilitarian program converges.

61 In Section 4.6, we focus upon the case when the discounted utilitarian program diverges,
 62 and we note that, if an overtaking criterion is applied to order divergent paths, then the EO
 63 would recommend almost starving the early generations. We contrast this with the discounted
 64 sustainabilitarian, who in this case recommends equal utility for all future generations. We find
 65 the latter recommendation much more appealing.

66 Section 5 concludes and offers some conjectures about the generalization of our results to the
 67 problem of intertemporal distribution in presence of global warming.

68 2. Ethical Observers

69 Consider an economy that will exist for an infinite number of generations; there is one repre-
 70 sentative agent at each date. Denote the generic utility stream by $(u_1, u_2, \dots) \equiv \{u_t\}_{t=1}^{\infty}$.

71 Let \mathbf{P} be an abstract set of feasible infinite utility streams, which may depend on a vector of
 72 initial conditions. A *social welfare function* is a real-valued function with domain \mathbf{P} . If the social
 73 welfare function of the planner, whom we call an Ethical Observer (EO), is $\Omega : \mathbf{P} \rightarrow \mathfrak{R}$, then she
 74 maximizes $\Omega(u_1, u_2, \dots)$ on \mathbf{P} .

75 For example, if the EO is *utilitarian*, then her maximization program is

76 **Program U.** $\max \sum_{t=1}^{\infty} u_t$ subject to $(u_1, u_2, \dots) \in \mathbf{P}$.

77 If the EO is a Rawlsian maximin (i. e., sustainabilitarian), then her maximization program
 78 is

$$\max \inf\{u_1, u_2, \dots\} \text{ subject to } (u_1, u_2, \dots) \in \mathbf{P},$$

79 which can also be written:

80 **Program SUS.** $\max \Lambda$ subject to $(u_1, u_2, \dots) \in \mathbf{P}, u_t \geq \Lambda, \forall t \geq 1$.

81 “SUS” stands for *sustainability*: the economy is sustainable if it chooses a path that guar-
 82 antees a certain level of human welfare forever. Note that in programs *U* and *SUS* there is no
 83 uncertainty concerning the existence of future generations.

84 We now introduce uncertainty by assuming that there is an exogenous probability $p \in (0, 1)$
 85 that mankind will become extinct at each date, if it has not done so already.

86 The exogeneity of p is a simplifying assumption: in many realistic applications, such as
 87 climate change, the policies adopted may well alter the probabilities of survival of mankind.
 88 Our postulate of an exogenous p implies that the EO cannot influence the length T of human
 89 history, i. e., the size of population across time, allowing us to focus on choosing potential utility
 90 levels, while T is randomly variable but exogenous. Whether a generation exists or not is, in our
 91 model, independent of the choices of the EO, enabling us to sidestep the well-known dilemmas
 92 of population ethics (see, e. g., Parfit, 1982, 1984).

93 We suppose that the preferences of the EO satisfy the expected utility hypothesis. An out-
 94 come (or ‘prize’) is defined by a date T , interpreted as the last date before extinction, and a util-
 95 ity vector (u_1, u_2, \dots, u_T) . Accordingly, her von Neumann-Morgenstern (vNM) utility function
 96 is defined on outcomes $(T; u_1, u_2, \dots, u_T)$, with vNM utility values $W(T; u_1, u_2, \dots, u_T)$. Under
 97 our assumption of exogenous probabilities, the EO’s choice of a path $(u_1, u_2, \dots) \in \mathbf{P}$ defines a
 98 lottery with expected utility

$$\begin{aligned} p W(1; u_1) &+ p(1-p) W(2; u_1, u_2) + p(1-p)^2 W(3; u_1, u_2, u_3) + \dots \\ &= p \sum_{t=1}^{\infty} (1-p)^{t-1} W(t; u_1, u_2, \dots, u_t). \end{aligned} \quad (1)$$

99 The vNM utility of a *utilitarian* EO if the world lasts T dates and she has chosen the path
 100 (u_1, u_2, \dots) is

$$W^U(T, u_1, \dots, u_T) \equiv \sum_{t=1}^T u_t,$$

101 and the expected utility of (u_1, u_2, \dots) is

$$p u_1 + (1-p)p(u_1 + u_2) + (1-p)^2 p(u_1 + u_2 + u_3) + \dots \quad (2)$$

102 By grouping the terms in (2), it becomes

$$\begin{aligned} &u_1 p(1 + (1-p) + (1-p)^2 + \dots) + \\ &u_2 (1-p)p(1 + (1-p) + (1-p)^2 + \dots) + \\ &u_3 (1-p)^2 p(1 + (1-p) + (1-p)^2 + \dots) + \dots \\ &= \sum_{t=1}^{\infty} (1-p)^{t-1} u_t. \end{aligned} \quad (3)$$

103 This immediately justifies the view that the utilitarian Ethical Observer should be, in the
 104 presence of uncertain future worlds, a *discounted utilitarian*, with the following optimization
 105 program.

106 **Program DU.** $\max \sum_{t=1}^{\infty} \varphi^{t-1} u_t$ subject to $(u_1, u_2, \dots) \in \mathbf{P}$, with $\varphi \equiv 1 - p$.

107 We believe this is, indeed, the most solid justification for the discounted-utilitarian ethic.³
 108 Note, however, that the discount factor, $\varphi \equiv 1 - p$, should be very close to one, assuming that p is
 109 very close to zero.⁴ Indeed, we cannot justify, using this approach, the relatively small discount
 110 factors that are often used in intergenerational welfare economics.

³Many economists attempt to justify the use of a discount factor on the grounds that individuals discount the utility they will receive at a later period in their lives. This fact can only justify using such a (subjective) discount factor in the context of a model with an infinite number of generations if we view the problem as isomorphic to a problem in which there is a single, infinitely lived agent. We cannot accept the plausibility of such an isomorphism. Just because an individual may today discount his future utility does not imply that ethical observers, today, are entitled to discount the utility of future generations. This point was clearly stated by Ramsey (1928) in his pioneering work on the theory of saving, who wrote, “One point should be emphasized more particularly; we do not discount later enjoyments in comparison with earlier ones, a practice which is ethically indefensible and arises merely from weakness of the imagination; we shall, however, in Section II, include such a rate of discount in some of our investigations.”

⁴Indeed the Stern Review (2007) chooses $\varphi = 0.999$ per annum, which we believe is reasonable. Nordhaus (2008), on the contrary, uses the low discount factor 0.985.

111 On the other hand, suppose that the EO is Rawlsian (or sustainabilitarian): she wishes to
 112 maximize the minimum utility of all individuals who ever live. In this case her vNM utility
 113 function is

$$W^R(T; u_1, \dots, u_T) = \min\{u_1, u_2, \dots, u_T\}, \quad (4)$$

114 and her expected utility associated with the path (u_1, u_2, \dots) is $p \sum_{t=1}^{\infty} (1-p)^{t-1} \min\{u_1, \dots, u_t\}$.
 115 Her optimization program is then the following one.

116 **Program R.** $\max p \sum_{t=1}^{\infty} (1-p)^{t-1} \min\{u_1, \dots, u_t\}$ subject to $(u_1, u_2, \dots) \in \mathbf{P}$.

117 Klaus Nehring, Andreu Mas-Colell and Geir Asheim have objected (in private communica-
 118 tions) to (4) for the following reason. Interpreting the vNM values as *ex post* utilities, the EO
 119 will never *ex post* prefer a longer time span to a shorter one with the same utility values for
 120 the dates present in both, i. e., she will *ex post* weakly prefer the outcome $(T; \bar{u}_1, \dots, \bar{u}_T)$ to
 121 the outcome $(T + \tau; \bar{u}_1, \dots, \bar{u}_T, u_{T+1}, \dots, u_{T+\tau})$, and she will actually prefer the shorter one if
 122 $u_t < \min\{\bar{u}_1, \dots, \bar{u}_T\}$ for some $t > T$. Consider for instance the outcomes $(5; \bar{u}, \bar{u}, \bar{u}, \bar{u}, \bar{u} - \varepsilon)$ and
 123 $(4; \bar{u}, \bar{u}, \bar{u}, \bar{u})$. In the second case, humans disappear at date 5; in the first case, at date 6, and the
 124 last generation has almost the utility of the previous ones. Yet the EO under formulation (4) must
 125 *ex post* prefer the second, shorter outcome. Note that this preference violates the “mere addition”
 126 desideratum in Parfit’s population ethics (Parfit, 1982).

127 As indicated, the difficulty is not critical under our assumption of an exogenous p , because
 128 our EO chooses, *ex ante*, lotteries with fixed probabilities, rather than outcomes. For instance,
 129 under our assumption of constant, exogenous probability, the EO would certainly choose the
 130 lottery $(\bar{u}, \bar{u}, \bar{u}, \bar{u}, \bar{u} - \varepsilon, 0, 0, \dots)$ over the lottery $(\bar{u}, \bar{u}, \bar{u}, \bar{u}, 0, 0, 0, \dots)$. But the problem would
 131 become serious were p endogenous. Indeed, the well-known criticisms of the maximin approach
 132 become more telling in the presence of an endogenously variable population.

133 Nehring’s suggestion is that we modify the vNM utility function to be

$$W^N(T; u_1, \dots, u_T) = T \min\{u_1, u_2, \dots, u_T\}. \quad (5)$$

134 Thus, in the example just given, the EO would *ex post* prefer the first outcome as long as $\varepsilon <$
 135 $\frac{\bar{u}}{5}$. Formulation (5) confers a powerful role to the length T of human history. But this too could
 136 be problematic were the probability of extinction endogenous and, accordingly, the EO could
 137 influence T : the resulting tradeoff between T and the sustainable utility level $\min\{u_1, u_2, \dots, u_T\}$
 138 could then lead to Parfit’s (1984) “repugnant conclusion.”⁵

139 More generally, the EO may adopt a vNM utility function of the form

$$W^\beta(T; u_1, \dots, u_T) = (1 + (T - 1)\beta) \min\{u_1, u_2, \dots, u_T\}, \quad (6)$$

140 with $\beta \in [0, 1]$, which reduces to (4) when $\beta = 0$ and to (5) when $\beta = 1$. An EO with the vNM
 141 utility function of (6) will be called an *Extended Rawlsian EO*.

142 We study the optimization programs of the various EO’s in two particular economic models:
 143 the cake-eating economy, and the education and capital economy, which yield quite different
 144 results. We will say that two programs are *equivalent* if one possesses a solution if and only if
 145 the other possesses a solution, and when both possess a solution, the solutions are the same.

⁵We are indebted to the referee for this comment.

146 Our main result in the cake-eating economy is the equivalence between programs DU and
 147 R : the Rawlsian (or sustainabilitarian) ethical observer and the utilitarian ethical observer make
 148 identical choices in the presence of uncertain future worlds.

149 In the education and capital economy, Program DU may diverge or converge: our main result
 150 there is that, if DU diverges, then, for any $\beta \in [0, 1]$, the EO's optimization problem under the
 151 vNM of (6), which, as noted, includes as special case Program R , is equivalent to the uncertainty-
 152 free program SUS : the Extended Rawlsian EO can then *ignore uncertainty*.

153 We conclude this section with a lemma.

154 **Lemma 1.** *If “ (u_1^R, u_2^R, \dots) solves Program $R \Rightarrow u_t^R \geq u_{t+1}^R, \forall t \geq 1$ ” and “ $(u_1^{DU}, u_2^{DU}, \dots)$ solves*
 155 *Program $DU \Rightarrow u_t^{DU} \geq u_{t+1}^{DU}, \forall t \geq 1$,” then Programs R and DU are equivalent.*

156 *Proof.* Note that $\min\{u_1, u_2, \dots, u_t\} = u_t, \forall t \geq 1$, if and only if $u_t \geq u_{t+1}, \forall t \geq 1$, in which case the
 157 objective function of Program R is $p \sum_{t=1}^{\infty} (1-p)^{t-1} u_t$, and Program R can be rewritten as

158 **Program CDU .** $\max p \sum_{t=1}^{\infty} (1-p)^{t-1} u_t$ subject to $u_t \geq u_{t+1}, \forall t \geq 1$ and $(u_1, u_2, \dots) \in \mathbf{P}$.

159 The objective function of Program CDU is that of Program DU multiplied by the positive
 160 constant p . If “ $(u_1^{DU}, u_2^{DU}, \dots)$ solves Program $DU \Rightarrow u_t^{DU} \geq u_{t+1}^{DU}, \forall t \geq 1$,” then the constraints
 161 $u_t \geq u_{t+1}$ can be added to Program DU , which then becomes equivalent to Program CDU . \square

162 **Remark.** Lemma 1 cannot cover the Extended Rawlsian EO with $\beta > 0$, who has a different
 163 objective function.

164 3. The cake-eating economy

165 Postulate an economy with a single good, non-producible and initially available in the amount
 166 ω . A consumption path is written (y_1, y_2, \dots) , where y_t is the consumption of the agent (or
 167 generation) alive at date t . For $t = 1, 2, \dots$, the utility function of Agent t is denoted $\tilde{u} : \mathfrak{R}_+ \rightarrow$
 168 $\mathfrak{R} : y_t \mapsto \tilde{u}(y_t)$, and assumed to be increasing. Hence, a consumption path (y_1, y_2, \dots) induces the
 169 utility path $(u_1, u_2, \dots) = (\tilde{u}(y_1), \tilde{u}(y_2), \dots)$. Taking $\omega = 1$, the set of feasible consumption paths
 170 is $\mathfrak{Y} \equiv \{(y_1, y_2, \dots) \in \mathfrak{R}_+^{\infty} : \sum_{t=1}^{\infty} y_t \leq 1\}$, with the set of feasible utility paths $\mathbf{P} = \{(u_1, u_2, \dots) \in$
 171 $\mathfrak{R}^{\infty} : \exists (y_1, y_2, \dots) \in \mathfrak{Y} \text{ such that } u_t = \tilde{u}(y_t), \forall t \geq 1\}$.

172 The *discounted utilitarian* program DU specializes to Program DU_1 , as follows, in the cake-
 173 eating economy.

174 **Program DU_1 .** $\max \sum_{t=1}^{\infty} \varphi^{t-1} \tilde{u}(y_t)$ subject to $\sum y_t \leq 1, y_t \geq 0, \forall t \geq 1$.

175 **Lemma 2.** *If $(y_1^{DU}, y_2^{DU}, \dots)$ solves Program DU_1 , then $y_t^{DU} \geq y_{t+1}^{DU}, \forall t \geq 1$.*

176 *Proof.* Suppose that for some $T, y_{T+1}^{DU} > y_T^{DU}$. Then switch these two terms, and the new policy
 177 strictly dominates $(y_1^{DU}, y_2^{DU}, \dots)$, because the coefficients of the objective function of DU_1 are
 178 strictly decreasing. Contradiction. \square

179 The Rawlsian Program R becomes, in the cake-eating economy, Program R_1 , as follows.

Program R_1 .

$$\begin{aligned} & \max \left\{ p\tilde{u}(y_1) + p(1-p) \min \{ \tilde{u}(y_1), \tilde{u}(y_2) \} + p(1-p)^2 \min \{ \tilde{u}(y_1), \tilde{u}(y_2), \tilde{u}(y_3) \} + \dots \right\} \\ & \text{subject to } \sum y_t \leq 1, y_t \geq 0, \forall t \geq 1. \end{aligned}$$

180 **Lemma 3.** *If (y_1^R, y_2^R, \dots) solves Program R_1 , then $y_t^R \geq y_{t+1}^R, \forall t \geq 1$.*

181 *Proof.* Appendix. □

182 **Theorem 1.** *Programs DU_1 and R_1 are equivalent, and $y_t \geq y_{t+1}, \forall t \geq 1$, at any solution.*

183 *Proof.* Immediate from Lemmas 1-3. □

184 Theorem 1, perhaps surprisingly, tells us that the Rawlsian EO behaves just like a discounted
185 utilitarian –and uses the same discount factor.

186 We now analyze the (common) solutions to programs DU_1 and R_1 .

187 **Theorem 2.** *Let \tilde{u} be concave, differentiable on \mathfrak{R}_{++} and increasing, and suppose that $\lim_{y \rightarrow 0} \tilde{u}'(y)$
188 $= \infty$ (i. e., \tilde{u} satisfies an “Inada condition”). If $(y_1^{DU}, y_2^{DU}, \dots)$ solves Program DU_1 , then
189 $y_t^{DU} > 0$ for all t .*

190 *Proof.* Appendix. □

191 Example 1 in the Appendix provides a utility function $\tilde{u} : \mathfrak{R}_{++}^2 \rightarrow \mathfrak{R}$ (concave, increasing,
192 differentiable) for which programs DU_1 and R_1 do *not* possess a solution.

193 The next theorem studies the case when the derivative of \tilde{u} at zero is finite.

194 **Theorem 3.** *Let \tilde{u} be strictly concave, increasing, and differentiable on \mathfrak{R}_+ , with $\tilde{u}'(0) = \gamma < \infty$.
195 Then Program R_1 possesses a unique solution (y_1^R, y_2^R, \dots) , and there is a date T such that $y_t^R = 0$
196 for all $t \geq T$.*

197 *Proof.* Appendix. □

198 We may thus summarize as follows, for functions \tilde{u} which are strictly concave, increasing,
199 and differentiable except perhaps at zero:

- 200 1. When programs DU_1 or R_1 have a solution, then the solution is unique and identical: the
201 Rawlsian EO and the discounted utilitarian EO make exactly the same recommendation
202 (Theorems 1-3).
- 203 2. If $\tilde{u}'(0) < \infty$, then a solution to programs DU_1 and R_1 does exist. Furthermore, there is a
204 T such that the optimal policy awards zero resource to all dates $t \geq T$: both the Rawlsian
205 EO and the discounted utilitarian EO prescribe zero consumption for all sufficiently distant
206 generations (Theorems 1-3).
- 207 3. If $\lim_{y \rightarrow 0} \tilde{u}'(y) = \infty$, and if there is a solution to programs DU_1 and R_1 , then the solution
208 implies $y_t > 0$ for all t : both the Rawlsian EO and the discounted utilitarian EO prescribe
209 positive consumption for all generations (Theorems 1-2).
- 210 4. There are functions $\tilde{u} : \mathfrak{R}_{++}^2 \rightarrow \mathfrak{R}$ with $\lim_{y \rightarrow 0} \tilde{u}(y) = -\infty$ for which programs DU_1 or R_1
211 have no solution. But if $\tilde{u}'(y_t)$ does not approach infinity too fast as y_t approaches zero,
212 then a solution exists (see Example 1 and its discussion in the Appendix).

213 **4. An economy with education and capital**

214 *4.1. The model*

215 At date t , the available amount of labor, measured in skill units and denoted x_t , is partitioned
216 into three parts: leisure (x_t^l), labor used in the production of commodities (x_t^c) and labor used
217 to educate the next generation (x_t^e). Utility depends on consumption (c_t) and leisure, and is
218 given by the function u : when no confusion is likely, we will denote $u_t = u(c_t, x_t^l)$. Physical
219 capital (s_t^k) and labor produce output according to the production function $f(s_t^k, x_t^c)$: output is
220 partitioned into consumption and investment (i_t). The initial endowment is the pair of stocks
221 $(x_0^e, s_0^k) \in \mathfrak{R}_{++}^2$. Given the initial endowment, a path for the economic variables is feasible if it
222 satisfies the following inequalities for $\forall t \geq 1$:

$$\begin{aligned} (1 - \delta)s_{t-1}^k + i_t &\geq s_t^k \text{ (law of motion of capital), where } \delta \in (0, 1) \text{ is the depreciation rate,} \\ f(s_t^k, x_t^c) &\geq c_t + i_t \text{ (technology for the production of output),} \\ \xi x_{t-1}^e &\geq x_t^e + x_t^c + x_t^l \text{ (education technology).} \end{aligned}$$

223 The last inequality models the technology of education: the quantity of skilled labor at the
224 next date t is simply a multiple ξ of the efficiency units of labor devoted to teaching at date $t - 1$.
225 Thus ξ , which will turn out to be a key parameter, is the rate at which skilled labor can reproduce
226 itself intergenerationally, or, in another locution, the student-teacher ratio.

227 The problem is non-traditional in one way: utility depends not upon raw leisure but upon
228 educated leisure. Thus, we assume that a person's leisure activities are more fulfilling, if she
229 is more highly educated. One might challenge this as an elitist view, but we insist upon it,
230 as we believe that education opens up for the individual increasing opportunities for the use of
231 leisure. We may think of education as permitting the diversification of the leisure resource, which
232 increases its usefulness. In the words of Martin Wolf (2007):

233 The ends people desire are, instead, what makes the means they employ valuable.
234 Ends should always come above the means people use. The question in education
235 is whether it, too, can be an end in itself and not merely a means to some other end
236 – a better job, a more attractive mate or even, that holiest of contemporary grails, a
237 more productive economy.

238 The answer has to be yes. The search for understanding is as much a defining char-
239 acteristic of humanity as is the search for beauty. It is, indeed, far more of a defining
240 characteristic than the search for food or for a mate. Anybody who denies its intrin-
241 sic value also denies what makes us most fully human.

242 On the role of education in production, we are reminded of the recent work of Goldin and
243 Katz (2008), who argue that the main reason for the excellent performance of the American
244 economy in the twentieth century was universal education. Similar points have been made with
245 respect to South Korea and Japan. Of course, the Goldin-Katz claim is somewhat different from
246 ours – theirs is based on the growth of consumption, while ours is based on the centrality of the
247 educational technology for growth of welfare.

248 We impose the following assumption. The Cobb-Douglas hypotheses could be dispensed
249 with in some of the results, but we adopt them for convenience and to shorten some of the
250 arguments.

251 **Assumption A.**

- 252 (a) Cobb-Douglas Utility Function: $u(c, x^l) = c^\alpha (x^l)^{1-\alpha}$, $\alpha \in (0, 1)$;
 253 (b) Cobb-Douglas Production Function: $f(s^k, x^c) = (s^k)^\theta (x^c)^{1-\theta}$, $\theta \in (0, 1)$;
 254 (c) $\xi > 1$.

255 The sustainability program SUS specializes to Program $SUS_2[x_0^e, s_0^k]$, as follows, in the edu-
 256 cation and capital economy.

Program $SUS_2[x_0^e, s_0^k]$.

max Λ subject to

- (v_t) $u(c_t, x_t^l) \geq \Lambda$, $t \geq 1$,
 (a_t) $(1 - \delta)s_{t-1}^k + i_t \geq s_t^k$, $t \geq 1$,
 (b_t) $f(s_t^k, x_t^c) \geq c_t + i_t$, $t \geq 1$,
 (d_t) $\xi x_{t-1}^e \geq x_t^e + x_t^c + x_t^l$, $t \geq 1$.

257 We have written the dual variables in parentheses for future use.

258 We state a turnpike theorem for the SUS_2 program.

259 **Theorem 4** (Turnpike Theorem).

- 260 A. *There is a ray $\Gamma \in \mathfrak{R}_+^2$ such that, if $(x_0^e, s_0^k) \in \Gamma$, then the solution path of Program*
 261 *$SUS_2[x_0^e, s_0^k]$ is stationary.*
 262 B. *If $(x_0^e, s_0^k) \notin \Gamma$, then along the solution path the sequence $((x_1^e, s_1^k), (x_2^e, s_2^k), \dots)$ converges*
 263 *to a point in Γ .*
 264 C. *Along the solution path, all constraints hold with equality (in particular, utility is constant*
 265 *over t).*
 266 D. *The solution to $SUS_2[x_0^e, s_0^k]$ is unique.*

267 *Proof.* Appendix. □

268 Figure 1 illustrates the Turnpike Theorem. The solution path determined by initial conditions
 269 off ray Γ has constant utility, and it has the property that, along this path, the sequence converges
 270 to a point in Γ .

271 **4.2. Discounted utilitarianism: the convergence condition $\varphi < 1/\xi$**

272 The discounted utilitarian program DU of Section 2 specializes to program $DU_2[\varphi, x_0^e, s_0^k]$,
 273 as follows, for the education and capital economy.

Program $DU_2[\varphi, x_0^e, s_0^k]$.

$$\begin{aligned} & \max \sum_{t=1}^{\infty} \varphi^{t-1} u(c_t, x_t^l) \text{ subject to} \\ & (1 - \delta)s_{t-1}^k + i_t \geq s_t^k, \quad t \geq 1, \\ & f(s_t^k, x_t^c) \geq c_t + i_t, \quad t \geq 1, \\ & \xi x_{t-1}^e \geq x_t^e + x_t^c + x_t^l, \quad t \geq 1. \end{aligned}$$

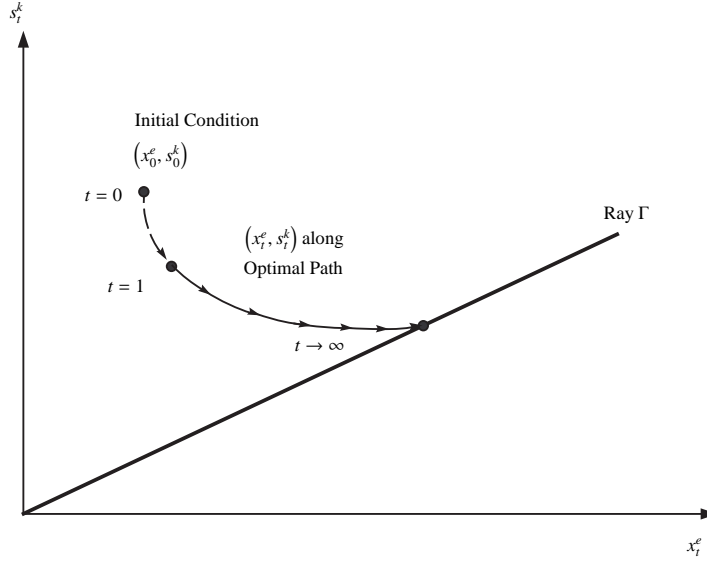


Figure 1: Convergence to ray Γ

274 Note that whether or not $DU_2[\varphi, x_0^e, s_0^k]$ converges depends only on φ and the initial ‘capital-
 275 labor ratio’ $\sigma = s_0^k/x_0^e$, by the homogeneity of the program. (The set of feasible paths is a convex
 276 cone.) We are interested in understanding the set $\{(\varphi, \sigma) \mid DU_2[\varphi, x_0^e, \sigma x_0^e] \text{ converges}\}$.

277 **Theorem 5.**

278 A. If $\varphi\xi > 1$, then Program $DU_2[\varphi, x_0^e, \sigma x_0^e]$ diverges for all $(x_0^e, \sigma x_0^e) \in \mathfrak{R}_{++}^2$.

279 B. If $\varphi\xi < 1$, then Program $DU_2[\varphi, x_0^e, \sigma x_0^e]$ converges for all $(x_0^e, \sigma x_0^e) \in \mathfrak{R}_{++}^2$.

280 *Proof.* Appendix. □

281 Theorem 5 is important for our theory, and perhaps surprising, for it says that the ‘power’ of
 282 the economy, in the sense of its capacity to cause the DU_2 program to diverge, depends only on
 283 the efficiency of the educational technology, namely, the coefficient ξ . In particular, we need no
 284 special assumptions on the technology f other than the standard ones in Assumption A.

285 The proof of Theorem 5 is not particularly transparent, and so we provide here a more intu-
 286 itive argument. Let $x_0^e = 1$. Suppose we can find positive numbers (σ, c, i, x^c, x^l) such that the
 287 following equations hold for some given positive g :

$$(g + \delta)\sigma = i, \tag{7}$$

$$f(\sigma, x^c) = c + i, \tag{8}$$

$$\xi = 1 + g + x^c + x^l. \tag{9}$$

288 Then, from an initial endowment of $(x_0^e, s_0^k) = (1, \sigma)$, we can produce a balanced growth path
 289 in which all variables grow by a rate g at each period. Just notice that the investment defined
 290 by (7) will make $s_1^k = (1 + g)\sigma$, that equation (9) says that $x_1^e = (1 + g)x_0^e$, and that the solution

291 (c, i, x^c, x^l) will grow at rate g from date one onwards, invoking the fact that all three equations
 292 are homogeneous of degree one in the five variables. Now, in order to solve these equations, it is
 293 obviously necessary that $1 + g < \xi$, for otherwise (9) would have no positive solution for (x^c, x^l) .
 294 The interesting fact is that the converse is true as well: as long as $1 + g < \xi$, we can produce
 295 the required solution, which would support a balanced growth path at growth rate g beginning at
 296 a capital-labor ratio σ . To see this, eliminate i using (7) (which will surely be positive for any
 297 positive σ); then we must find (σ, c, x^c, x^l) positive such that:

$$\begin{aligned} f(\sigma, x^c) &= c + (g + \delta)\sigma, \\ \xi - (1 + g) &= x^c + x^l, \end{aligned}$$

298 which is equivalent to finding (σ, x^c) such that

$$\begin{aligned} f(\sigma, x^c) &> (g + \delta)\sigma, \\ 0 < x^c &< \xi - (1 + g). \end{aligned}$$

299 But this can be accomplished if and only if there exists $\sigma > 0$ such that

$$f(\sigma, \xi - (1 + g)) > (g + \delta)\sigma,$$

300 or, invoking the fact that f is one-homogeneous, if and only if:

$$f\left(1, \frac{\xi - (1 + g)}{\sigma}\right) > g + \delta.$$

301 But since f increases without bound as we increase its labor argument, we can surely find σ
 302 sufficiently small that this is true. Let the value of such an admissible σ be denoted $\hat{\sigma}$.

303 Now beginning with an arbitrary positive endowment vector (x_0^e, s_0^k) , we can reach the capital-
 304 labor ratio $\hat{\sigma}$ in a finite number of steps; from there we take off at any desired growth rate $g < \xi - 1$.
 305 Since utility is also homogenous of degree one in (c, x^l) , it grows at that rate too. So the growth
 306 factor of utility is $(1 + g) < \xi$. It is now clear that Program DU_2 diverges if and only if $\varphi\xi > 1$.

307 The reason the above argument is only an intuition for, rather than a proof of, Theorem 5, is
 308 that a proof cannot limit itself to studying only balanced growth paths.

309 We remind the reader that Theorem 5 depends, as well, on our assumption that the leisure argu-
 310 ment of the utility function is measured in quality units, one that we strongly defend, although
 311 it may be somewhat controversial.

312 4.3. *The divergence of discounted utilitarianism and the sustainability of the extended Rawlsian* 313 *path*

314 Consider the Extended Rawlsian EO, i. e., with vNM utility function given by (6) above,
 315 with $\beta \in [0, 1]$. Her optimization program for the education and capital economy can be written
 316 as follows.

Program $R^\beta[\varphi, x_0^e, s_0^k]$.

$$\max \left\{ u_1 + \varphi(1 + \beta) \min \{u_1, u_2\} + \varphi^2(1 + 2\beta) \min \{u_1, u_2, u_3\} + \dots \right\}$$

subject to $u_t \equiv u(c_t, x_t^l)$ and

$$(1 - \delta)s_{t-1}^k + i_t \geq s_t^k, \quad t \geq 1,$$

$$f(s_t^k, x_t^c) \geq c_t + i_t, \quad t \geq 1,$$

$$\xi x_{t-1}^e \geq x_t^e + x_t^c + x_t^l, \quad t \geq 1.$$

317 **Lemma 4.** For any path $(u_1, u_2, \dots) \in \mathfrak{R}_+^\infty$, the sum $\sum_{t=1}^\infty \varphi^{t-1} (1 + (t-1)\beta) \min\{u_1, \dots, u_t\}$
 318 converges.

319 *Proof.* Appendix. □

320 **Lemma 5.** If (u_1, u_2, \dots) solves Program $R^\beta[\varphi, s_0^k, x_0^e]$, then $u_t \geq u_{t+1}$ for all t .

321 *Proof.* Suppose to the contrary that $u_2 > u_1$. Then it follows that $u_1 = \min\{u_1, u_2\}$. We can
 322 distribute back a small amount of resources from date 2 to date 1: reduce by a small amount ε
 323 the value of x_1^e , increase x_1^l by ε , and decrease x_2^l by $\xi\varepsilon$, making the date 2 agent take the reduction
 324 of his skilled labor supply entirely in a reduction of leisure. This will increase the values of u_1
 325 and $(1 + \beta) \min\{u_1, u_2\}$ and will leave all other numbers $(1 + (t-1)\beta) \min\{u_1, \dots, u_t\}$ unchanged
 326 or possibly greater. Hence, since the objective was finite by Lemma 4, it is now increased, a
 327 contradiction. The general claim follows from an induction argument. □

328 We now state our main theorem:

329 **Theorem 6.** Let $(x_0^e, s_0^k) \in \Gamma$. If $\varphi\xi \geq 1$, then for $\beta \in [0, 1]$, any solution to Program $R^\beta[\varphi, x_0^e, s_0^k]$
 330 is the solution to Program $SUS_2[x_0^e, s_0^k]$. Since the solution to $SUS_2[x_0^e, s_0^k]$ is unique, so is the
 331 solution to $R^\beta[\varphi, x_0^e, s_0^k]$.

332 *Proof.* Appendix. □

333 Combined with Theorem 5, we have that, if Program DU_2 diverges, then the Extended Rawl-
 334 sian EO can *ignore uncertainty* in choosing the optimal path (at least in the case when the initial
 335 endowment vector lies on the ray Γ). We conjecture that Theorem 6 is true even if the initial
 336 endowment is not on the ray Γ .

337 4.4. The case where discounted utilitarianism converges

338 This section focuses on the case $\beta = 0$, for which Program R^β is just the application of the
 339 Rawlsian Program R of Section 2 above to the education and capital economy: let us refer to
 340 it as Program $R_2[\varphi, x_0^e, s_0^k]$, or simply Program R_2 . We expect that, if $\varphi\xi < 1$, then the solution
 341 to Program R_2 will not be the solution to Program SUS_2 , which is to say that the inequalities
 342 $u_t \geq u_{t+1}$ of Lemma 5 will not all be satisfied with equality. Thus, the solution to the Rawlsian
 343 EO's problem under uncertainty R_2 may involve decreasing utilities over time. Indeed this is true
 344 for φ sufficiently close to zero, as the following simple result shows.

345 **Theorem 7.** Given (x_0^e, s_0^k) , there is a number $\bar{\varphi} > 0$ such that, if $\varphi < \bar{\varphi}$, then the solution to
 346 $R_2[\varphi, x_0^e, s_0^k]$ entails $u_1 > u_2$ on the solution path.

347 *Proof.* Appendix. □

348 Moreover, a consequence of Theorem 8 below is that, under our Cobb-Douglas assumptions,
 349 for any $\varphi < 1/\xi$, if the capital-labor ratio s_0^k/x_0^e is sufficiently high, then utilities are strictly
 350 monotone decreasing on the optimal path.

351 We now ask: If the $DU_2[\varphi, x_0^e, s_0^k]$ program converges, is its solution the same as the solution
 352 to $R_2[\varphi, x_0^e, s_0^k]$? By lemmas 1 and 5, this will be the case if, at the solution to $DU_2[\varphi, x_0^e, s_0^k]$,
 353 utilities are weakly decreasing with time.

354 For an initial condition (s_0^k, x_0^e) , define the ‘capital-labor ratio’ $\sigma_0 = s_0^k/x_0^e$. Recall that
 355 $u(c, x) = c^\alpha x^{1-\alpha}$, and $f(s, x) = s^\theta x^{1-\theta}$. Define the following variables:

$$\begin{aligned} E &= (\varphi\xi)^{1/(\alpha\theta)}(1-\delta), \\ \tilde{x}_1^c &= \frac{(\xi-E)\alpha(1-\theta)}{1-\alpha\theta}, \quad \tilde{x}_1^l = \frac{(\xi-E)(1-\alpha)}{1-\alpha\theta}, \quad \tilde{x}_t^e = E^t, \\ \tilde{c}_1 &= \sigma_0^\theta (\tilde{x}_1^c)^{1-\theta} (1-\delta)^\theta, \\ \tilde{s}_t^k &= (1-\delta)^t s_0^k, \quad \tilde{x}_t^c = \tilde{x}_1^c E^{t-1}, \quad \tilde{x}_t^l = \tilde{x}_1^l E^{t-1}, \quad \tilde{c}_t = \tilde{c}_1 ((1-\delta)^\theta E^{1-\theta})^{t-1}. \end{aligned}$$

356 **Theorem 8.** Suppose that $\varphi\xi < 1$, and that $s_0^k/x_0^e = \sigma_0 \geq \sigma^*$ where σ^* is the root of the equation

$$1 - \theta \left(\frac{\tilde{x}_1^c}{(1-\delta)\sigma} \right)^{1-\theta} = \frac{\tilde{c}_1 (\varphi\xi)^{1/(\alpha\theta)} \theta (1-\alpha)}{\sigma (1-\alpha\theta)}.$$

357 Then the solution to $DU_2[\varphi, x_0^e, s_0^k]$ is given by the geometric sequence: $s_t^k = \tilde{s}_t^k x_0^e$, $x_t^e = \tilde{x}_t^e x_0^e$,
 358 $x_t^l = \tilde{x}_t^l x_0^e$, $x_t^c = \tilde{x}_t^c x_0^e$, $i_t = 0$ for all $t \geq 1$.

359 *Proof.* Appendix. □

360 **Corollary.** If $\varphi\xi < 1$ and $\sigma_0 \geq \sigma^*$, then Programs $DU_2[\varphi, x_0^e, s_0^k]$ and $R_2[\varphi, x_0^e, s_0^k]$
 361 are equivalent.

362 *Proof.* Along the solution to Program DU_2 , we have that

$$\tilde{u}_t = \tilde{u}_1 \left(((1-\delta)^\theta E^{1-\theta})^\alpha E^{1-\alpha} \right)^{t-1},$$

363 where $\tilde{u}_1 = \tilde{c}_1^\alpha (\tilde{x}_1^l)^{1-\alpha}$; thus utilities are strictly decreasing with time because $E < 1$. The result
 364 then follows from lemmas 1 and 5. □

365 What happens when $\sigma_0 < \sigma^*$? The solution to DU_2 will not be the well-behaved solution
 366 of geometric decay of Theorem 8. Will, nevertheless, utilities still be monotone decreasing on
 367 the optimal path? Perhaps, surprisingly, the answer is in general negative. Example 2 in the
 368 Appendix has the property that, along the solution path to Program DU_2 , $u_2 > u_1$, whereas the
 369 utilities from date 2 onwards decay geometrically as in Theorem 8.

370 How do the solutions to DU_2 and R_2 compare when they are different and DU_2 converges?
 371 To see this, we calculate the solution to R_2 for Example 2 in the Appendix. There, the Rawlsian
 372 EO gives higher utility to the first generation than the utilitarian EO, but the reverse is true for all
 373 dates after that. In fact, the ratio of utilities for the two programs is constant for dates 2 and later
 374 at 1.015, with the larger utility associated with DU_2 : this is perhaps a surprise.

375 This concludes our discussion of the relationship between the DU_2 and R_2 programs in the
 376 case where DU_2 converges. Unlike the cake-eating problem, the solutions to these two pro-
 377 grams are not always identical –although they are identical when the initial capital-labor ratio is
 378 sufficiently large.

379 Based on Example 2 in the Appendix, we may conjecture what the general solution to
 380 $DU_2[\varphi, x_0^e, s_0^k]$ looks like in the convergent case. There will be a sequence of numbers $\tilde{\sigma} >$
 381 $\sigma^* > \sigma_1 > \sigma_2 > \dots > 0$, where σ^* is given in Theorem 8, where, if $\sigma_T > \sigma_0 > \sigma_{T+1}$, the first
 382 T dates will have $i_t > 0$, and at date $T + 1$, the capital-labor ratio will be $\tilde{\sigma}$, at which point the
 383 geometric-decay solution of Theorem 8 takes over. The same pattern should be true in the solu-
 384 tion to Program $R_2[\varphi, x_0^e, s_0^k]$, except that utility will be equal for all the dates when investment is
 385 positive.

386 4.5. Growth

387 Some may find sustainability, in the sense of program *SUS*, to be too stark, as it leads to
 388 a constant level of human welfare until the disappearance of the species. If, however, we treat
 389 resources, such as the biosphere, as of limited capacity, then sustainability may be the best we
 390 can hope for. Nevertheless, we now introduce a program which permits the growth of welfare.

Program *g-SUS* $[x_0^e, s_0^k]$.

max Λ subject to

$$[(r_t)] \quad u(c_t, x_t^l) \geq (1 + g)^{t-1} \Lambda, \quad t \geq 1,$$

$$[(a_t)] \quad f(s_t^k, x_t^c) \geq c_t + i_t, \quad t \geq 1,$$

$$[(b_t)] \quad (1 - \delta)s_{t-1}^k + i_t \geq s_t^k, \quad t \geq 1,$$

$$[(d_t)] \quad \xi x_{t-1}^e \geq x_t^e + x_t^c + x_t^l, \quad t \geq 1.$$

391 Program *g-SUS* maximizes date-1 welfare subject to assuring that welfare grows at rate g
 392 forever. Obviously, Program *g-SUS* becomes *SUS*₂ when $g = 0$.

393 What is the largest g for which Program *g-SUS* possesses a solution? We give a partial answer
 394 with the next theorem.

395 **Definition.** A balanced growth path at rate g is a path satisfying the (a_t) , (b_t) and (d_t) constraints
 396 of Program *g-SUS* $[x_0^e, s_0^k]$ such that:

$$\begin{aligned} s_t^k &= (1 + g)s_{t-1}^k \text{ and } x_t^e = (1 + g)x_{t-1}^e, \text{ for } t \geq 1, \\ z_t &= (1 + g)z_{t-1} \text{ for all other variables } z \in \{x^c, x^l, i, c\}, \text{ for } t \geq 2. \end{aligned}$$

397 **Theorem 9.** Suppose that $0 \leq g < \xi - 1$ and $x_0^e = 1$. Then there exists a value s_0^k such that the
 398 solution to Program *g-SUS* $[x_0^e, s_0^k]$ is a balanced growth path at rate g . Conversely, if $g \geq \xi - 1$,
 399 then there exists no such path for any value of s_0^k .⁶

400 *Proof.* Appendix. □

401 We expect that a turnpike theorem holds for the *g-SUS* model as well, and so, if and only if
 402 $0 \leq g < \xi - 1$, and given any value of s_0^k , Program *g-SUS* will possess a solution at which all
 403 constraints bind, which converges to a balanced growth path at rate g .

404 4.6. Social choice when $DU_2[\varphi, x_0^e, s_0^k]$ diverges

405 According to Theorem 5, $DU_2[\varphi, x_0^e, s_0^k]$ diverges when $\varphi\xi > 1$. The usual way of choosing
 406 among paths in the case of divergence is to use a version of the overtaking criterion: the latest
 407 proposal that we have seen along these lines is that of Basu and Mitra (2007). The utility path
 408 $(\bar{u}_1, \bar{u}_2, \dots)$ is at least as good as the utility path $(\bar{u}_1, \bar{u}_2, \dots)$ according to the overtaking criterion
 409 if there exists a T such that $\sum_{t=1}^{T-1} \varphi^{t-1} \bar{u}_t \geq \sum_{t=1}^{T-1} \varphi^{t-1} \bar{u}_t$ and $t \geq T \Rightarrow \bar{u}_t \geq \bar{u}_t$. This defines a pre-
 410 order (i. e., an incomplete order) on feasible paths when a program diverges.

411 The proof of Theorem 9 showed that balanced growth paths exist for the education and capital
 412 economy as long as $g < \xi - 1$. The condition for a divergence of such a path in Program DU_2 is
 413 $\varphi(1 + g) \geq 1$. This condition surely holds when g is close to $\xi - 1$ because $\varphi(1 + (\xi - 1)) = \varphi\xi > 1$.

⁶We are not interested in the problem with negative g .

414 Let $(\bar{u}_1, \bar{u}_2, \dots)$ and $(\bar{\bar{u}}_1, \bar{\bar{u}}_2, \dots)$ be two feasible balanced-growth paths for a given initial
 415 endowment (x_0^e, s_0^k) which grow at rates g_1 and g_2 , respectively, where $g_2 > g_1$. It is easy to see
 416 that $(\bar{\bar{u}}_1, \bar{\bar{u}}_2, \dots)$ is better than the utility path $(\bar{u}_1, \bar{u}_2, \dots)$ according to the overtaking criterion.
 417 But it is also the case that utility will be smaller for the early date(s) on the preferred path. (To
 418 grow forever faster requires making early sacrifices.) This is interesting, because discounted
 419 utilitarianism is usually associated with implying that the later generations sustain low utility.
 420 *This, however, is only the case when the program converges.* Indeed, as the proof of Theorem 9
 421 shows, as the growth rate g approaches its unattainable supremum $(\xi - 1)$ (and these high-growth-
 422 rate paths are the most desirable paths according to the overtaking criterion), the utility of the
 423 first generation approaches zero. We do not take this as a criticism of overtaking: rather, it is a
 424 criticism of discounted utilitarianism.

425 In contrast, as Theorem 6 showed, if $\varphi\xi \geq 1$, then the solution to Program $R_2[\varphi, x_0^e, s_0^k]$
 426 entails constant utility for all generations, at the highest possible level at which such a level can
 427 be sustained. We find this distinctly superior, from the ethical viewpoint, to the recommendation
 428 of the discounted utilitarian.

429 Finally, we note that the case of divergence may be the salient one. By definition, $\xi =$
 430 $x_t/x_{t-1}^e = x_t/(\tau^e x_{t-1})$, where τ^e is the fraction of the labor force of generation $t - 1$ that is devoted
 431 to teaching. As a rough approximation, assume that population growth is zero and that skill
 432 growth is zero; then $x_t = x_{t-1}$ and so, if $\tau^e \approx 0.05$, we have $\xi \approx 20$. Since we have suggested,
 433 following Stern (2007), that $\varphi = 0.999$ is appropriate, we have that $\varphi\xi$ is substantially larger than
 434 one.

435 5. Conclusion

436 In the cake-eating problem, we showed that two Ethical Observers, facing uncertain possible
 437 future worlds, who have utilitarian and Rawlsian von Neumann Morgenstern preferences over
 438 risk, respectively, would recommend the same allocation of the exhaustible resource over future
 439 generations. At first blush, it seems surprising that these two Observers, with apparently very
 440 different preferences, would agree on the recommended path. The best analogy we can think
 441 of is with the solution to the problem with *no uncertainty* concerning the existence of future
 442 generations, and a *finite* horizon. The utilitarian and the Rawlsian will recommend the same
 443 allocation of the exhaustible resource in this case –namely, split it equally among all generations.
 444 This solution is unique only if u is strictly concave –if u is linear, then the utilitarian is indifferent
 445 among all possible distributions of the resource.

446 We then introduced a generalization of the classical growth model, which includes an educa-
 447 tion sector. Moreover, we postulated that welfare depends on consumption and educated leisure.
 448 Now, the program of the utilitarian Ethical Observer, in the presence of uncertainty, does not
 449 always converge, while the program of the sustainabilitarian (i. e., Rawlsian) does. We charac-
 450 terized when the former program converges (Theorem 5), and we showed that when it does *not*
 451 converge, the (extended) sustainabilitarian proposes the same path as she would if there were
 452 no uncertainty (Theorem 6). We believe this is an important result, as parameter values in the
 453 real world are likely to be such that the discounted utilitarian program does not converge (see
 454 Section 4.6). Moreover, we argued that if this is the case, then the most desirable paths accord-
 455 ing to the discounted-utilitarian objective would leave the early generations with very low utility.
 456 (This conclusion is very different from the recommendation of discounted utilitarianism in the
 457 convergent case.) In contrast, when the discounted utilitarian program diverges, as we said, the
 458 sustainabilitarian recommends equal welfare for all generations.

459 Finally, we showed that when the discounted utilitarian program converges, it is not generally
 460 the case that the two Ethical Observers will recommend the same paths, although they do if
 461 the capital-labor ratio of the initial endowment vector is sufficiently large (Theorem 8 and its
 462 Corollary).

463 In our companion paper Llavador et al. (2009), we study a model which is a ramification of
 464 the model of Section 4 of the present paper, one which articulates the issue of global warming.
 465 In that model, production of the consumption-investment good affects negatively the quality of
 466 the biosphere (carbon emissions increase global temperature), and the quality of the biosphere
 467 enters into the utility of individuals. As well as a production and education sector, that model
 468 also contains an R&D sector, where research produces knowledge that both improves the tech-
 469 nology of commodity production, and enters directly into the utility of people. (Knowledge and
 470 biospheric quality are global public goods.) We know that with appropriate parameter values,
 471 the discounted utilitarian program of the more ramified model diverges; we do not know whether
 472 analogues of the theorems presented here continue to hold. Naturally, we would be interested in
 473 eventually extending the present analysis to that model: we propose to think of the central results
 474 of the model of Section 4 as conjectures concerning the global-warming model. In particular, if
 475 the discounted-utilitarian objective function diverges on the set of paths defined for the global-
 476 warming model, then we conjecture that the sustainabilitarian can ignore the kind of uncertainty
 477 studied in the present paper (Theorem 6). However, we must say that there is another kind of
 478 uncertainty, not discussed here, which is more the focus of current discussions of global warm-
 479 ing: the uncertainty about the relationship between atmospheric carbon and global temperature
 480 (biospheric quality). That kind of uncertainty involves quite different considerations from those
 481 studied here.

482 Appendix A. Proofs and Examples

483 *Proof of Lemma 3*

484 We claim that for every T , $y_T^R = \min\{y_1^R, y_2^R, \dots, y_T^R\}$. For suppose this were not the case,
 485 for some T . Then let $\varepsilon = y_T^R - \min\{y_1^R, y_2^R, \dots, y_{T-1}^R\}$. By hypothesis, $\varepsilon > 0$. Define the path
 486 $(\bar{y}_1, \bar{y}_2, \dots)$ as follows:

$$\begin{aligned}\bar{y}_T &= y_T^R - \frac{\varepsilon}{2}, \\ \bar{y}_t &= y_t^R + \frac{\varepsilon}{2(T-1)}, \text{ for } 1 \leq t \leq T-1, \\ \bar{y}_t &= y_t^R, \text{ for } t > T.\end{aligned}$$

487 Obviously $(\bar{y}_1, \bar{y}_2, \dots)$ is feasible for Program R_1 . In the move from (y_1^R, y_2^R, \dots) to $(\bar{y}_1, \bar{y}_2, \dots)$,
 488 the first T terms in the objective function of Program R_1 all (strictly) increase. Furthermore, all
 489 terms greater than the T th term either increase or stay the same. Notice that \bar{y}_T remains at least $\frac{\varepsilon}{2}$
 490 greater than the minimum of $\{\bar{y}_1, \bar{y}_2, \dots, \bar{y}_t\}$ for all $t > T$, since that minimum is bounded above
 491 by $\min\{\bar{y}_1, \dots, \bar{y}_{T-1}\}$. So $\tilde{u}(\bar{y}_T)$ is never the minimum in any of the terms of the objective with
 492 $t > T$. Consequently, the objective function of Program R_1 (obviously bounded) attains a higher
 493 value at $(\bar{y}_1, \bar{y}_2, \dots)$ than at (y_1^R, y_2^R, \dots) , a contradiction. \square

494 *Proof of Theorem 2*

495 Step 0. Since $\tilde{u}(0)$ is finite, w.l.o.g., we take $\tilde{u}(0) = 0$.

496 Step 1. Let $(y_1^{DU}, y_2^{DU}, \dots)$ solve Program DU_1 . Suppose there is a T such that $y_t^{DU} = 0$. Then
 497 T must be greater than one. For if $y_1^{DU} = 0$, simply define a new path $(\bar{y}_1, \bar{y}_2, \dots)$ by $\bar{y}_t = y_{t+1}^{DU}$ for
 498 all $t = 1, 2, \dots$. This path increases the value of the objective function in DU_1 , an impossibility.
 499 Therefore $T > 1$.

500 Step 2. Now let T be the smallest date for which $y_t^{DU} = 0$. Then it must be the case that for
 501 any sufficiently small $\varepsilon > 0$, we have $\tilde{u}(y_{T-1}^{DU} - \varepsilon) + \varphi\tilde{u}(\varepsilon) \leq \tilde{u}(y_{T-1}^{DU})$, for otherwise, a transfer of
 502 ε from date $T - 1$ to date T would increase the value of the objective function in Program DU_1 .
 503 But this inequality can be written $\varphi\tilde{u}(\varepsilon) \leq \tilde{u}(y_{T-1}^{DU}) - \tilde{u}(y_{T-1}^{DU} - \varepsilon)$. Dividing both sides by ε and
 504 letting ε approach zero, this implies that $\varphi \lim_{\varepsilon \rightarrow 0} \frac{\tilde{u}(\varepsilon)}{\varepsilon} \leq \tilde{u}'(y_{T-1}^{DU})$. But $\lim_{\varepsilon \rightarrow 0} \frac{\tilde{u}(\varepsilon)}{\varepsilon} = \infty$, which gives the
 505 desired contradiction. \square

506 *Example 1*

507 This is an example of a function \tilde{u} for which Program DU_1 has no solution. Consider the
 508 function $\tilde{u} : \mathfrak{R}_{++} \rightarrow \mathfrak{R} : \tilde{u}(y) = \int e^{1/y} dy - y$. We have $\tilde{u}'(y) = e^{1/y} - 1$, $\tilde{u}''(y) = -\frac{e^{1/y}}{y^2}$. Thus,
 509 \tilde{u} is an increasing, concave function on the positive real line, and the Inada condition holds.
 510 The function \tilde{u} cannot be continuously defined at zero, as it approaches negative infinity as y
 511 approaches zero.

512 If the path $(y_1^{DU}, y_2^{DU}, \dots)$ solves problem DU_1 for this \tilde{u} , then $(y_1^{DU}, y_2^{DU}, \dots)$ must be strictly
 513 positive because the domain of \tilde{u} is \mathfrak{R}_{++} . It follows that the first-order Kuhn-Tucker conditions
 514 hold –there is a number $\lambda > 0$ such that $e^{1/y_t} = 1 + (\lambda/\varphi^{t-1})$ for all t . But this implies that
 515 $y_t = 1/\log[1 + \lambda/\varphi^{t-1}]$, and so it must be the case that $\sum_{t=1}^{\infty} \{1/(\log[1 + \lambda/\varphi^{t-1}])\} = 1$. For
 516 large t , we can approximate the denominator in the terms in this series by $\log \lambda/\varphi^{t-1} = \log \lambda +$
 517 $(t - 1)\log(1/\varphi)$. But these terms grow like $k(t - 1)$, where $k = \log(1/\varphi)$, and so the series
 518 grows like $1/(k(t - 1))$, and therefore it does not converge, a contradiction. Therefore there is no
 519 solution to program DU_1 , and hence to the Program R_1 , for this \tilde{u} .

520 The intuition here is that the derivative of \tilde{u} is increasing too fast (exponentially) as y ap-
 521 proaches zero. Let $V^R(y_1, y_2, \dots)$ be the value of the objective function of Program R_1 at
 522 path (y_1, y_2, \dots) . The result is perhaps surprising, because it is easy to see that the function
 523 $V^R(y_1, y_2, \dots)$ is bounded on the feasible set. Hence, it must be the case for this \tilde{u} that the fi-
 524 nite supremum of $\{V^R(y_1, y_2, \dots) \mid (y_1, y_2, \dots) \text{ is feasible}\}$ is never attained. It is easy to check
 525 that if $\tilde{u}(y) = \frac{y^r}{r}$, for any $r \in (-\infty, 1)$, then Program DU_1 has a solution. The Inada condition
 526 holds for these functions, and the first order-conditions can be solved for a positive path whose
 527 components sum to unity.

528 *Proof of Theorem 3*

529 Step 1. We introduce the following sequence of programs. Define Program DU^T as:

$$\begin{aligned} & \max \sum_{t=1}^T \varphi^{t-1} \tilde{u}(y_t) \\ & \text{subject to } \sum_{t=1}^T y_t \leq 1, \\ & \quad y_t \geq 0, \forall t \geq 1. \end{aligned}$$

530 Step 2. Note that for sufficiently large T , it must be the case that the solution (z_1, z_2, \dots, z_T)
 531 to Program DU^T , which of course exists, has $z_T = 0$. For if not, and $(z_1, z_2, \dots, z_T) \gg 0$, then
 532 there are first order conditions of the form:

$$\varphi^{t-1} \tilde{u}'(z_t) = \lambda, \text{ some positive number } \lambda.$$

533 Of course it follows, from the usual argument, that (z_1, z_2, \dots, z_T) is a weakly decreasing se-
 534 quence, and consequently, by choosing a large T , we can guarantee that z_T is bounded above by
 535 an arbitrarily small number, because of the cake-eating constraint. Consequently λ must be very
 536 close to $\varphi^{T-1} \gamma$, and hence must be arbitrarily small. But since $\tilde{u}'(z_1) = \lambda$, this implies that z_1
 537 becomes arbitrarily large, contradicting the fact that $\sum z_t = 1$. Thus there is a date T such that
 538 the solution to Program DU^T has $z_T = 0$.

539 Step 3. Now let T be the smallest date such that $z_T = 0$; denote the solution to Program DU^T
 540 by $(\bar{z}_1, \bar{z}_2, \dots, \bar{z}_T)$. We will assume that $z_t > 0$ for $t < T$, but the proof can be modified in an
 541 obvious way if this is not the case. Then the following Kuhn-Tucker (K-T) conditions must hold
 542 for the (concave) Program DU^T :

543 There are non-negative numbers λ, μ_T such that:

$$\begin{aligned} \varphi^{t-1} \tilde{u}'(z_t) - \lambda &= 0, \text{ for } t < T, \\ \varphi^{T-1} \gamma - \lambda + \mu_T &= 0. \end{aligned}$$

544 Step 4. We claim that the path $z^{T+1} \equiv (z_1, \dots, z_T, 0)$ is the solution to Program DU^{T+1} . To
 545 see this, write down the K-T conditions for this program, namely:

546 There are non-negative numbers $(\lambda, \mu_T, \mu_{T+1})$ such that:

$$\begin{aligned} \varphi^{t-1} \tilde{u}'(z_t) - \lambda &= 0, \text{ for } t < T, \\ \varphi^{T-1} \gamma - \lambda + \mu_T &= 0, \\ \varphi^T \gamma - \lambda + \mu_{T+1} &= 0. \end{aligned}$$

547 We note that the values of λ and μ_T continue to solve these FOCs, for the vector z^{T+1} , and
 548 we define the new shadow price by

$$\mu_{T+1} = \lambda - \varphi^T \gamma > 0.$$

549 Thus, since we have a concave program, we have shown that z^{T+1} is its solution.

550 Step 5. We continue in this manner to show that the vector $z^S = (z, 0, 0, \dots, 0)$ is the solution
 551 for Program DU^S for any $S > T$. The new Lagrangian multiplier at each step is defined by:

$$\mu_S = \lambda - \varphi^{S-1} \gamma,$$

552 and so we note, for use below, that $\lim_{S \rightarrow \infty} \mu_S = \lambda$.

553 Step 6. We now claim that the vector $(z_1^\infty, z_2^\infty, \dots) \equiv (z, 0, 0, \dots)$ solves Program DU_1 .
 554 We proceed by contradiction. Denote by $V^{DU}(y_1, y_2, \dots)$ the value of the objective function
 555 of Program DU_1 at the path (y_1, y_2, \dots) . Suppose the claim were false, and there is a path
 556 (y_1, y_2, \dots) which $V^{DU}(y_1, y_2, \dots) > V^{DU}(z_1^\infty, z_2^\infty, \dots)$. Write $y_t = z_t^\infty + g_t$ for all t ; of course,
 557 $\sum g_t = 0$. We define a function $H : \mathfrak{R} \rightarrow \mathfrak{R}$ as follows:

$$H(\varepsilon) = \sum_{t=1}^{T-1} \varphi^{t-1} \tilde{u}(z_t^\infty + \varepsilon g_t) + \sum_{t=T}^{\infty} \varphi^{t-1} \tilde{u}(0 + \varepsilon g_t) + \lambda \left(1 - \sum_{t=1}^{\infty} (z_t^\infty + \varepsilon g_t) \right) + \sum_{t=T}^{\infty} \mu_t (0 + \varepsilon g_t).$$

558 Verify that $H(0) = V^{DU}(z^\infty)$ and that $H(1) \geq V^{DU}(y_1, y_2, \dots)$, which follows from the fact
 559 that (y_1, y_2, \dots) is feasible and that the Lagrangian multipliers are all non-negative. Suppose we
 560 can show that H is maximized at zero: then we will know that $H(0) \geq H(1)$, which implies that
 561 $V^{DU}(z^\infty) \geq V^{DU}(y_1, y_2, \dots)$, which is the desired contradiction.

562 Step 7. It therefore remains to show that zero maximizes H . Note that H is a concave
 563 function, so it suffices to show that $H'(0) = 0$. We compute:

$$H'(0) = \sum_{t=1}^{T-1} \varphi^{t-1} \tilde{u}'(z_t^\infty) g_t + \sum_{t=T}^{\infty} \varphi^{t-1} \gamma g_t - \lambda \sum_{t=1}^{\infty} g_t + \sum_{t=T}^{\infty} \mu_t g_t.$$

564 Grouping together all terms associated with the same g_t , we see that for $t < T$, the coefficient
 565 of g_t is $\varphi^{t-1} \tilde{u}'(z_t^\infty) - \lambda = 0$, and for $t \geq T$ the coefficient of g_t is $\varphi^{t-1} \gamma - \lambda + \mu_t = 0$. Thus the
 566 derivative vanishes at zero, as required.

567 Step 8. There is a final, transversality condition: We must show that the function H is well-
 568 defined on the interval $[0,1]$. The only term that might cause concern is the last one, which is
 569 $\varepsilon \sum_{t=1}^{\infty} \mu_t g_t$. But since $\mu_t \rightarrow \lambda$ and $g_t \rightarrow 0$ and $\sum_{t=T}^{\infty} g_t = -\sum_{t=1}^{T-1} g_t$, it follows that $\sum_{t=1}^{\infty} \mu_t g_t$
 570 converges, and the proof is complete.

571 Step 9. The uniqueness of the solution follows from the strict concavity of \tilde{u} . □

572 *Proof of Theorem 4 (The Turnpike Theorem)*

573 *The program*

574 Recall that we aim at finding the maximum level of sustainable utility for a fairly simple
 575 infinitely lived economy. Formally:

576 Program SUS_2

max Λ subject to

(P1) $c_t^\alpha (x_t^l)^{1-\alpha} \geq \Lambda, \quad t \geq 1,$

(P2) $\xi x_{t-1}^e \geq x_t^e + x_t^l + x_t^c, \quad t \geq 1,$

(P3) $(s_t^k)^\theta (x_t^c)^{1-\theta} \geq c_t + i_t, \quad t \geq 1,$

(P4) $(1 - \delta) s_{t-1}^k + i_t \geq s_t^k, \quad t \geq 1.$

577 The initial endowment is a vector (x_0^e, s_0^k) .

578 The *value function* of the program maps the initial endowment into the value Λ ; thus we write
 579 $V(x_0^e, s_0^k) = \Lambda$.

580 Define $F^\Lambda = \{(x_0^e, s_0^k) \mid V(x_0^e, s_0^k) = \Lambda\}$. This is the set of initial endowments that generate
 581 the same value for SUS_2 .

582 We define a *feasible path* as a set of sequences $\{x_t^e\}_{t=0,1,2,\dots}, \{s_t^k\}_{t=0,1,2,\dots}$ and all other variables
 583 beginning at $t = 1$, such that inequalities (P2), (P3), and (P4) hold. Denote the set of feasible
 584 paths by \mathcal{P} .

585 Denote the set of feasible paths beginning at a given initial vector (x_0^e, s_0^k) by $P(x_0^e, s_0^k)$.

586 **Proposition 1.** *The set \mathcal{P} is a closed convex cone. The set $P(x_0^e, s_0^k)$ is closed and convex.*

587 *Proof.* Easy. □

588 **Proposition 2.** *At any solution to Program SUS_2 , all the constraints (P1)-(P4) bind at all dates.*
 589 *The solution to SUS_2 is unique.*

590 *Proof.* Step 1. It is obvious that (P2)-(P4) bind. What requires proof is that $u(c_t, x_t^l) = \Lambda$ for all
591 t . We first prove this is the case for $t = 1$. Suppose, to the contrary, that at an optimal solution,
592 $u(c_1, x_1^l) > \Lambda$. Reduce x_1^l by ε and increase each of x_1^e and x_1^c by $\frac{\varepsilon}{2}$ so that $u(c_1, x_1^l - \varepsilon) \equiv \Lambda' > \Lambda$.
593 Now define (i_1', s_1^k) to be the simultaneous solution of the two equations:

$$c_1 + i_1' = f\left(s_1^k, x_1^c + \frac{\varepsilon}{2}\right),$$

$$(1 - \delta)s_0^k + i_1' = s_1^k.$$

594 Obviously, $s_1^k > s_1^k$; therefore $(x_1^e + \frac{\varepsilon}{2}, s_1^k) \gg (x_1^e, s_1^k)$. It follows that, with this altered vector
595 of endowments, $(x_1^e + \frac{\varepsilon}{2}, s_1^k)$, for the program beginning at date 2, the value of the program
596 beginning at date 2 is greater than Λ , since the value function of the program is homogeneous
597 of degree one in its endowment vector. Let the value of the program, beginning at date 2, be
598 $\Lambda^* > \Lambda$. We have now produced a feasible path where for all t , $u(\hat{c}_t, \hat{x}_t^l) \geq \min(\Lambda', \Lambda^*) > \Lambda$. This
599 contradiction proves that $u(c_1, x_1^l) = \Lambda$.

600 Step 2. Assume now that in any optimal solution, for $1 \leq t < T$, $u(c_t, x_t^l) = \Lambda$, but there
601 is an optimal solution for which $u(c_T, x_T^l) > \Lambda$. Reduce x_{T-1}^e by ε/ξ and increase x_{T-1}^l by the
602 same amount, increasing utility at date $T - 1$, which is now greater than Λ . This decreases
603 $x_T \equiv x_T^e + x_T^l + x_T^c$ by ε , and let this decrease be implemented by decreasing x_T^l by ε , which
604 may be chosen small enough that utility at date T is still greater than Λ . We have now produced
605 an optimal path for the program for which $u(c_{T-1}, x_{T-1}^l) > \Lambda$, which contradicts the induction
606 hypothesis. This proves that for all t , $u_t = \Lambda$.

607 Step 3. We next show that the solution to SUS_2 is unique. Any two solutions must have the
608 same values of $\{(c_t, x_t^l)\}$: for if not, take any non-trivial convex combination of the two solutions,
609 producing another optimal solution for which the constraints (P1) do not bind (using the Cobb-
610 Douglas form of u); this contradicts what has been proved above. In like manner, the values
611 $\{(x_t^e, s_t^k)\}$ must be the same in the two solutions, since otherwise a convex combination of them
612 would produce an optimal solution in which the constraints (P3) do not bind. But if the dated
613 capital-stocks are identical in the two solutions, so must be the dated investments. Since the
614 values $\{(x_t^c, x_t^l)\}$ are identical in the two solutions, we see, by iteration, that the values of $\{x_t^e\}$ are
615 also identical. This proves the claim. \square

616 **Proposition 3.**

- 617 A. Let $(\tilde{x}_0^e, \tilde{s}_0^k) \gg (x_0^e, s_0^k)$. Then $V(\tilde{x}_0^e, \tilde{s}_0^k) \gg V(x_0^e, s_0^k)$.
- 618 B. Along the optimal path beginning at (x_0^e, s_0^k) , there is no T such that $(x_T^e, s_T^k) \gg (x_0^e, s_0^k)$.
- 619 C. Let $(x_{0j}^e, s_{0j}^k) \in F^\kappa$ be an infinite sequence of points in F^κ , for some fixed κ , such that
620 $x_{0j}^k \rightarrow \infty$. Then $s_{0j}^k \rightarrow 0$.

621 *Proof.* A. If $(\tilde{x}_0^e, \tilde{s}_0^k) \gg (x_0^e, s_0^k)$, then there is a positive number δ^* such that $(\tilde{x}_0^e, \tilde{s}_0^k) \gg (1 + \delta^*)(x_0^e, s_0^k)$.
622 Since \mathcal{P} is a cone, and the utility of Generation t is homogenous of degree 1 in its arguments, it
623 follows immediately that $V(\tilde{x}_0^e, \tilde{s}_0^k) > (1 + \delta^*)V(x_0^e, s_0^k)$.

624 B. Suppose that there is a T such that $(x_T^e, s_T^k) \gg (x_0^e, s_0^k)$. Let the value of the program be
625 κ . By Part A, the value of the sub-program that begins at date T is strictly greater than κ . This
626 contradicts the fact that the constraints (P1) are binding for all t .

627 C. Suppose the premise were false; then there is a subsequence $s_{0j}^k \rightarrow S > 0$, some S . We
 628 can choose a number $\hat{S} > S$ and a number \hat{x} such that $V(\hat{x}, \hat{S}) = \hat{\kappa} > \kappa$. We can also choose an
 629 index j such that the program beginning with the endowments (x_{0j}^e, s_{0j}^k) possesses a feasible path
 630 that, at its first step, has three properties:

- (i) $s_1^k > \hat{S}$,
- (ii) $x_1^e > \hat{x}$,
- (iii) $c_1^\alpha (x_1^l)^{1-\alpha} > \hat{\kappa}$.

631 (This is obvious from examining the technology.) It therefore follows that $V(x_1^e, s_1^k) > \hat{\kappa}$: invoke
 632 Part A of this proposition. But this is a contradiction, because $V(x_{0j}^k, s_{0j}^k) = \kappa < \hat{\kappa}$. \square

633 Since all the constraints of SUS_2 bind, we can write down the Kuhn-Tucker conditions for this
 634 concave program. It turns out that these conditions imply only three new pieces of information,
 635 which are:

$$(D1) \quad \frac{x_t^l}{c_t} = \frac{1-\alpha}{\alpha(1-\theta)} \frac{x_t^c}{c_t + i_t}, \quad t \geq 1;$$

$$(D2) \quad \frac{x_{t+1}^l}{c_{t+1}} = \frac{x_t^l}{c_t} \frac{\xi}{1-\delta} \left(1 - \frac{\theta(c_t + i_t)}{s_t^k} \right), \quad t \geq 1.$$

$$(D3) \quad \sum_t \left(\frac{1}{\xi} \right)^t x_t^l \text{ converges.}$$

636 The other Kuhn-Tucker conditions just define the various Lagrangian multipliers, which are
 637 all non-negative.

638 It follows that: *A feasible path and a number κ for which all the primal constraints bind at*
 639 *all t , and for which (D1), (D2) and (D3) hold, is an optimal solution.*⁷

640 *The stationary ray*

641 We ask: Is there a ray of initial endowments in \mathfrak{R}_+^2 for which the optimal solution is *station-*
 642 *ary*, that is, for which all variables are constant over time? We study this by writing down the
 643 primal constraints and equations (D1) and (D2) for a hypothetical stationary ray, and see what
 644 they imply. Indeed, we can solve them: there is a unique such ray for the initial condition. The
 645 ray passes through the following point:

$$x_0^e = 1, s_0^k = (\xi - 1) \left(\frac{\xi\theta}{\xi + \delta - 1} \right)^{\frac{1}{1-\theta}} x^{c*},$$

$$\text{where } x^{c*} = \frac{\alpha(1-\theta)(\xi + \delta - 1)}{\alpha(1-\theta)(\xi + \delta - 1) + (1-\alpha)(\xi + \delta - 1 - \xi\delta\theta)}.$$

646 Indeed, we can compute the values of all the variables on this ray. Call these the *stationary state*
 647 *values*. Of course they are defined up to a multiplicative constant. Let us denote this ray by Γ .

⁷One may ask, conversely: Does the optimal solution have to satisfy these equations? The answer to this must be affirmative: there is an infinite dimensional version of the Kuhn-Tucker theorem, using the Hahn-Banach theorem, which tells us that this is so.

648 *The Turnpike Theorem*

649 It is very difficult to actually compute the optimal path, if we begin from an endowment
 650 vector off the stationary ray Γ . We shall, however, prove (Proposition 4 below) that from any
 651 initial vector (x_0^e, s_0^k) , the optimal solution to SUS_2 converges to a point on Γ .

652 In the following, given any two variables a_t and b_t , we use the notation for ratios: $\frac{a_t}{b_t} = \left(\frac{a}{b}\right)_t$.

653 **Lemma 6.** *Suppose that, in the optimal solution, the limit of the sequence $\left\{\left(x^l/c\right)_{t=1,2,\dots}\right\}$ exists
 654 and is finite. Then the solution converges to the stationary state values.*

655 *Proof.* Step 1. Denote the limit of the sequence $\left\{\left(x^l/c\right)_{t=1,2,\dots}\right\}$ by $\bar{\lambda}$. We first argue that $\bar{\lambda} \neq 0$. If
 656 $\bar{\lambda} = 0$, then $\lim(c/x^l)_t = \infty$. By (D1), $\lim(x^c/(c+i))_t = 0$, and so $\lim(x^c/s^k)_t = 0$, by invoking
 657 (P3). Now $\theta(c_t+i_t)/s_t^k = \theta(x_t^c/s_t^k)^{1-\theta}$, so $\lim\theta(c_t+i_t)/s_t^k = 0$, which means, by (D2), that
 658 $(c/x^l)_{t+1}/(c/x^l)_t \rightarrow ((1-\delta)/\xi) < 1$, because $\xi > 1$. It is therefore impossible that $\lim\left(\frac{c}{x^l}\right)_t = \infty$.
 659 Therefore $\bar{\lambda} > 0$.

660 Step 2. By (P1), $x_t^l(c_t/x_t^l)^\alpha = \kappa$ for all t . Therefore $\lim x_t^l = \kappa\bar{\lambda}^\alpha$ and so $\lim c_t = \kappa\bar{\lambda}^{\alpha-1}$. From
 661 (D2), it also follows that $\frac{\xi}{1-\delta} \lim\left(1 - \frac{\theta(c_t+i_t)}{s_t^k}\right) = 1$; therefore $\lim((c+i)/s^k)_t$ has the value of the
 662 ratio of $(c+i)/s^k$ in the stationary state. Therefore $\lim(x^c/s^k)_t$ has the same value as the ratio of
 663 those variables in the stationary state. By (D1) it now follows that $\bar{\lambda}$ is also the ratio of x^l/c in
 664 the stationary state.

665 Step 3. Suppose that there were a subsequence of $\{s_t^k\}$ that diverged to infinity. Since
 666 $\lim(x^c/s^k)_t$ is finite, it follows that the same subsequence of $\{x_t^c\}$ diverges to infinity. It fol-
 667 lows from (P2) that the same subsequence of $\{x_t^e\}$ diverges to infinity. In particular, there exists
 668 a T such that $(x_T^e, s_T^k) \gg (x_0^e, s_0^k)$. But this contradicts Part B of Proposition 3. Therefore
 669 the sequence $\{s_t^k\}$ is bounded. It immediately follows that the sequence $\{x_t^c\}$ is bounded, since
 670 $\lim(x^c/s^k)_t$ exists and is finite; and since $\lim((c+i)/s^k)_t$ also exists and is finite, the sequence
 671 $\{i_t\}$ is bounded.

672 Thus all the sequences of variables, except possibly for $\{x_t^e\}$, are bounded. Therefore we
 673 can choose a single subsequence of all the variables (except possibly of $\{x_t^e\}$) which converges to
 674 values $(\bar{s}^k, \bar{x}^c, \bar{i})$ and we have already shown that $\{x_t^l\}, \{c_t\}$ converge to values \bar{x}^l and \bar{c} . Furthermore
 675 we know that $\{s_t^k\}$ converges to a positive number, because $\lim(\theta(c_t+i_t)/s_t^k)$ has the value of the
 676 same ratio in the stationary state and $\{c_t\}$ converges to a positive number.

677 It now follows, by invoking Proposition 3, Part C, that $\{x_t^e\}$ does not diverge to infinity –since
 678 $(x_t^e, s_t^k) \in F^k$ for all t . So there is a subsequence of the original sequence such that *all* variables
 679 converge.

680 We proceed to show that this subsequence of variables converges to stationary state values.
 681 Denote the limits:

$$\bar{\lambda}_1 = \lim \frac{c_t + i_t}{x_t^c} = \lim \frac{\bar{c} + \bar{i}}{\bar{x}^c}, \quad (\text{A.1})$$

$$\bar{\lambda}_2 = \lim \left(\frac{s^k}{x^c}\right)_t. \quad (\text{A.2})$$

682 We have shown that $\bar{\lambda}_1$ and $\bar{\lambda}_2$ are the values of the corresponding ratios in the stationary

683 state. Now from (P3) we have:

$$x_t^c \bar{\lambda}_2^\theta - i_t \rightarrow \bar{c}. \quad (\text{A.3})$$

684 Note that equations (A.1) and (A.3) comprise two simultaneous equations, in the limit, for the
 685 limits of the variables x^c and i . Hence the sequences $\{x_t^c\}$ and $\{i_t\}$ must converge, and to stationary
 686 state values, since these same two equations hold for the stationary state variables. We therefore
 687 have, by (A.2), that $\{s_t^k\}$ also converges to the appropriate stationary state value. Likewise with
 688 $\{x_t^e\}$.

689 Finally, indeed the *whole* sequence of variables converges to the same stationary state: for if
 690 not, there would be another limit point approached simultaneously by some other subsequence
 691 of the variables, to a stationary state. But since the stationary ray is unique, that limit of (x_t^e, s_t^k)
 692 must also be on the ray Γ . However, we cannot have two subsequences approaching different
 693 points on the ray: that would violate Proposition 3, Part B. \square

694 **Proposition 4.** *From any initial vector (x_0^e, s_0^k) , the optimal solution to SUS_2 converges to a point*
 695 *on Γ .*

696 *Proof.*⁸

697 Step 1. On the optimal path, the sequence $\{(x_t^l/c)\}_{t=1,2,\dots}$ does not diverge to infinity. Suppose it
 698 did diverge to infinity. Then from (D1), the sequence $x_t^c/(c_t + i_t)$ diverges to infinity also. But,
 699 invoking (P3), $x_t^c/(c_t + i_t) = (x_t^c/s_t^k)^\theta$, and so $x_t^c/s_t^k \rightarrow \infty$. Now $\theta(c_t + i_t)/s_t^k = \theta(x_t^c/s_t^k)^{1-\theta}$, and
 700 so it follows that $\theta(c_t + i_t)/s_t^k$ diverges to infinity. But this contradicts (D2), for it would mean
 701 that eventually the ratio x_t^l/c_t is negative.

702 Step 2. Hence it follows that, on the optimal path, the sequence $\{(x_t^l/c)\}_{t=1,2,\dots}$ has a (finite)
 703 limit point. If the sequence $\{(x_t^l/c)\}_{t=1,2,\dots}$ indeed converges to this limit point, then the theorem is
 704 proved, by Lemma 6.

705 Step 3. Thus, the remainder of the proof will show that the limit point of the sequence
 706 $\{(x_t^l/c)\}_{t=1,2,\dots}$ is unique, and hence it is the limit of the sequence.

707 By exploiting equations (D1) and (P3), we can rewrite (D2) as follows:

$$(D2^*) \quad \left(\frac{x^l}{c}\right)_{t+1} = \left(\frac{x^l}{c}\right)_t \frac{\xi}{1-\delta} \left(1 - \theta \left(\frac{\alpha(1-\theta)}{1-\alpha}\right)^{\frac{1-\theta}{\theta}} \left(\frac{x^l}{c}\right)_t^{\frac{1-\theta}{\theta}}\right).$$

708 It will be convenient to define the function: $f^*(x) = ax(1 - bx^r)$, where $a = \frac{\xi}{1-\delta}$, $b =$
 709 $\theta \left(\frac{\alpha(1-\theta)}{1-\alpha}\right)^{\frac{1-\theta}{\theta}}$, and $r = (1 - \theta)/\theta$. Thus (D2*) says that

$$f^*\left(\frac{x_t^l}{c_t}\right) = \frac{x_{t+1}^l}{c_{t+1}} \text{ for all } t.$$

710 Compute that $\frac{d^2 f^*}{dx^2} = -rab(1+r)x^{r-1}$, and so f^* is a concave function on \mathfrak{R}_+ . Let A^* be the value
 711 of the ratio x^l/c in the stationary state. Then we have: $f^*(A^*) = A^*$ and $f^*(0) = 0$. The first claim
 712 follows since the equation (D2*) holds, of course, at the stationary state as well.

713 Finally, note that another root of f^* is given by $x^* = (1/b)^{1/r}$. Concavity implies that f^* has
 714 only the two fixed points 0 and A^* .

⁸Thanks to Cong Huang, who completed and simplified this proof.

715 Because $\left\{\left(\frac{d}{c}\right)_{t=1,2,\dots}\right\}$ is bounded, it possesses a lim inf and a lim sup. For convenience, denote
 716 $y_t = \left(\frac{d}{c}\right)_t$, and define

$$\sigma = \liminf y_t, \quad \sigma^* = \limsup y_t.$$

717 Since $f^*(y_t) = y_{t+1}$, we have $\inf f^*(y_t) = \sigma$, and by the continuity of f^* , $\inf f^*(y_t) = f^*(\inf y_t) =$
 718 $f^*(\sigma) = \sigma$, so σ is a fixed point of f^* . In like manner, σ^* is a fixed point of f^* .

719 If we can establish that $\sigma \neq 0$, then we must have $\sigma = A^* = \sigma^*$, and hence the limit of $\{y_t\}$
 720 exists. But this is established by an argument that mimics Step 1 of the proof of Lemma 6, as
 721 follows.

722 If $\sigma = 0$, then, by (D1), $\liminf (x^c/(c+i))_t = 0$, and so $\liminf (x^c/s^k)_t = 0$, by invoking
 723 (P3). Now $\theta(c_t + i_t)/s_t^k = \theta(x_t^c/s_t^k)^{1-\theta}$, so $\liminf \theta(c_t + i_t)/s_t^k = 0$, which means, by (D2), that
 724 $\liminf y_{t+1}/y_t = \xi/(1-\delta) > 1$, because $\xi > 1$. But this immediately implies that $\liminf y_t > 0$,
 725 a contradiction. Therefore $\sigma \neq 0$, and Proposition 4 is proved. \square

726 The proof of the Turnpike Theorem follows from the previous discussion, in particular from
 727 propositions 2 and 4. \square

728 *Proof of Theorem 5*

729 *Part A*

730 Step 1. Let $x_0^e = 1$. We claim that for any small $\varepsilon > 0$, we can find values σ and i such that:

$$\begin{aligned} i &= (\xi - \varepsilon + \delta - 1)\sigma, \\ i &= f((\xi - \varepsilon)\sigma, \varepsilon). \end{aligned}$$

731 By plotting the graphs of these two functions in (σ, i) space, we can observe that they cross
 732 at the origin and at some positive value of i –by assumption A(b).

733 Step 2. Let $\varepsilon < (\varphi\xi - 1)/\varphi$, and let σ be chosen to satisfy the equations in Step 1, thus
 734 defining investment at date 1 when

$$x_1^c = \varepsilon, \quad x_1^e = \xi - \varepsilon, \quad c_1 = 0 = x_1^l.$$

735 Note from Step 1 that we may take $s_1^k = (\xi - \varepsilon)\sigma$. Let $V(x_0^e, s_0^k)$ be the value function of
 736 Program $DU_2[x_0^e, s_0^k]$, if it converges. Then we must have, by consideration of the choice of date
 737 1 values above, $V(1, \sigma) \geq 0 + (\xi - \varepsilon)\varphi V(1, \sigma)$. But $(\xi - \varepsilon)\varphi > \xi\varphi - (\xi\varphi - 1) = 1$, implying
 738 that the last equation stated cannot hold, and hence Program DU_2 must diverge beginning with
 739 endowment $(1, \sigma)$.

740 Step 3. It immediately follows that Program DU_2 diverges for $\hat{\sigma} > \sigma$. (Just throw away some
 741 capital at date 1 and reduce the capital-labor ratio to σ .) Moreover, the program must diverge for
 742 $0 < \hat{\sigma} < \sigma$ as well (at date 1, invest very little in education, thus increasing the capital-labor ratio
 743 at date 2 to a value $s_1^k/x_1^e \geq \sigma$). \square

744 *Part B*

745 Step 1. Let $x_0^e = 1$. The largest possible investment that can be made at date 1 if $s_0^k = \sigma$ is
 746 given by $I(\sigma)$, defined by the equation:

$$f((1 - \delta)\sigma + I(\sigma), \xi) = I(\sigma),$$

747 because $x_1^c \leq \xi$. Define σ^* such that:

$$f((1 - \delta), \frac{\xi}{\sigma^*}) / (\xi - (1 - \delta)) = 1 - f_1((1 - \delta)\sigma^*, \xi),$$

748 where $f_1(s, x) = \frac{\partial f}{\partial s}(s, x)$. A monotonicity argument, invoking the intermediate value theorem,
749 shows that σ^* exists uniquely.

750 Let $m = f_1((1 - \delta)\sigma^*, \xi)$ and note that $0 < m < 1$.

751 Step 2. The graph of the function

$$z(i) = f((1 - \delta)\sigma^* + i, \xi)$$

752 lies everywhere on or below the graph of the function

$$y(i) = f((1 - \delta)\sigma^*, \xi) + mi,$$

753 and $y(0) = z(0)$ (by the concavity of f). The second graph meets the 45° ray in (i, y) space at the
754 point $i = f((1 - \delta)\sigma^*, \xi) / (1 - m)$. Therefore

$$I(\sigma^*) \leq \frac{f((1 - \delta)\sigma^*, \xi)}{1 - m}.$$

755 Step 3. Hence, beginning at $s_0^k = \sigma^*$:

$$\begin{aligned} s_1^k &\leq (1 - \delta)\sigma^* + I(\sigma^*) \leq (1 - \delta)\sigma^* + f((1 - \delta)\sigma^*, \xi) / (1 - m) \\ &\leq (1 - \delta)\sigma^* + \sigma^*(\xi - (1 - \delta)) = \xi\sigma^*. \end{aligned}$$

756 Therefore:

$$u(c_1, x_1^l) \leq u(f(s_1^k, \xi), \xi) \leq u(f(\xi\sigma^*, \xi), \xi) \leq \xi N,$$

757 where $N \equiv u(f(\sigma^*, 1), 1)$.

758 Step 4. For any number $\psi > 1$, we have:

$$f((1 - \delta)\psi\sigma^* + \psi I(\sigma^*), \xi) < f((1 - \delta)\psi\sigma^* + \psi I(\sigma^*), \psi\xi) = \psi I(\sigma^*).$$

759 Consider the function $\Psi(x) = x - f((1 - \delta)\psi\sigma^* + x, \xi)$; note that $\Psi'(x) > 0$ (since $m < 1$).
760 We have (from the above) that $\Psi(\psi I(\sigma^*)) > 0$, and by definition, $\Psi(I(\psi\sigma^*)) = 0$. It follows that
761 $I(\psi\sigma^*) < \psi I(\sigma^*)$.

762 Step 5. Now compute that

$$\begin{aligned} s_2^k &\leq (1 - \delta)s_1^k + I(s_1^k) \leq (1 - \delta)\xi\sigma^* + I(\xi\sigma^*) \leq (1 - \delta)\xi\sigma^* + \xi I(\sigma^*), \\ &= \xi((1 - \delta)\sigma^* + I(\sigma^*)) \leq \xi^2\sigma^*, \end{aligned}$$

763 which follows by invoking the definition of $I(\cdot)$, and steps 3 and 4.

764 By induction we have $s_t^k \leq \xi^t \sigma^*$. But $x_t^c \leq \xi^t$ and $x_t^l \leq \xi^t$ as well, and so $u(c_t, x_t^l) \leq$
765 $u(f(s_t^k, \xi^t), \xi^t) \leq \xi^t N$. It follows that $\sum \varphi^{t-1} u_t \leq \xi \sum (\varphi\xi)^{t-1} N < \infty$.

766 Step 6. Now suppose that $\sigma > \sigma^*$; let $\sigma = \psi\sigma^*$, $\psi > 1$. Then beginning at $s_0^k = \sigma$:

$$\begin{aligned} s_1^k &\leq (1 - \delta)\sigma + I(\sigma) = (1 - \delta)\psi\sigma^* + I(\psi\sigma^*) \\ &< \psi((1 - \delta)\sigma^* + I(\sigma^*)) && \text{[by Step 4]} \\ &\leq \psi\xi\sigma^* = \xi\sigma. \end{aligned}$$

767 And so $u(c_t, x_t^j) \leq u(f(s_t^k, \xi_t^e), \xi_t^l) \leq \xi_t^l u(f(\sigma, 1), 1)$, and as before:

$$\sum (\varphi \xi)^{t-1} u_t < \infty.$$

768 **Step 7.** Therefore DU_2 converges for $\sigma \geq \sigma^*$. *A fortiori*, it converges for $\sigma < \sigma^*$, by the free
769 disposal of capital. \square

770 *Proof of Lemma 4*

771 For any (u_1, u_2, \dots) , $\min\{u_1, \dots, u_t\} \geq \min\{u_1, \dots, u_{t+1}\}$. Therefore

$$\begin{aligned} \sum_1^\infty \varphi^{t-1} (1 + (t-1)\beta) \min\{u_1, \dots, u_t\} &\leq u_1 \sum_1^\infty \varphi^{t-1} (1 + (t-1)\beta) \\ &= u_1 [\varphi^0 + \varphi^1 + \varphi^2 + \varphi^3 + \dots \\ &\quad + \beta\varphi^1 + \beta\varphi^2 + \beta\varphi^3 + \dots \\ &\quad + \beta\varphi^2 + \beta\varphi^3 + \dots \\ &\quad + \beta\varphi^3 + \dots \\ &\quad + \dots] \\ &= u_1 \left(\sum_1^\infty \varphi^{t-1} + \beta \left[\sum_2^\infty \varphi^{t-1} + \sum_3^\infty \varphi^{t-1} + \dots \right] \right) = u_1 \left(\frac{1}{1-\varphi} + \beta \left(\frac{\varphi}{1-\varphi} + \frac{\varphi^2}{1-\varphi} + \dots \right) \right) \\ &= u_1 \left(\frac{1}{1-\varphi} + \frac{\beta}{1-\varphi} \left(\frac{\varphi}{1-\varphi} \right) \right) < \infty. \end{aligned}$$

772 Hence, the sum $\sum_{t=1}^\infty \varphi^{t-1} [1 + (t-1)\beta] \min\{u_1, \dots, u_t\}$ of nonnegative terms converges. \square

773 *Proof of Theorem 6*

774 Consider the *constrained discounted utility program* $CDU_2[\varphi, x_0^e, s_0^k]$ (which specializes program *CDU* in the proof of Lemma 1 to the education and capital economy), as follows.
775

Program $CDU_2[\varphi, x_0^e, s_0^k]$.

$$\begin{aligned} \max \sum_{t=1}^\infty \varphi^{t-1} u(c_t, x_t^j) \text{ subject to :} \\ (1 - \delta)s_{t-1}^k + i_t \geq s_t^k, \\ f(s_t^k, x_t^j) \geq c_t + i_t, \\ \xi x_{t-1}^e \geq x_t^e + x_t^c + x_t^j, \\ u(c_t, x_t^j) \geq u(c_{t+1}, x_{t+1}^j), \quad t \geq 1. \end{aligned}$$

776 Note that Program CDU_2 is not concave, because of the last constraint, which is not quasi-
777 concave. (The last constraint is quasi-concave only if u is linear.) Hence we cannot immediately
778 use concave optimization theory to analyze Program CDU_2 .

779 **Lemma 7.** *The solution to $R_2[\varphi, x_0^e, s_0^k]$ is also the solution to $CDU_2[\varphi, x_0^e, s_0^k]$.*

780 *Proof.* Immediate from Lemma 5. □

781 But the solution to CDU_2 is in general *different* from the solution to DU_2 , the last being
782 sometimes unbounded, while CDU_2 is surely bounded.

783 Now if $DU_2[\varphi, x_0^e, s_0^k]$ diverges, then utility is unbounded above over time. It seems reason-
784 able to conjecture that, in this case, the last constraint of $CDU_2[\varphi, x_0^e, s_0^k]$ will bind at every date.
785 But if this is the case, then the solution to $CDU_2[\varphi, x_0^e, s_0^k]$ is just the solution to $SUS_2[x_0^e, s_0^k]$,
786 which means that the egalitarian ethical observer in the environment with uncertain worlds will
787 behave just as if there were no uncertainty.

788 We now prove that this conjecture is true. To do so we make use of the following program.

Program $PP[\varphi, x_0^e, s_0^k]$.

$$\max \left\{ \frac{\Lambda}{1-\varphi} - \sum_2^{\infty} \varphi^{t-1} \lambda_t \right\} \text{ subject to:}$$

$$\lambda_1 \equiv 0,$$

$$(v_t) \quad u(c_t, x_t^l) \geq \Lambda - \lambda_t, \quad t \geq 1,$$

$$(m_t) \quad \lambda_{t+1} \geq \lambda_t, \quad t \geq 1,$$

$$(a_t) \quad f(s_t^k, x_t^c) \geq c_t + i_t, \quad t \geq 1,$$

$$(b_t) \quad (1 - \delta)s_{t-1}^k + i_t \geq s_t^k, \quad t \geq 1,$$

$$(d_t) \quad \xi x_{t-1}^e \geq x_t^c + x_t^l + x_t^e, \quad t \geq 1.$$

789 Dual variables are stated to the left of the constraints. The primal variables in Program PP
790 are all the usual economic variables, plus the variables $\Lambda, \lambda_2, \lambda_3, \dots$. We call the usual economic
791 variables of a feasible path in Program PP the *economic part* of the path. Note that PP is a
792 concave program, so it may be solved with traditional methods.

793 **Lemma 8.** *Let $(x_0^e, s_0^k) \in \Gamma$.⁹ If $\varphi \xi \geq 1$, then the solution to Program $SUS_2[x_0^e, s_0^k]$ forms the*
794 *economic part of the solution to Program $PP[\varphi, x_0^e, s_0^k]$.*

795 *Proof.*

796 Step 1. We first write down the Kuhn-Tucker conditions which characterize the solution to Pro-
797 gram $SUS_2[x_0^e, s_0^k]$. These are:

$$(SUS1) \quad (\partial \Lambda) : \quad 1 = \sum_1^{\infty} v_t,$$

$$(SUS2) \quad (\partial c_t) : \quad v_t u_1[t] = a_t,$$

$$(SUS3) \quad (\partial x_t^l) : \quad v_t u_2[t] = d_t,$$

$$(SUS4) \quad (\partial s_t^k) : \quad a_t f_1[t] + b_{t+1}(1 - \delta) - b_t = 0,$$

$$(SUS5) \quad (\partial i_t) : \quad a_t = b_t,$$

$$(SUS6) \quad (\partial x_t^c) : \quad a_t f_2[t] = d_t,$$

$$(SUS7) \quad (\partial x_t^e) : \quad \xi d_{t+1} = d_t,$$

⁹Recall the definition of Γ in the statement of the Turnpike Theorem.

798 where we use the notation $u_1[t] \equiv \frac{\partial}{\partial c_t} u(c_t, x_t^l)$, $u_2[t] \equiv \frac{\partial}{\partial x_t^l} u(c_t, x_t^l)$, etc. At the solution to SUS_2 ,
799 non-negative dual variables satisfying the above conditions exist and all the primal constraints
800 are binding. Denote the primal (economic) variables at the solution by $(\hat{\Lambda}, \{\hat{c}_t, \hat{x}_t, \dots\}_{t=1}^{\infty})$. If
801 $(x_0^e, s_0^k) \in \Gamma$, then, because the solution is stationary, $u_1[t] = u_1[1]$ for all t , and likewise for the
802 other derivatives of u and f .

803 Step 2. Define $\hat{\lambda}_t = 0$, for all $t \geq 1$. We wish to show that $\hat{\Phi} = (\hat{\Lambda}, \{\hat{c}_t, \hat{x}_t, \dots\}_{t=1}^{\infty}, \{\hat{\lambda}_t\}_{t=1}^{\infty})$ is
804 the solution to Program $PP[\varphi, x_0^e, s_0^k]$. Let $\Phi = (\Lambda, \{c_t, x_t, \dots\}_{t=1}^{\infty}, \{\lambda_t\}_{t=1}^{\infty})$ be the purported optimal
805 path for Program $PP[\varphi, x_0^e, s_0^k]$. Denote the difference between these two paths by:

$$\Delta\Lambda = \Lambda - \hat{\Lambda}, \quad \Delta c_t = c_t - \hat{c}_t, \quad \Delta x_t = x_t - \hat{x}_t, \dots, \quad \Delta\lambda_t = \lambda_t - \hat{\lambda}_t = \lambda_t,$$

806 that is, schematically, $\Delta\Phi \equiv \Phi - \hat{\Phi}$.

807 Define:

$$\begin{aligned} \hat{a}_t &= a_t/(1-\varphi), & t \geq 1, \\ \hat{b}_t &= b_t/(1-\varphi), & t \geq 1, \\ \hat{v}_t &= v_t/(1-\varphi), & t \geq 1, \\ \hat{d}_t &= d_t/(1-\varphi), & t \geq 1. \end{aligned}$$

808 Now define the following function of a real variable:

$$\begin{aligned} \Theta(\varepsilon) &= \frac{\hat{\Lambda} + \varepsilon\Delta\Lambda}{1-\varphi} - \sum_2^{\infty} \varphi^{t-1} (\hat{\lambda}_t + \varepsilon\Delta\lambda_t) + \sum_1^{\infty} \hat{v}_t (u(\hat{c}_t + \varepsilon\Delta c_t, \hat{x}_t^l + \varepsilon\Delta x_t^l) - (\hat{\Lambda} + \varepsilon\Delta\Lambda) + (\hat{\lambda}_t + \varepsilon\Delta\lambda_t)) \\ &+ \sum_1^{\infty} m_t ((\hat{\lambda}_{t+1} + \varepsilon\Delta\lambda_{t+1}) - (\hat{\lambda}_t + \varepsilon\Delta\lambda_t)) + \sum_1^{\infty} \hat{a}_t (f(\hat{s}_t^k + \varepsilon\Delta s_t^k, \hat{x}_t^c + \varepsilon\Delta x_t^c - (\hat{c}_t + \varepsilon\Delta c_t) - (\hat{i}_t + \varepsilon\Delta i_t)) \\ &+ \sum_1^{\infty} \hat{b}_t ((1-\delta)(\hat{s}_{t-1}^k + \varepsilon\Delta s_{t-1}^k) + (\hat{i}_t + \varepsilon\Delta i_t) - (\hat{s}_t^k + \varepsilon\Delta s_t^k)) \\ &+ \sum_1^{\infty} \hat{d}_t (\xi(\hat{x}_{t-1}^e + \varepsilon\Delta x_{t-1}^e) - (\hat{x}_t^e + \varepsilon\Delta x_t^e) - (\hat{x}_t^c + \varepsilon\Delta x_t^c) - (\hat{x}_t^l + \varepsilon\Delta x_t^l)). \end{aligned}$$

809 All the variables in this function are defined except for the sequence of numbers (m_1, m_2, \dots) .
810 Note that Θ is a concave function, a consequence of the concavity of u and f . Note that Θ is
811 defined on $[0, 1]$, since the feasible set of Program PP is convex. Suppose we can produce a non-
812 negative sequence (m_1, m_2, \dots) such that the derivative of Θ exists and is zero at $\varepsilon = 0$. Then Θ
813 will be maximized at zero, and so in particular, $\Theta(0) \geq \Theta(1)$. Now note that $\Theta(0) = \frac{\hat{\Lambda}}{1-\varphi}$, which
814 is the value of the objective function of Program PP at the path $\hat{\Phi}$; all the other terms vanish,
815 since all the primal constraints of Program SUS_2 are binding on this path, and $\hat{\lambda}_t = 0$ for all t .
816 We also have: $\Theta(1) = \frac{\Lambda}{1-\varphi} - \sum_2^{\infty} \varphi^{t-1} \lambda_t +$ non-negative terms. It will therefore follow that

$$\frac{\hat{\Lambda}}{1-\varphi} \geq \frac{\Lambda}{1-\varphi} - \sum_2^{\infty} \varphi^{t-1} \lambda_t,$$

817 proving that the value of the objective function of Program PP at $\hat{\Phi}$ weakly dominates the value
818 at any other feasible path, and hence $\hat{\Phi}$ is a solution to Program PP .

819 Step 3. We now evaluate $\Theta'(0)$, by taking the derivative of Θ w. r. t. ε term by term,
820 gathering terms together. Indeed, what we are doing is re-deriving the Kuhn-Tucker conditions:

821 we are going through this process because there is a step at which we must deviate from the usual
 822 procedure. We compute:

$$\begin{aligned} \Theta'(0) = & \Delta\Lambda \left(\frac{1}{1-\varphi} - \sum_1^\infty \hat{v}_t \right) + \sum_1^\infty \Delta c_t (\hat{v}_t u_1[t] - \hat{a}_t) + \sum_1^\infty \Delta x_t^l (\hat{v}_t u_2[t] - \hat{d}_t) \\ & + \sum_1^\infty \Delta x_t^c (\hat{a}_t f_2[t] - \hat{d}_t) + \sum_1^\infty \Delta x_t^e (\xi \hat{d}_{t+1} - \hat{d}_t) + \sum_1^\infty \Delta i_t (\hat{b}_t - \hat{a}_t) \\ & + \sum_1^\infty \Delta s_t^k \left((1-\delta)\hat{b}_{t+1} - \hat{b}_t + \hat{a}_t f_1[t] \right) + \left\{ \sum_1^\infty m_t (\Delta\lambda_{t+1} - \Delta\lambda_t) - \sum_2^\infty \varphi^{t-1} \Delta\lambda_t + \sum_1^\infty \hat{v}_t \Delta\lambda_t \right\}. \end{aligned}$$

823 Notice that all terms on the r. h. s. of this equation except the last bracketed term vanish by
 824 conditions (SUS1)-(SUS7) of Step 1, and the definition of the dual variables. Furthermore, it is
 825 legitimate to collect and recombine terms as we have, because all the relevant series converge.
 826 The point at which care must be taken is *not* to attempt to recombine terms in the bracketed term,
 827 because the series in the bracketed term may not converge.

Step 4. It follows that we will have shown $\Theta'(0) = 0$ if we can produce a non-negative sequence (m_1, m_2, \dots) such that

$$\sum_1^\infty m_t (\Delta\lambda_{t+1} - \Delta\lambda_t) - \sum_2^\infty \varphi^{t-1} \Delta\lambda_t + \sum_2^\infty \hat{v}_t \Delta\lambda_t = 0,$$

828 which is the same equation as:

$$\sum_1^\infty m_t (\lambda_{t+1} - \lambda_t) - \sum_2^\infty \varphi^{t-1} \lambda_t + \sum_2^\infty \hat{v}_t \lambda_t = 0, \quad (\text{A.4})$$

829 since $\Delta\lambda_t = \lambda_t$ for all $t \geq 1$.

830 If the sequence $(\lambda_1, \lambda_2, \dots)$ is identically zero, then obviously any choice of (m_1, m_2, \dots) will
 831 guarantee (A.7). So suppose this is not the case. Then for some $T \geq 1$, $(\lambda_{T+1} - \lambda_T) > 0$ (recall
 832 that $\lambda_1 = 0$) and all terms $(\lambda_{t+1} - \lambda_t) \geq 0$ (see the constraint in Program *PP*). Consequently, by
 833 choosing $m_T \geq 0$ appropriately, and $m_t = 0$ for all $t \neq T$, we can make the sum $\sum_1^\infty m_t (\lambda_{t+1} - \lambda_t)$
 834 equal *any* desired non-negative number. Hence we can solve (A.4) if (and only if):

$$- \sum_2^\infty \varphi^{t-1} \lambda_t + \sum_1^\infty \hat{v}_t \lambda_t = \sum_2^\infty \lambda_t (\hat{v}_t - \varphi^{t-1}) \leq 0. \quad (\text{A.5})$$

835 Note that both series on the l. h. s. of (A.5) converge, since $(\lambda_1, \lambda_2, \dots)$ is bounded above by
 836 Λ (since if $\lambda_t > \Lambda$ for any t , then one can replace λ_t with Λ , and the new path remains feasible
 837 while the objective function of Program *PP* increases), and \hat{v}_t is a geometric series converging to
 838 zero (see below), so it is permissible to add these two series together term-wise.

839 We now invoke the premise that the solution $\hat{\Phi}$ is stationary. Using this fact, we can solve the
 840 Kuhn-Tucker conditions in Step 1 and compute that $\hat{v}_t = \left(\frac{1}{\xi}\right)^t \frac{\xi-1}{1-\varphi}$.

841 Now observe that

$$\begin{aligned} \sum_2^\infty (\hat{v}_t - \varphi^{t-1}) &= \sum_2^\infty \left(\left(\frac{1}{\xi}\right)^t \frac{\xi-1}{1-\varphi} - \varphi^{t-1} \right) \\ &= \frac{\xi-1}{\xi(1-\varphi)} \frac{1/\xi}{1-1/\xi} - \frac{\varphi}{1-\varphi} = \frac{1/\xi}{1-\varphi} - \frac{\varphi}{1-\varphi} < 0, \end{aligned}$$

842 where the last inequality follows because $\varphi\xi > 1$.¹⁰ Note that the terms in this sum are surely
 843 positive for small values of t (at least for $t = 1$), but eventually they turn negative and stay
 844 negative forever. This is clear if we note that the sign of the t th term is the same as the sign of

$$\frac{\xi - 1}{(1 - \varphi)\xi} - (\xi\varphi)^{t-1},$$

845 which becomes negative at some t because $(\xi\varphi)^{t-1}$ grows without bound.

846 Let us denote $\zeta_t = \left(\left(\frac{1}{\xi}\right)^t \frac{\xi-1}{1-\varphi} - \varphi^{t-1}\right)$. We have shown that $\sum_2^\infty \zeta_t < 0$. Let T be the largest
 847 integer for which ζ_t is non-negative. Then we may write:

$$\sum_2^\infty \lambda_t \zeta_t = \sum_2^T \lambda_t \zeta_t + \sum_{T+1}^\infty \lambda_t \zeta_t \leq \sum_2^T \lambda_T \zeta_t + \sum_{T+1}^\infty \lambda_T \zeta_t = \lambda_T \sum_2^\infty \zeta_t \leq 0,$$

848 where we have invoked the fact that $(\lambda_1, \lambda_2, \dots)$ is a weakly increasing non-negative sequence.
 849 This proves (A.5), and hence the lemma, except for the case $\varphi\xi = 1$.

850 If $\varphi\xi = 1$, then $\zeta_t = 0$ for all t , and (A.5) obviously holds. \square

851 **Lemma 9.** *If the solution to Program $SUS_2[x_0^e, s_0^k]$ is the economic part of the solution to Pro-*
 852 *gram $PP[\varphi, x_0^e, s_0^k]$, then it is also the solution to Program $CDU_2[\varphi, x_0^e, s_0^k]$.*

853 *Proof.* Denote the solution to Program SUS_2 by $\hat{\Phi}$, as in the proof of Lemma 8. Denote the
 854 solution to Program CDU_2 by $\tilde{\Phi} = \{\tilde{c}_t, \tilde{x}_t, \dots\}$. We can extend the path $\tilde{\Phi}$ to a feasible path for
 855 Program PP by defining $\tilde{\lambda} = \tilde{u}_1$ and $\tilde{\lambda}_t = \tilde{u}_1 - \tilde{u}_t$. The path $\hat{\Phi}$ is extended in like manner to a
 856 feasible path for Program SUS_2 (and in fact its solution path, by the premise) by letting $\hat{\lambda}_t = 0$
 857 for all t . If the solution to Program CDU_2 were not the economic part of the solution to Program
 858 PP , then we would have:

$$\frac{\hat{u}_1}{1 - \varphi} > \frac{\tilde{u}_1}{1 - \varphi} - \sum_2^\infty \varphi^{t-1}(\tilde{u}_1 - \tilde{u}_t) = \sum_1^\infty \varphi^{t-1} \tilde{u}_t,$$

859 for the left-hand side of this inequality is the value of PP , by the premise of the lemma, and the
 860 right-hand side is the value of the objective of PP at a non-optimal, feasible solution. But note
 861 that this inequality says:

$$\frac{\hat{u}_1}{1 - \varphi} > \sum_1^\infty \varphi^{t-1} \tilde{u}_t.$$

862 However the solution to SUS_2 –path $\hat{\Phi}$ – is a feasible path for CDU_2 ; thus, the last equation
 863 contradicts the optimality of the $\tilde{\Phi}$ path for CDU_2 . This contradiction proves the lemma. \square

864 **Lemma 10.** *Let $(x_0^e, s_0^k) \in \Gamma$. If $\varphi\xi \geq 1$, then the solution to Program $R_2[\varphi, x_0^e, s_0^k]$ is the solution*
 865 *to Program $SUS_2[x_0^e, s_0^k]$.*

866 *Proof.* Follows immediately from lemmas 7-9. \square

¹⁰We deal with the boundary case $\varphi\xi = 1$ below.

867 We now proceed to the proof of Theorem 6.
 868 Step1. From Lemma 5, we can write Program R^β as follows:

$$\begin{aligned} \max \quad & \sum_{t=1}^{\infty} \varphi^{t-1} (1 + (t-1)\beta) u_t \\ \text{subject to } & u \in P, \\ & u_t \geq u_{t+1}, \text{ for } t \geq 1. \end{aligned}$$

869 Since the value of the program is finite (by Lemma 4), we can break up the series in the
 870 objective function, and write it as:

$$\begin{aligned} & u_1 + \varphi u_2 + \varphi^2 u_3 + \varphi^3 u_4 + \dots \\ & + \beta \varphi u_2 + \beta \varphi^2 u_3 + \beta \varphi^3 u_4 + \dots \\ & + \beta \varphi^2 u_3 + \beta \varphi^3 u_4 + \dots \\ & + \beta \varphi^3 u_4 + \dots \\ & = \sum_1^{\infty} \varphi^{t-1} u_t + \beta \left[\sum_2^{\infty} \varphi^{t-1} u_t + \sum_3^{\infty} \varphi^{t-1} u_t + \sum_4^{\infty} \varphi^{t-1} u_t + \dots \right]. \end{aligned} \tag{A.6}$$

871 Step 2. Suppose, contrary to the claim, that (u_1^*, u_2^*, \dots) solves Program $R^\beta[\varphi, x_0^e, s_0^k]$ but
 872 not Program $SUS_2[x_0^e, s_0^k]$, whose solution has constant utilities at the level denoted by Λ^* . Be-
 873 cause Program $DU_2[x_0^e, s_0^k]$ diverges, we know by lemmas 4 and 10 that the solution to Program
 874 SUS_2 is the same as the solution to program R_2 , which, by Lemma 5, is equivalent to Program
 875 $CDU_2[x_0^e, s_0^k]$:

$$\begin{aligned} \max \quad & \sum_1^{\infty} \varphi^{t-1} u_t \\ \text{subject to } & u \in P, \\ & u_t \geq u_{t+1}. \end{aligned}$$

876 Hence, the solution to Program CDU_2 is $(\Lambda^*, \Lambda^*, \dots)$. The assumption that (u_1^*, u_2^*, \dots) is
 877 not the solution to Program SUS_2 then implies that (u_1^*, u_2^*, \dots) is not the solution to Program
 878 $CDU_2[x_0^e, s_0^k]$ either, i. e.,

$$\sum_1^{\infty} \varphi^{t-1} u_t^* < \sum_1^{\infty} \varphi^{t-1} \Lambda^* = \frac{\Lambda^*}{1-\varphi} \tag{A.7}$$

879 (since (u_1^*, u_2^*, \dots) is feasible for Program SUS_2 and the solution to that program is unique), i. e.,
 880 the first term in (A.6) evaluated at (u_1^*, u_2^*, \dots) is less than $\frac{\Lambda^*}{1-\varphi}$.

881 Step 3. The proof will be completed after showing that (A.7) implies that the value of the
 882 objective function of R^β at $(\Lambda^*, \Lambda^*, \dots)$ is higher than at (u_1^*, u_2^*, \dots) , and, hence, (u_1^*, u_2^*, \dots)
 883 does not solve R^β , contrary to hypothesis. If $\beta = 0$, then from (A.6) the value of the objective
 884 function of R^β at (u_1^*, u_2^*, \dots) is $\sum_1^{\infty} \varphi^{t-1} u_t^*$, by (A.7) less than $\sum_1^{\infty} \varphi^{t-1} \Lambda^*$, which is the desired
 885 contradiction. So let $\beta > 0$. Again by (A.7), the first term of (A.6) is less than $\frac{\Lambda^*}{1-\varphi}$. Suppose now
 886 that the second term in (A.6) evaluated at (u_1^*, u_2^*, \dots) is greater than $\beta \frac{\varphi}{1-\varphi} \Lambda^*$, which, because
 887 $\beta > 0$, implies that

$$\sum_2^{\infty} \varphi^{t-1} u_t^* > \sum_1^{\infty} \varphi^t \Lambda^*. \tag{A.8}$$

888 If we had that $u_1^* \geq \Lambda^*$, then, by (A.8), $u_1^* + \sum_2^\infty \varphi^{t-1} u_t^* > \Lambda^* + \sum_1^\infty \varphi^t \Lambda^* = \Lambda^* + \frac{\varphi}{1-\varphi} \Lambda^* = \frac{1}{1-\varphi} \Lambda^*$,
889 contradicting (A.7). Thus, $u_1^* < \Lambda^*$ and, therefore, by Lemma 5, $u_t^* \leq u_1^* < \Lambda^*$ for all t , and
890 so (A.8) would be false. Therefore the second term of (A.6) when evaluated at (u_1^*, u_2^*, \dots) is
891 $\beta \sum_{t=2}^\infty \varphi^{t-1} u_t^*$, which is less than or equal to $\beta \frac{\varphi}{1-\varphi} \Lambda^*$. By induction, we see that for all values
892 $\tau \geq 2$:

$$\beta \sum_{t=\tau}^\infty \varphi^{t-1} u_t^* \leq \beta \sum_{t=\tau}^\infty \varphi^{t-1} \Lambda^*.$$

893 Hence, $(\Lambda^*, \Lambda^*, \dots)$ dominates (u_1^*, u_2^*, \dots) in Program R^β while satisfying its constraints, a
894 contradiction which establishes the theorem. \square

895 *Proof of Theorem 7*

896 Step 1. It is obvious that if $\varphi = 0$, then the solution to Program R_2 requires simply maximizing
897 the utility of the first generation. In particular, this will require $x_1^e = 0$ and hence $u_2 = 0$.

898 Step 2. More generally, suppose that in the solution to Program R_2 , we have $u_1 = u_2 > 0$.
899 Then if we reduce x_2^l by $\xi \varepsilon$, we can increase x_1^l by ε . This leaves all variables after date 2
900 unchanged, since Generation 2 continues to pass down the same endowment to Generation 3. It
901 therefore must be the case that this change does not increase the value of $u_1 + \varphi u_2$; therefore we
902 must have:

$$\frac{\partial u(c_1, x_1^l)}{\partial x_1^l} - \varphi \frac{\partial u(c_2, x_2^l)}{\partial x_2^l} \xi \leq 0.$$

903 Choosing

$$\bar{\varphi} = \frac{\partial u(c_1, x_1^l)}{\partial x_1^l} \left/ \left(\frac{\partial u(c_2, x_2^l)}{\partial x_2^l} \xi \right) \right.$$

904 therefore proves the theorem. \square

905 *Proof of Theorem 8*

906 Step 1. Without loss of generality, we assume that $x_0^e = 1$, and so $s_0^k = \sigma_0$. Since the set of
907 feasible paths is a convex cone, the primal variables at the solution of the general problem where
908 $x_0^e \neq 1$ are simply the ones computed here, multiplied by x_0^e .

909 We write the DU_2 program with its dual variables:

$$\begin{aligned} \max \sum_1^\infty \varphi^{t-1} u(c_t, x_t^l) \text{ subject to} \\ (C1): \quad (1 - \delta) s_{t-1}^k + i_t \geq s_t^k, \quad (a_t) \\ (C2): \quad f(s_t^k, x_t^c) \geq c_t + i_t, \quad (b_t) \\ (C3): \quad \xi x_{t-1}^e \geq x_t^e + x_t^l + x_t^c, \quad (d_t) \\ (C4): \quad i_t \geq 0. \quad (e_t) \end{aligned}$$

910 The Kuhn-Tucker conditions for a solution to this program where all the constraints bind are:

$$\begin{aligned}
(KT1) \quad (\partial c_t) &: \quad \varphi^{t-1} u_1[t] = b_t, \\
(KT2) \quad (\partial x_t^l) &: \quad \varphi^{t-1} u_2[t] = d_t, \\
(KT3) \quad (\partial x_t^e) &: \quad d_t = (1/\xi)^{t-1} d_1, \\
(KT4) \quad (\partial x_t^c) &: \quad b_t f_2[t] = d_t, \\
(KT5) \quad (\partial s_t^k) &: \quad (1 - \delta) a_{t+1} = a_t - b_t f_1[t], \\
(KT6) \quad (i_t) &: \quad a_t = b_t - e_t,
\end{aligned}$$

911 where all equations hold for $t = 1, 2, 3, \dots$. Again, $u_j[t]$ and $f_j[t]$ are the j th partial derivatives
912 of the utility function u and the production function f for $j = 1, 2$.

913 We will show that there exist non-negative dual variables such that the proposed path satisfies
914 all the Kuhn-Tucker constraints. All the relevant infinite series converge, so that the satisfaction
915 of the K-T constraints suffices to prove optimality of this infinite program.

916 Step 2. Our method will be to substitute the values on the proposed solution path into the
917 primal and dual constraints, and to show that non-negative values of all dual variables can be
918 computed. To this end, the educational constraint (C3) gives us:

$$\xi - E = x_1^l + x_1^c, \quad (A.9)$$

919 recalling that $x_0^e = 1$.

920 Step 3. The dual K-T constraints imply the following:

$$u_2[t] = f_2[t] u_1[t], \quad (A.10)$$

$$\varphi \xi u_2[t+1] = u_2[t], \quad (A.11)$$

$$e_t - (1 - \delta) e_{t+1} = (1 - f_1[t]) b_t - (1 - \delta) b_{t+1}. \quad (A.12)$$

921 The remaining dual constraints simply define (non-negative) values of the dual variables.

922 Step 4. Equation (A.11) says that

$$\frac{1 - \alpha}{\alpha} \frac{c_t}{x_t^l} = (1 - \theta) \left(\frac{(1 - \delta)^t \sigma_0}{E^{t-1} x_1^c} \right);$$

923 substituting \tilde{c}_t for c_t allows us to reduce this equation to:

$$\frac{(1 - \alpha) x_1^c}{\alpha(1 - \theta)} = x_1^l. \quad (A.13)$$

924 Equations (A.12) and (A.13) comprise two linear equations in (x_1^c, x_1^l) , which solve to give

$$x_1^c = \tilde{x}_1^c, \quad x_1^l = \tilde{x}_1^l,$$

925 as required.

926 Step 5. We next analyze equation (A.12), which says:

$$(\varphi \xi) \left(\frac{c_{t+1}}{x_{t+1}^l} \right)^\alpha \left(\frac{x_t^l}{c_t} \right)^\alpha = 1.$$

927 Substituting in the values \tilde{c}_t and \tilde{x}_t^l gives us an equation in the variable E :

$$\varphi\xi\left(\frac{1-\delta}{E}\right)^{\alpha\theta} = 1,$$

928 which solves to give the prescribed value for E . Note that $E < 1$ since $\varphi\xi < 1$.

929 Step 6. The prescribed values of all primal variables have been verified. The Kuhn-Tucker
 930 equations (KT1-3) give us non-negative solutions for b_t and d_t . It is left only to solve for e_t and
 931 to show that for all t , $b_t \geq e_t$, which will give non-negative values for a_t .

932 Step 7. Define the new variables:

$$m_t = (1 - f_1[t])\varphi^{t-1}u_1[t] - (1 - \delta)\varphi^t u_1[t + 1].$$

933 We show in this step that there exists a number $\hat{\sigma}$ such that if $\sigma_0 \geq \hat{\sigma}$, then $m_t \geq 0$ for all
 934 $t \geq 1$. The desired result is equivalent to:

$$(\forall t \geq 1) \quad 1 - \theta \left(\frac{\tilde{x}_1^c E^{t-1}}{\sigma(1-\delta)^t} \right)^{1-\theta} \geq? (1-\delta)\varphi \left(\frac{E}{1-\delta} \right)^{(1-\alpha)\theta}. \quad (\text{A.14})$$

935 Since $\frac{E}{1-\delta} < 1$, the l. h. s. of (A.14) is increasing in t ; thus we need only verify (A.14) for $t = 1$,
 936 which is to say, to verify that:

$$1 - \theta \left(\frac{\tilde{x}_1^c}{\sigma(1-\delta)} \right)^{1-\theta} \geq? (1-\delta)\varphi \left(\frac{E}{1-\delta} \right)^{\theta(1-\alpha)},$$

937 an inequality which holds for sufficiently large σ if and only if:

$$1 > (1-\delta)\varphi \left(\frac{E}{1-\delta} \right)^{\theta(1-\alpha)},$$

938 which is immediately seen to be true from the definition of E .

939 Step 8. Now note that equation (A.12) can be written

$$e_t - (1-\delta)e_{t+1} = m_t, \quad t \geq 1.$$

940 This system of difference equations yields the following solution:

$$e_T = \frac{e_1}{(1-\delta)^{T-1}} - \sum_{t=1}^{T-1} m_t (1-\delta)^{t-T}, \quad T = 2, 3, \dots$$

941 Now choose $e_1 = \sum_{t=1}^{\infty} (1-\delta)^{t-1} m_t$. (We note that this series converges.) To verify that $e_T \geq$
 942 0 for all $T \geq 1$ we must show that

$$T \geq 2 \Rightarrow e_1 \geq \sum_{t=1}^{T-1} m_t (1-\delta)^{t-1},$$

943 a fact which follows from the definition of e_1 and the fact that (m_1, m_2, \dots) is a non-negative
 944 sequence.

945 Step 9. The final step is to show that $a_t \geq 0$ where $a_t = b_t - e_t$. It suffices to show that for all
 946 $T \geq 1$, $(1 - \delta)^{T-1} b_T \geq (1 - \delta)^{T-1} e_T$, or that:

$$(1 - \delta)^{T-1} b_T \geq? \sum_{t=T}^{\infty} m_t (1 - \delta)^{t-1}.$$

947 The r. h. s. of this inequality can be shown (with some algebra) to equal

$$((1 - \delta)\varphi)^{T-1} (1 - f_1[T]) u_1[T] - \sum_{T+1}^{\infty} f_1[t] ((1 - \delta)\varphi)^{t-1} u_1[t];$$

948 since $b_T = \varphi^{T-1} u_1[T]$, our desired inequality reduces to showing that

$$((1 - \delta)\varphi)^{T-1} u_1[T] \geq? ((1 - \delta)\varphi)^{T-1} (1 - f_1[T]) u_1[T] - \text{a positive term},$$

949 which is surely true. This concludes the demonstration that all the K-T conditions hold with the
 950 dual variables as defined.

951 Step 10. Finally, we derive the critical value σ^* . The infinite-series expression for e_1 can be
 952 expanded and reduced (with much algebra) to show that

$$e_1(\sigma_0) = (1 - f_1[1]) u_1[1] - \left(\frac{\tilde{c}_1}{x_1^e} \right)^\alpha \frac{\alpha \theta}{\sigma_0} (\varphi \xi)^{\frac{1}{\alpha \theta}} \frac{1 - \alpha}{1 - \alpha \theta}, \quad (\text{A.15})$$

953 which we write as a function of the initial capital-labor ratio. The reader should note, from
 954 the K-T conditions (KT1-6) in Step 1 that the dual variables are functions only of the *marginal*
 955 utilities and productivities at the various dates, which are, for the Cobb-Douglas case, functions
 956 of *ratios* of the primal variables. Therefore the dual variables are independent of the scale of the
 957 endowment vector (i. e., the value of x_0^e).

958 The critical value of σ_0 is that number σ^* for which $e_1(\sigma^*) = 0$: for if $e_1(\sigma_0) > 0$ then a
 959 slight decrease in σ_0 will still deliver a positive value of e_1 , and all the other e_t . But this would
 960 mean that investment is identically zero on the optimal path. The zero of equation (A.15) is the
 961 solution to the equation in the statement of the theorem, which concludes the proof. \square

962 *Example 2*

963 This is an example of an education and capital economy where, along the solution path to
 964 Program DU_2 , $u_2 > u_1$, whereas the utilities from date 2 onwards decay geometrically. The
 965 example is presented in lemmas 11 and 12 below.

966 **Lemma 11.** Let $(\alpha, \theta, \delta, \xi, \varphi) = (0.66, 0.25, 0.1, 1.1, 0.9)$ and $(x_0^e, s_0^k) = (1, 0.15)$. In particular,
 967 $\varphi \xi < 1$. Then $\sigma^* = 0.186198$ and so $\sigma_0 = 0.15 < \sigma^*$. The solution to DU_2 is given by $(c_1,$
 968 $x_1^l, x_1^c, x_1^e, i_1, s_1^k) = (0.192294, 0.0482943, 0.870989, 0.154375, 0.0746361, 0.114795)$. We have
 969 $\sigma_1 = 0.1979 > \sigma^*$ and the variables from date 2 onwards are given by:
 970 $t \geq 2$: $i_t = 0$, $s_t^k = x_1^e \tilde{s}_t^k$, $x_t = x_1^e \tilde{x}_t$, $c_t = x_1^e \tilde{c}_t$. In particular, $u_1 = 0.1138$ and $u_2 = 0.1169 > u_1$.
 971 The utilities from date 2 onwards decay geometrically as in Theorem 8.

972 *Proof.* Step1. We will produce the example by finding an initial endowment vector (x_0^e, s_0^k) such
 973 that $\sigma_0 < \sigma^*$ and the solution to $DU_2[\varphi, x_0^e, s_0^k]$ has the following property: on the optimal path,
 974 at date 1, we have $\sigma_1 = s_1^k/x_1^e > \sigma^*$. For we then know what the optimal path is from date 1
 975 onwards: it is just the path stipulated in Theorem 8. Our strategy will be to find such values of
 976 (x_0^e, s_0^k) , where, on the optimal path, we have $u_1 < u_2$.

977 We write down the program we wish to solve, where (x_0^e, s_0^k) is now an unknown endowment.

Program PP^* $[\varphi, x_0^e, s_0^k]$.

$$\begin{aligned} & \max \sum_1^{\infty} \varphi^{t-2} u(c_t, x_t^l) \text{ subject to :} \\ & (a_t) \quad (1 - \delta)s_{t-1}^k + i_t \geq s_t^k, \quad t \geq 1, \\ & (b_t) \quad f(s_t^k, x_t^e) \geq c_t + i_t, \quad t \geq 1, \\ & (d_t) \quad \xi x_{t-1}^e \geq x_t \equiv x_t^e + x_t^l + x_t^c, \quad t \geq 1, \\ & (e_t) \quad i_t \geq 0, \quad t \geq 1. \end{aligned}$$

978 Note that we have factored out φ from the usual statement of the objective function. Of
979 course this makes no difference to the solution. The reason for doing so will become apparent
980 momentarily.

981 We are searching for a solution such that $\sigma_1 > \sigma^*$, $i_1 > 0$, and it follows (by Theorem 8)
982 that $i_t = 0, t \geq 2$. Hence all constraints of program PP^* will bind except for the first investment
983 constraint. This gives the following K-T conditions:

$$\begin{aligned} (\partial c_t) : & \quad \varphi^{t-2} u_1[t] = b_t, \quad t \geq 1, \\ (\partial x_t^l) : & \quad \varphi^{t-2} u_2[t] = a_t, \quad t \geq 1, \\ (\partial x_t^e) : & \quad b_t f_2[t] = d_t, \quad t \geq 1, \\ (\partial x_t^c) : & \quad \xi d_{t+1} = d_t, \quad t \geq 1, \\ (\partial i_t) : & \quad a_1 = b_1, \\ & \quad a_t = b_t - e_t, \quad t > 1, \\ (\partial s_t^k) : & \quad (1 - \delta)a_{t+1} = a_t - b_t f_1[t], \quad t \geq 1. \end{aligned}$$

984 Step 2. In Theorem 8, we solved the DU_2 problem with the normalization $x_0^e = 1, \sigma_0 = s_0^k$.
985 Recall from Step 10 of the proof of Theorem 8 that the values of the dual variables of that
986 program are functions *only* of σ_0 : that is, they depend only on the capital-labor ratio at date 0,
987 not on the scale of the initial endowment vector.

988 Step 3. Program PP^* beginning at date 1 (not date 0) is *exactly* the program solved in The-
989 orem 8. (That is why we factored out φ from the objective.) Since $\sigma_1 > \sigma^*$ in the solution
990 we are looking for, it follows that the dual variables from date 1 on in Program PP^* are exactly
991 the dual variables computed in Theorem 8, where the initial capital – labor ratio is σ_1 , and the
992 primal variables from date 1 are exactly the tilde primal variables of Theorem 8, multiplied by
993 x_1^e , whatever that turns out to be.

994 Denote the dual variables computed in the proof of Theorem 8 with tildes $\tilde{a}_t(\sigma), \tilde{b}_t(\sigma)$, etc.,
995 where σ is the initial capital-labor ratio of that program.

996 Step 4. We now compute what information is contained in the K-T constraints for Program
997 PP^* . First, we know that $d_2 = \tilde{d}_1(\sigma_1)$: this follows from the above discussion. But $d_2 = \frac{1}{\xi} d_1 =$
998 $\varphi^{-1} u_2[1]$ and therefore:

$$u_2[1] = \varphi \xi \tilde{d}_1(\sigma_1). \quad (\text{A.16})$$

999 From Theorem 8, we know that $\tilde{d}_1(\sigma_1) = \tilde{u}_2[1] = (1 - \alpha) \left(\frac{\tilde{c}_1(\sigma_1)}{\tilde{x}_1(\sigma_1)} \right)^\alpha$, and we therefore can

1000 write, manipulating equation (A.16):

$$\frac{c_1}{x_1^l} = (\varphi\xi)^{1/\alpha} \left(\frac{\tilde{c}_1(\sigma_1)}{\tilde{x}_1^l(\sigma_1)} \right), \quad (\text{A.17})$$

1001 where c_1, x_1^l are the date 1 values on the optimal path for Program PP^* .

1002 Our second equation is

$$\frac{u_2[1]}{u_1[1]} = f_2[1],$$

1003 which comes from the first three K-T constraints of Program PP^* . This gives:

$$\frac{1 - \alpha}{\alpha} \frac{c_1}{x_1^l} = (1 - \theta) \frac{f(s_1^k, x_1^c)}{x_1^c} = (1 - \theta) \left(\frac{s_1^k}{x_1^c} \right)^\theta. \quad (\text{A.18})$$

1004 The next three equations simply restate the primal constraints:

$$(1 - \delta)s_0^k + i_1 = s_1^k, \quad (\text{A.19})$$

$$\xi x_0^e = x_1^e + x_1^l + x_1^c, \quad (\text{A.20})$$

$$f(s_1^k, x_1^c) = c_1 + i_1. \quad (\text{A.21})$$

1005 Equation (A.21) comes from the (∂s_1^k) K-T condition. As before, we know that $b_2 = \tilde{b}_1(\sigma_1)$
 1006 and $e_2 = \tilde{e}_1(\sigma_1)$ and so $a_2 = b_2 - e_2 = \tilde{b}_1(\sigma_1) - \tilde{e}_1(\sigma_1)$. Thus, we may write that K-T condition
 1007 as:

$$(1 - \delta)(\tilde{b}_1(\sigma_1) - \tilde{e}_1(\sigma_1)) = \alpha\varphi^{-1} \left(\frac{x_1^l}{c_1} \right)^{1-\alpha} \left(1 - \theta \left(\frac{x_1^c}{s_1^k} \right)^{1-\theta} \right). \quad (\text{A.22})$$

1008 The six equations (A.17)-(A.22) are equations in the six unknowns $x_1^e, x_1^l, x_1^c, i_1, c_1, s_1^k$ when
 1009 the endowment (x_0^e, s_0^k) is given. Of course, $\sigma_1 = s_1^k/x_1^e$. We know the expressions for all the tilde
 1010 variables from Theorem 8, as functions of σ_1 .

1011 Indeed, these six equations contain all the new information about the solution to Program PP^*
 1012 –the remaining K-T conditions simply emulate the solution of the program from date 1 onwards,
 1013 which we know from Theorem 8.

1014 We now show how to solve these six equations. Define two new variables:

$$A = \frac{c_1}{x_1^l}, \quad B = \frac{s_1^k}{x_1^c}$$

1015 Note that equations (A.17), (A.18) and (A.22) above are simultaneous equations in the three
 1016 unknowns A, B and σ_1 . Hence we can solve for these three variables (which we will do in an
 1017 example, given below). Now, knowing these three variables, we can write all the information re-
 1018 maining in the six equations as the following system of six linear equations in the six unknowns:

$$\begin{aligned} s_1^k &= \sigma_1 x_1^e, \\ (1 - \delta)s_0^k + i_1 &= s_1^k, \\ x_1^c f(B, 1) &= c_1 + i_1, \\ c_1 &= A x_1^l, \\ \xi x_0^e &= x_1^e + x_1^l + x_1^c, \\ s_1^k &= B x_1^c. \end{aligned}$$

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We write these equations in matrix form $Mz = Q$, where

$$M = \begin{pmatrix} 0 & 0 & 0 & -\sigma_1 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ -1 & 0 & f(B, 1) & 0 & -1 & 0 \\ -1 & A & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & B & 0 & 0 & -1 \end{pmatrix}, Q = \begin{pmatrix} 0 \\ (1 - \delta)s_0^k \\ 0 \\ 0 \\ \xi x_0^e \\ 0 \end{pmatrix}.$$

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(The order of the variables is $(c_1, x_1^l, x_1^c, x_1^e, i_1, s_1^k)$.) Hence we can compute the solution $z = M^{-1}Q$. If we insert an endowment vector (x_0^e, s_0^k) with $\sigma_0 < \sigma^*$ and the solution Q generated is a positive vector, and $\frac{s_0^k}{x_1^e} \geq \sigma^*$, then we have a solution to PP^* of the required form. For it will immediately follow that all the dual variables are non-negative, from the K-T conditions, and so we have produced a path where all the K-T conditions hold —by again invoking Theorem 8.

Step 5. Examine the *Mathematica* program (available from the authors) which calculates this solution for several numerical values. In particular, an instance is provided in which $u_1 < u_2$ on the optimal path, which proves the lemma. \square

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Lemma 12. The solution to $R_2[\varphi, x_0^e, s_0^k]$ with the data of the premise of Lemma 11 is given by: $(s_1^k, i_1, c_1, x_1^e, x_1^c, x_1^l, \sigma_1) = (0.167583, 0.0325828, 0.131227, 0.847974, 0.162572, 0.894544, 0.197627)$ and for $t > 1$: $i_t = 0$, $s_t^k = x_1^e \tilde{s}_t^k$, $x_t = x_1^e \tilde{x}_t$, $c_t = x_1^e \tilde{c}_t$. At the solution, $u_1 = u_2$. Indeed, the utilities at the solutions of DU_2 and R_2 for this economy are given by:

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| | u_1 | u_2 | $u_t, t > 2$ |
|--------|--------|--------|-----------------|
| DU_2 | 0.1138 | 0.1169 | geometric decay |
| R_2 | 0.1152 | 0.1152 | geometric decay |

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Proof. **Step 1.** We will find a solution to Program $PP[\varphi, x_0^e, s_0^k]$: this will also be a solution to $CDU_2[\varphi, x_0^e, s_0^k]$ and hence to $R_2[\varphi, x_0^e, s_0^k]$. Recall that Program $PP[\varphi, x_0^e, s_0^k]$ is:

$$\max \left\{ \frac{\Lambda}{1 - \rho} - \sum_2^{\infty} \varphi^{t-1} \lambda_t \right\} \quad \text{subject to}$$

$$(v_t) \quad u(c_t, x_t^l) \geq \Lambda - \lambda_t, \quad t \geq 1,$$

$$(m_t) \quad \lambda_{t+1} \geq \lambda_t, \quad t \geq 1,$$

$$(b_t) \quad f(s_t^k, x_t^c) \geq c_t + i_t, \quad t \geq 1,$$

$$(a_t) \quad (1 - \delta)s_{t-1}^k + i_t \geq s_t^k, \quad t \geq 1,$$

$$(d_t) \quad \xi x_{t-1}^e \geq x_t^c + x_t^l + x_t^e, \quad t \geq 1,$$

$$(e_t) \quad i_t \geq 0, \quad t \geq 1,$$

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where $\lambda_1 \equiv 0$. For the specified economy, we conjecture a solution where $u_1 = u_2 > u_3 > \dots$ and where the geometric-decay solution begins at date 2. Thus, of the set of m_t constraints, only the m_1 constraint will bind, and so $m_t = 0$ for $t > 1$. The e_1 constraint will be slack, since we conjecture that $i_1 > 0$. All other constraints will bind at the solution. The K-T conditions are

1039 therefore:

$$\begin{aligned}
(\partial\Lambda) : \quad & \frac{1}{1-\varphi} = \sum_1^{\infty} v_t, \\
(\partial\lambda_2) : \quad & -\varphi + m_1 + v_2 = 0, \\
(\partial\lambda_t) : \quad & v_t = \varphi^{t-1}, \quad t > 2, \\
(\partial c_t) : \quad & v_t u_t = b_t, \quad t \geq 1, \\
(\partial x_t^j) : \quad & v_t u_t = d_t, \quad t \geq 1, \\
(\partial x_t^c) : \quad & b_t f_2[t] = d_t, \quad t \geq 1, \\
(\partial x_t^e) : \quad & \xi d_{t+1} = d_t, \quad t \geq 1, \\
(\partial s_t^k) : \quad & a_{t+1}(1-\delta) - a_t + b_t f_1[t] = 0, \quad t \geq 1, \\
(\partial i_1) : \quad & a_1 = b_1, \\
(\partial i_t) : \quad & a_t = b_t - e_t, \quad t > 1.
\end{aligned}$$

1040 Step 2. We can reduce the first three dual K-T conditions to the equations:

$$v_2 = 1 + \varphi - v_1, \quad m_1 = v_1 - 1,$$

1041 thus eliminating the variables v_2 and m_1 . We must, after finding a value for v_1 , check that v_2 and
1042 m_1 are non-negative.

1043 For $t \geq 2$, we define all the dual variables to equal the dual variables of the geometric-decay
1044 solution which begins at date 2 with the endowment (x_1^e, s_1^k) , multiplied by λ . Denoting the latter
1045 variables with tildes, we therefore define for $t \geq 2$:

$$a_t = \varphi \tilde{a}_{t-1}, \quad b_t = \varphi \tilde{b}_{t-1}, \quad d_t = \varphi \tilde{d}_{t-1}, \quad e_t = \varphi \tilde{e}_{t-1}.$$

1046 Then all the dual constraints which involve these variables are satisfied where the primal
1047 variables for dates $t \geq 2$ are given by the geometric-decay solution to Theorem 8. For this to
1048 be a solution, we must check that $\sigma_1 \equiv s_1^k/x_1^e \geq \sigma^*$. We are left only with the dual constraints
1049 associated with date 1, which are:

$$\begin{aligned}
v_1 u_1[1] &= b_1, \\
v_1 u_2[1] &= d_1, \\
u_2[1] &= f_2[1] u_1[1], \\
\xi d_2 &= d_1, \\
a_2(1-\delta) &= b_1(1-f_1[1]).
\end{aligned}$$

1050 The first two of the above constraints simply define b_1 and d_1 . Thus we are left with three
1051 substantive equations. Substituting in for the values of d_2 and a_2 , these become:

$$u_2[1] = f_2[1] u_1[1], \tag{A.23}$$

$$\varphi \xi \tilde{d}_1 = v_1 u_2[1], \tag{A.24}$$

$$\varphi(\tilde{b}_1 - \tilde{e}_1)(1-\delta) = v_1 u_1[1](1-f_1[1]). \tag{A.25}$$

1052 Recall from the proof of Theorem 8 that the expressions for $\tilde{d}_1, \tilde{b}_1, \tilde{e}_1$ are known functions of
 1053 $\sigma_1 \equiv s_1^k/x_1^e$. In particular, we have:

$$\tilde{b}_1 = \alpha \left(\frac{\tilde{c}_1(\sigma_1)}{\tilde{x}_1^l} \right)^{\alpha-1}, \tilde{d}_1 = (1-\alpha) \left(\frac{\tilde{c}_1(\sigma_1)}{\tilde{x}_1^l} \right)^\alpha,$$

1054 while the expression for \tilde{e}_1 is given as equation (A.15) in Step 10 of the proof of Theorem 8.

1055 In addition we have the primal constraints:

$$u(c_1, x_1^l) = x_1^e u(\tilde{c}_1(\sigma_1), \tilde{x}_1^l) \quad (\text{i. e. } , u_1 = u_2), \quad (\text{A.26})$$

$$x_1^c f(s_1^k/x_1^c, 1) = c_1 + i_1, \quad (\text{A.27})$$

$$(1-\delta)s_0^k = s_1^k - i_1, \quad (\text{A.28})$$

$$\xi x_0^e = x_1^e + x_1^c + x_1^l. \quad (\text{A.29})$$

1056 The seven equations (A.23)-(A.29) define a system of seven equations in the seven unknowns
 1057 $(s_1^k, i_1, c_1, x_1^e, x_1^c, x_1^l, v_1)$.

1058 Step 3. We proceed to solve these equations as follows. Recall that $A = c_1/x_1^l, B = s_1^k/x_1^c$.
 1059 Rewriting equation (A.23) as

$$\frac{1-\alpha}{\alpha} A = (1-\theta)B^\theta, \quad (\text{A.30})$$

1060 allows us express A as a function of B :

$$A[B] = \frac{\alpha(1-\theta)}{1-\alpha} B^\theta. \quad (\text{A.31})$$

1061 We define the following mapping. Begin with an arbitrary positive value for B . Then compute
 1062 A by (A.31). Now equations (A.24) and (A.25) comprise two simultaneous equations in (σ_1, v_1) .
 1063 Solve them. This leaves us with the four equations (A.26)-(A.29), which are now linear equations
 1064 in the primal variables, once A, B and σ_1 are specified constants. To these, append the equations:

$$\sigma_1 x_1^e = s_1^k, \quad A x_1^l = c_1.$$

1065 We now have a linear system of six equations in the six date-one primal variables. Solve
 1066 them, and define $\hat{B} = s_1^k/x_1^c$. A fixed point of the mapping $B \rightarrow \hat{B}$ generates a solution to the
 1067 seven equations (A.23-A.29) in the six primal variables plus v_1 .

1068 We find the fixed point of this mapping for the stipulated economy. (See the available *Math-*
 1069 *ematica* program.) We find that $v_1 = 1.01304$, and it follows that v_2 and m_1 are positive and
 1070 $\sigma_1 = 0.1976 > \sigma^*$. Hence we have a solution to all the Kuhn-Tucker conditions, and hence,
 1071 since PP is a concave program, to Program PP . The solution is reported in the lemma's state-
 1072 ment. \square

1073 *Proof of Theorem 9*

1074 Step 1. We first write down the Kuhn-Tucker conditions for a solution to Program g - SUS .

$$(\partial\Lambda) : \quad 1 = \sum_1^\infty r_t(1+g)^{t-1},$$

$$(\partial c_t) : \quad r_t u_1[t] = a_t,$$

$$(\partial x_t^l) : \quad r_t u_2[t] = d_t,$$

$$(\partial x_t^e) : \quad \xi d_{t+1} = d_t,$$

$$(\partial x_t^c) : \quad a_t f_2[t] = d_t,$$

$$(\partial s_t^k) : \quad a_t f_1[t] + (1-\delta)b_{t+1} = b_t,$$

$$(\partial i_t) : \quad a_t = b_t.$$

1075 In addition, let all the primal constraints hold with equality. We shall attempt to solve all
1076 these equations for a balanced growth path.

1077 On such a path, $u_j[t] = u_j[1]$ and $f_j[t] = f_j[1]$ for $j = 1, 2$ and $t \geq 1$. The primal and dual
1078 equations yield the following substantive relations on a balanced growth path for the economic
1079 variables:

$$i_1 = (g + \delta)s_0^k, \quad (\text{A.32})$$

$$\xi - (1 + g) = x_1^c + x_1^l, \quad (\text{A.33})$$

$$f_2[1] = \frac{u_2[1]}{u_1[1]}, \quad (\text{A.34})$$

$$\xi = \frac{1 - \delta}{1 - f_1[1]}, \quad (\text{A.35})$$

$$f((1 + g)s_0^k, x_1^c) = c_1 + i_1. \quad (\text{A.36})$$

1080 The other dual constraints simply define non-negative dual variables in terms of the primal
1081 variables, with one exception: we must verify that the series in the $(\partial\Lambda)$ constraint converges.
1082 Thus, given g , if we can solve the five equations (A.32)-(A.36) for $(s_0^k, x_1^c, x_1^l, c_1, i_1)$ and the series
1083 in $(\partial\Lambda)$ converges, then the balanced growth path at rate g defined by these values, along with the
1084 associated dual variables, solves the Kuhn-Tucker constraints. Modulo transversality conditions,
1085 which we will comment upon below, and since g -*SUS* is a concave program, the theorem will be
1086 demonstrated.

1087 Step 2. From the dual K- T conditions, we deduce that $r_t = \frac{d_1}{u_2[1]} \left(\frac{1}{\xi}\right)^{t-1}$. Consequently the
1088 series in the $(\partial\Lambda)$ K- T condition defines a value for d_1 if and only if $\frac{1+g}{\xi} < 1$. This is true because
1089 by hypothesis, $g < \xi - 1$.

1090 Step 3. Thus, it remains to solve the five equations (A.32)-(A.36). Specializing to Cobb-
1091 Douglas, we re-write the five equations as follows.

$$i_1 = (g + \delta)s_0^k, \quad (\text{A.37})$$

$$\xi - (1 + g) = x_1^c + x_1^l, \quad (\text{A.38})$$

$$(1 - \theta) \left(\frac{(1 + g)s_0^k}{x_1^c} \right)^\theta = \frac{(1 - \alpha)c_1}{\alpha x_1^l}, \quad (\text{A.39})$$

$$\theta \left(\frac{x_1^c}{(1 + g)s_0^k} \right)^{1-\theta} = \frac{\xi - (1 - \delta)}{\xi}, \quad (\text{A.40})$$

$$((1 + g)s_0^k)^\theta (x_1^c)^{1-\theta} = c_1 + (g + \delta)s_0^k. \quad (\text{A.41})$$

1092 Step 4. Now denote $X = \frac{x_1^c}{s_0^k}$, $Y = \frac{c_1}{x_1^l}$. Solve (A.39) and (A.40) for X and Y :

$$X = (1 + g) \left(\frac{\xi - (1 - \delta)}{\theta \xi} \right)^{1/(1-\theta)},$$

$$Y = \frac{\alpha(1-\theta)}{1-\alpha} \left(\frac{\xi - (1 - \delta)}{\beta \xi} \right)^{-\theta/(1-\theta)}.$$

1093 Next, divide equation (A.41) through by s_0^k , giving:

$$\frac{c_1}{s_0^k} = (1 + g)^\theta X^{1-\theta} - (g + \delta), \quad (\text{A.42})$$

1094 which generates a *necessary condition*:

$$(1 + g)^\theta X^{1-\theta} > (g + \delta). \quad (\text{A.43})$$

1095 Now, noting that $XY = \frac{c_1}{s_0^k} \frac{x_1^c}{x_1^j}$, and using (A.43), we have:

$$Z \frac{x_1^c}{x_1^j} = XY, \text{ where } Z \equiv (1 + g)^\theta X^{1-\theta} - (g + \delta),$$

1096 or $x_1^c = \frac{x_1^j XY}{Z}$. Using (A.38), and substituting this value for x_1^c , we can solve for x_1^j :

$$x_1^j = \frac{Z}{Z + XY} (\xi - (1 - \delta)).$$

1097 Consequently, from (A.38), $x_1^c = \frac{XY}{XY+Z} (\xi - (1 - \delta))$. Thus both x_1^c and x_1^j are positive numbers.
1098 We can now use the equations to solve quickly for positive values of s_0^k , i_1 and c_1 .

1099 Step 5. We now verify (A.43). Define the function $\Upsilon(g) = (1 + g)^{\frac{\xi - (1 - \delta)}{\theta \xi}} - (g + \delta)$. Check
1100 that $\Upsilon(0) > 0$ if and only if $\xi > \frac{1 - \delta}{1 - \theta \delta}$; but this is true because $\xi > 1$. Check that $\Upsilon(\xi - 1) =$
1101 $(\xi - (1 - \delta))^{\frac{1 - \theta}{\theta}} > 0$. Since Υ is linear, it follows that $\Upsilon(g) > 0$ on the interval $[0, \xi - 1]$,
1102 demonstrating (A.43).

1103 Step 6. We finally remark that all the transversality conditions hold because each sequence of
1104 dual variables (e. g., (a_1, a_2, \dots)) converges to zero geometrically. This proves the first direction
1105 of the theorem.

1106 Step 7. To prove the converse, let $g = \xi - 1$. On a balanced growth path, we therefore require
1107 $x_1^e = (1 + g)x_0^e = \xi x_0^e$, which implies that $x_1^c = x_1^j = 0$. So no balanced growth path can be
1108 supported at the rate $g = \xi - 1$. It is obvious, *a fortiori*, that no such path exists for $g > \xi - 1$. \square

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