Constant Interest Rate Projections without the Curse of Indeterminacy: A Note [*]

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Abstract

Constant interest rate (CIR) projections are often criticized on the grounds that they are inconsistent with the existence of a unique equilibrium in a variety of forward-looking models. This note shows how to construct CIR projections that are not subject to that criticism, using a standard New Keynesian model as a reference framework.

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1 Introduction

Constant interest rate (CIR) projections (i.e. forecasts conditional on the assumption of an unchanged interest rate) have been for some time one of the tools used by inflation targeting central banks to base their decisions and to explain the latter to the public.[1] Such projections, however, have been criticized on the grounds that they are based on an assumption about future policy that is inconsistent with the existence of unique equilibrium in a variety of forward-looking models, including the new vintage of optimizing models commonly used for monetary policy analysis.[2]

An approach that has been proposed in the literature to overcome that problem consists of assuming an interest rate rule that guarantees a unique equilibrium while assuming the presence of "exogenous" shocks to the rule taking the values needed to keep the interest rate constant during the period considered. A potential problem with that rule is that the shocks the size and pattern of the shocks required to keep the interest rate constant may be implausible or hard to reconcile with the assumption of rational expectations.[3]

The purpose of the present note is to illustrate, in the context of a canonical version of the New Keynesian model, how CIR projections can be computed, in a way consistent with the model, while guaranteeing a unique equilibrium and without the need to rely on exogenous policy shocks. In particular, we show how a version of a Taylor interest rate rule that has the central bank respond to both inflation and output growth with coefficients that satisfy a certain proportionality condition can guarantee a unique equilibrium characterized by a constant nominal rate, and can thus be the basis of CIR projections.

The note is organized as follows. Section 2 revisits the basic problem of indeterminacy under an interest rate peg and discusses its consequences for CIR projections. Section 3 presents and analyzes a rule that overcomes

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[1]Thus, for instance, CIR projections constituted the base-case projections presented in the Inflation Reports of the Bank of England until August 2004 and, until very recently, those of of the Sveriges Riksbank as well.

[2]See, e.g., Honkapohja and Mitra (2005) and Woodford (2005) for a discussion of the problem of indeterminacy under an interest rate peg in the context of the New Keynesian model. An early statement of that problem in the context of a simple ad-hoc macro model can be found in Sargent and Wallace (1975).

[3]See, e.g. Leeper and Zha (2003) and Waggoner and Zha (1999) for a discussion of that approach and its shortcomings.

the indeterminacy problem. Section 4 offers some caveats and concluding remarks.

2 Constant Interest Rate Projections and the Curse of Indeterminacy

In this section we describe the nature of the problem associated with constant interest rate policies and, as a result, with the projections based on those policies. We start by briefly reviewing the canonical New Keynesian model that will serve us as a reference framework.

2.1 A Baseline Model

The model is a baseline version of the New Keynesian framework used in much recent work in monetary economics. The non-policy block of the model is made up of the following two equations:

$$y_t = E_t\{y_{t+1}\} - \frac{1}{\sigma}\left(i_t - E_t\{\pi_{t+1}\} - \rho\right) \tag{1}$$

$$\pi_t = \beta \ E_t\{\pi_{t+1}\} + \kappa \ x_t \tag{2}$$

where y_t denotes (log) output, i_t is the short term nominal interest rate, $\pi_t \equiv p_t - p_{t-1}$ is the rate of inflation between $t-1$ and t (with p_t denoting the log of the price level), and $x_t \equiv y_t - y_t^n$ is the output gap (i.e. the log deviation of output from its natural level y_t^n). Equation (1) can be obtained by log-linearizing the representative household's Euler equation, and imposing a market clearing condition requiring that consumption equals output at all times. Equation (2) is a version of the so-called New Keynesian Phillips curve, which can be derived by aggregating the price-setting decisions of monopolistically competitive firms subject to Calvo-type constraints on the frequency of price adjustment, combined with standard assumptions on technology and labor markets. In that context, parameter σ corresponds to the coefficient of relative risk aversion, β is the household's discount factor (with $\rho \equiv -\log \beta$ being the corresponding discount rate), and κ is a coefficient which is inversely related to the degree of price rigidities.[4]

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[4]The reader is referred to King and Wolman (1996), Woodford (2003) or Galí (2008) for a detailed description of the model and a derivation of its key equations.

We can rewrite (1) in terms of the output gap, as follows:

$$x_t = E_t\{x_{t+1}\} - \frac{1}{\sigma}\left(i_t - E_t\{\pi_{t+1}\} - \rho\right) + E_t\{\Delta y_{t+1}^n\} \tag{3}$$

Henceforth we assume that (log) natural output follows a stationary AR(1) process in first differences, i.e.

$$\Delta y_t^n = \rho_y \ \Delta y_{t-1}^n + \varepsilon_t$$

where $\rho_y \in [0,1)$ and $\{\varepsilon_t\}$ is a white noise process. The evolution of y_t^n summarizes the impact of a variety of exogenous real forces impinging on the economy.

2.2   Constant Interest Rate Policies and Equilibrium Indeterminacy

Given a path for the nominal interest rate, any non-explosive solution to the system of difference equations (2) and (3) constitutes an equilibrium of our model economy. Thus, one would think that a natural approach to the construction of CIR projections at any finite horizon, as of period t (which is taken to be the period when the projection is made), would require solving for $\{x_{t+k}, \ \pi_{t+k}\}$ given a hypothetical constant interest rate path $i_{t+k} = i$ chosen by the monetary authority, given the sequence of forecasts for natural output growth $E_t\{\Delta y_{t+k}^n\} = \rho_y^k \ \Delta y_t^n$, all for $k = 0, 1, 2, ...$

The difficulty with that approach has to do with the fact that if we set the nominal interest rate equal to a constant, the system of difference equations (2)-(3) has a multiplicity of solutions.[5] To see this note that if we substitute the "rule" $i_{t+k} = i$ into (3), we can write the equilibrium conditions from period t onward as a system

$$\begin{bmatrix} \widehat{\pi}_{t+k} \\ \widehat{x}_{t+k} \end{bmatrix} = \mathbf{A}_0 \begin{bmatrix} E_{t+k}\{\widehat{\pi}_{t+k+1}\} \\ E_{t+k}\{\widehat{x}_{t+k+1}\} \end{bmatrix} + \mathbf{B}_0 \ \Delta y_{t+k}^n \tag{4}$$

for $k = 0, 1, 2, ...$where

$$\mathbf{A}_0 \equiv \begin{bmatrix} \beta + \frac{\kappa}{\sigma} & \kappa \\ \frac{1}{\sigma} & 1 \end{bmatrix} \quad ; \quad \mathbf{B}_0 \equiv \begin{bmatrix} \kappa\rho_y \\ \rho_y \end{bmatrix}$$

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[5]This will also be the case if the nominal interest rate varies as a function of exogenous shocks only or if it responds too weakly to some endogenous variables.

and where $\widehat{\pi}_{t+k} \equiv \pi_{t+k} - (i_t - \rho)$ and $\widehat{x}_{t+k} = x_{t+k} - \kappa^{-1}(1-\beta)(i-\rho)$ represent the deviations of inflation and the output gap from their steady state values consistent with a constant nominal rate i.

Iterating forward on (4), we see that the CIR projections for the output gap and inflation at a horizon k are given, as of time t, by

$$\left[\begin{array}{c} E_t\{\widehat{\pi}_{t+k}\} \\ E_t\{\widehat{x}_{t+k}\} \end{array} \right] = \mathbf{A}_0^{-k} \left(\left[\begin{array}{c} \widehat{\pi}_t \\ \widehat{x}_t \end{array} \right] - \left[\mathbf{I} + \rho_y\mathbf{A}_0 + (\rho_y\mathbf{A}_0)^2 + ...(\rho_y\mathbf{A}_0)^{k-1} \right] \ \mathbf{B}_0 \ \Delta y_t^n \right) \tag{5}$$

Thus, in order for the CIR projections at any $k-$horizon to be well defined, the current values of $\widehat{x}_t$ and $\widehat{\pi}_t$ must be determined uniquely. But that requires that the solution to (4) exist and be unique, for which a necessary and sufficient condition is that the two eigenvalues of $\mathbf{A}_0$ lie within the unit circle, given that both $\widehat{x}_{t+k}$ and $\widehat{\pi}_{t+k}$ are non-predetermined variables (see, e.g. Blanchard and Kahn (1980)).[6] The following lemma shows that the previous condition is not satisfied.

Lemma 1 *Let λ_1 and λ_2 denote the eigenvalues of $\mathbf{A}_0$, where $\lambda_1 \leq \lambda_2$. Both λ_1 and λ_2 are real and satisfy $0 < \lambda_1 < 1 < \lambda_2$.*

Proof: see appendix.

It follows from Lemma 1 that a multiplicity of non-explosive solutions to (4) exist, implying that the equilibrium values $\widehat{x}_t$ and $\widehat{\pi}_t$ are not uniquely determined. Given (5), the CIR projections $E_t\{\widehat{x}_{t+k}\}$ and $E_t\{\widehat{\pi}_{t+k}\}$ are not uniquely determined either.

3 Constant Interest Rate Projections without the Curse of Indeterminacy

Consider next an interest rate rule of the form

$$\widehat{i}_{t+k} = \phi \ (\widehat{\pi}_{t+k} + \sigma \ \Delta y_{t+k}) \tag{6}$$

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[6]Throughout the present paper the analysis is restricted to equilibria in a neighborhood of the steady state being considered. See Benhabib, Schmitt-Grohé and Uribe (2001) for a critical view of that approach, and an example of global multiplicity of equilibria coexisting with local uniqueness.

for $k = 0, 1, 2, ...$where $\widehat{i}_{t+k} \equiv i_{t+k} - i$ and, as before, $\widehat{\pi}_{t+k} \equiv \pi_{t+k} - (i - \rho)$. We assume ϕ is a constant coefficient satisfying $\phi > 1$.[7] Combining (6) with (1) we can derive the difference equation:

$$\widehat{i}_{t+k} = \frac{1}{\phi} \ E_{t+k}\{\widehat{i}_{t+k+1}\} \tag{7}$$

Under our assumption that $\phi > 1$ the only non-explosive solution to (7) is $\widehat{i}_{t+k} = 0$ for $k = 0, 1, 2, ...$Equivalently, by following rule (6) from period t onward the central bank supports a constant interest rate policy

$$i_{t+k} = i$$

for $k = 0, 1, 2, ...$for any interest rate $i \geq 0$ of its choice.

The equilibrium dynamics under (6) from period t onward are described by (2), which we rewrite here in terms deviations from means as

$$\widehat{\pi}_{t+k} = \beta \ E_{t+k}\{\widehat{\pi}_{t+k+1}\} + \kappa \ \widehat{x}_{t+k} \tag{8}$$

together with

$$\widehat{x}_{t+k} = E_{t+k}\{\widehat{x}_{t+k+1}\} - \frac{1}{\sigma} \ [\phi \ (\widehat{\pi}_{t+k} + \sigma \ \Delta\widehat{x}_{t+k}) - E_{t+k}\{\widehat{\pi}_{t+k+1}\}] - (\phi - \rho_y) \ \Delta y_t^n \tag{9}$$

The previous equilibrium conditions can be obtained by combining (3) with (6), together with the definition of the output gap and our assumption on the process followed by y_t^n. Equivalently, and more compactly, we have

$$\begin{bmatrix} \widehat{\pi}_{t+k} \\ \widehat{x}_{t+k} \\ \widehat{x}_{t+k-1} \end{bmatrix} = \mathbf{A} \begin{bmatrix} E_{t+k}\{\widehat{\pi}_{t+k+1}\} \\ E_{t+k}\{\widehat{x}_{t+k+1}\} \\ \widehat{x}_{t+k} \end{bmatrix} + \mathbf{B} \ \Delta y_{t+k}^n \tag{10}$$

for $k = 0, 1, 2, ..$ and where

$$\mathbf{A} \equiv \begin{bmatrix} \beta & 0 & \kappa \\ 0 & 0 & 1 \\ \frac{\beta\phi - 1}{\sigma\phi} & -\frac{1}{\phi} & 1 + \frac{1}{\phi} + \frac{\kappa}{\sigma} \end{bmatrix} \quad ; \quad \mathbf{B} \equiv \begin{bmatrix} 0 \\ 0 \\ 1 - \frac{\rho_y}{\phi} \end{bmatrix}$$

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[7]See Galí (2003) for a comparison of the welfare implications of such a rule relative to other simple rules.

Given that the system (10) involves two non-predetermined and one predetermined variables, it has a unique non-explosive solution if and only if two eigenvalues of $\mathbf{A}$ lie inside, and one outside, the unit circle. The following proposition establishes a necessary and sufficient condition for that property to hold.

Proposition 2 *A necessary an sufficient condition for (10) to have a unique non-explosive solution is given by* $\phi > 1$

Proof. The characteristic polynomial of $\mathbf{A}$ is given by

$$
\begin{aligned}
p_A(z) &= z^3 - \left(1 + \beta + \frac{\kappa}{\sigma} + \frac{1}{\phi}\right) z^2 + \left(\frac{1}{\phi}\left(1 + \beta + \frac{\kappa}{\sigma}\right) + \beta\right) z - \frac{\beta}{\phi} \\
&= (z - \lambda_1)\ (z - \lambda_2)\ \left(z - \phi^{-1}\right)
\end{aligned}
$$

where $\lambda_1 \equiv \frac{1+\beta+\frac{\kappa}{\sigma} - \sqrt{(1+\beta+\frac{\kappa}{\sigma})^2 - 4\beta}}{2}$ and $\lambda_2 \equiv \frac{1+\beta+\frac{\kappa}{\sigma} + \sqrt{(1+\beta+\frac{\kappa}{\sigma})^2 - 4\beta}}{2}$. Using the same logic as in the proof of Lemma 1 we conclude that both λ_1 and λ_2 are real, and satisfy the inequality $0 < \lambda_1 < 1 < \lambda_3$. Thus, as long as $\phi > 1$, two eigenvalues of $\mathbf{A}$ lie inside, and one outside, the unit circle. *QED.*

Next we show how CIR projections can be computed. We note first that under the CIR rule we must have $\widehat{\pi}_{t+k} + \sigma\ \Delta y_{t+k} = 0$ or, equivalently,

$$
\widehat{\pi}_{t+k} = -\sigma\ (\Delta\widehat{x}_{t+k} + \Delta y^n_{t+k}) \tag{11}
$$

for $k = 0, 1, 2, \dots$ [8]

Substituting the previous expression into (8) we obtain, after some algebraic manipulation, the second order difference equation

$$
\widehat{x}_{t+k} = a\ \widehat{x}_{t+k-1} + a\beta\ E_{t+k}\{\widehat{x}_{t+k+1}\} - a(1 - \beta\rho_y)\ \Delta y^n_{t+k}
$$

for $k = 0, 1, 2, \dots$ where $a \equiv \frac{\sigma}{\sigma(1+\beta)+\kappa} \in \left(0, \frac{1}{1+\beta}\right)$. The corresponding stationary solution is given by

$$
\widehat{x}_{t+k} = \delta\ \widehat{x}_{t+k-1} - \gamma\ \Delta y^n_{t+k} \tag{12}
$$

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[8]Note that in the case of logarithmic consumption utility ($\sigma = 1$), the rule considered here implies full stabilization of nominal income growth $\pi_{t+k} + \Delta y_{t+k}$.

where $\delta \equiv \frac{1-\sqrt{1-4a^2\beta}}{2a\beta} \in (0,1)$ and $\gamma = \frac{\delta(1-\beta\rho_y)}{1-\delta\beta\rho_y} \in (0,1)$.

Combining (11) and (12) we can obtain the stationary solution for inflation under the unchanged interest rate policy

$$\widehat{\pi}_{t+k} = \sigma(1-\delta)\,\widehat{x}_{t+k-1} - \sigma(1-\gamma)\,\Delta y^n_{t+k}$$

More compactly,

$$\left[\begin{array}{c} \widehat{\pi}_{t+k} \\ \widehat{x}_{t+k} \end{array}\right] = \mathbf{C}\left[\begin{array}{c} \widehat{\pi}_{t+k-1} \\ \widehat{x}_{t+k-1} \end{array}\right] - \mathbf{D}\,\Delta y^n_{t+k} \tag{13}$$

for $k = 0, 1, 2, ...$where

$$\mathbf{C} \equiv \left[\begin{array}{cc} 0 & \sigma(1-\delta) \\ 0 & \delta \end{array}\right] \quad ; \quad \mathbf{D} \equiv \left[\begin{array}{c} \sigma(1-\gamma) \\ \gamma \end{array}\right]$$

Note that the equilibrium dynamics of inflation and the output gap are independent of the particular choice of coefficient ϕ by the monetary authority, as long as $\phi > 1$.

Let $E_t\{\pi_{t+k}|i\}$ and $E_t\{x_{t+k}|i\}$ denote the CIR projection for inflation and the output gap, at a k-period horizon, given the constant interest rate $i \geq 0$. Iterating over (13) we obtain

$$\left[\begin{array}{c} E_t\{\pi_{t+k}|i\} \\ E_t\{x_{t+k}|i\} \end{array}\right] = \left(\mathbf{I}-\mathbf{C}^{k+1}\right)\left[\begin{array}{c} \pi \\ x \end{array}\right] + \mathbf{C}^{k+1}\left[\begin{array}{c} \pi_{t-1} \\ x_{t-1} \end{array}\right] - \left(\mathbf{I}\rho_y^k + \mathbf{C}\rho_y^{k-1} + ... + \mathbf{C}^k\right)\mathbf{D}\,\Delta y^n_t$$

and where $\pi \equiv i - \rho$ and $x \equiv \kappa^{-1}(1-\beta)(i-\rho)$ are the steady state values of inflation and the output gap consistent with a constant interest i. The previous expression makes it clear that the CIR projections are well defined and unique, for any horizon $k \geq 0$.

More generally, one can show that the equilibrium of the basic New Keynesian model considered here will be determinate if the central bank follows an interest rate rule of the form

$$\widehat{i}_{t+k} = \phi\,\widehat{\pi}_{t+k} + \gamma\,\Delta y_{t+k} \tag{14}$$

as long as $\phi > 1$, i.e. as long as the central bank adjusts the interest rate more than one-for-one in response to changes in inflation, thus satisfying the so-called Taylor principle. Rule (6) is a particular case of (14) corresponding

to $\gamma = \phi\sigma$, and for which the implied nominal rate remains constant *in equilibrium,* as shown above. One should emphasize, however, that it is important for the central bank to make its policy rule transparent, for it is the public's perception that the central bank will respond to inflation fluctuations that makes the equilibrium determinate.

4   Summary and Caveats

The present paper has shown, in the context of a baseline New Keynesian model, the possibility of pursuing a monetary policy characterized by a constant nominal interest rate, without that policy leading to a multiplicity of equilibria. The existence of a unique equilibrium associated with any given constant nominal rate makes it possible to construct forecasts of any variable conditional on a given choice of that rate, i.e. constant interest rate (CIR) projections. Thus, our analysis provides a counterexample to concerns often found in the literature about the feasibility of those projections. Those concerns are expressed on the grounds that in a variety of forward-looking models (including the one used here) constant interest rate policies are associated with an indeterminate equilibrium and, hence, with the lack of a well defined projection of any macroeconomic variable conditional on a given constant level of the interest rate.

Of course, solving the problem of indeterminacy does not automatically make CIR projections desirable as a tool to help inform policy decisions. A discussion of their potential usefulness, however, falls beyond the scope of the present paper, which has focused on the narrower issue of whether they are well defined at all. But there is one feature that may be worth pointing out, and that would seem to make those CIR projections unappealing at least to an inflation targeting central bank: in most monetary models there is generally only one level of the nominal interest rate that is consistent with a given long-run inflation target, so it is not clear why the central bank would want to consider the implications of policies involving a constant interest rate path at a level different from the one consistent with the desired target for inflation. A possible response to that concern is that in practice a central bank would never intend to keep the nominal rate constant *forever* (at whichever level is being considered), but only for a finite amount of time (e.g. until the reference horizon for the projections), and then revert back to some rule that would be consistent with the desired inflation target.

But it should be clear that, in a forward-looking model, neither current nor projected outcomes will be invariant to the specific rule to which the central bank is assumed to revert at some point in the future, even if the latter is not expected to be in place until after the horizon date for the projections. Hence, the specification of a constant interest rate path would have to be supplemented with a description of the rule that will eventually replace it and a specification of the timing of that replacement. Needless to say, the introduction of policy switches of that kind could complicate the analysis considerably, but it would lead to CIR projections consistent with an unchanged long-run inflation target.

Appendix

Proof of Lemma

From the properties of matrices and their eigenvalues, we have $tr(\mathbf{A}_0) = 1 + \beta + \frac{\kappa}{\sigma} = \lambda_1 + \lambda_2$ and $\det(\mathbf{A}_0) = \beta = \lambda_1\lambda_2$. Thus we have $\lambda_1 = \frac{1+\beta+\frac{\kappa}{\sigma}-\sqrt{(1+\beta+\frac{\kappa}{\sigma})^2-4\beta}}{2}$ and $\lambda_2 = \frac{1+\beta+\frac{\kappa}{\sigma}+\sqrt{(1+\beta+\frac{\kappa}{\sigma})^2-4\beta}}{2}$. Note that

$$
\begin{aligned}
\Delta &\equiv (1 + \beta + \frac{\kappa}{\sigma})^2 - 4\beta \\
&> (1 + \beta)^2 - 4\beta \\
&= (1 - \beta)^2
\end{aligned}
\tag{15}
$$

from which it follows that both eigenvalues are real. Also, (15) implies $\lambda_2 > \frac{1+\beta+\frac{\kappa}{\sigma}+\sqrt{(1-\beta)^2}}{2} > 1$.

Note also that $\sqrt{(1 + \beta + \frac{\kappa}{\sigma})^2 - 4\beta} < 1 + \beta + \frac{\kappa}{\sigma}$, from which we have $\lambda_1 > 0$. Finally, $\lambda_1 < 1$ follows from the fact that $\lim_{\frac{\kappa}{\sigma}\to 0} \lambda_1 = \beta < 1$, and $\frac{\partial \lambda_1}{\partial(\kappa/\sigma)} < 0$. QED.

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