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ARCH Patterns in Cointegrated Systems

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Abstract

We introduce a class of cointegrated models that allows for conditional heteroskedasticity. Conditions for covariance stationarity and strict stationarity are explored by means of Monte Carlo simulation. Simulation techniques are also used to highlight the finite-sample properties of the maximum-likelihood estimator and the influence of rank restrictions. Forecasting properties are illustrated using an exemplary data set.

1. The basic model

Many systems of economic time series variables are known to be cointegrated in the sense of Engle and Granger (1987). At the same time, in particular if the variables are financial high-frequency series, there is widespread evidence on fluctuations in conditional variance. The most commonly used model for this phenomenon of time-changing conditional variances and clustered volatility is the ARCH model by Engle (1982) and its variants such as e.g. GARCH (Bollerslev 1986) and ARMA-ARCH (Weiss 1984). Hence, the interaction of the two apparitions certainly deserve some closer examination.

As long as ARCH effects remain "well-behaved" and do not entail the violation of fourth-moments conditions in the marginal distribution, it can be shown that the influence of ARCH on cointegration tests and parameter estimates disappears asymptotically even if such an influence were present in smaller samples. Such general derivations which continue to hold in "mild ARCH" situations have e.g. been presented by Park and Phillips (1988). If fourth-moments conditions are violated, some of the rationales for central limit theorems break down and the ground becomes more difficult to explore. Hence, such exploration is typically done via Monte Carlo simulations (Kim and Schmidt 1993, Kunst 1993a, Hecq and Urbain 1993). If ARCH effects become so strong that even second moments cease being finite, handling of the situation becomes even more difficult, as in these situations cumulated sums of processes cannot be readily distinguished from the series proper on the basis of moments properties (also compare Sampson 1990). Some theoretical work in this direction has been done by Hansen (1992) who calls his model "bi-integrated" as it is integrated in its mean as well as in its ARCH-like dependence structure of its volatility.[1] Fortunately, these "IGARCH" processes do not constitute natural boundaries of economic reality as e.g. Nelson (1990)

[1] Hansen's bi-integrated model is not really an IGARCH model but similar. Hansen assumes that errors variance follows a random walk which may create problems as then variances can become negative with positive probability. In contrast, the IGARCH model does not permit negative variance by construction.

has shown, and strictly stationary ARCH processes can be found in an area where even moments of order 2- λ do not exist. It is tempting to view such processes in the stable laws framework (see Phillips (1990) and Phillips and Loretan (1990) for recent extensions of the stable laws model to integrated processes, Kunst (1993a) for a Monte Carlo misspecification analysis) but the high serial dependence prevents even non-standard stable CLTs from being applicable.

To highlight the features at stake rather than to funnel the investigation through a narrow parameterization, we suggest the following model:

$$\begin{aligned} \Delta X_t &= \alpha_1(X_{t-1} + \beta Y_{t-1}) + \varepsilon_{1t} \\ \Delta Y_t &= \alpha_2(X_{t-1} + \beta Y_{t-1}) + \varepsilon_{2t} \\ E\left(\varepsilon_{1t}^2 | I_t\right) &= a_0 + a_1\varepsilon_{1t-1}^2 + a_2\varepsilon_{2t-1}^2 (= h_{1t}) \\ E\left(\varepsilon_{2t}^2 | I_t\right) &= b_0 + b_1\varepsilon_{1t-1}^2 + b_2\varepsilon_{2t-1}^2 (= h_{2t}) \end{aligned} \tag{1.1}$$

Alternatively, (1.1) can be written as

$$\begin{bmatrix} X_t \\ Y_t \end{bmatrix} = \begin{bmatrix} 1+\alpha_1 & \alpha_1\beta \\ \alpha_2 & 1+\alpha_2\beta \end{bmatrix} \begin{bmatrix} X_{t-1} \\ Y_{t-1} \end{bmatrix} + \begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{bmatrix}$$

$$\begin{bmatrix} h_{1t} \\ h_{2t} \end{bmatrix} = \begin{bmatrix} a_0 \\ b_0 \end{bmatrix} + \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} \begin{bmatrix} \varepsilon_{1,t-1}^2 \\ \varepsilon_{2,t-1}^2 \end{bmatrix}$$

The bivariate process (1.1) is known to be covariance-stationary within the boundaries prescribed by the upper two linear equations and the lower two quadratic equations separately. It follows immediately that the whole process is covariance-stationary if the matrices $\mathbb{M}_1$ and $\mathbb{M}_2$ with

$$\begin{aligned} \mathbb{M}_1 &= \begin{bmatrix} 1+\alpha_1 & \alpha_1\beta \\ \alpha_2 & 1+\alpha_2\beta \end{bmatrix} \\ \mathbb{M}_2 &= \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} \end{aligned} \tag{1.2}$$

have all their eigenvalues strictly inside the unit circle. If $\mathbb{M}_1$ has one or two eigenvalues on the unit circle, the bivariate process is stationary in its first differences. If only one of the two $\mathbb{M}_1$ eigenvalues is on the unit circle - in that case, only +1 and -1 are possible -

a simple change of co-ordinates produces a system with two variables, one of which is stationary and the other one is difference-stationary. The eigenvalues of $\mathbf{M}_2$ do not have any direct impact on this property as long as they stay in their stability area, i.e., within the unit circle.

Because the system (1.1) changes its shape if transformations of variables or innovations series are allowed, it pays to look at the more general system class which was in the focus of the factor-ARCH structures by Engle and co-authors. Under co-ordinate transformations, (1.1) becomes a special case of the general model

$$\begin{aligned}
\Delta X_t &= \alpha_1(X_{t-1} + \beta Y_{t-1}) + \varepsilon_{1t} \\
\Delta Y_t &= \alpha_2(X_{t-1} + \beta Y_{t-1}) + \varepsilon_{2t} \\
E(\varepsilon_{1t}^2 | I_{t-1}) &= a_0 + a_1\varepsilon_{1t-1}^2 + a_2\varepsilon_{2t-1}^2 + a_3\varepsilon_{1t-1}\varepsilon_{2t-1} \\
E(\varepsilon_{2t}^2 | I_{t-1}) &= b_0 + b_1\varepsilon_{1t-1}^2 + b_2\varepsilon_{2t-1}^2 + b_3\varepsilon_{1t-1}\varepsilon_{2t-1} \\
E(\varepsilon_{1t}\varepsilon_{2t} | I_{t-1}) &= c_0 + c_1\varepsilon_{1t-1}^2 + c_2\varepsilon_{2t-1}^2 + c_3\varepsilon_{1t-1}\varepsilon_{2t-1}
\end{aligned} \tag{1.3}$$

Note that the *transformed* system is not of the type (1.1) but that it allows for dependence of conditional variances h_{it} on lagged squared errors as well as on cross-products. It is easy to show - maintaining the assumption of conditional Gaussian distributions - that such cross-terms do not exert any influence on conditions of covariance stationarity. It is less easy to show that such cross-terms *do* exert some influence on strict stationarity properties. (compare Kunst 1993b and the strand of literature related to random coefficients models such as Tsay 1987, Bera et al. 1992)

The general model (1.3), however, is too general for practical purposes, particularly as it does not accommodate for the restrictions on the coefficients a_i, b_i, c_i imposed by the non-negative definiteness conditions on covariance matrices. Some necessary conditions have been stated by Engle et al. (1984). They are non-linear inequality constraints and appear rather cumbersome for practical applications. In contrast, all definiteness restrictions are naturally contained in the matrix-vector form

$$E(\mathbf{e}_t\mathbf{e}_t' | I_{t-1}) = \Sigma_0 + \Lambda \begin{bmatrix} \mathbf{e}_{t-1}' & 0 \\ 0 & \mathbf{e}_{t-1}' \end{bmatrix} \begin{bmatrix} \mathbf{A} & 0 \\ 0 & \mathbf{B} \end{bmatrix} \begin{bmatrix} \mathbf{e}_{t-1} & 0 \\ 0 & \mathbf{e}_{t-1} \end{bmatrix} \Lambda' \tag{1.4}$$

where the bold-faced $\mathbf{e}_t = (\varepsilon_{1t},\varepsilon_{2t})'$, Σ_0 is a non-negative definite 2 ×2-matrix containing the unconditional variance matrix of $\mathbf{e}_t$, and $\mathbb{A}$ is a general non-singular matrix of dimension 2 ×2 used for transforming the original variates. $\mathbb{A}$ and $\mathbb{B}$ are non-negative definite 2 ×2-matrices.

Note that (1.4) restricts the general form (1.3) because of the block-diagonality of the "inner" matrix. All features treated below, however, are difficult to extend to the case of a general inner matrix, hence (1.4) will be the model of interest in what follows. Additionally, definiteness and identification restrictions become very involved for the extended form. Engle and Bollerslev (1986, p.11) also considered (1.4) but did not elaborate on it. Later, a different form was suggested by Engle and Kroner (1993), whose so-called BEKK representation relies on the principle of expressing a quadratic form as a sum of squared linear factors. The number of factors can be increased until the BEKK representation encompasses the full form (1.3). The BEKK form is practical as it naturally contains all definiteness constraints and identification can also be imposed easily by zero restrictions. However, in case of a factor deficiency, it excludes certain models from consideration whose value is difficult to assess.

It is worth while to examine some of the possible rank deficiencies in the matrices $\mathbb{A}$ and $\mathbb{B}$. If both of them are zero matrices, i.e. $\mathrm{rk}\mathbb{A}=\mathrm{rk}\mathbb{B}=0$, then there are no ARCH effects in the system, no matter which co-ordinate representation is chosen. If $\mathrm{rk}\mathbb{A}=0$ but $\mathrm{rk}\mathbb{B}=2$, then all ARCH effects just depend on one factor of the type

$$b_{11}\varepsilon_{1t-1}^2 + 2b_{12}\varepsilon_{1t-1}\varepsilon_{2t-1} + b_{22}\varepsilon_{2t-1}^2 \qquad (1.5)$$

In consequence, there is a linear combination of the $\mathbf{e}_t$ representation at hand, provided by $\mathbb{A}^{-1}\mathbf{e}_t$, which has its first component non-ARCH and the second one depending on the factor. This is a typical "common features" event in the sense of Engle and Kozicki (1993).

Next, let $\mathrm{rk}\mathbb{A}=0$ but $\mathrm{rk}\mathbb{B}=1$. Then, there is a way to simplify the factor in (1.5) to a simple square of a linear combination of $\mathbf{e}_{t-1}$. Though singular, $\mathbb{B}$ still has a well-defined Banachiewicz decomposition of the form $\mathbb{B}=\mathbb{L}\mathbb{D}\mathbb{L}'$ with $\mathbb{L}$ non-singular with a unit diagonal and **D** a diagonal matrix with $d_{11}=0$. The transformation $\mathbf{e}_t^* = \mathbf{L}'\mathbf{e}_t$ "rotates" the system into a position where all heteroskedasticity is explained by $(\varepsilon_{2,t-1}^*)^2$. Although that transformation may appear to yield a natural transformation of the system, the transformation $\Lambda^{-1}\mathbf{e}_t$ is maybe even more attractive as it renders one of the co-ordinates as non-ARCH. The condition for the two "natural" rotations to be equivalent amounts to a reduced-rank condition on Λ^{-1} and **B** jointly. For more applications of the "canonical" Λ^{-1} transformation, see Section 2.

In the case of rk**A**=rk**B**=1, two "natural factors", i.e., linear combinations of the original $\mathbf{e}_t$, determine the system's ARCH properties. Hence, the system can be rotated in such a way that its conditional covariance matrix just depends on past squares of the two natural factors. Schematically,

$$\begin{aligned} &\tilde{\mathbf{e}}_t = \tilde{\Gamma}\mathbf{e}_t \qquad \hat{\mathbf{e}}_t = \hat{\Gamma}\mathbf{e}_t \\ &E\hat{e}_{1,t}^2 = \sigma_{11}^{(0)} + a_1\tilde{e}_{1,t-1}^2 \\ &E\hat{e}_{2,t}^2 = \sigma_{22}^{(0)} + a_{22}\tilde{e}_{2,t-1}^2 \end{aligned} \tag{1.6}$$

Note that this case only obtains if two rank restrictions are valid. Whenever one of the two matrices has full rank and the other one at least rank 1, such a representation is impossible and there is no co-ordinate transformation which enables a representation of the ARCH structure in individual past squares alone. From linear algebra (compare e.g. Gel'fand (1965)) it is known that two separate quadratic forms can always be brought into diagonal forms by coordinate transformations. Hence, the following representation is always possible:

$$\begin{aligned} &\tilde{\mathbf{e}}_t = \tilde{\Gamma}\mathbf{e}_t \qquad \hat{\mathbf{e}}_t = \hat{\Gamma}\mathbf{e}_t \\ &E\hat{e}_{1,t}^2 = \sigma_{11}^{(0)} + a_1\tilde{e}_{1,t-1}^2 + a_{12}\tilde{e}_{1,t-1}^2 \\ &E\hat{e}_{2,t}^2 = \sigma_{22}^{(0)} + a_2\tilde{e}_{1,t-1}^2 + a_{22}\tilde{e}_{2,t-1}^2 \end{aligned} \tag{1.7}$$

What then is the difference between rk**A**=1 or rk**A**=2 given that **B** is of full rank? In the first case, three factors suffice to represent structures for both variates while in the second case full four factors are needed. In detail, rk**A**=1 permits a representation of the form

$$\tilde{e}_t = \tilde{\Gamma} e_t \qquad \hat{e}_t = \hat{\Gamma} e_t$$
$$E\hat{e}^2_{1,t} = \sigma^{(0)}_{11} + a_1\tilde{e}^2_{1,t-1} \qquad (1.8)$$
$$E\hat{e}^2_{2,t} = \sigma^{(0)}_{22} + a_2\tilde{e}^2_{1,t-1} + a_{22}\tilde{e}^2_{2,t-1}$$

It appears that these features are easier to interpret from the form (1.4) than from vectorizations such as (1.3) even though the latter form is given preference in the literature (see, e.g., Bollerslev and Engle (1993) or Engle et al. (1984)).

An important requirement from parameterized models such as (1.4) is identifiability. Clearly, the given model allows for scaling up $\mathbb{A}$ and $\mathbb{B}$ with separate constants and multiplying the corresponding diagonal scaling matrix into $\mathbb{A}$. This issue of non-identification can, however, be resolved by normalizing $\mathbb{A}$, which can be attained by e.g. $\mathrm{tr}\mathbb{A}=2$ and thus including the identity matrix among the possible candidates. It is less immediate how to avoid non-identification due to rotations between the two quadratic forms $e'\mathbb{A}e$ and $e'\mathbb{B}e$. A somehow artificial restriction would be $a_{11} \geq a_{22}$ or $a_{11} \geq b_{11}$, thus expressing a tendency for $\mathbb{A}$ to represent the heteroskedasticity in e_{1t} and $\mathbb{B}$ to represent heteroskedasticity in e_{2t}.

In conjunction with the linear cointegrated part of the model, *three* different axis transformations deserve consideration. Firstly, the rotation of the vector autoregression into a decomposition where one variate is stationary and the other one is still integrated. This rotation is prescribed by the solution vectors of a canonical correlation problem (compare Johansen (1988)). Secondly, the transformation of the bivariate ARCH model into the block-diagonal form where there is no conditional heteroskedasticity in the cross-terms. This transformation is given by $\mathbb{A}$. Thirdly, a possible axis transformation into positions where rank restrictions on $\mathbb{A}$ and $\mathbb{B}$ come out clearly, as in (1.6)-(1.8).

2. Stability conditions

The matrix Λ can be used to rotate the system in such a way that the covariance of the errors is no more dependent on previous information. This representation is particularly appropriate for determining whether the system is stationary. Let us write

$$\tilde{\varepsilon}_t = \Lambda^{-1}\varepsilon_t \qquad \tilde{\Sigma}_0 = \Lambda^{-1}\Sigma_0\Lambda^{-T}$$

with "T" for the transpose due to technical reasons. Then, (1.4) becomes

$$\begin{aligned} E(\tilde{\mathbf{e}}_t\tilde{\mathbf{e}}_t'|I_{t-1}) - \tilde{\Sigma}_0 &= I \otimes \mathbf{e}_{t-1}'\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} I \otimes \mathbf{e}_{t-1} = \\ &= I \otimes \tilde{\mathbf{e}}_{t-1}'\Lambda'\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} I \otimes \Lambda\tilde{\mathbf{e}}_{t-1} = I \otimes \tilde{\mathbf{e}}_{t-1}'\begin{bmatrix} \Lambda' A \Lambda & 0 \\ 0 & \Lambda' B \Lambda \end{bmatrix} I \otimes \tilde{\mathbf{e}}_{t-1} = \\ &= I \otimes \tilde{\mathbf{e}}_{t-1}'\begin{bmatrix} \tilde{A} & 0 \\ 0 & \tilde{B} \end{bmatrix} I \otimes \tilde{\mathbf{e}}_{t-1} = \begin{bmatrix} \tilde{\mathbf{e}}_{t-1}'\tilde{A}\,\tilde{\mathbf{e}}_{t-1} & 0 \\ 0 & \tilde{\mathbf{e}}_{t-1}'\tilde{B}\,\tilde{\mathbf{e}}_{t-1} \end{bmatrix} \end{aligned} \tag{2.1}$$

Collecting the time-constant part of the difference equation and taking unconditional expectations, we obtain

$$\begin{bmatrix} E\varepsilon_{1t}^2 \\ E\varepsilon_{2t}^2 \end{bmatrix} = \begin{bmatrix} \sigma_{01} \\ \sigma_{02} \end{bmatrix} + \begin{bmatrix} \tilde{a}_{11} & \tilde{a}_{22} \\ \tilde{b}_{11} & \tilde{b}_{22} \end{bmatrix} \begin{bmatrix} E\varepsilon_{1t-1}^2 \\ E\varepsilon_{2t-1}^2 \end{bmatrix} \tag{2.2}$$

where the coefficient matrix must have both its eigenvalues less than one, hence (2.1) has a stationary solution satisfying (2.2). Expressing such a condition in the parameters of the original system (1.4) appears cumbersome, whereas the transformation to (2.1) can be conducted quickly.

As with all ARCH models where heteroskedasticity merely depends on previous innovations but not on the process proper, stability conditions for the ARCH part (2.1) and the linear VAR part do not interact. Models with parameter interaction, as those of Weiss (1984), Tsay (1987) or Kunst (1993b), necessarily have much more intricate stability conditions. Here, covariance-stationary members of the model class are simply described by the intersection of stable bivariate VAR systems and stable (1.4) ARCH systems. It appears questionable, however, whether covariance stationarity is the natural characterization of stability in a non-linear environment. Strict stationarity conditions, however, are possibly intractable in the suggested model class.

Monte Carlo simulation techniques can be used to explore strict stationarity in those regions where theoretical results do not permit insight. For instance, Kunst (1993b) reports simulations to fix the stability boundaries of a bivariate ARCH model with conditional variance depending on observations. In Kunst (1994), these results are extended to some univariate CHARMA-type models (compare Tsay 1987). Also by simulation, Kleibergen and VanDijk (1993) replicate the theoretical stability boundary of the GARCH(1,1) model previously established by Nelson (1990).

Figure 1 gives the simulated stationarity boundaries for a simple bivariate structure of type (1.4), in detail:

$$\begin{bmatrix} h_{1t} \\ h_{2t} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + I \otimes e'_{t-1} \begin{bmatrix} a_1 & \tau_1 a_1 & 0 & 0 \\ \tau_1 a_1 & a_1 & 0 & 0 \\ 0 & 0 & a_2 & \tau_2 a_2 \\ 0 & 0 & \tau_2 a_2 & a_2 \end{bmatrix} I \otimes e_{t-1} \tag{2.3}$$

In Figure 1, the innermost line $a_1+a_2=1$ is the natural boundary for the covariance-stationary parameterizations. For the inner curve, $\mathbb{A}$ and $\mathbb{B}$ are diagonal matrices with constant diagonal ($\tau_1=\tau_2=0$). In this case, also for strict stationarity, conditions can be calculated, using the multivariate version of the results by Bougerol and Picard (1992). For more general cases, however, with non-diagonal $\mathbb{A}$ or $\mathbb{B}$, such results do not exist.

The outer of the two central curves in Figure 1 was then found by simulating the previous model with diagonal $\mathbb{B}$ but non-diagonal $\mathbb{A}$, i.e. $\tau_1 \neq 0$. Note that non-negative definiteness requires that τ be in the closed interval [-1,1]. For the outer curve, τ was set at +1. Stability properties are unaffected by the sign of τ. For some more τ with $0<\tau<1$, simulated boundaries were found to lie in the region between the two shown curves. The strict stationarity area was found to be strictly monotonously increasing with τ.

The outermost line corresponds to the model with $\tau_1=\tau_2=1$. In this case, the ARCH model becomes essentially univariate since

$$E_{t-1}(\varepsilon_{1t}+\varepsilon_{2t})^2 = E_{t-1}\varepsilon_{1t}^2 + E_{t-1}\varepsilon_{2t}^2 = \\ = 2 + (a_1 + a_2)(\varepsilon_{1t-1}+\varepsilon_{2t-1})^2 \tag{2.4}$$

and the condition for strict stationarity becomes simply for the sum a_1+a_2 to be less than the univariate boundary value given by Nelson (1990) as 3.5... In consequence, though both τ_1 as well as τ_2 help expanding the stationarity area, reaction is much more sensitive if both **A** and **B** are allowed to become non-diagonal.

For higher-order ARCH models or for general matrices **A** and **B**, the high dimensionality of the parameter space makes graphical representations such as Figure 1 impossible. In general, a rather wide area of strictly stationary models without finite second moments embeds the area of covariance stationarity. A somehow arbitrary experiment is shown in Figure 2. The matrix **B** is scalar and **A** is assumed as diagonal but with possibly different elements. a_{11} is set at the fixed value of 0.8 while a_{22} is allowed to vary according to the parameterization $a_{22} = 0.8*p_3$ (p_3 stands for the third parameter). The simulated stationarity boundary in the (b_{11},p_3) plane is shown in Figure 2. For b_{11}=0, the boundary becomes a vertical asymptote, as in this case any value of p_3 just increases the variance of the process without affecting its existence properties.

3. The multivariate model

(1.4) can be generalized immediately to represent a multivariate model with dimension k. Such model could be written as

$$E(\mathbf{e}_t\mathbf{e}_t'|I_{t-1}) = \\ = \Sigma_0 + \Lambda \begin{bmatrix} \mathbf{e}_{t-1}' & 0 & \cdots & 0 \\ 0 & \mathbf{e}_{t-1}' & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{e}_{t-1}' \end{bmatrix} \begin{bmatrix} \mathbf{A}_1 & 0 & \cdots & 0 \\ 0 & \mathbf{A}_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{A}_k \end{bmatrix} \begin{bmatrix} \mathbf{e}_{t-1} & 0 & \cdots & 0 \\ 0 & \mathbf{e}_{t-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{e}_{t-1} \end{bmatrix} \Lambda' \tag{3.1}$$

Similar to the bivariate model, it appears natural to normalize the matrix $\mathbb{A}$ or to impose conditions on the $\mathbb{A}_i$ to warrant identifiability of the problem. Note that the number of parameters contained in (3.1) is

$$k(\frac{k(k+1)}{2}-1)+k^2=\frac{k^3}{2}+\frac{3k^2}{2}-k \tag{3.2}$$

(not counting Σ_0). In contrast, multivariate versions of the general model (1.3) would imply $(\frac{k(k+1)}{2})^2$ parameters which are, however, restricted by the definiteness conditions. For k=2, the difference in the two numbers is small, with (3.2) yielding 8 parameters and (1.3) containing 9. For larger k, the difference becomes substantial. Additional to the identifying restrictions, definiteness requires the diagonals of the Cholesky factors in $\mathbb{A}_i = \mathbb{L}_i\mathbb{L}_i'$ to be non-negative. This condition can be imposed easily during estimation.

4. Estimation

4.1 The maximum likelihood estimator

Assuming conditional normality, the log-likelihood of the model (1.4) can be developed from the log-likelihood of the errors as

$$\ell(\mathbb{A},\mathbb{B},\mathbb{A};\mathbf{e}) =$$

$$= -\tfrac{T}{2}\log(2\pi) - \tfrac{1}{2}\sum_{t=1}^{T-1}\log\left|\Sigma_0 + (\mathbb{A}\otimes\mathbf{e}_t')\mathrm{diag}(\mathbb{A},\mathbb{B})(\mathbb{A}'\otimes\mathbf{e}_t)\right| - \tag{4.1}$$

$$-\tfrac{1}{2}\sum_{t=2}^{T}\mathbf{e}_t'\left(\Sigma_0 + (\mathbb{A}\otimes\mathbf{e}_{t-1}')\mathrm{diag}(\mathbb{A},\mathbb{B})(\mathbb{A}'\otimes\mathbf{e}_{t-1})\right)^{-1}\mathbf{e}_t$$

where the notations $\mathbf{e} = (\varepsilon_{11},\varepsilon_{21},\varepsilon_{12},\ldots\varepsilon_{1T},\varepsilon_{2T})'$ and $\mathbf{e}_t = (\varepsilon_{1t},\varepsilon_{2t})'$ are used for convenience. To obtain the process likelihood from (4.1), let us adopt Johansen's (1988) notation $\alpha\beta'$ for the cointegrating matrix but keep Σ for the errors covariance matrix. Neglecting the constant part of the likelihood, this yields for a homoskedastic system

$$\ell(\alpha,\beta,\Sigma;X) = const - \tfrac{T}{2}\log|\Sigma| - \tfrac{1}{2}(\Delta X + \alpha\beta X_{-1})'(I\otimes\Sigma^{-1})(\Delta X + \alpha\beta X_{-1}) \tag{4.2}$$

In an ARCH system, (4.1) and (4.2) can be merged into the full process likelihood

$$
\begin{aligned}
&\ell(\mathbf{A},\mathbf{B},\Lambda,\alpha,\beta;X) = \\
&= const - \tfrac{1}{2}\sum_{t=2}^{T-1}\log\left|\Sigma_0 + \Lambda\otimes(\Delta X_t' - X_{t-1}'\beta\alpha')\mathrm{diag}(\mathbf{A},\mathbf{B})\,\Lambda'\otimes(\Delta X_t - \alpha\beta' X_{t-1})\right| - \\
&\quad - \tfrac{1}{2}\sum_{t=3}^{T}(\Delta X_t' - X_{t-1}'\beta\alpha')\left[\Sigma_0 + \Lambda\otimes(\Delta X_{t-1}' - X_{t-2}'\beta\alpha')\times\right. \\
&\quad \left.\times\,\mathrm{diag}(\mathbf{A},\mathbf{B})\,\Lambda'\otimes(\Delta X_{t-1} - \alpha\beta' X_{t-2})\right]^{-1}(\Delta X_t - \alpha\beta' X_{t-1})
\end{aligned}
\qquad (4.3)
$$

and the profile likelihoods in the directions of the linear and ARCH parameters look like (4.2) and (4.1), respectively. Due to an argument parallel to Engle's (1982) regularity condition for ARCH models, the information matrix is block-diagonal with the respective blocks (α,β) and $(\mathbf{A},\mathbf{B},\Lambda)$. Hence, maximization of the likelihood can be conducted efficiently via iterations between the linear and the ARCH part. The profile steps into the ARCH direction are somewhat complicated but are basically reminiscent of least-squares regression (compare Engle (1982)). The profile steps into the direction of (α,β) are best described as weighted reduced-rank regression.

4.2 Ordinary least squares

It is tempting to use standard ordinary least squares (OLS) in estimation, at least for the linear part of the model. Unrestricted OLS should be consistent under rather general conditions for the ARCH errors model and asymptotically normal at least for "weak" ARCH structures (compare Weiss (1984)). Ordinary least squares estimation for the ARCH model structure itself is certainly less recommendable and needs stringent moment conditions even in rather simple models (eighth moments in the Weiss (1984) model). However, even for ARCH structures, OLS is generally considered as a reliable first indicator for the true structure and could be used at least as a starting value for an iterative procedure.

We report a small-scale Monte Carlo experiment which highlights the fact that OLS for the ARCH model can be severely misleading. However, it cannot be claimed in general that preliminary estimation by OLS is fruitless per se. Point estimates of all coefficients are certainly extremely unreliable.

We generated 100 replications of 1000 observations each from the following bivariate structure

$$\begin{bmatrix}\Delta X_t\\ \Delta Y_t\end{bmatrix}=\begin{bmatrix}-0.5\\ 0.7\end{bmatrix}\begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}X_{t-1}\\ Y_{t-1}\end{bmatrix}+\begin{bmatrix}\varepsilon_{1t}\\ \varepsilon_{2t}\end{bmatrix} \tag{4.4}$$
$$E_{t-1}\varepsilon_t\varepsilon_t' = 0.1I_2 + (\varepsilon_{t-1}'\otimes I_2)\mathrm{diag}(0.4,0.2,0,0.9)(\varepsilon_{t-1}\otimes I_2)$$

(4.4) does not contain any Λ rotation. The cointegrating vector influences both variates by error correction. Heteroskedasticity is regular in the first and strong in the second factor. The results are rather robust to the specification of the loading vector (-0.5 0.7)' as long as both loadings are different from 0.

Figure 3a shows a cross-plot between OLS estimates of the slope and intercept of the cointegrating regression $Y_t = \alpha + \beta X_t + u_t$. Although the cointegrating regression is inefficient and kurtosis is high because of the strong ARCH in the second factor, results appear satisfactorily clustered around the true values of (0.2, 1.0). Correlation between the two estimates is negative: if the slope is erroneously low, this is counteracted by an increased intercept and *vice versa*. Still, some intercept estimates are as high as 0.4 but the main mass of the distribution is in the right place.

In striking contrast, OLS estimates of ARCH parameters even from the full first-order VAR - which is correctly specified - are unreliable as shown in Figures 3b and 3c. The true value of 0.9 in the second factor is in the upper decile of the sampling distribution, mean and median are around 0.5. Many estimates are negative, as they have not been restricted a priori. However, even a priori restriction would not prevent the occurrence of entirely erratic realizations of the spurious coefficients on lagged cross terms such as $\varepsilon_{1,t-1}\varepsilon_{2,t-1}$.

4.3 Maximum likelihood estimation

Returning to the maximum likelihood estimator described in Section 4.1, the experiment in Section 4.2 was repeated using straightforward maximization of the likelihood. To keep the system identified, the cointegrating vector was parameterized as $(1, \beta)'$ and the matrix Λ was restricted to be unit-diagonal symmetric with off-diagonal parameter λ. The non-negative definiteness of the matrices $\mathbf{A}$, $\mathbf{B}$, Σ_0 was imposed by the Banachiewicz decomposition, e.g.:

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ a_3 & 1 \end{bmatrix} \begin{bmatrix} a_1^2 & 0 \\ 0 & a_2^2 \end{bmatrix} \begin{bmatrix} 1 & a_3 \\ 0 & 1 \end{bmatrix}$$

This way, the new parameters a_1, a_2, a_3 can vary freely and still generate non-negative definite matrices.

Some of the results are depicted in Figures 4a-4c. Figure 4a cross-plots the estimates of the loading coefficients α_1 and α_2. The observations are scattered in an area which is rather tight around the true values (-0.5, 0.7), there is no obvious correlation between coefficient estimates. The numerical summary in Table 1 shows that the cointegrating coefficient β yielded the most accurate estimate with a standard deviation of only 5×10^{-4} whereas most of the other parameter estimates still have approximately 2×10^{-2}. This difference in orders of magnitude corresponds well to the presumption that β estimation may be consistent of order T and of order $T^{-1/2}$ for all other estimates. Anyway, the gain in precision relative to the OLS estimates reported in the last section appears convincing.

Incidents of spurious λ rotation were scarce and its standard error corresponded in magnitude to the others at 0.03. Matrices $\mathbf{A}$ and $\mathbf{B}$ were also estimated satisfactorily, with slightly higher standard errors of order $\theta \times 10^{-2}$, i.e., proportional to the true value, and a slight downward bias, at least for large true values. The behavior of these estimates is also shown in Figures 4b-c. Estimates for the individual entries of Σ_0 appear unbiased with standard errors of around 0.01, i.e., smaller than the ARCH coefficients and the loadings.

We remark that behavior of the ML estimator depends critically on well-specified starting values, particularly for Σ_0. Too small unconditional variances entail frequent convergence of the iteration toward solutions with large ARCH and persistently small Σ_0, i.e., the true solution is not found.

4.4 Restricted maximum likelihood estimation

In one of the designs used above, the rank of the second-variate ARCH matrix $\mathbb{B}$ was 1 in the generated data but this restriction was not imposed during the estimation stage. It may be interesting to know if imposing such restriction modifies the performance of ML estimation. It may also be interesting to investigate the properties of likelihood-ratio tests with the restricted model as null hypothesis. According to conventional statistical theory, $2LR$ withe LR denoting the log-likelihood ratio statistic should be - at least asymptotically - distributed chi-square with one degree of freedom.

With respect to the first question, some evidence is provided in Table 2. The unsrestricted estimation results have been repeated from Table 1 to facilitate a comparison. It is seen that the beneficial effects of imposing the (correct) restriction $\text{rk}\,\mathbb{B}=1$ are small and sample standard deviations or measured biases hardly change from the unrestricted version. This means that, at least for a sample size of 1000 or more, keeping an additional spurious parameter in the ARCH matrices does not have any impact on the precision of ML estimation.

With respect to the second question, more than 100 replications would certainly be necessary to enable the calculation of reliable significance points for the LR test. Within the limits of the simulation exercise, the empirical 95% fractile was 4.13 which is not too different from the theoretical chi-square value of 3.81. The empirical 99% fractile was 6.73 (theoretical 6.63) and the empirical 90% point was 3.05 (theoretical

2.71). The larger difference in the 90% points can be explained by the fact that, probably due to the above-mentioned starting value problems, in approximately 20% of the cases, the restricted likelihood was *better* than the unrestricted one which is impossible and indicates that unrestricted estimation did not attain the global maximum.

4.5 Weakly non-stationary cases

The basic design contains an ARCH effect in the second component that is very strong but still fulfills the restrictions of covariance stationarity. Table 3 gives the results of an experiment in which the value of 0.9 was increased to exactly 1.0 so that the whole system became non-stationary in the sense of covariance stationarity, though it still was strictly stationary. The limiting behavior of all estimates is not known in the univariate IGARCH model and the same holds for our multivariate model. Nonetheless, changes between the covariance stationary case in Table 2 and the covariance non-stationary one in Table 3 are only slight. Also in this case, restricted ML estimation is not very different from unrestricted ML. Only the estimate for b_{11} becomes more precise which is certainly to be expected from the experimental design. In contrast, the estimate for b_{12} is less precise in the restricted version because of its exact dependence on the (small but unrestrictedly estimated) first diagonal b_{11} and the (large) b_{22}.

With regard to the likelihood-ratio test on the rank of B, again chi-square(1) appears to be a good approximation. It is questionable whether detailed sample fractiles should be taken seriously as the number of replications is rather small at 100. However, if one is willing to do so, then deviations from the theoretical distribution now point into the reverse direction, with the empirical fractiles at 2.79 (90%, theoretical 2.71), 3.18 (95%, theoretical 3.84) and 5.95 (99%, theoretical 6.63). Hence, the (true) null hypothesis of rk B=1 is rejected less frequently in the tails (5% and 1%) than would be appropriate and test power against the alternative rk B=2 is probably affected adversely.

4.6 Less error-correcting influence, misspecification of the cointegrating rank

The error-correcting influence - i.e., the "strength" of cointegration - in the basic design is rather pronounced. To check on the way a weakening of the error-correcting influence affects parameter estimation, three deviations from the basic design have been considered:

(1) Reducing α_1 from 0.5 to 0. Only the second component is affected by the error-correcting effects.

(2) Setting $\alpha_1=\alpha_2=0$. β becomes undefined and the cointegrating rank becomes 0, i.e., there is no cointegration. In this regard, it may be interesting to see whether the LR statistic on the cointegrating rank is affected by the ARCH effects. This LR statistic is known not to be chi-square distributed (see Johansen (1988) and Johansen and Juselius (1990)).

(3) A second cointegrating vector y_1+y_2 influences the second component. The whole system is now stationary and the cointegrating rank is 2. The remark concerning case (2) again applies.

The results for experiment (1) are summarized in Table 4. The precision of the estimates appears comparable to the previous experiments. What is surprising, however, is the extreme reaction of the likelihood-ratio test to this very design. Empirical fractiles exceed the chi-square(1) fractiles substantially and this phenomenon does not seem to be explicable by starting value effects (significance points are 4.68, 32.59, and 105.14). It is rather the asymmetry between the two components which could play a role as the LR behavior in case (2) is less conspicuous. The rank restriction within the ARCH matrix of the second component coincides with the fact that it is this component which carries the error-correcting behavior.

Table 5 shows the results of the non-cointegrating design (2). Here, of course, the estimates of the cointegrating parameter β become unreliable, as this parameter becomes not identified. All other parameter estimates are only marginally affected. The LR statistic of the null hypothesis rk B=1 now has the properties known from Tables 1-3 again. The significance points are 3.88, 5.27, and 7.47, slightly exceeding the theoretical fractiles. What is perhaps more interesting, is the LR statistic of the hypothesis rk $\Pi=0$.

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A PPENDIX

Lemma: The representation (1.4) yields the "diagonal ARCH model" by Engle et al. (1984) in trivial cases only, where conditional covariances h_{12} are time-constant.

Proof: We first note that, according to the suggested model (1.4)

$$\begin{bmatrix} h_{1t} & h_{12t} \\ h_{12t} & h_{2t} \end{bmatrix} = \Sigma_0 + \begin{bmatrix} \lambda_{11} \mathbf{e}'_{t-1}\mathbf{A}\mathbf{e}_{t-1} & \lambda_{12} \mathbf{e}'_{t-1}\mathbf{B}\mathbf{e}_{t-1} \\ \lambda_{21} \mathbf{e}'_{t-1}\mathbf{A}\mathbf{e}_{t-1} & \lambda_{22} \mathbf{e}'_{t-1}\mathbf{B}\mathbf{e}_{t-1} \end{bmatrix} \begin{bmatrix} \lambda_{11} & \lambda_{21} \\ \lambda_{12} & \lambda_{22} \end{bmatrix}$$

Hence, the "diagonal model" entails the six conditions

$$\begin{aligned}
\lambda_{11}^2 a_{12} + \lambda_{12}^2 b_{12} &= 0 \\
\lambda_{11}^2 a_{22} + \lambda_{12}^2 b_{22} &= 0 \\
\lambda_{21}^2 a_{11} + \lambda_{22}^2 b_{11} &= 0 \\
\lambda_{21}^2 a_{12} + \lambda_{22}^2 b_{12} &= 0 \\
\lambda_{11}\lambda_{21} a_{11} + \lambda_{12}\lambda_{22} b_{11} &= 0 \\
\lambda_{11}\lambda_{21} a_{22} + \lambda_{12}\lambda_{22} b_{22} &= 0
\end{aligned}$$

or, in matrix notation:

$$[0 \quad 0] = [\lambda_{21}^2 \quad \lambda_{22}^2] \begin{bmatrix} a_{11} & a_{12} \\ b_{11} & b_{12} \end{bmatrix}$$

$$[0 \quad 0] = [\lambda_{11}^2 \quad \lambda_{12}^2] \begin{bmatrix} a_{12} & a_{22} \\ b_{12} & b_{22} \end{bmatrix}$$

$$[0 \quad 0] = [\lambda_{11}\lambda_{21} \quad \lambda_{12}\lambda_{22}] \begin{bmatrix} a_{11} & a_{22} \\ b_{11} & b_{22} \end{bmatrix}$$

Remembering that we assumed Λ to represent a non-singular transformation, we now separate cases according to the following criterion:

Case A: All $\lambda_{ij} \neq 0$ Case B: Some $\lambda_{ij} = 0$

ad Case A: If the first two right-hand-side matrices formed from a and b elements are non-singular, then Λ becomes singular. This even happens if only one of the two matrices is non-singular. Therefore, both matrices must be singular. This implies that either there are entire rows or columns of zeros or **B** is proportional to **A**. We first concentrate on this proportional case. From the three equation systems, the proportionality constant κ must be

$$\kappa = -\frac{\lambda_{21}^2}{\lambda_{22}^2} = -\frac{\lambda_{11}^2}{\lambda_{12}^2} = -\frac{\lambda_{11}\lambda_{21}}{\lambda_{12}\lambda_{22}}$$

Trying to solve all three conditions at once yields a singular Λ, contradicting the assumption.

ad Case B: For example, assume $\lambda_{11}=0$. Then, since $a_{22} \geq 0$ and $b_{22} \geq 0$, from the second equation system $b_{22}=0$ or $\lambda_{12}=0$. The latter yields a singular Λ and is impossible. The first implies $b_{12}=0$ and, from the first equation, as λ_{21} must not be 0, $a_{12}=0$ and $a_{11}=0$. The product element from the first equation $\lambda_{22}^2 b_{11}$ must also be 0 and there are two cases. $b_{11}=0$ gives the following situation:

$\mathbb{A}$ is non-singular with $\lambda_{11}=0$, $\mathbb{A}$ only contains the non-zero element a_{22}, and $\mathbb{B}=0$. All heteroskedasticity depends on $\varepsilon_{2,t-1}^2$. It is easily seen that conditional heteroskedasticity cannot influence the covariances.

On the other hand, $\lambda_{22}=0$ yields the following situation:

$\mathbb{A}$ is anti-diagonal, $\mathbb{A}$ only contains a_{22}, and $\mathbb{B}$ only contains b_{11}. Here, ARCH structures of the two series ε_{1t} and ε_{2t} are mutually independent. For reasons of identification, it may be wiser to exclude this case from investigation.

Now assume $\lambda_{12}=0$ (the cases $\lambda_{21}=0$ and $\lambda_{22}=0$ are then essentially covered). Then λ_{11} and λ_{22} are both non-zero to warrant non-singularity of Λ. From the second equation, $a_{12}=a_{22}=0$. From the first equation, $b_{11}=b_{12}=0$. Two cases are possible: $a_{11}=0$ or $\lambda_{21}=0$ (or both).

If $a_{11}=0$, then all $\mathbb{A}=0$, $\mathbb{B}$ just contains b_{22}, and Λ is non-singular with $\lambda_{12}=0$. All heteroskedasticity depends on $\varepsilon_{2,t-1}^2$ and cannot influence covariances.

If $\lambda_{21}=0$, then Λ is diagonal. $\mathbb{A}$ just contains a_{11} and $\mathbb{B}$ just b_{22}. ARCH structures are again mutually independent. With respect to identifying conditions, this is the more natural representation of the system. □

TABLE 1: Summary of results of maximum-likelihood estimation for cointegrated bivariate ARCH models. 100 replications of processes of length 1000.

	Design 1			Design 2		
	est. mean	est.st.dev.	true value	est. mean	est.st.dev.	true value
μ_1	-.001	.018		.001	.020	
μ_2	.242	.019		.240	.023	
α_1	-.500	.021	-.5	-.501	.020	-.5
α_2	.699	.018	.7	.699	.024	.7
β	-1.000	.0005	-1.0	-1.000	.0006	-1.0
λ	.001	.029	0	-.003	.035	0
a_{11}	.393	.050	.4	.394	.055	.4
a_{12}	.000	.038	0	-.002	.037	0
a_{22}	.199	.039	.2	.200	.040	.2
b_{11}	.005	.008	0	.279	.076	.3
b_{12}	.001	.013	0	-.007	.070	0
b_{22}	.879	.071	.9	.586	.068	.6
σ_{11}	.102	.009	.1	.102	.010	.1
σ_{12}	-.001	.011	0	.001	.013	0
σ_{22}	.100	.008	.1	.105	.016	.1

TABLE 2: Summary of results of restricted and unrestricted maximum-likelihood estimation for cointegrated bivariate ARCH models. 100 replications of processes of length 1000.

	unrestricted			restricted		
	est. mean	est.st.dev.	true value	est. mean	est.st.dev.	true value
μ_1	-.001	.018		-.000	.018	
μ_2	.242	.019		.242	.019	
α_1	-.500	.021	-.5	-.500	.021	-.5
α_2	.699	.018	.7	.699	.017	.7
β	-1.000	.0005	-1.0	-1.000	.0005	-1.0
λ	.001	.029	0	.001	.029	0
a_{11}	.393	.050	.4	.393	.051	.4
a_{12}	.000	.038	0	.001	.038	0
a_{22}	.199	.039	.2	.199	.039	.2
b_{11}	.005	.008	0	.001	.002	.0
b_{12}	.001	.013	0	-.001	.033	0
b_{22}	.879	.071	.9	.880	.070	.9
σ_{11}	.102	.009	.1	.102	.009	.1
σ_{12}	-.001	.011	0	-.001	.011	0
σ_{22}	.100	.008	.1	.101	.008	.1

TABLE 3: Summary of results of restricted and unrestricted maximum-likelihood estimation for cointegrated bivariate IARCH models. 100 replications of processes of length 1000.

	unrestricted			restricted		
	est. mean	est.st.dev.	true value	est. mean	est.st.dev.	true value
μ_1	-0.000	0.018		-0.000	0.018	
μ_2	0.242	0.019		0.242	0.011	
α_1	-0.500	0.021	-.5	-0.500	0.021	-.5
α_2	0.699	0.017	.7	0.700	0.017	.7
β	-1.000	0.0005	-1.0	-1.000	0.0005	-1.0
λ	0.001	0.027	0	0.001	0.027	0
a_{11}	0.394	0.056	.4	0.392	0.051	.4
a_{12}	0.001	0.036	0	0.001	0.036	0
a_{22}	0.197	0.039	.2	0.199	0.037	.2
b_{11}	0.007	0.016	0	0.001	0.002	.0
b_{12}	-0.001	0.015	0	-0.000	0.034	0
b_{22}	0.976	0.075	1.0	0.980	0.074	1.0
σ_{11}	0.102	0.009	.1	0.101	0.008	.1
σ_{12}	-0.000	0.011	0	-0.001	0.010	0
σ_{22}	0.100	0.009	.1	0.101	0.008	.1

TABLE 4: Summary of results of restricted and unrestricted maximum-likelihood estimation for cointegrated bivariate ARCH models. Only one component suffers error-correcting influence. 100 replications of processes of length 1000.

	unrestricted			restricted		
	est. mean	est.st.dev.	true value	est. mean	est.st.dev.	true value
μ_1	0.099	0.015		0.099	0.015	
μ_2	0.242	0.020		0.242	0.011	
α_1	-0.004	0.021	0	-0.004	0.021	0
α_2	0.700	0.018	.7	0.700	0.018	.7
β	-1.000	0.0005	-1.0	-1.0000	0.0005	-1.0
λ	0.000	0.032	0	0.000	0.033	0
a_{11}	0.382	0.089	.4	0.393	0.051	.4
a_{12}	0.002	0.041	0	0.001	0.036	0
a_{22}	0.203	0.042	.2	0.199	0.039	.2
b_{11}	0.007	0.015	0	0.001	0.002	.0
b_{12}	-0.002	0.026	0	-0.001	0.033	0
b_{22}	0.875	0.071	.9	0.871	0.071	.9
σ_{11}	0.104	0.013	.1	0.102	0.009	.1
σ_{12}	-0.000	0.012	0	-0.000	0.012	0
σ_{22}	0.100	0.009	.1	0.101	0.008	.1

TABLE 5: Summary of results of restricted and unrestricted maximum-likelihood estimation for not cointegrated bivariate ARCH models. 100 replications of processes of length 1000.

	unrestricted			restricted		
	est. mean	est.st.dev.	true value	est. mean	est.st.dev.	true value
μ_1	0.103	0.035		0.107	0.037	
μ_2	0.105	0.024		0.106	0.025	
α_1	-0.001	0.002	0	-0.001	0.002	0
α_2	0.001	0.001	0	0.001	0.002	0
β	0.193	13.174	-	3.437	45.361	-
λ	0.001	0.031	0	0.003	0.035	0
a_{11}	0.390	0.066	.4	0.396	0.091	.4
a_{12}	0.000	0.037	0	0.002	0.0310	0
a_{22}	0.200	0.040	.2	0.204	0.052	.2
b_{11}	0.006	0.012	0	0.001	0.002	.0
b_{12}	0.002	0.021	0	-0.002	0.032	0
b_{22}	0.878	0.0693	.9	0.877	0.078	.9
σ_{11}	0.102	0.0118	.1	0.102	0.0118	.1
σ_{12}	-0.001	0.012	0	-0.002	0.013	0
σ_{22}	0.091	0.008	.1	0.101	0.008	.1

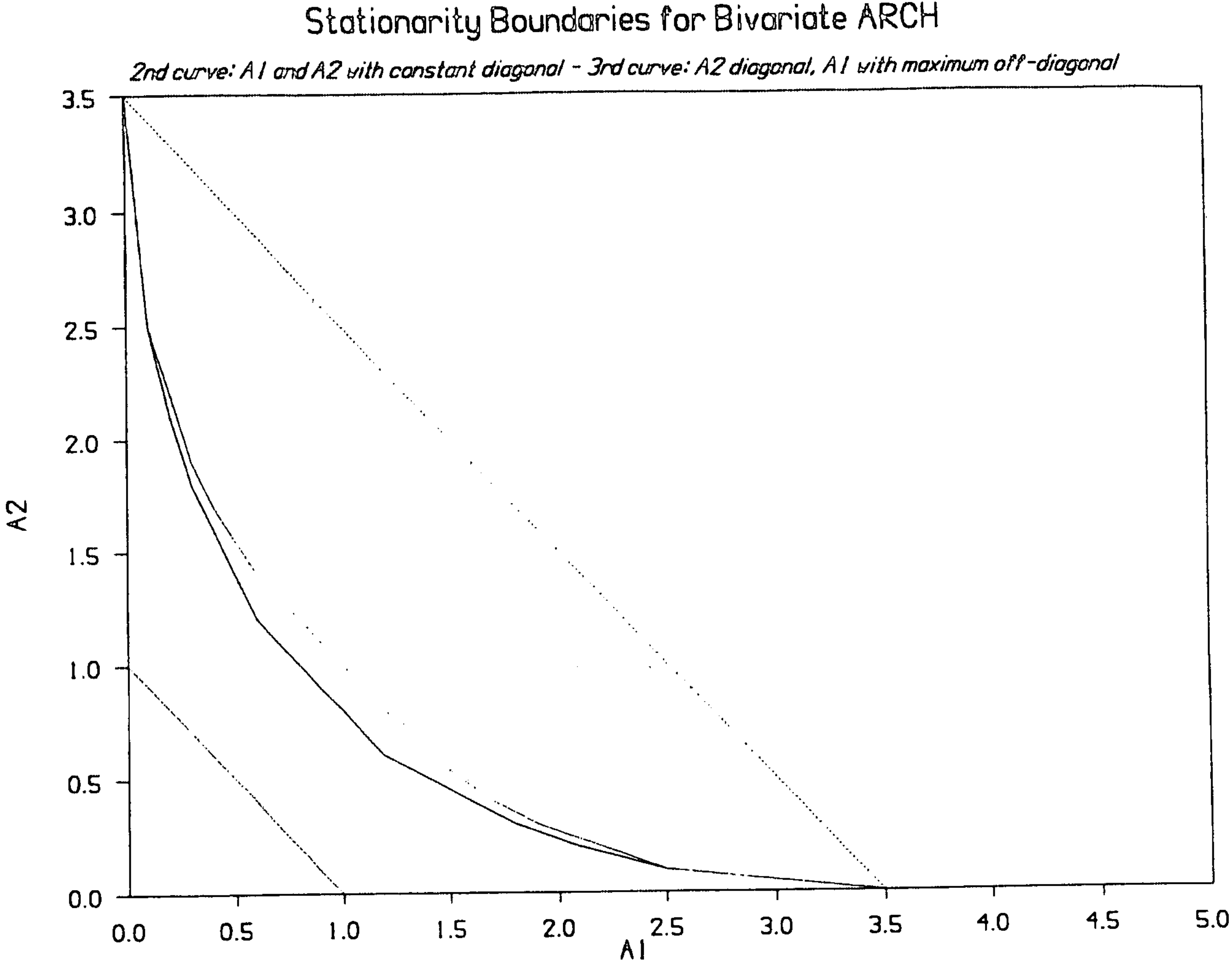

FIGURE 1

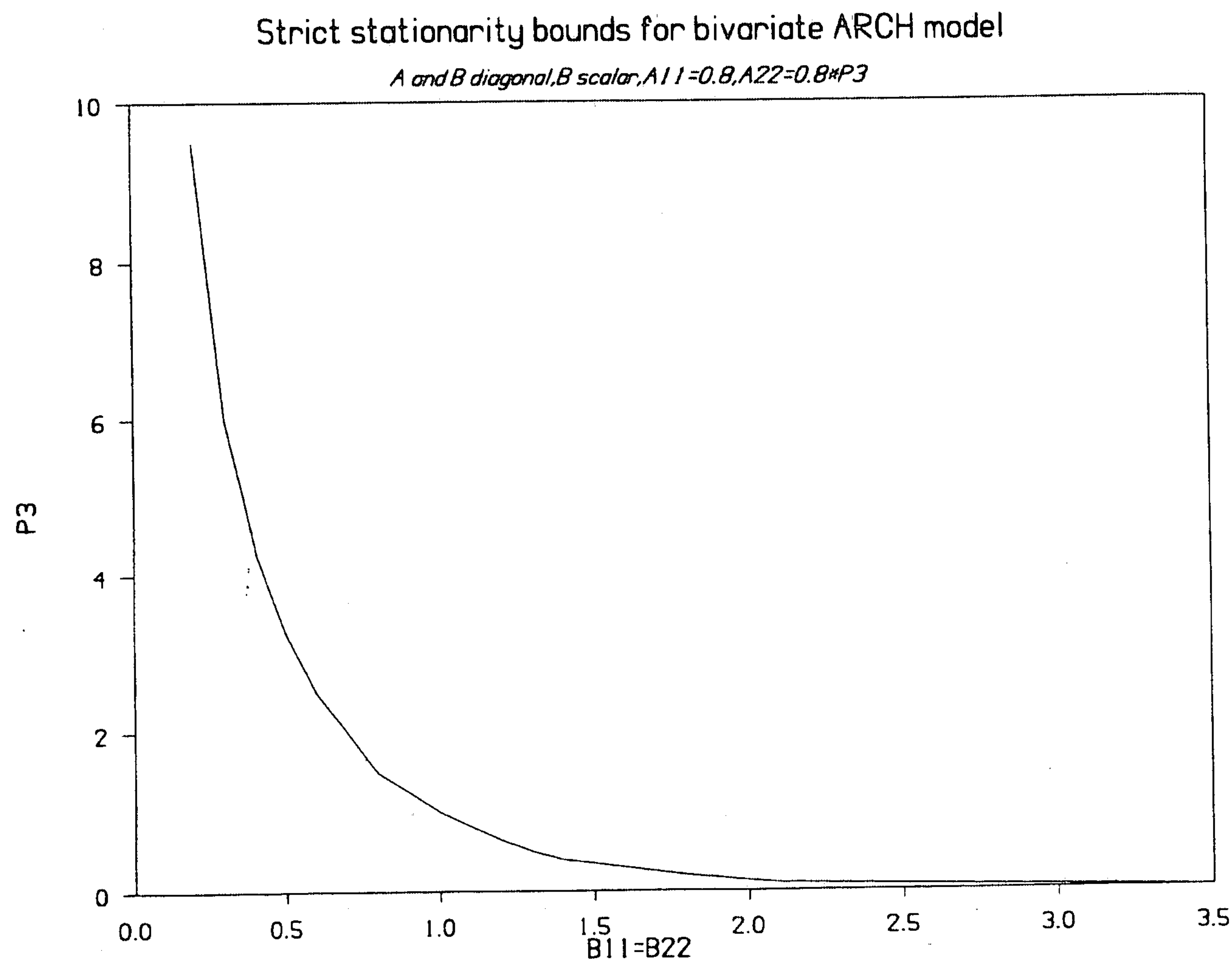

FIGURE 2

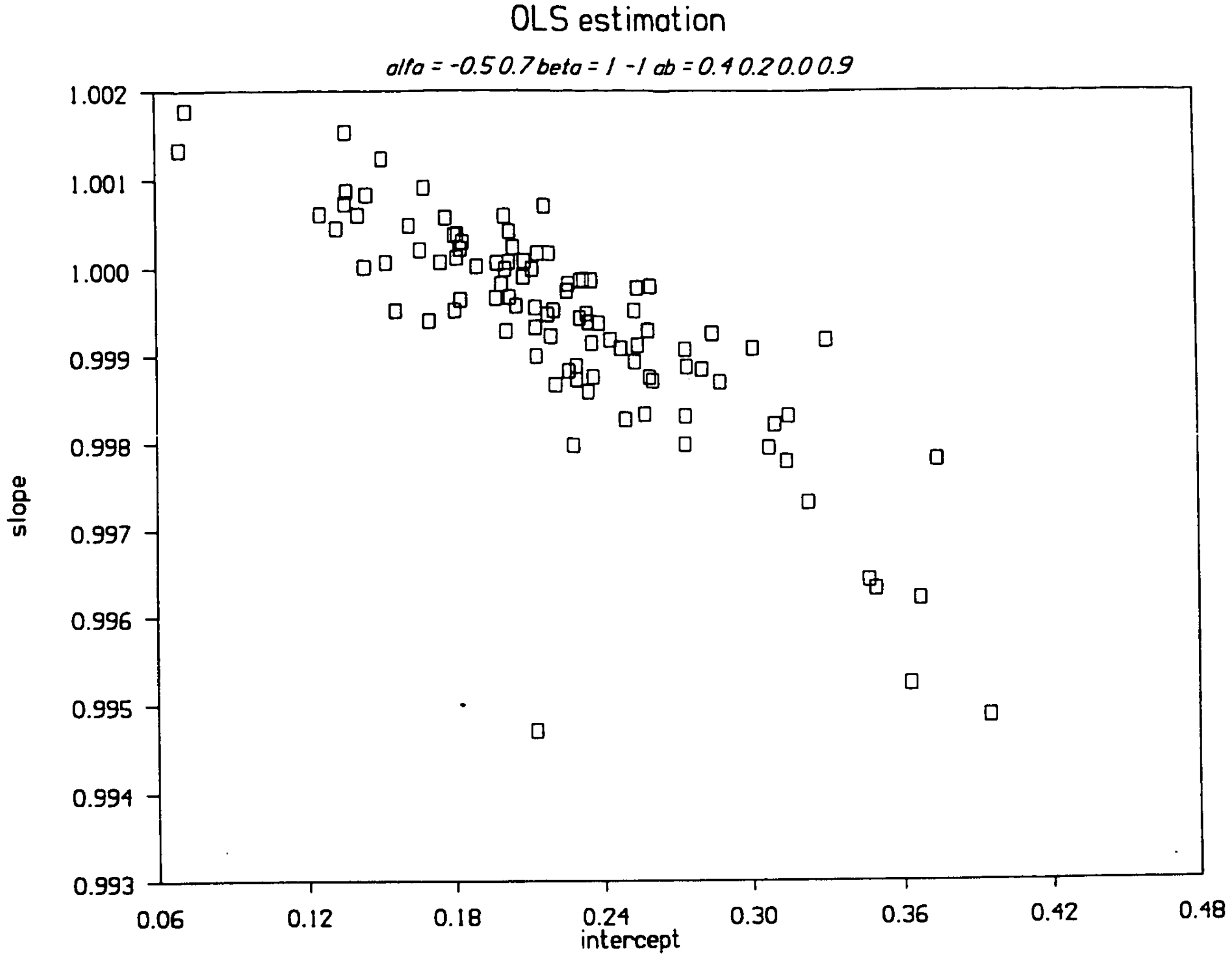

FIGURE 3a

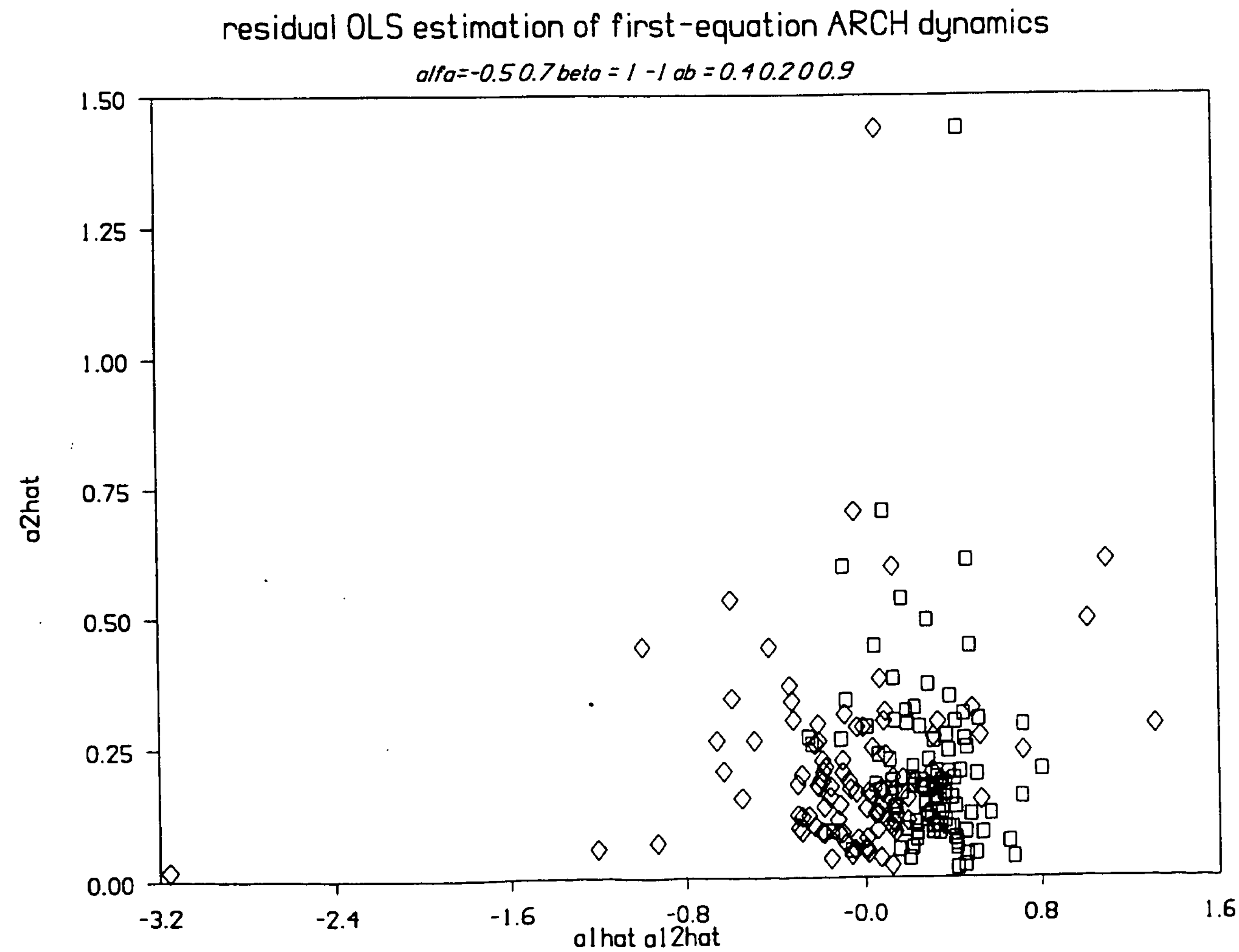

FIGURE 3b

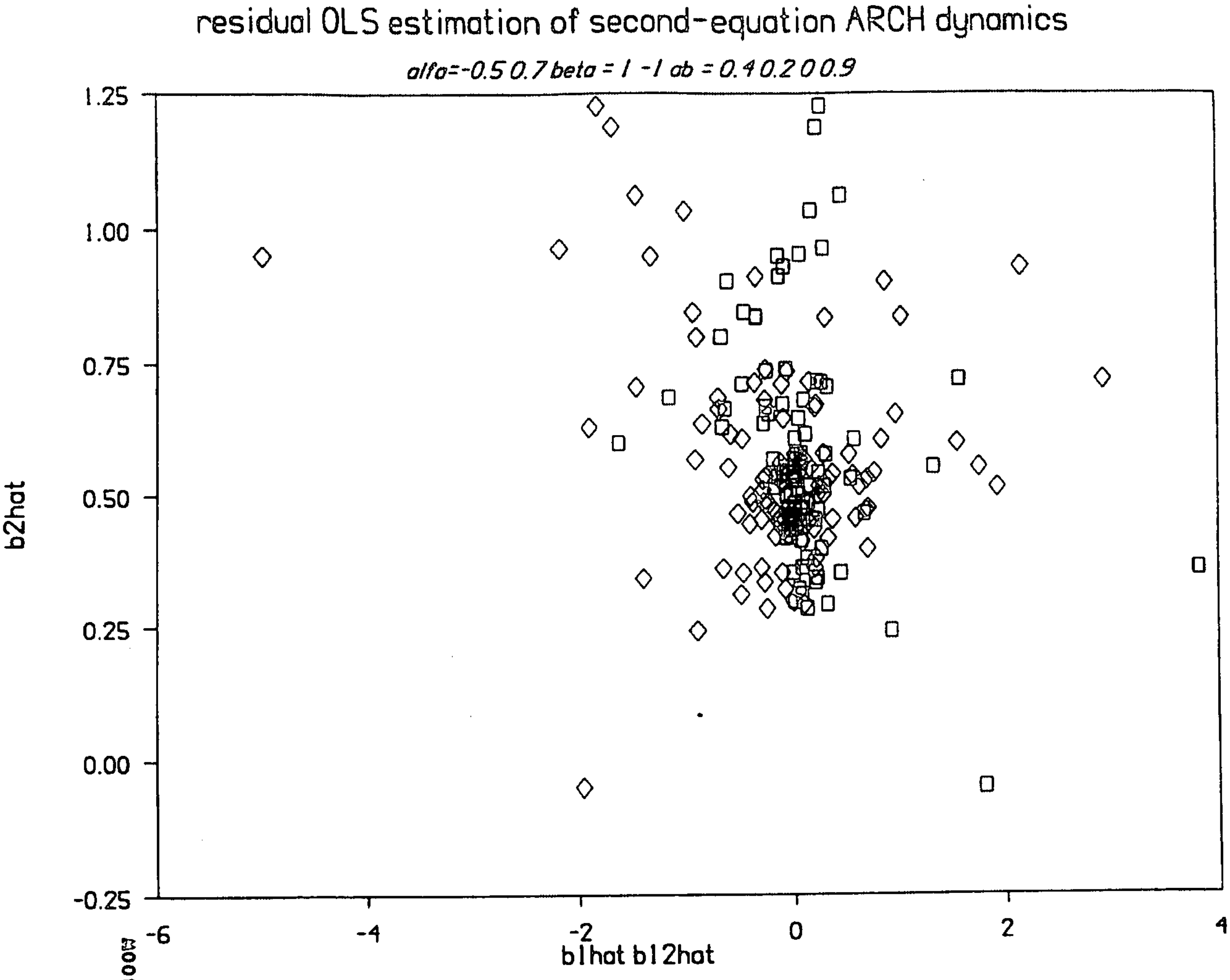

FIGURE 3c

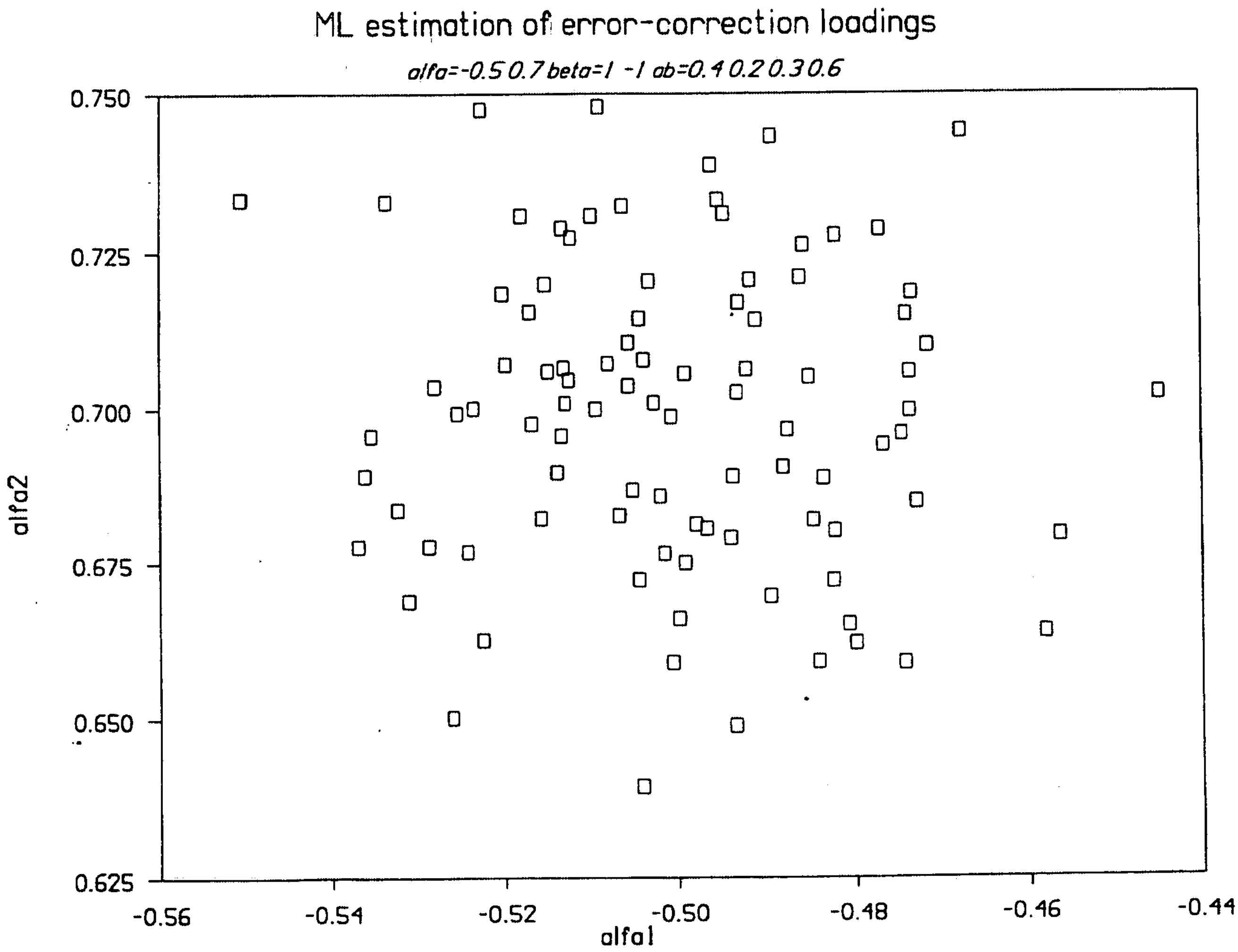

FIGURE 4a

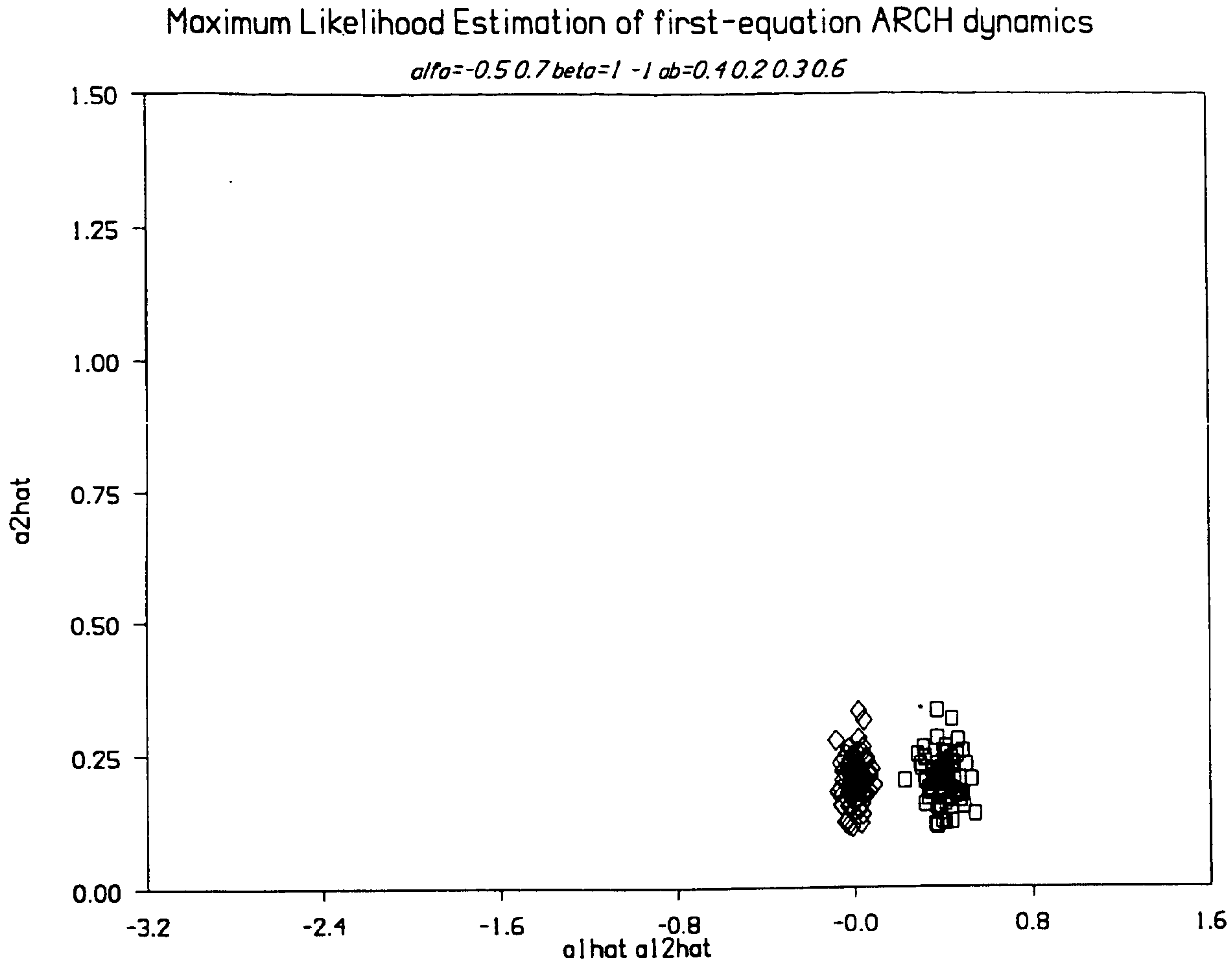

FIGURE 4b

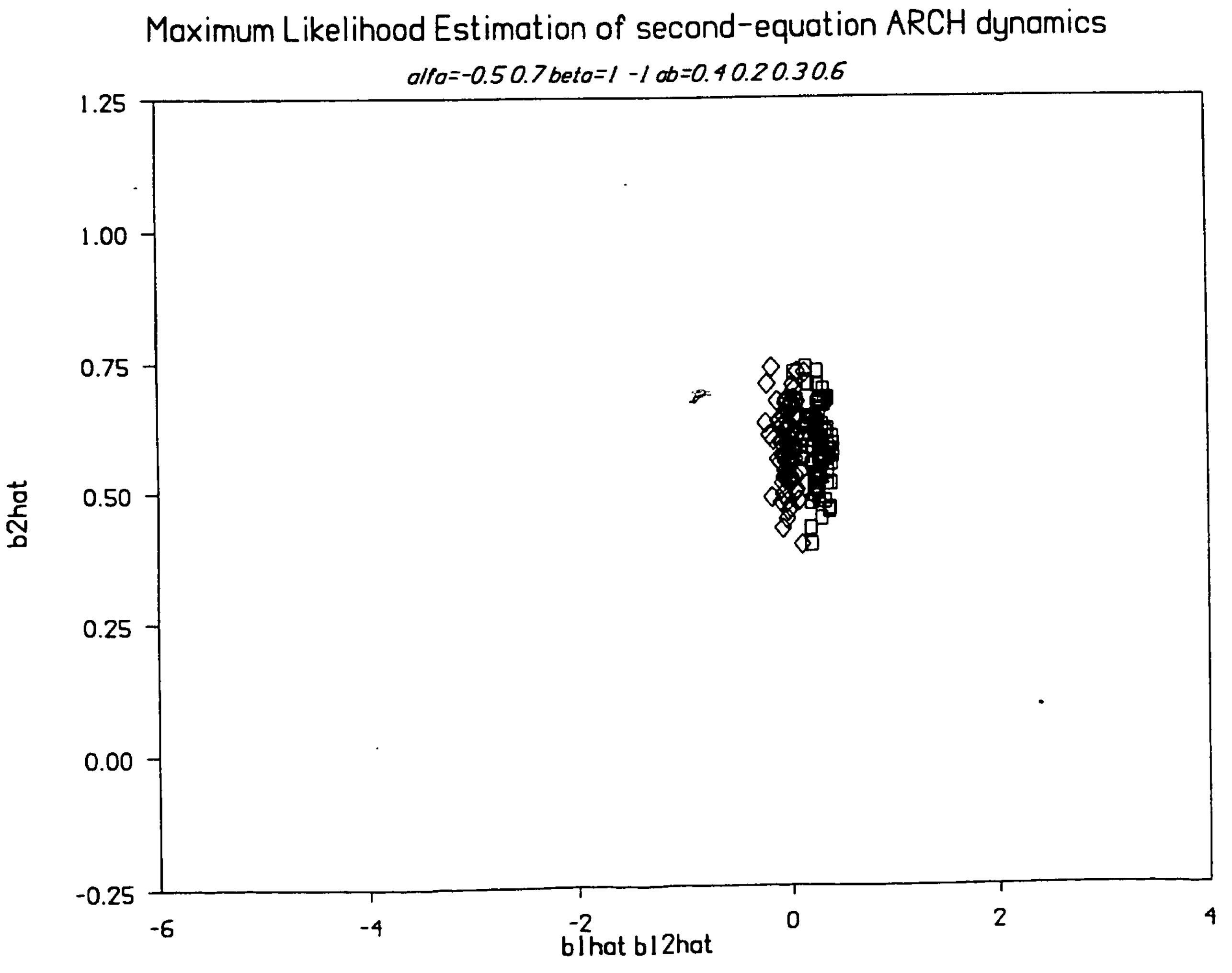

FIGURE 4c

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