

# Contribution Biplots

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**Abstract:** In order to interpret the biplot it is necessary to know which points – usually variables – are the ones that are important contributors to the solution, and this information is available separately as part of the biplot’s numerical results. We propose a new scaling of the display, called the contribution biplot, which incorporates this diagnostic directly into the graphical display, showing visually the important contributors and thus facilitating the biplot interpretation and often simplifying the graphical representation considerably. The contribution biplot can be applied to a wide variety of analyses such as correspondence analysis, principal component analysis, log-ratio analysis and the graphical results of a discriminant analysis/MANOVA, in fact to any method based on the singular-value decomposition. In the contribution biplot one set of points, usually the rows of the data matrix, optimally represent the spatial positions of the cases or sample units, according to some distance measure that usually incorporates some form of standardization unless all data are comparable in scale. The other set of points, usually the columns, is represented by vectors that are related to their contributions to the low-dimensional solution. A fringe benefit is that usually only one common scale for row and column points is needed on the principal axes, thus avoiding the problem of enlarging or contracting the scale of one set of points to make the biplot legible. Furthermore, this version of the biplot also solves the problem in correspondence analysis of low-frequency categories that are located on the periphery of the map, giving the false impression that they are important, when they are in fact contributing minimally to the solution.

**Keywords:** Biplot, contributions, correspondence analysis, discriminant analysis, log-ratio analysis, MANOVA, principal component analysis, scaling, singular value decomposition, weighting.

## 1. INTRODUCTION

The *biplot*, as defined by Gabriel (1971) in the context of principal component analysis, is a graphical display of the rows and columns of a data matrix as points (or, equivalently, as vectors) in a low-dimensional Euclidean space, usually of dimensionality two or three. Any matrix  $\mathbf{T}$  ( $I \times J$ ) can be decomposed as the product of two matrices  $\mathbf{A}$  ( $I \times K$ ) and  $\mathbf{B}^T$  ( $K \times J$ ):

$$\mathbf{T} = \mathbf{A} \mathbf{B}^T. \quad (1)$$

We call  $\mathbf{T}$  the *target matrix*, and  $\mathbf{A}$  and  $\mathbf{B}$  the *left* and *right matrices*, respectively, of the decomposition. This decomposition is not unique, since for any given decomposition (1),  $\mathbf{T}$  can be equivalently written as  $\mathbf{T} = \mathbf{A} \mathbf{Q} \mathbf{Q}^{-1} \mathbf{B}^T = [\mathbf{A} \mathbf{Q}] [\mathbf{B} (\mathbf{Q}^T)^{-1}]^T$  for any  $K \times K$  non-singular matrix  $\mathbf{Q}$  – hence there are infinite possible decompositions of a particular target.

A convenient way of obtaining a particular form of (1), with convenient properties, is to use the singular value decomposition (SVD) of  $\mathbf{T}$ :

$$\mathbf{T} = \mathbf{U} \mathbf{\Gamma} \mathbf{V}^T, \text{ where } \mathbf{U}^T \mathbf{U} = \mathbf{V}^T \mathbf{V} = \mathbf{I}, \mathbf{\Gamma} = \text{diag}(\gamma_1 \geq \gamma_2 \geq \dots \geq \gamma_K > 0). \quad (2)$$

Then  $\mathbf{A}$  and  $\mathbf{B}$  can be defined as:

$$\mathbf{A} = \mathbf{U} \mathbf{\Gamma}^\alpha \quad \mathbf{B} = \mathbf{V} \mathbf{\Gamma}^{1-\alpha}. \quad (3)$$

for any value of  $\alpha$ . The advantage of obtaining the decomposition via the SVD is the well-known Eckart-Young theorem which states that the least-squares matrix approximation of the target matrix  $\mathbf{T}$  can be obtained for any given rank  $K^*$  of the approximating matrix, using the first (i.e., largest)  $K^*$  singular values in the diagonal of  $\mathbf{\Gamma}$ , and the corresponding first  $K^*$  left and right singular vectors in  $\mathbf{U}$  and  $\mathbf{V}$  (Eckart and Young 1936).

Thus a target matrix  $\mathbf{T}$  can be approximated by its ‘closest’ (least-squares) approximation of rank 2, for example:

$$[\mathbf{u}_1 \quad \mathbf{u}_2] \begin{bmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{bmatrix} [\mathbf{v}_1 \quad \mathbf{v}_2]^T \quad (4)$$

and then  $\mathbf{A}$  and  $\mathbf{B}$  can be defined, for example with  $\alpha = 1$  in (3), as

$$\mathbf{A} = [\mathbf{u}_1 \quad \mathbf{u}_2] \begin{bmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{bmatrix} \quad \mathbf{B} = [\mathbf{v}_1 \quad \mathbf{v}_2] \quad (5)$$

The rows of  $\mathbf{A}$  and of  $\mathbf{B}$  provide two-dimensional coordinates for points representing the rows and columns of the target matrix in a planar display, so that the scalar product between the  $i$ -th row point and the  $j$ -th column point approximates the  $(i,j)$ -th element  $t_{ij}$  of the target matrix. The reproduction of the original element by the scalar product is as good as the overall “variance explained” by the solution, measured by  $(\gamma_1^2 + \gamma_2^2) / \sum_k \gamma_k^2$  in the two-dimensional case.

Since scalar products between two vectors  $\mathbf{a}$  and  $\mathbf{b}$  are equal to the length of the perpendicular projection of  $\mathbf{a}$  onto  $\mathbf{b}$  multiplied by the length of  $\mathbf{b}$  (or vice versa), it follows that the direction vectors representing the columns (for example) can be calibrated so that the approximated values of the target matrix can be read off by projecting the row points perpendicularly onto each column vector. In this case we refer to the calibrated column vectors as *biplot axes*. For more details about biplots, see Gabriel and Odoroff (1990), Greenacre (1993), Gower and Hand (1996), Greenacre (2007: chap.15) and Greenacre (2010).

In spite of the simplicity of the biplot, its use often causes confusion because of the many ways in which  $\mathbf{A}$  and  $\mathbf{B}$  can be formed from the SVD, and the corresponding variety of ways it can be interpreted. While the scalar product rule and the calibration of axes holds for all biplots, the use

of different values of  $\alpha$  produces biplots with different within-set interpretations of the clouds of row and column points. The usual choices  $\alpha=1$  and  $\alpha=0$  give biplots in which distances are approximated between rows and between columns respectively. Often the two sets of plotted points are on such different scales that they are plotted on two different scales (see, for example, the `biplot` function in the R package). To alleviate this, the choice  $\alpha=1/2$  is sometimes recommended so that row and column points have similar dispersions in the display, but this choice optimizes neither the distances between rows nor the distances between columns. This last choice, called the *symmetric biplot* by Greenacre (2007) has the least benefit for the user, since data matrices are invariably interpreted asymmetrically, with at least one of the sets of distances being important to visualize (see Greenacre (2007: Epilogue) for further discussion about this aspect). Another alternative, which is not a biplot in the strict sense of the definition, is to plot  $\mathbf{U} \mathbf{\Gamma}$  and  $\mathbf{V} \mathbf{\Gamma}$  in a joint plot, called the *symmetric map* by Greenacre (2007) – here the word “map” is used when distances between at least one set of points are being displayed. In the symmetric map the distances in both the row and column displays are displayed, but not their scalar products, unless the singular values on the axes are not too different – see Gabriel (2002) for a thorough treatment of this subject.

In the usual biplot displays up till now, it is difficult for the user to know which points – usually the variables – are driving the solution and thus should be used to interpret the results. Such information is available, but in numerical form, and thus has to be considered separately. The proposal of this paper is to introduce an alternative scaling of a biplot display which specifically visualizes the important variables for the interpretation. The proposed biplot is thus called the *contribution biplot* and is universally applicable across several different types of multivariate methods based on the SVD, for example correspondence analysis, principal component analysis, log-ratio analysis and discriminant analysis. As a spin-off to this proposal it will be

demonstrated that this new biplot display solves two other practical problems. First, a common scale for the row and column points in the display is usually possible, thus avoiding the two scales mentioned previously. And second, in the case of methods such as correspondence analysis and log-ratio analysis which involve point weighting, outlying points with low weight, which appear to be important but in fact are contributing very little to the solution, are pulled towards the centre of the biplot and are immediately seen to be relatively unimportant to the interpretation.

To appreciate the usefulness of the contribution biplot straight away, consider Figure 1, showing a regular correspondence analysis biplot (Figure 1(a)) and the corresponding contribution biplot (Figure 1(b)) for a data set on fatty acid (lipid) compositions in a sample of 40 fish from a food web study of Arctic fish. There are three groups of fish, which can be seen in both displays. On the left it is not clear which fatty acids are responsible for the separation of the fish, whereas in the contribution biplot on the right it is immediately clear that there are mainly three fatty acids separating out the three respective groups – notice that the fatty acid  $16:1(n-7)$  that separates out the group at bottom left in the displays is not the one that stands out at bottom left in the regular biplot in Figure 1(a).

Contribution biplots will be defined first in the context of correspondence analysis (Section 2) and then applied to principal component analysis (Section 3), log-ratio analysis (Section 4) and discriminant analysis/MANOVA (Section 5). Section 6 shows how contribution biplots are related to linear regression, and Section 7 concludes with a discussion.

## **2. CONTRIBUTION BILOTS FOR CORRESPONDENCE ANALYSIS**

In the contribution biplot the weighting of each point is important in determining the contributions made by each point to the solution. Since differential weighting is inherent in

correspondence analysis (CA), we choose to use CA to introduce contribution biplots. To fix notation, we describe the theory of CA briefly. Given a contingency table  $\mathbf{N}$ , and associated correspondence table  $\mathbf{P} = (1/n) \mathbf{N}$  of relative frequencies, where  $n$  is the grand total of  $\mathbf{N}$ , let  $\mathbf{r}$  and  $\mathbf{c}$  be the row and column marginal totals of  $\mathbf{P}$  (called row and column *masses* in CA terminology). Let  $\mathbf{D}_r$  and  $\mathbf{D}_c$  be the diagonal matrices of these masses. CA can be defined as approximating the target matrix of standardized residuals based on the relative frequencies in  $\mathbf{P}$ :

$$\mathbf{T} = \mathbf{D}_r^{-1/2} (\mathbf{P} - \mathbf{r}\mathbf{c}^T) \mathbf{D}_c^{-1/2} \quad (6)$$

which can be written equivalently as:

$$\mathbf{T} = \mathbf{D}_r^{1/2} (\mathbf{D}_r^{-1} \mathbf{P} - \mathbf{1}\mathbf{c}^T) \mathbf{D}_c^{-1/2} \quad (7)$$

It is this latter form (7) that is useful here, because it shows that CA is the analysis of the row profiles (rows of frequencies relative to their totals, in the matrix  $\mathbf{D}_r^{-1} \mathbf{P}$ ), centred with respect to the average row profile  $\mathbf{c}^T$ , with row profiles weighted by the masses in  $\mathbf{r}$  and the distances between row profiles defined by chi-square distances using the metric  $\mathbf{D}_c^{-1}$  (see, for example, Greenacre 2007: Theoretical Appendix). After decomposing  $\mathbf{T}$  by the SVD, as in (2):  $\mathbf{T} = \mathbf{U} \mathbf{\Gamma} \mathbf{V}^T$ , where  $\mathbf{U}^T \mathbf{U} = \mathbf{V}^T \mathbf{V} = \mathbf{I}$ , the rows and columns are often plotted according to the so-called *principal coordinates* of the rows:  $\mathbf{F} = \mathbf{D}_r^{-1/2} \mathbf{U} \mathbf{\Gamma}$  and the *standard coordinates* of the columns:  $\mathbf{Y} = \mathbf{D}_c^{-1/2} \mathbf{V}$ . The rationale for this choice of coordinates is as follows: the row points defined by  $\mathbf{F}$  visualize approximate chi-square distances between row profiles that are weighted by the row masses; the column points defined by  $\mathbf{Y}$  are the projections of the unit profiles, or vertex points of the simplex space of the profiles; and row points are weighted averages of the

column vertex points, using the elements of the row profiles as weights:  $\mathbf{F} = \mathbf{D}_r^{-1} \mathbf{P} \mathbf{Y}$ . The joint display of  $\mathbf{F}$  and  $\mathbf{Y}$  is a biplot of the target matrix  $(\mathbf{D}_r^{-1} \mathbf{P} - \mathbf{1} \mathbf{c}^T) \mathbf{D}_c^{-1}$ , since from (7):

$$(\mathbf{D}_r^{-1} \mathbf{P} - \mathbf{1} \mathbf{c}^T) \mathbf{D}_c^{-1} = \mathbf{D}_r^{-1/2} \mathbf{U} \mathbf{\Gamma} \mathbf{V}^T \mathbf{D}_c^{-1/2} = \mathbf{F} \mathbf{Y}^T \quad (8)$$

Hence scalar products between row and column points in a low-dimensional display (using the first columns of  $\mathbf{F}$  and  $\mathbf{Y}$ ) are approximating the centred profiles relative to their average values:  $(p_{ij}/r_i - c_j)/c_j$ . This joint plot is also called the (row principal) asymmetric map – see Greenacre (1993, 2007).

The fact that the row profiles and the extreme unit profiles (vertices) lie in the same multidimensional space is an attractive property of the asymmetric map, but in practice it is problematic when the variance in the data, called *inertia* in CA, is low. For example, the ‘author’ data set, available in the **ca** package of Nenadić and Greenacre (2007), consists of the counts of the 26 letters of the alphabet in 12 samples of text from books by famous English authors – these data are reproduced in Greenacre and Lewi (2009). The profiles of the books across the letters are very similar, as expected, and the inertia is consequently very low. The asymmetric map of this table (Figure 2) shows the 12 books clustered so tightly at the centre of the map that they cannot be labelled. This is clearly not a good biplot and the scale of one of the sets of points needs to be changed to make the joint plot legible. The directions defined by the column points are the correct ones, but they are too long. Rather than simply reduce their overall scale, which is the usual pragmatic solution, the proposed contribution biplot reduces their length by multiplying each column point (representing a letter in the example) by the square root of the mass,  $c_j^{1/2}$ , associated with the column point. Thus all letters are pulled in, but low frequency letters such as *z* and *q* are pulled in a lot more than high frequency letters such as *a* and *e*. This is a premultiplication of  $\mathbf{Y}$  by  $\mathbf{D}_c^{1/2}$  and converts (8) into the following biplot:

$$(\mathbf{D}_r^{-1}\mathbf{P} - \mathbf{1}\mathbf{c}^T)\mathbf{D}_c^{-1/2} = \mathbf{F}(\mathbf{D}_c^{1/2}\mathbf{Y})^T \quad (9)$$

The elements of the target matrix being approximated (shown in the target matrix on the right of (9)) are now the standardized differences  $(p_{ij}/r_i - c_j)/c_j^{1/2}$  between the profile and its average, standardized by dividing by the square root of the average, as in the chi-square metric. The right matrix  $\mathbf{D}_c^{1/2}\mathbf{Y}$  of the biplot in (9), representing the columns, is the original matrix  $\mathbf{V}$  of right singular vectors – we call these the *contribution coordinates*.

To show the connection with the contributions, the parts of inertia due to each column along the principal axes are, in scalar form for column  $j$  along axis  $k$ :  $c_j y_{jk}^2$ , relative to the sum of these quantities, which is equal to 1 from the definition of  $\mathbf{Y}$ :  $\mathbf{Y}^T \mathbf{D}_c \mathbf{Y} = \mathbf{V}^T \mathbf{V} = \mathbf{I}$ . Thus by ‘shrinking’ each  $y_{jk}$  by  $c_j^{1/2}$ , the square of the new coordinate is exactly the contribution of the  $j$ -th point to the inertia along the  $k$ -th axis.

The contribution CA biplot for the ‘author’ data is given in Figure 3, and clearly the problem of the difference in scales has been cleared up. But, more importantly, the lengths of each vector joining the letters to the origin are related to the contributions that the letters make to the principal axes. Thus it can be seen immediately that it is the letters  $d$ ,  $h$ ,  $w$  and  $c$  have high contributions to the first axis, and  $y$  is especially dominant on the second – the closer letters are to the centre the less important they are to the interpretation. Table 1 lists the contributions for each letter to the two axes. Thus,  $d$ , for example, has contributions of 0.1704 to axis 1 and 0.0593 to axis 2, and the square roots of these values are  $0.1704^{1/2} = 0.413$  and  $0.0593^{1/2} = 0.244$ , respectively. The coordinates of  $d$  in Figure 3 are (again from Table 1, the square roots of the masses multiplied by the standard CA coordinates)  $0.046^{1/2} \times (-1.9256) = -0.413$  and  $0.046^{1/2} \times 1.1354 = 0.244$  – this is an empirical check that the squares of the coordinates are the contributions. Notice, however, that the squared lengths of the vectors, equal to the sum of their



squared lengths along the axes, are not equal to their contributions to the two-dimensional map, since the inertias along the axes are not the same. As an alternative, we could define the lengths of the vectors to be related to their contributions to the two-dimensional map, at the expense of losing their contributions on each dimension as in the contribution biplot.

Video 1 in the supplementary multimedia material is a dynamic graphic which shows the transition from Figure 2 to Figure 3, as the biplot vectors for the letters are gradually shrunk proportional to the square roots of their respective masses. The size of the triangular symbol for each letter is proportional to the mass of the letter, so it is the points with smaller triangles that are pulled in the most by the standard biplot scaling. During the transition the row (book) points are in fixed positions and the scale of the map is adjusted as the letter points shrink differentially towards the centre – the dashed box shown throughout the animation actually delimits the area of the final frame of the animation in which the contribution CA biplot resides.

The contribution CA biplot functions equally well for a table with very low inertia, such as the data set ‘author’, as for a table with high inertia. The data set ‘benthos’ (reported in Greenacre and Lewi (2009) and also analysed in Greenacre (2007: chapter 10)) consists of the counts of 92 marine species at 13 different locations on the North Sea bed, 11 of which lie close to an oil-drilling platform that causes sea-bed pollution, while the other two (R40 and R42) are reference stations far away from the pollution source. These data have high inertia, which is characteristic of ecological abundance data of this type. Thus, the station points are much more spread out in the profile space with respect to the extreme vertex points, as shown in the asymmetric CA map of Figure 4. Figure 5 shows the CA contribution biplot of the same data and video 2 of the multimedia material shows the dynamic transition from the Figure 4 to Figure 5. Again each species point is shown with a symbol proportional to its overall abundance. Only the top 10 contributing species have been labelled – the remaining 82 unlabelled species are all pulled in

close to the centre of the contribution biplot, contributing collectively only 15% to the solution. Showing the species' vectors in terms of their contributions assists and clarifies the interpretation by concentrating attention on the most influential species. In Figure 5 it can be clearly seen that there is a group of species chiefly responsible for the separation to the right of the unpolluted reference stations R40 and R42 (these would be the species most affected and reduced by the pollution). There is mainly one species (*Chaetozona setosa*) that dominates the polluted stations, especially station S15 which is one of the closest to the oil-drilling platform and most contaminated. Station S24 is a special case because there is an unusually high occurrence of *Myriochele oculata*, but this is unrelated to the pollution gradient.

Notice that in Figure 3 the book points show less dispersion than the station points in Figure 5, relative to the spread of the 'variables' in each biplot (letters in Figure 2, species in Figure 5, respectively). This partly testifies to the lower inertia in the 'author' data set, compared to the 'benthos' data set, although the impression of the spread of the rows relative to the columns is also governed by how many variables there are. The more variables there are, the more the numerical values of the contributions are reduced as a group, because the contributions sum to 1 for each axis.

### 3. CONTRIBUTION BILOTS FOR PRINCIPAL COMPONENT ANALYSIS

In the case of principal component analysis (PCA), where data are assumed on interval scales, there are two distinct situations: variables that need to be standardized, and variables that are all on the same scale and that need no standardization. The first situation is the most common, when variables are on different measurement scales. Table 2 shows the 'environ' data set consisting of  $J=10$  variables measured at the same  $I=13$  locations as the previous 'benthos' data set. The usual PCA approach is to standardize the variables, in which case the total variance is equal to 10, the number of variables. In calculating the variances of each variable

with a view to standardization, we use  $I$ , not  $I-1$ , to average the squared deviations from the mean. In other words, we weight each sample by a constant  $1/I$ , which mimics the row weighting idea in CA, except in the CA case there are different weights for each row. We could, however, weight the samples using the CA masses, which is what is done if we used the environmental variables as constraints in a canonical correspondence analysis (CCA) (ter Braak, 1986).

Distances between samples are Euclidean on the standardized scales and these distances will increase as the number of variables in the analysis increases. Since we wish to limit the variation in the samples so that we can eventually plot sample distances and variable contributions on the same biplot, a convenient rescaling is to calculate Euclidean distances using average squared differences between (standardized) variables, not sum of squared differences. For a table  $\mathbf{X}$  of standardized data, the distance between samples  $i$  and  $i'$  is:

$$\sqrt{(1/J) \sum_{j=1}^J (x_{ij} - x_{i'j})^2} \quad (10)$$

This again mimics the idea in CA of weighting the columns of the matrix – in PCA all the columns now receive an equal weight of  $1/J$  compared to the differential weighting in CA. The SVD for PCA then multiplies the data by the square roots of the row and column masses (assuming  $\mathbf{X}$  is already column-centred, so that all variables have mean 0 and variance 1):

$$\sqrt{(1/I)} \sqrt{(1/J)} \mathbf{X} = \sqrt{(1/IJ)} \mathbf{X} = \mathbf{U} \mathbf{F} \mathbf{V}^T \quad (11)$$

Hence the sum of the eigenvalues (that is, the sum of squared singular values) is equal to 1, and not  $J$  (the number of variables) as in the usual definition of PCA. The biplot coordinates are  $\mathbf{F} = \sqrt{I} \mathbf{U} \mathbf{F}$  (for the rows, where  $\sqrt{I}$  is the inverse of the row masses, as in CA) and  $\mathbf{Y} = \mathbf{V}$  (for

the columns, where there is no rescaling by the masses, as in the standard CA biplot). Again the squared lengths of the column coordinates are equal to the contributions of the respective variables to the respective principal axes.

Figure 6 shows the PCA contribution biplot of the ‘environ’ data set of Table 2. All the variables measuring pollution (THC=Total Hydrocarbon Content, and the metals) point to the left and contribute the most to the horizontal axis, which separates the most polluted station S15 on the left (one of the closest to the oil rig) from the unpolluted reference stations R40 and R42 on the right. Distance versus depth contribute strongly to the second axis, with the TOM (=Total Organic Material) biplot axis positive – both the highly polluted station S15 and the two reference stations are high on TOM.

The reference stations are at shallower depths than the others, but only by a few metres, and this difference is accentuated by the standardization. Also, because the variables are highly skew, a logarithmic transformation might be considered appropriate – the log-transformed data are then not standardized, so that the multiplicative differences are visualized. Figure 7 shows the PCA contribution biplot of the log-transformed data, and the role of the variable depth is considerably reduced, with most of the variance now concentrated on the first axis which is the pollution gradient. This also shows that the contribution biplot functions well for PCA of unstandardized log-transformed data as well. The only slightly problematic case would be when all the original variables are on the same scale, and standardization is not required, but the common scale has a very small or very large range – for example, when all data are ratings from 0 to 100. Then the scale of the row points will be quite different to that of the column points, but this can be rectified by some appropriate overall rescaling of the original data, for example to rescale the ratings from 0 to 10, or from 0 to 1.

It is important to realize that the PCA contribution biplot described above is not the same as the biplot in which the variables are depicted by their correlations with the principal axes, and thus lie within a unit circle in the two-dimensional solution, for example. This latter biplot associates the singular values with the column coordinates, depicting rows (cases) with coordinates in  $\sqrt{I}\mathbf{U}$  (see the SVD in (11)) and the columns (variables) with coordinates in  $\mathbf{V}\mathbf{\Gamma}$ . The squares of the correlations are the parts of the (unit) variance of that variable explained by the axes, and can be summed for the two axes, hence the drawing of a unit circle which represents perfect explanation of the variables. The average sum of squared column coordinates on an axis is then the part of the total variance on that axis, while the cases are represented by coordinates that have average sum of squares equal to 1 on every principal axis.

#### 4. CONTRIBUTION BIPLOTS FOR LOG-RATIO ANALYSIS

Log-ratio analysis is the visualization of a table of strictly positive data, log-transformed and double-centred, again using the SVD. The method is popular in the analysis of compositional data (Aitchison 1986). There are two forms, an unweighted form (see Aitchison 1983, Aitchison and Greenacre 2002) and a weighted form, the latter also known as ‘spectral mapping’ (Lewi 1976, Greenacre and Lewi 2009). The row and column weights commonly used in weighted log-ratio analysis are the same as the masses in CA, that is the relative values of the column margins of the original table. Using the same notation as in CA for the row and column masses  $\mathbf{r}$  and  $\mathbf{c}$ , and  $\log(\mathbf{N})$  to denote the matrix of log-transformed data, the target matrix and its SVD are:

$$\mathbf{T} = \mathbf{D}_r^{1/2}(\mathbf{I} - \mathbf{1}\mathbf{r}^\top)\log(\mathbf{N})(\mathbf{I} - \mathbf{1}\mathbf{c}^\top)^\top \mathbf{D}_c^{1/2} = \mathbf{U}\mathbf{\Omega}\mathbf{V}^\top \quad (12)$$

(just substitute  $\mathbf{r} = (1/I)\mathbf{1}$  and  $\mathbf{c} = (1/J)\mathbf{1}$  for the unweighted form). Assuming rows were cases/samples and the columns variables/components, the standard log-ratio biplot would then

use the principal coordinates  $\mathbf{D}_r^{-1/2}\mathbf{U}\mathbf{\Omega}$  for the rows and  $\mathbf{V}$  for the columns. Some care is needed here, since the analysis is often performed to detect equilibrium in a subset of components, which can be diagnosed by some components lining up in straight lines (see, for example, Aitchison and Greenacre 2002). But the contribution biplot normalization, while having the advantage of showing which components are important for the interpretation of the biplot, has the disadvantage of destroying this property. A dynamic graphic, which alternates between the usual map where the columns are shown in principal or standard coordinates and the contribution biplot, can be useful to diagnose models in subcompositions as well as see which components are determinant in the solution – video 3 in the supplementary material shows an example of this for the ‘author’ data, starting with a LRA solution that allows model diagnosis and ending with the contribution biplot that represents component contributions and thus easier for descriptive interpretation.

## 5. CONTRIBUTION BILOTS FOR DISCRIMINANT ANALYSIS / MANOVA

Linear discriminant analysis (LDA) and multivariate analysis of variance (MANOVA) rely on a common theory, which reduces to a SVD of the matrix of group centroids, weighted by their respective group sizes, in a space structured by the Mahalanobis metric – see, for example, Mardia, Kent and Bibby (1979). The resultant principal axes are often called canonical axes in this context, and the terminology canonical variate analysis (CVA) is a common synonym as well. This is another case where the contribution biplot can be used to represent the centroids by their projections onto the optimal plane along with the points for the variables such that their squared lengths along a canonical axis are the contributions to the group discrimination along the axis. The idea of a MANOVA biplot originates in the work of Gabriel (1972).

Suppose that there are  $G$  groups of cases (rows) in a cases-by-variables data matrix, and that the  $G$  vectors of variable averages are contained in the  $G \times J$  matrix  $\mathbf{G}$ . The averages are centred

with respect to the global averages of the variables across all the cases – these global averages are identical to the centroid of the group averages, where each group is weighted proportionally to the number of cases in it. A  $J \times J$  covariance matrix  $\mathbf{S}_g$  is computed within each group – as in PCA the averaging of the squares and cross-products in the computation is performed by dividing by the number of cases  $n_g$  in each group, not  $n_g - 1$ . The estimated common covariance matrix  $\mathbf{S}$  is then computed as  $\sum_g (n_g/n) \mathbf{S}_g$ . Weights of  $n_g/n$  are assigned to each group average, and these weights form the diagonal of the diagonal matrix  $\mathbf{D}_w$ .

The weighted PCA of the group averages, in the Mahalanobis metric  $\mathbf{S}^{-1}$ , equivalent to LDA/MANOVA/CVA is achieved by the following SVD, imitating the procedure followed for CA and PCA:

$$\mathbf{T} = \mathbf{D}_w^{1/2} \mathbf{G} \mathbf{S}^{-1/2} \sqrt{1/J} = \mathbf{U} \mathbf{\Gamma} \mathbf{V}^T \quad (13)$$

Thus the contribution biplot would visualize the rows by the principal coordinates of the group centroids in  $\mathbf{D}_w^{-1/2} \mathbf{U} \mathbf{\Gamma}$  and the coordinates  $\mathbf{V}$  of the columns (variables) that depict the contributions to the principal axes. Figure 8 shows the contribution biplot in the two-dimensional plane of the three centroids of the famous Fisher ‘iris’ data, where it can be seen immediately which of the variables is responsible for the separation of the three groups.

## 6. CONTRIBUTION BILOTS AND REGRESSION

An enlightening way of thinking about the variables displayed as vectors in a biplot is in terms of linear regression: the coordinates of the vectors can be shown to be regression coefficients of the variables on the principal axes. Consider the general weighted least-squares regression (WLS) of a set of variables in the columns of a matrix  $\mathbf{T}$  (we intentionally use the notation of the target matrix) on a set of predictors in the columns of a matrix  $\mathbf{Z}$ , where the “cases” (rows) are

weighted by positive numbers in the diagonal matrix  $\mathbf{D}_w$ . Writing the regression model as  $\mathbf{T} = \mathbf{Z}\Psi + \mathbf{E}$ , the matrix  $\Psi$  of regression coefficients is estimated as:  $\Psi = (\mathbf{Z}^T \mathbf{D}_w \mathbf{Z})^{-1} \mathbf{Z}^T \mathbf{D}_w \mathbf{T}$ .

Applying this result to CA, where  $\mathbf{T} = (\mathbf{D}_r^{-1} \mathbf{P} - \mathbf{1c}^T) \mathbf{D}_c^{-1/2}$  (hence the variables are the standardized profiles),  $\mathbf{Z} = \mathbf{F} = \mathbf{D}_r^{-1/2} \mathbf{U}\mathbf{\Gamma}$  (the row principal coordinates), and  $\mathbf{D}_w = \mathbf{D}_r$  (the row masses), the regression coefficients are:

$$\begin{aligned} (\mathbf{F}^T \mathbf{D}_r \mathbf{F})^{-1} \mathbf{F}^T \mathbf{D}_r (\mathbf{D}_r^{-1} \mathbf{P} - \mathbf{1c}^T) \mathbf{D}_c^{-1/2} &= (\mathbf{\Gamma}^2)^{-1} (\mathbf{\Gamma} \mathbf{U}^T \mathbf{D}_r^{-1/2}) \mathbf{D}_r (\mathbf{D}_r^{-1/2} \mathbf{U} \mathbf{V}^T) \\ &= \mathbf{V}^T \end{aligned}$$

since  $\mathbf{U}^T \mathbf{U} = \mathbf{I}$ . So the regression coefficients are exactly the contribution coordinates, and the vectors using these coordinates in two-dimensional biplots, for example, are equivalent to WLS regression planes above the biplot, the direction vectors being the gradients of the planes showing the directions of steepest ascent. The contours of these planes are at right-angles to these gradient vectors, which shows that projecting points onto biplot vectors is simply estimating their values according to their regression relationships with the principal axes.

Similar arguments pertain to the contribution biplot in the other applications.

## 7. DISCUSSION AND CONCLUSIONS

The subject of the scaling in biplots, and in particular in CA, has been the subject of much discussion and controversy, so we need a good reason to propose yet another one. Amongst all biplot options for a typical cases  $\times$  variables data set, what we are really interested in is the direction made by each ‘variable’ of the data set, so the proposal of this paper is to maintain these directions while making the lengths of the biplot vectors both meaningful and also on a scale which allows joint representation of the ‘case’ points without the need for two sets of scales for the row and column points. Both these objectives are attained by the contribution



biplot. The case points are always in their principal coordinate positions, so that distances between case points are interpretable. The variables are indicated by lines or arrows coinciding with the biplot axes, which could optionally be calibrated so that the projections of the case points on a biplot axis approximate the values for that variable. The lengths of the lines along each principal axis of the biplot are related to the contribution made by that variable to the solution.

Although motivated principally in correspondence analysis because of the problem of differential masses of the row and column points, it has also been shown that the contribution biplot functions well in standardized and unstandardized principal component analysis, log-ratio analysis and discriminant analysis. We believe that the contribution biplot can become a standard way of showing the results of methods based on the singular-value decomposition, since it is directly related to the biplot's correct interpretation.

## COMPUTING NOTE

The **ca** package (Nenadić and Greenacre 2007) in R (R development core team, 2008) was used to perform the CA calculations in this article, while all others were computed using matrix functions in R function, notably `svd`. The standard biplot option for CA already exists in the **ca** package, for example `map="rowgreen"` for the standard (row principal) biplot. The option `map="rowgab"` is the one proposed by Gabriel and Odoroff (1990) where the standard coordinates of ‘variable’ points are multiplied by their masses, not the square roots of the masses as in the standard biplot.

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## SUPPLEMENTAL MATERIAL

*Video 1:* Dynamic transition between the asymmetric CA map (Figure 2) and the CA contribution biplot (Figure 3) of the data set ‘author’. Abbreviations for the books are used – see Figure 3 for full titles.

*Video 2:* Dynamic transition between the asymmetric CA map (Figure 4) and the CA contribution biplot (Figure 5) of the data set ‘benthos’.

*Video 3:* Dynamic transition between the asymmetric LRA map and its contribution biplot version, for the data set ‘author’. In the first frame one can see the lining up of three letters,  $k$ ,  $y$  and  $x$ , which indicates an equilibrium model for these letters (given in Greenacre and Lewi, 2009), but which cannot be diagnosed in the contribution biplot.

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*Table 1:* Masses (relative frequencies), contributions (relative to total), and contribution coordinates of the 26 letters of the ‘author’ data set, for the two principal axes of Figures 2 and 3. The coordinates of the letters in the standard biplot of Figure 3 are the square roots of the masses multiplied by the standard coordinates. The squares of these coordinates are equal to the contributions.

	<b>Masses</b>	<b>Contributions</b>		<b>Standard coordinates</b>	
		<i>Axis 1</i>	<i>Axis 2</i>	<i>Axis 1</i>	<i>Axis 2</i>
<b>a</b>	0.0798	0.0000	0.0082	0.0176	0.3203
<b>b</b>	0.0157	0.0152	0.0025	0.9845	0.3980
<b>c</b>	0.0228	0.1020	0.0430	2.1150	1.3734
<b>d</b>	0.0460	0.1704	0.0593	-1.9256	1.1354
<b>e</b>	0.1271	0.0010	0.0596	0.0867	0.6848
<b>f</b>	0.0194	0.0317	0.0104	1.2765	0.7330
<b>g</b>	0.0200	0.0209	0.0025	-1.0207	-0.3530
<b>h</b>	0.0649	0.1463	0.0059	-1.5013	0.3024
<b>i</b>	0.0701	0.0050	0.0555	0.2675	-0.8895
<b>j</b>	0.0008	0.0002	0.0007	0.4533	-0.9160
<b>k</b>	0.0092	0.0697	0.0139	-2.7552	-1.2316
<b>l</b>	0.0427	0.0442	0.0012	1.0183	0.1650
<b>m</b>	0.0255	0.0130	0.0500	0.7127	-1.4010
<b>n</b>	0.0690	0.0028	0.0120	-0.2004	0.4173
<b>o</b>	0.0766	0.0009	0.0312	-0.1085	-0.6380
<b>p</b>	0.0152	0.0393	0.0512	1.6108	1.8379
<b>q</b>	0.0007	0.0111	0.0025	4.0798	1.9148
<b>r</b>	0.0519	0.0181	0.0280	0.5914	0.7342
<b>s</b>	0.0607	0.0449	0.0100	0.8602	-0.4056
<b>t</b>	0.0930	0.0009	0.0038	-0.1005	-0.2031
<b>u</b>	0.0298	0.0008	0.0308	0.1633	-1.0171
<b>v</b>	0.0096	0.0500	0.0003	2.2813	-0.1770
<b>w</b>	0.0258	0.1614	0.0021	-2.4992	0.2847
<b>x</b>	0.0012	0.0129	0.0206	3.3405	-4.2154
<b>y</b>	0.0219	0.0000	0.4851	0.0015	-4.7061
<b>z</b>	0.0008	0.0371	0.0099	6.8081	3.5092

Table 2: Data set ‘environ’ of 10 variables measured at 13 locations in the North Sea. Distance is from the oil drilling platform (the source of the pollution). The two reference stations R40 and R42 are 10 kms away and reflect an unpolluted environment.

	<i>DEPTH</i>	<i>Ba</i>	<i>Cd</i>	<i>Cu</i>	<i>Fe</i>	<i>Pb</i>	<i>Zn</i>	<i>THC</i>	<i>TOM</i>	<i>DISTANCE</i>
<b>S4</b>	76	1656	0.02	1.3	2022	11.7	16.1	8	0.6	750
<b>S8</b>	74	1373	0.02	1.1	2398	8.9	13.6	9	0.7	1800
<b>S9</b>	76	3680	0.07	3.9	2985	34.4	45.7	103	0.8	800
<b>S12</b>	72	2094	0.04	1.2	2535	21.3	15.1	6	0.7	2500
<b>S13</b>	72	2813	0.04	1.6	2612	17.3	18	27	1	1300
<b>S14</b>	76	4493	0.06	3.3	2515	28.8	26.7	546	1	850
<b>S15</b>	76	6466	0.14	6.2	3421	61.4	72.5	952	1.1	500
<b>S18</b>	72	1661	0.02	1.3	2381	19.8	13.8	10	0.9	2500
<b>S19</b>	72	3580	0.05	2.4	3452	33.7	28.9	32	1	1200
<b>S23</b>	76	2247	0.02	1.5	3457	21.4	14.9	16	0.7	1000
<b>S24</b>	76	2034	0.05	2.7	2311	15.5	16.8	11	0.7	450
<b>R40</b>	67	40	0.01	0	1804	5.9	5.9	3	0.8	10000
<b>R42</b>	67	85	0.01	0.17	1815	6.7	5.9	3	1	10000

*Figure 1:* Correspondence analysis of the fatty acid data set, showing (a) the biplot with rows in principal coordinates and columns in standard coordinates (row principal asymmetric map), with two different scales for row and column points; (b) the contribution biplot, using one common scale, showing the labels of only those fatty acids that are the most outlying and which contribute more than average to the principal axes.

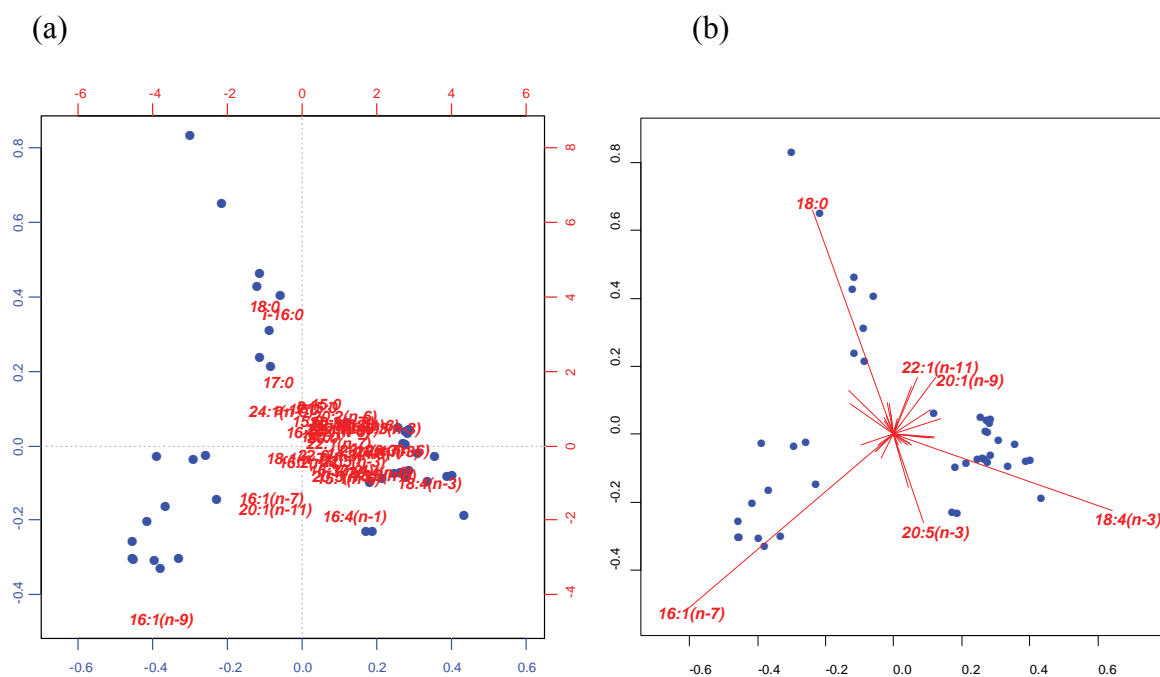


Figure 2: Correspondence analysis of data set ‘author’, with rows in principal coordinates and columns in standard coordinates (row principal asymmetric map), showing row profiles bunched up near the origin of the biplot.

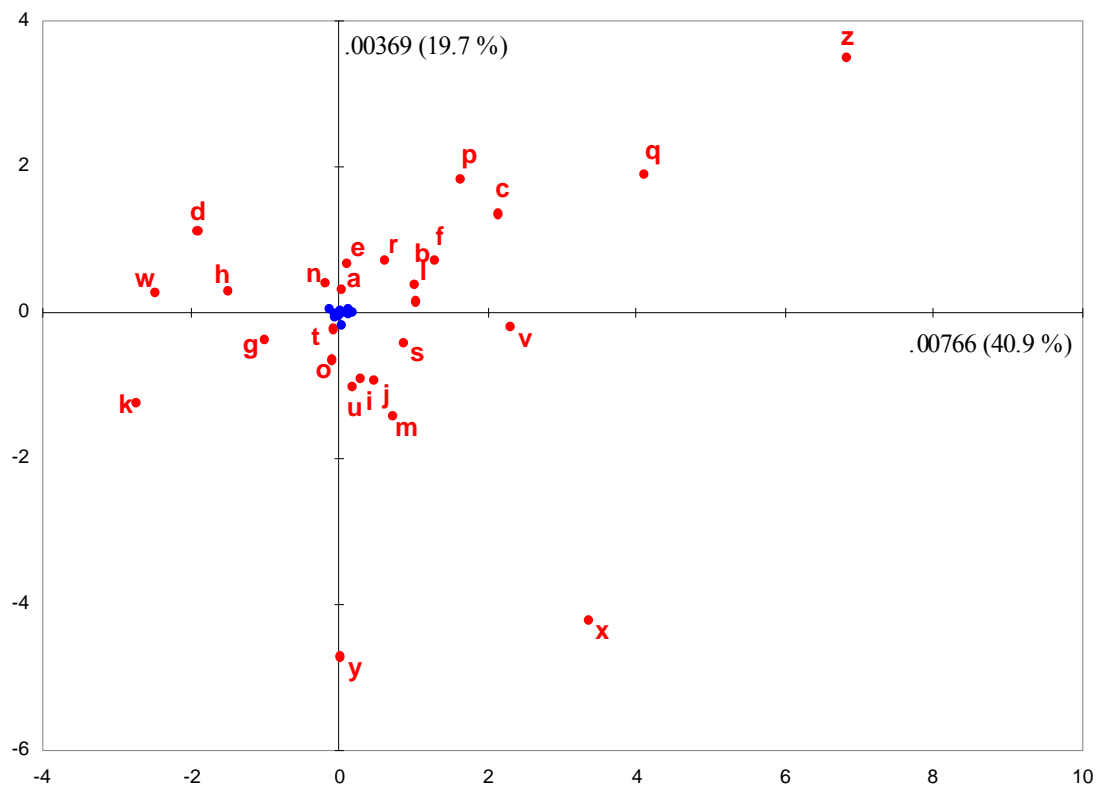




Figure 3: Contribution biplot of the correspondence analysis of ‘author’ data, showing the letters as the same directions as in Figure 1, but multiplied by the square roots of their masses: the squared coordinates of each letter on an axis is equal to its contribution to that axis.

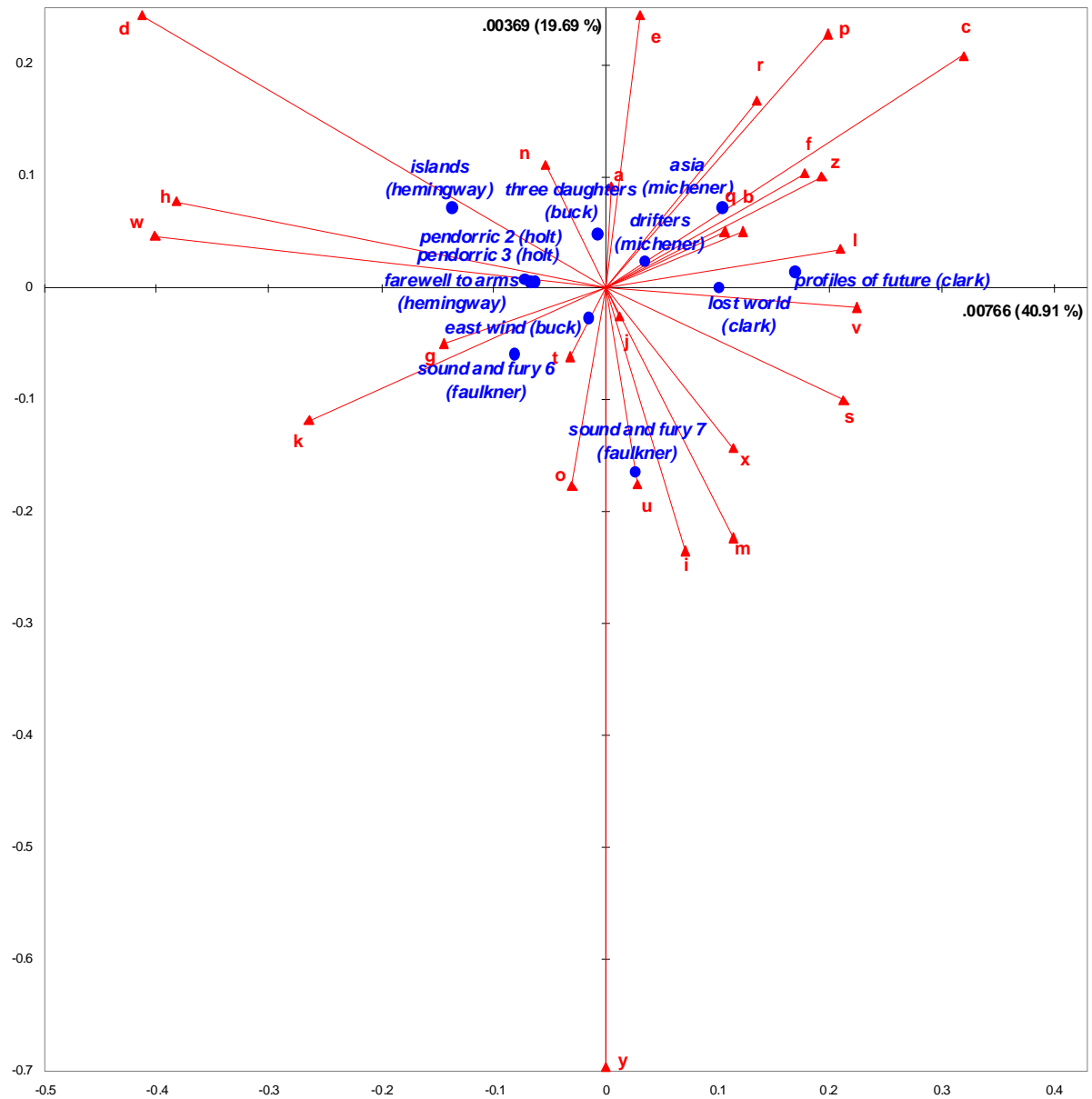


Figure 4: Asymmetric CA map of ‘benthos’ data set, with stations in principal coordinates, and species in standard coordinates (this is the first frame of the animation in Figure 6). The top 10 contributing species are labelled, accounting for 85% of the inertia of the two-dimensional solution. These are all highly abundant species while the remaining 82 lower abundance species collectively contribute only 15% to the solution. Notice that many of these low-contributing species are outlying in this map. The triangular symbols for the species are proportional to the overall species abundance. Inertias on axes 1 and 2 are 0.2457 (31.4%) and 0.2043 (26.1%) respectively.

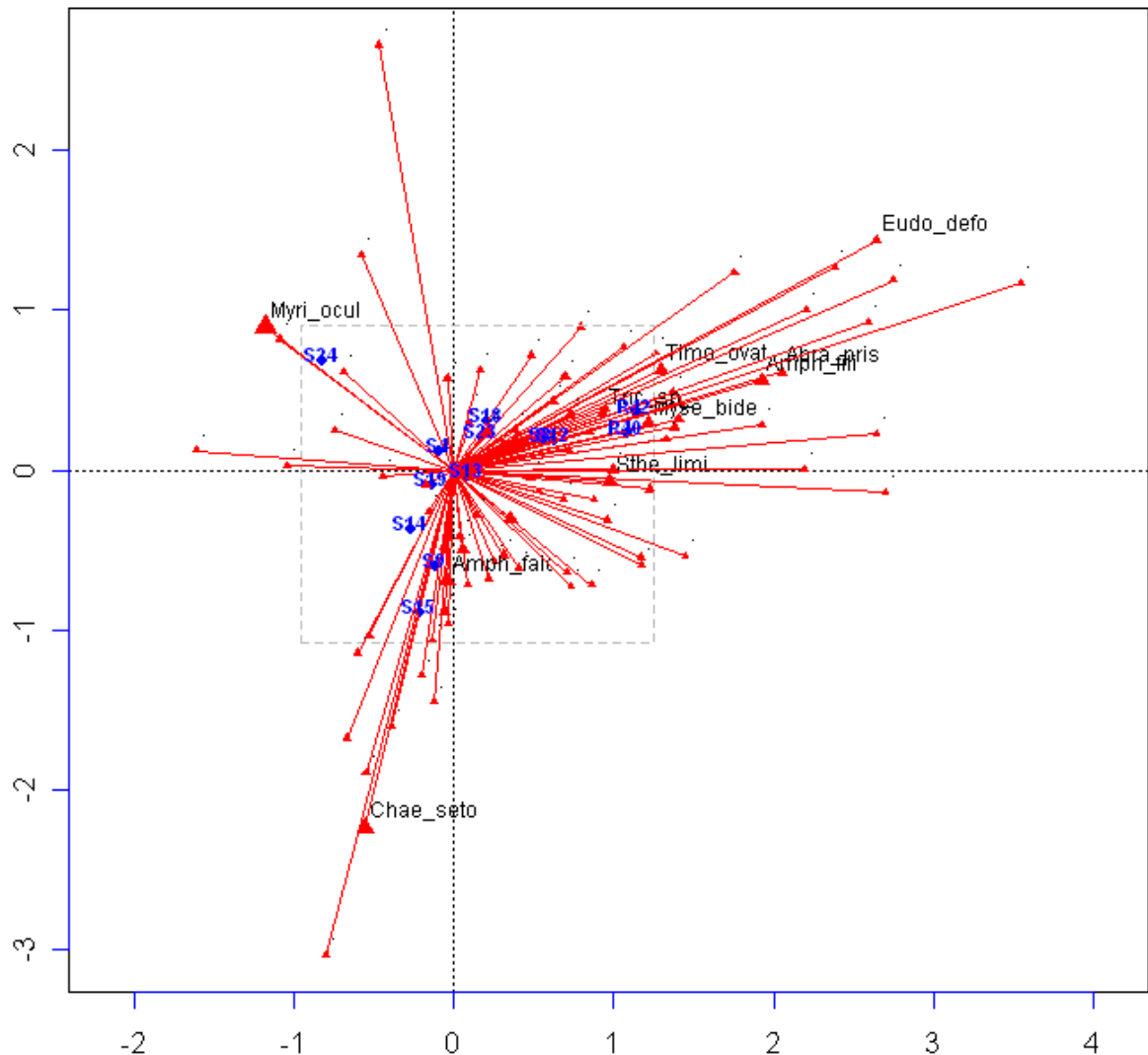


Figure 5: Contribution biplot of the CA of the ‘benthos’ data set. The top 10 contributing species are now the most outlying while the remaining 82 lower abundance species that collectively contribute very little all lie towards the centre of the biplot. The station points are in the same positions as in Figure 4 and have inertias 0.2457 (31.4%) and 0.2043 (26.1%) on axes 1 and 2 respectively. This is the last frame of the animation of Video 2, and its boundaries are given by the dashed box of Figure 4, showing how much we have “zoomed in” on the configuration of station points.

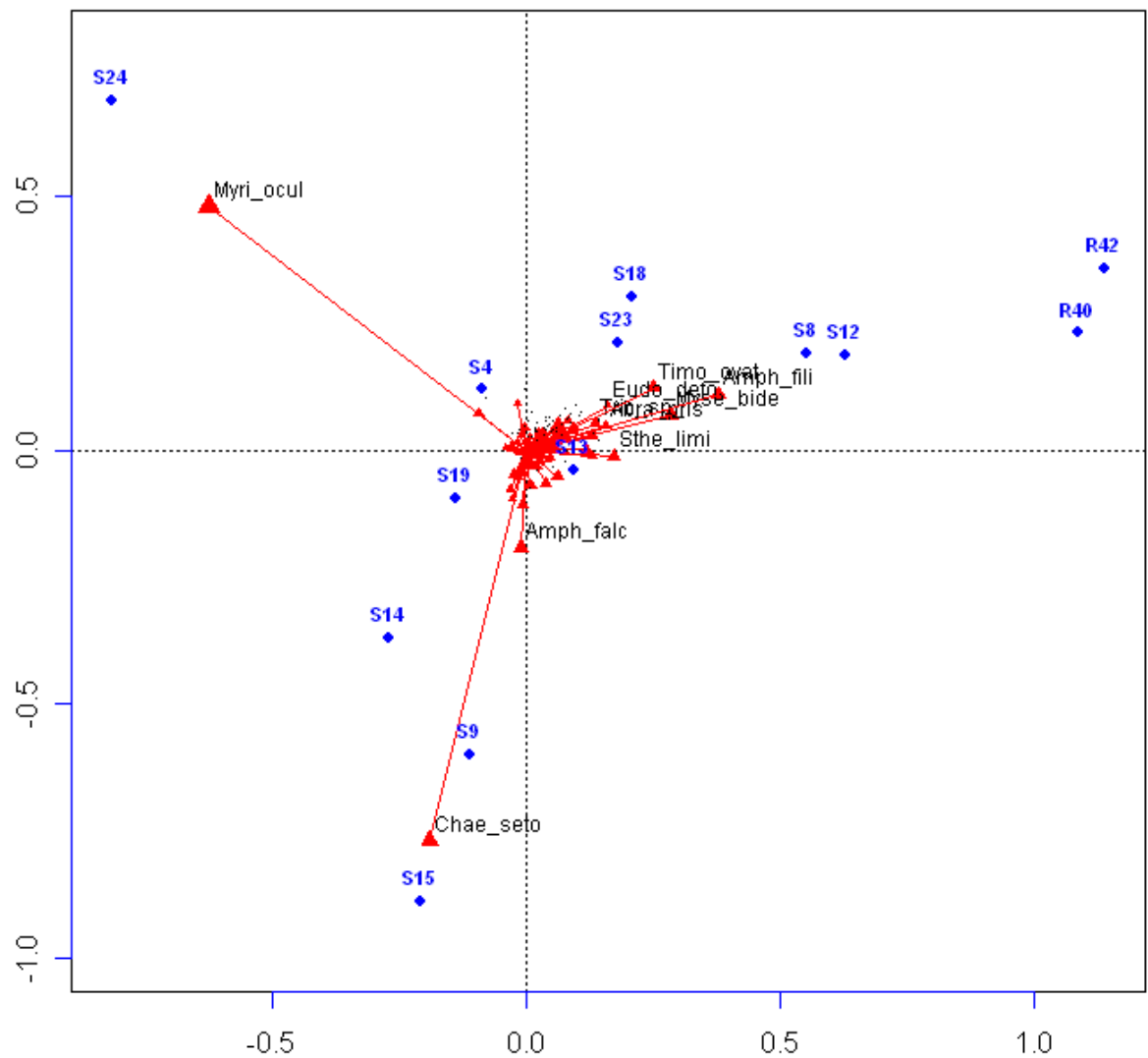
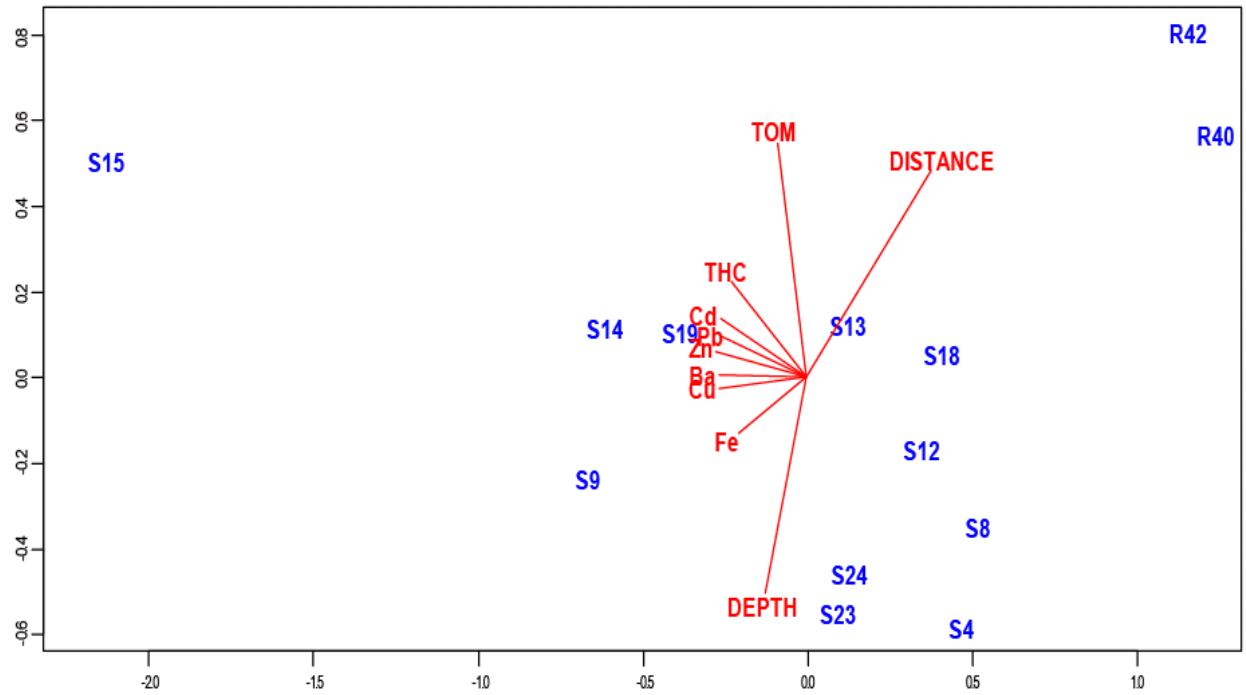
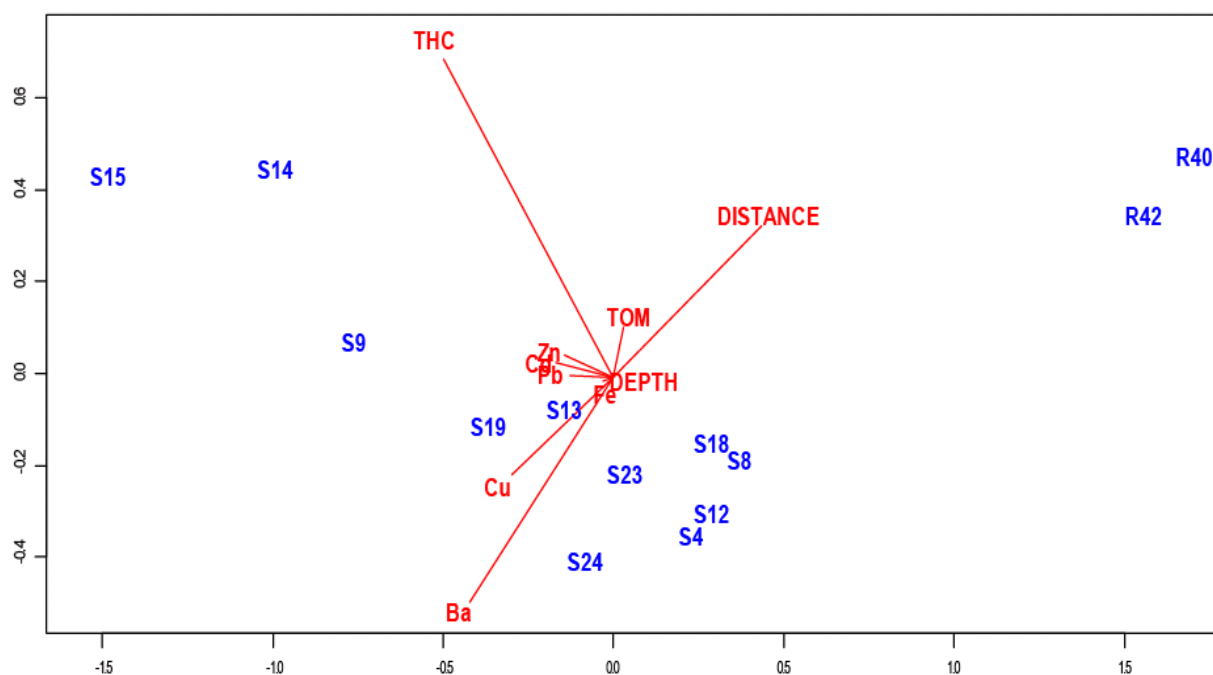


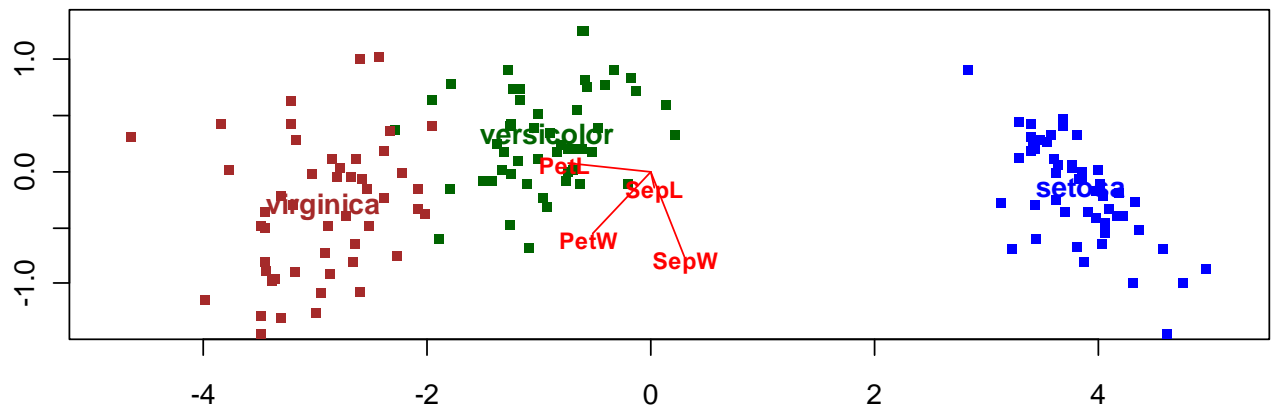
Figure 6: PCA contribution biplot of ‘environ’ data set of Table 2. Average sum-of-squares on the axes are 0.7008 and 0.1791 respectively (these are also proportions of variance accounted for, since total variance is scaled to be 1).



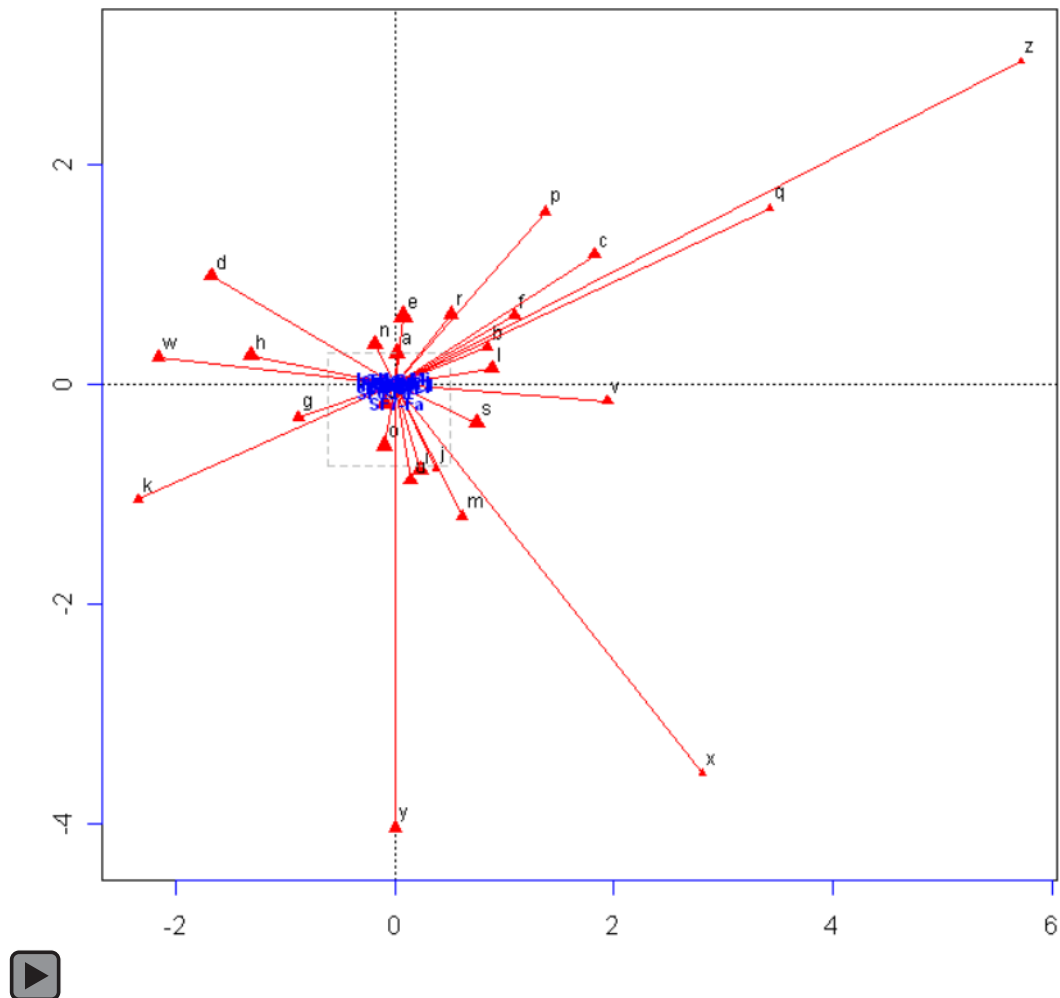
*Figure 7: The PCA contribution biplot of the log-transformed (and unstandardized) data of Table 2. Total variance (which is again the average of the individual variances) is 0.8692, and the percentages of variance explained on axes 1 and 2 are 84.5% and 11.0% respectively.*



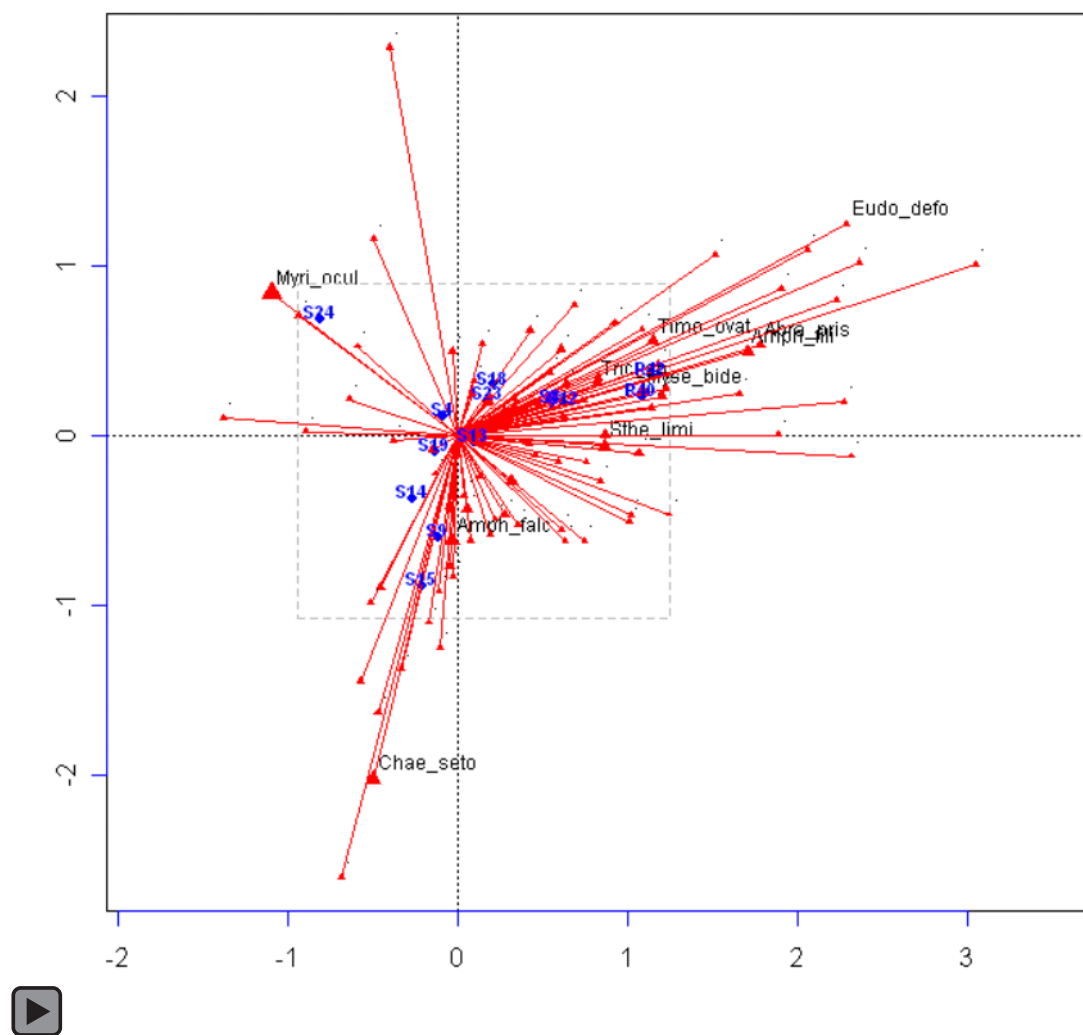
*Figure 8:* The MANOVA contribution biplot of the three groups in the classic ‘iris’ data set. Total variance of the three group centroids in the Mahalanobis space is 8.119, of which 99.1% is accounted for by the first canonical axis. The variable Petal Length (*PetL*) has the biggest contribution to the strong horizontal separation of the groups, while Sepal Length (*SepL*) has almost no contribution to the discriminant space.



*Video 1: Dynamic transition between the asymmetric CA map (Figure 2) and the CA contribution biplot (Figure 3) of the data set ‘author’. Abbreviations for the books are used – see Figure 3 for full titles. **Click on graphic to animate...***



Video 2: Dynamic transition between the asymmetric CA map (Figure 4) and the CA contribution biplot (Figure 5) of the data set ‘benthos’. *Click on graphic to animate...*





Video 3: Dynamic transition between the asymmetric LRA map and its contribution biplot version, for the data set ‘author’. In the first frame one can see the lining up of three letters,  $k$ ,  $y$  and  $x$ , which indicates an equilibrium model for these letters (given in Greenacre and Lewi, 2009), but which cannot be diagnosed in the contribution biplot. *Click on graphic to animate...*

